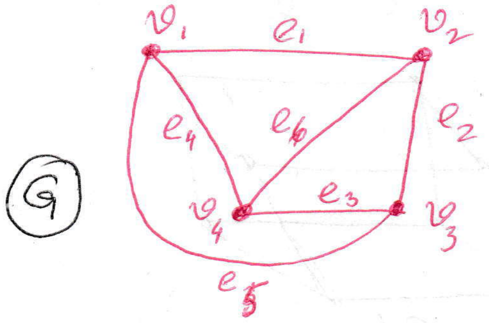
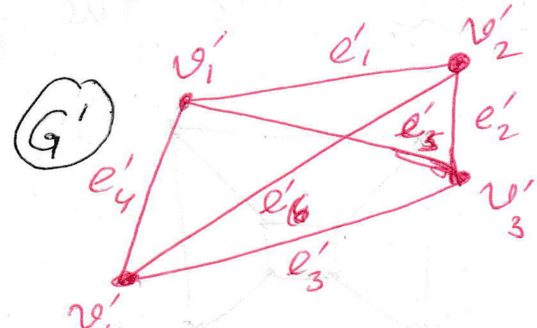
Planar and dual graphs

A graph is called a planar graph (or plane graph) if it can be drawn on a plane in such a way that any two of its edges either meet only at their end vertices or do not meet at all. So no two of its edges cross-over. A graph which is isomorphic to a plane graph is called a planar graph.





In the above figure, G is a planar graph as no two edges of G cross over. However, G' is also a planar graph as it is isomorphic to graph G.

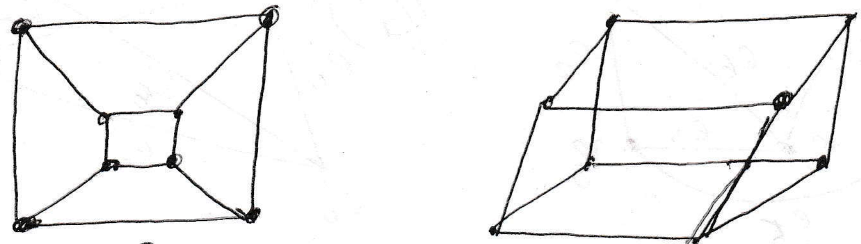
Two graphs are thought as equivalent and called isomorphic if they have identical behaviour in terms of graph-theoretic properties. More precisely, two graphs G and G' are said to be isomorphic to each other if there is one-to-one correspondence between their vertices and between their edges such that the incidence relationship is preserved. For example if edge e_i is incident on vertices v_1 and v_2 in graph G, then there is a corresponding edge e_i' in G' which is incident on the vertices v_1' and v_2' corresponding to v_1 and v_2 respectively.

To verify that two graphs G and G′ are isomorphic, the following correspondence between the two graphs have to be verified:

(i) The vertices v_1, v_2, v_3 and v_4 of G correspond to vertices v_1', v_2', v_3' and v_4' of G′

(ii) The edges $e_1, e_2, e_3, e_4, e_5, e_6$ of G correspond to edges $e_1', e_2', e_3', e_4', e_5', e_6'$ of G′.

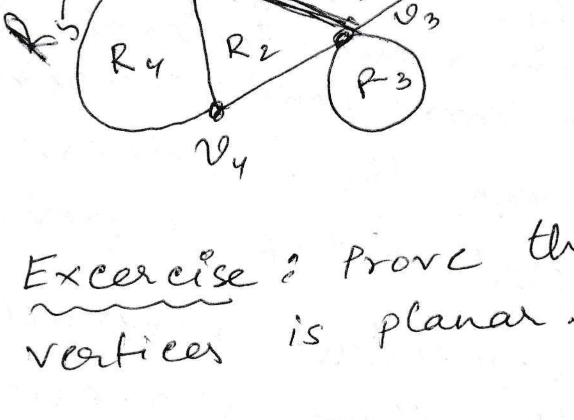
Question : Are the following graphs isomorphic?



G G′

A graph which is not planar is called a non-planar graph.

A planar graph divides a plane into multiple regions. In the graph below, the planar graph has divided the plane into multiple regions, namely R_1, R_2, R_3 and R_4.

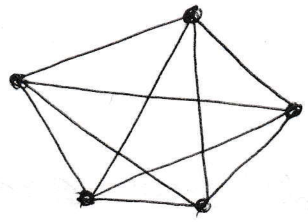


Excercise : Prove that a complete graph of 4 vertices is planar.

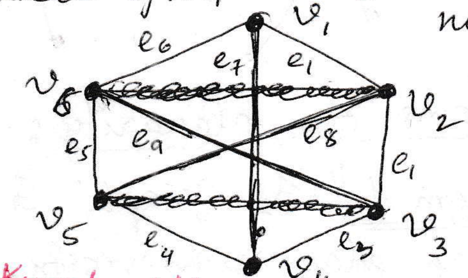
Kuratowski's graph:

First graph: A complete graph of 5 vertices or K_5. It is a non-planar graph with smallest number of vertices.

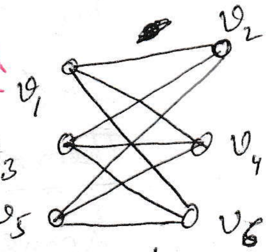
Second graph: A complete bipartite graph of 6 vertices or $K_{3,3}$. It is a non-planar graph with smallest number of edges.



Kuratowski's First Graph



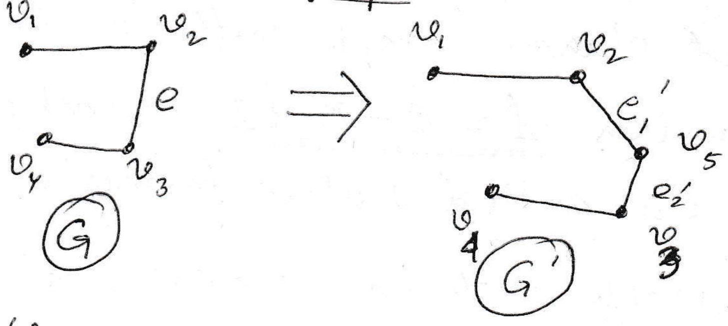
Kuratowski's Second Graph



Excercise for students:

Prove that Kuratowski's first graph is non-planar.

Homeomorphic graph:



If the edge of a graph is deleted and replaced by one new vertex and two new edges, this replacement operation is termed as edge subdivision. In the above graphs G and G', edge e in G is replaced by new vertex v_5 and new edges e_1' and e_2'. This is an example of edge subdivision.

A graph obtained from another graph by one or more such edge subdivision is called homeomorph. In the above example, G' is a homeomorph of G, or in other words G and G' are called homeomorphic graphs.

Theoreme on planarity:

Theorem 1// If graph G' is obtained from graph G by an edge subdivision, then graph G is planar if and only if G' is planar.

Theorem 2// A graph is planar if and only if every homeomorph of it is planar.

Theorem 3// Every subgraph of a planar graph is planar.

* Theorem 4// A connected planar graph with 'n' vertices and 'e' edges determines $f = e - n + 2$ number of regions. This is also called Euler's formula.

* Theorem 5// A planar graph with 'n' vertices, 'e' edges and 'k' connected components determines $f = e - n + k + 1$ number of regions.

This is generalized form of Euler's formula.

* Theorem 6// In a simple connected planar graph G with 'n' vertices, 'e' edges, 'f' regions. Then

(a) $e \geq \frac{3}{2} f$ (b) $e \leq 3n - 6$.

Theorem 7// All drawings of a connected planar graph having vertex connectivity 3 are same.

Theorem 8// A complete graph of 5 vertices is non-planar.

Theorem 9/ K_6 is non-planar.

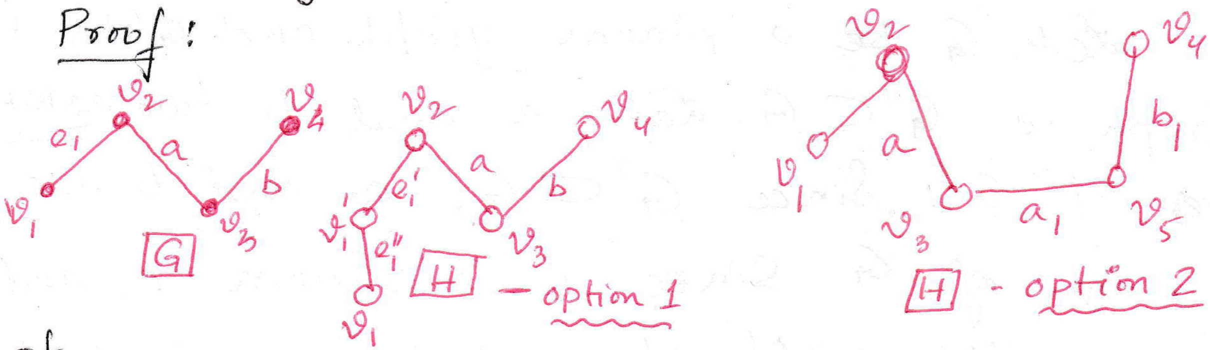
Theorem 10/ Both K_5 and K_6 are regular.

Theorem 11// Removal of one vertex or edge makes both K_5 and K_6 planar.

Proof of theorems on planarity –

Theorem 1: If a graph H is obtained from the graph G by an edge subdivision, then G is planar if and only if H is planar.

Proof:



Proof: Let H be planar. Let a, and b be two arbitrary edges of G. If neither of these edges are subdivided, then a and b are also edges of graph H. Also, either they will meet at vertices or will not meet at all. If any of these two edges, say b, is sub-divided into two edges a_1 and b_1 (say) in H, then each of these pairs a, a_1 and a, b_1 either meet at end vertexes or do not meet at all. Hence in G a and b also either meet at end vertex or does not meet at all. Hence G is planar.

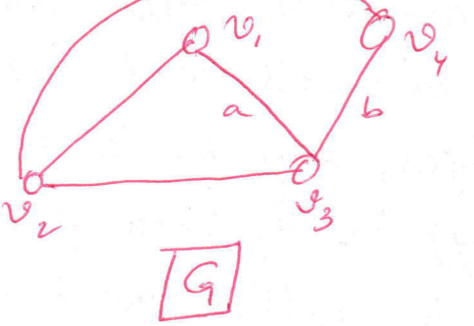
Theorem 2: A graph is planar if and only if every homeomorph of it is planar.

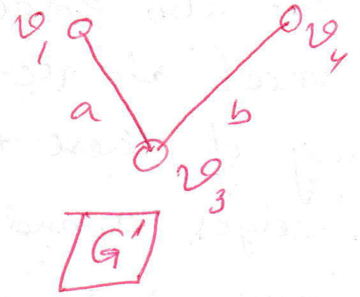
Proof: Let G be a graph and H be its homeomorph. We can get H from G by a finite sequence of edge subdivisions (since they are homeomorphic), $G \to G_1 \to G_2 \to \cdots \to G_n \to H$. Thus G_1 is obtained from G by an edge subdivision. So by Theorem 1, G_0 is planar if and only if G_1 is planar. Same way G_1 is planar if and only if G_2 is planar.

This goes on for all other subsequent homeomorphic graphs namely $G_2, G_3, \ldots, G_n, H$. Hence proved.

Theorem 3| Every subgraph of a planar graph is a planar.

Proof: Let G be a planar graph and G' be its subgraph i.e. $G' \subset G$. Let a and b two arbitrary edges of G'. Since $G' \subset G$, a and b are also edges of G. Since G is planar, a and b can either meet at a vertex or do not meet at all. Hence, G' is a planar graph.

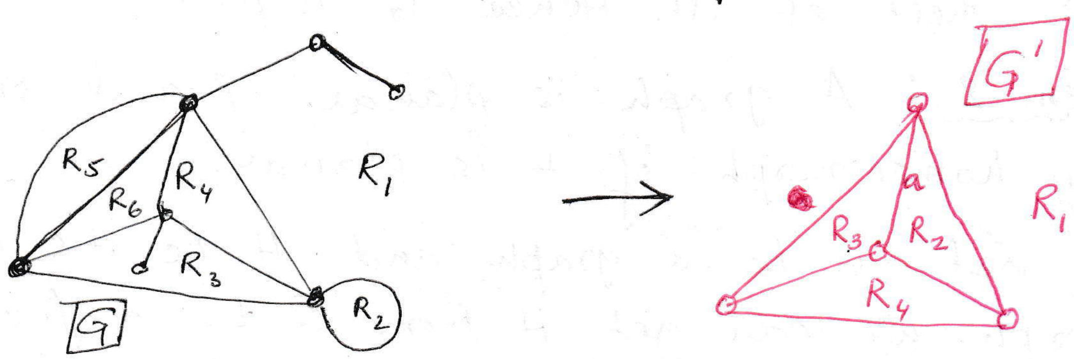




Euler's Formula

Theorem 4| A connected planar graph with 'n' vertices and 'e' edges has $f = e - n + 2$, regions.

Proof:



Assumption 1 :→ We can assume the graph as a simple graph without any self-loop or parallel edge. If the proof applies for such a simple graph, it will apply it general for any planar graph. Reason is self-loop or parallel edge adds a region to the graph but at the same time the number of edges also gets increased by 1.

Assumption 2 :- We can also disregard or remove all edges which do not form boundaries of any region. Removal of such edge decreases e by one but also decreases n by 1. Hence, $(e-n)$ remains constant.

So, instead of graph G, we can prove the theorem with graph G' which is a simple graph.

Now, if $f=1$, i.e. G has only one region, there cannot be any circuit and graph G has to be a tree. In case of a tree, $e = n-1$.

$$\therefore \quad n - e + f = n - (n-1) + 1 = 2$$

$$\text{or, } 2 = 2$$

$\therefore$ The theorem holds for $f=1$.

For $f>1$, G cannot be a tree. There will be at least one circuit. Let 'a' be an edge of a circuit. Then $(G-a)$ is still connected. Since G is planar, so $(G-a)$ is also planar as $(G-a) \subset G$ [by theorem 3]. Due to removal of edge 'a', two regions of G (namely R_2 and R_3) will be combined into one. So the number of regions of graph $(G-a)$ will be $(f-1)$ and the number of edges will be $(e-1)$.

$\therefore$ We get for graph $(G-a)$, $n-(e-1)+(f-1)=2$, or, $n-e+f=2$ (assuming for graph G, $n-e+f=2$)

So, by method of induction, we can say that the relation $n - e + f = 2$ is valid for all simple planar graphs or for any planar graph.

$\therefore$ $\boxed{f = e - n + 2}$ holds for any planar graph.

Corollary / In any simple, connected planar graph with f regions, n vertices and e edges $(e > 2)$,

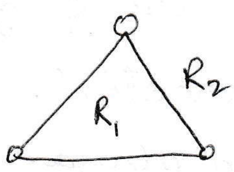
(i) $e \geq \frac{3}{2} f$ (ii) $e \leq 3n - 6$.

Proof: Since each region is bound by at least 3 edges, and each edge belongs to exactly 2 regions,

$$2e \geq 3f$$

or, $e \geq \frac{3}{2} f$ ((i) proved)

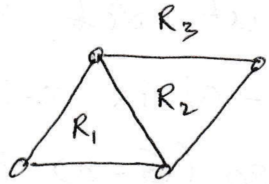
Example :



for $f = 2$

$e = 3$

$\therefore$ $2e = 3f$

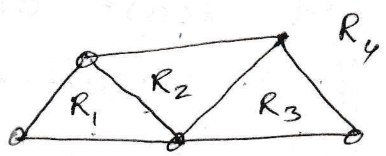


For $f = 3$

$e = 5$

$2e = 10, 3f = 9$

$\therefore$ $2e > 3f$



For $f = 4$

$e = 7$

$2e = 14, 3f = 12$

$\therefore$ $2e > 3f$

If we substitute f from Euler's inequality (i.e. $f = e - n + 2$)

$e \geq \frac{3}{2}(e - n + 2)$

or, $2e \geq 3e - 3n + 6$

or, $e \leq 3n - 6$ (proved)

The inequality ($e \leq 3n-6$) is often useful in finding out whether a graph is planar. For example, in case of Kuratowski's First graph K_5, $n=5$, $e=10$

$3n-6 = 3.5-6 = 9 < e$. So the graph violates the inequality. Hence, it is non-planar.

However, this inequality is a necessary condition but <u>not</u> a sufficient condition for deciding the planarity of a graph. This means, every graph to be planar must be satisfying this inequality. But satisfying the inequality doesn't ensure the fact that it is planar. A case in point is Kuratowski's Second graph, $K_{3,3}$.

For $K_{3,3}$, $n=6$, $e=9$

$\therefore 3n-6 = 3.6-6 = 12 > e$.

But still, even after satisfying the inequality, $K_{3,3}$ in non-planar.

To prove the non-planarity of $K_{3,3}$, we can make use of the additional fact that no region in this graph is bounded by fewer than 4 edges. Hence, if this graph is planar, we would have $2e \geq 4f$

or, $2e \geq 4(e-n+2)$

or, $2.9 \geq 4(9-6+2)$

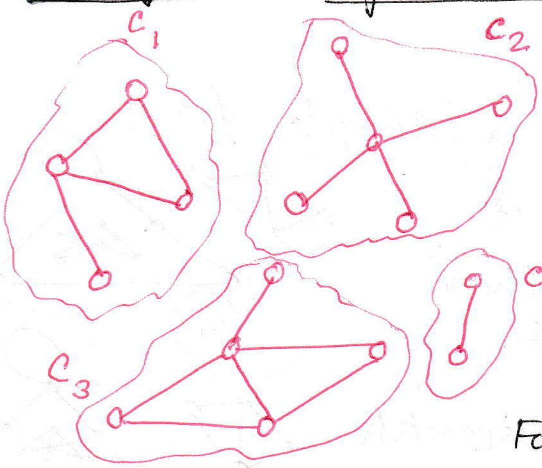
or, $18 \geq 20$, which is a contradiction.

Hence, $K_{3,3}$ is not planar.

Theorem-5 (Generalized Euler's Formula)

A planar graph with 'n' vertices, 'e' number of edges and 'k' number of connected components has $f = e - n + k + 1$ number of regions.

Definition of connected components:



In graph theory, a connected component is a subgraph in which any two vertices are connected to each other by paths but not connected to any vertex in the supergraph. For example, in the adjacent diagram, there are 4 connected components.

Proof Let the graph G have k connected components, $G_1, G_2, \ldots, G_k$. Let C_i represent the i-th component which has n_i vertices, e_i edges and f_i regions. Then by Euler's theorem for a connected plane graph,

$$f_i = e_i - n_i + 2$$

Since exterior region of all components are commone, therefore number of regions without exterior component is $(f_i - 1)$

$\therefore$ Interior regions for all components in G

$$= \sum_{i=1}^{k} (f_i - 1) = \sum_{i=1}^{k} f_i - \sum_{i=1}^{k} 1$$

$$= \sum_{i=1}^{k} (e_i - n_i + 2) - k$$

$$= \sum_{i=1}^{k} e_i - \sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 2 - k$$

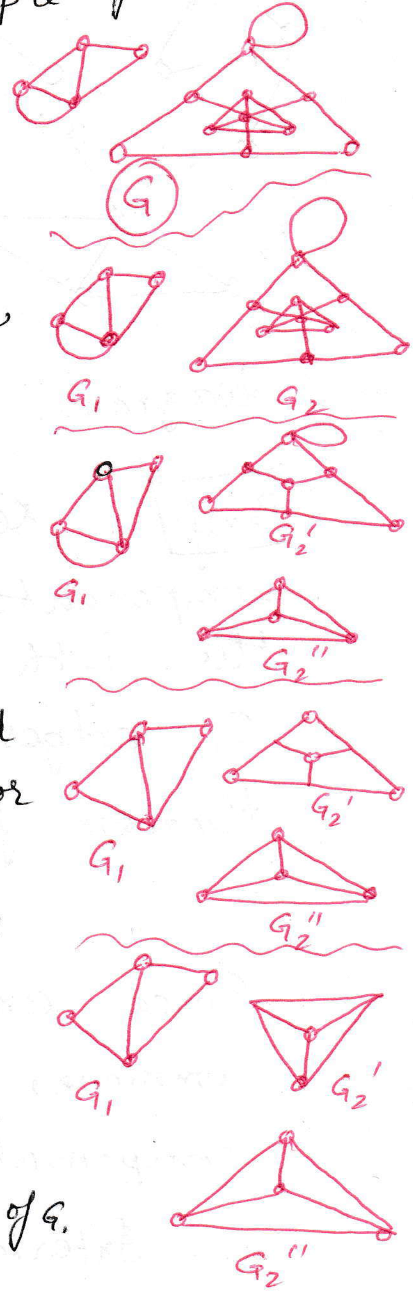
$$= e - n + 2k - k$$

$$= e - n + k$$

Including one exterior region, total number of region of G, ie. $\boxed{f = e - n + k + 1}$ (proved).

Detection of planarity:

Given a graph G, reduce it to a simple form through the following steps. Then it becomes easy to determine whether the graph is planar or not.



Step 1/ If a graph has several components, consider one component at a time.

Step 2/ If any component is separable, it would have several blocks. Consider one block at a time. For example, graph G_2 has got a cut vertex. So it is separable into two blocks.

Step 3/ Remove all self-loops and parallel edges from the individual components or blocks of the graph.

Step 4/ If any of the components or block of G have 2 edges having exactly one vertex in common and the vertex is of degree 2, then eliminate such vertex as because such elimination does not affect planarity of G. This is done in the block G_2'.

Step 5/ Repeat the above steps as long as possible. Then review the planarity of each block or component. If all blocks or components of a graph are planar, the graph is also planar.

Excercise for students :

① Remove an edge of Kuratowski's First graph and show that it becomes planar.

② Remove a vertex from Kuratowski's First graph and show that is becomes planar.

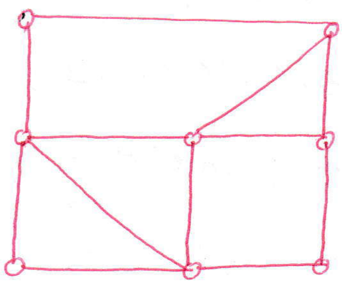
Theorem 12] Kuratowski's First graph is non-planar graph with the smallest number of vertices.

Proof] Obvious, as any graph with 4 or less number of vertices, is planar; even if they are complete graphs.

Theorem 13] Kuratowski's Second graph is the non-planar graph with the smallest number of edges.

Proof not in scope.

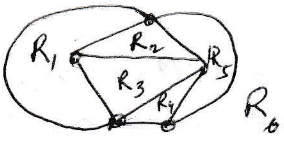
③ Verify Euler's theorem for the following graph.



Theorem 14] (Kuratowski's Theorem)

A graph G is planar if and only if G does not contain either of the Kuratowski's two graphs.

Note/ Non-planar graphs technically do not have faces in cases when edges cross one another. Hence Kuratowski's First graph would have 6 faces.



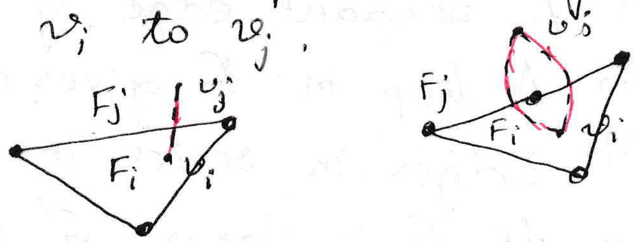
Hence Euler's condition $f = e - n + 2$ will not satisfy as $f = 6$ and $e - n + 2$
$= 10 - 5 + 2$
$= 7$.

Dual of a graph:

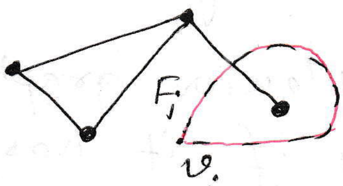
If G is a graph with 'n' number of regions, $F_1, F_2, \ldots, F_n$, place 'n' number of points $v_1, v_2, \ldots, v_n$, in each of ~~the~~ the regions. Then join these 'n' points in the following way –

(i) If two regions F_i and F_j are adjacent (i.e. they have a common edge), v_i and v_j are joined by a line segment intersecting the common edge between F_i and F_j exactly once.

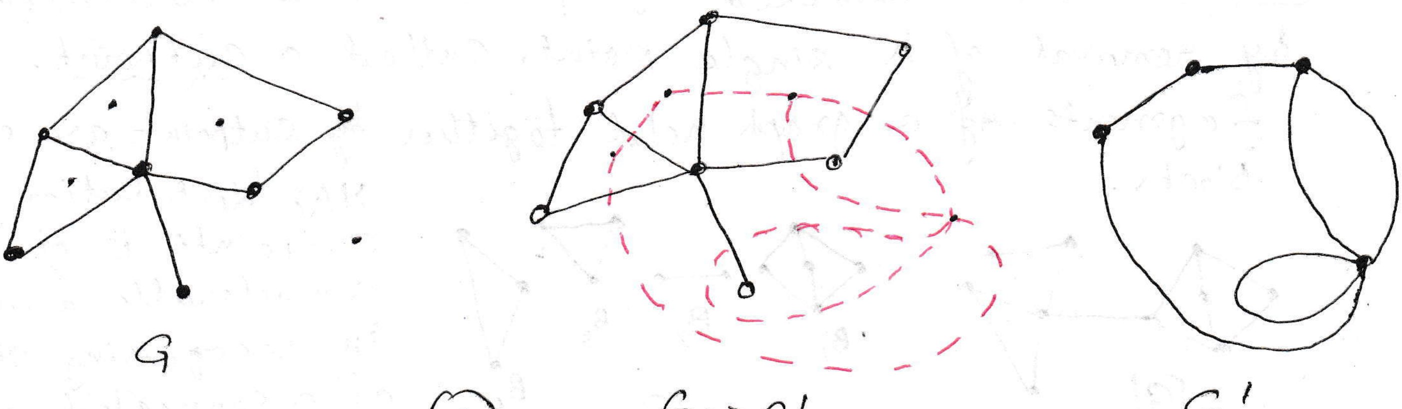
If there are more than one common edges, one line segment is drawn for each of the common edges from v_i to v_j.



(ii) If an edge e of G lies entirely in one region, say F_i, a self-loop is to be drawn at v_i intersecting the edge e exactly once.



The new graph G' thus formed from G is called <u>dual</u> or <u>geometric dual</u> of G.



G $G \to G'$ G'

Points to be noted:

① Since one edge of G' intersects one and only one edge of G, there is always a one-to-one Correspondence between the edges of G and those of its dual G'.

② Every planar graph has dual and vice versa.

Relation between a graph and its dual:

Let G be a graph and G' be its dual. Some relations between G and G' are noted ~~to~~ below-

(i) A pendant edge of G gives a loop in G'

(ii) A loop in G gives a pendant edge in G'

(iii) Edges in series in G ~~give~~ produces parallel edges.

(iv) If G is planar, G' is planar.

(v) (a) Number of vertices of G' = Number of regions in G

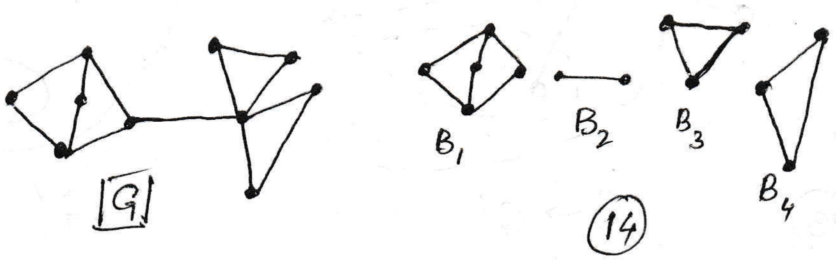
(b) Number of edges in G' = Number of edges of G

(c) Number of regions of G' = Number of vertices of G.

Theorem A planar graph has unique dual if and only if it has a unique drawing on a plane.

Proof Follows from the method of construction.

Note: Some connected graphs can be disconnected by removal of a single point, called a cutpoint. The fragments of a graph held together by cutpoints are called blocks.

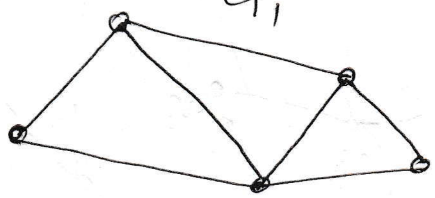


The distribution of cutpoints is of considerable assistance in recognising structure of a connected graph.

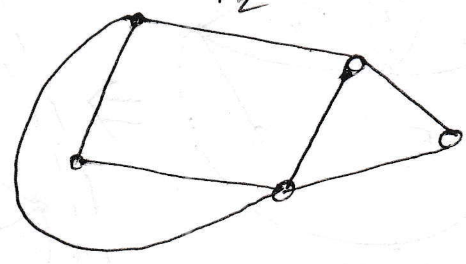
Problem/ Give an example to show a graph that is drawn in two different ways as planar graph. Show that their duals are different. (WBUT 2014)

Soln/

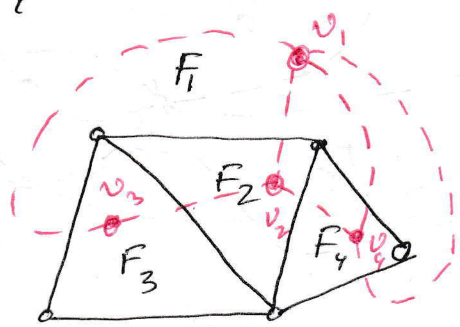
G_1



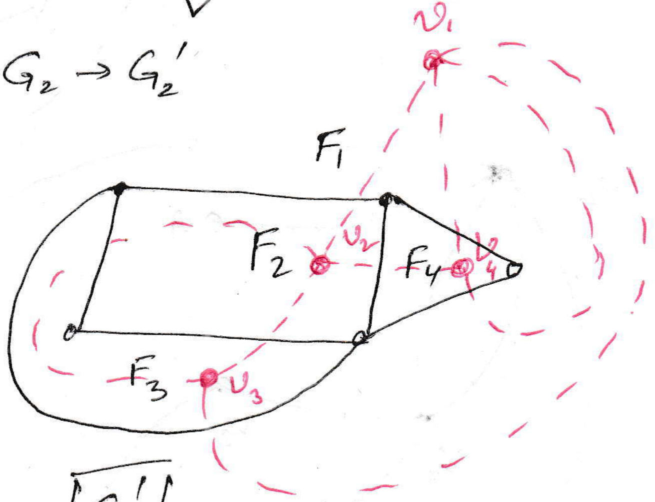
G_2



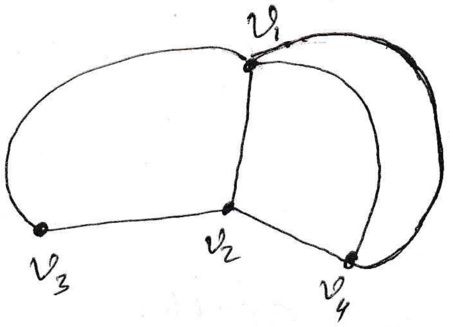
$G_1 \to G_1'$



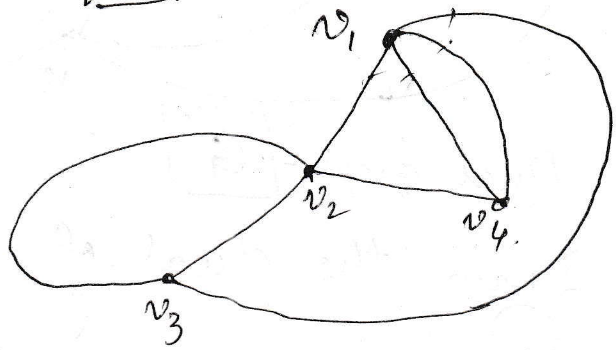
$G_2 \to G_2'$



$\boxed{G_1'}$



$\boxed{G_2'}$

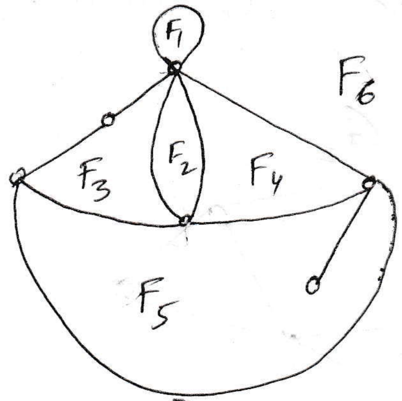


It is evident that G_1' and G_2' are not similar. Also, there is at least one vertex, v_1, in G_2' which has degree 4. However, in G_1' there is no vertex with degree 4, the highest degree in G_1' being 3. Hence, G_1' and G_2' are not even isomorphic.

Practice example for students:

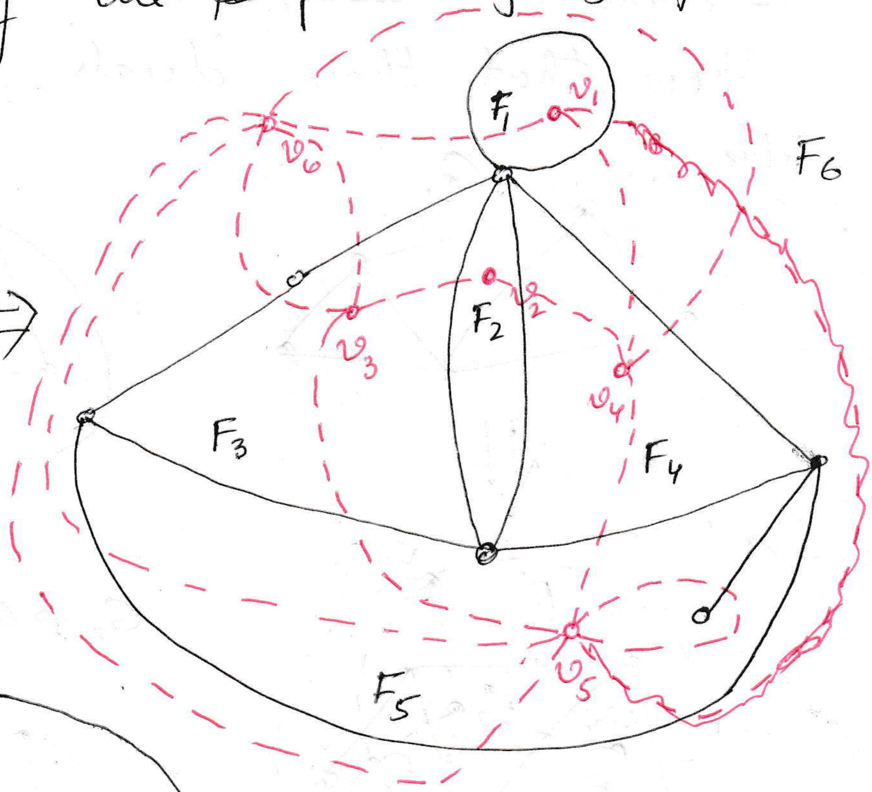
① Draw the dual of the following graph.

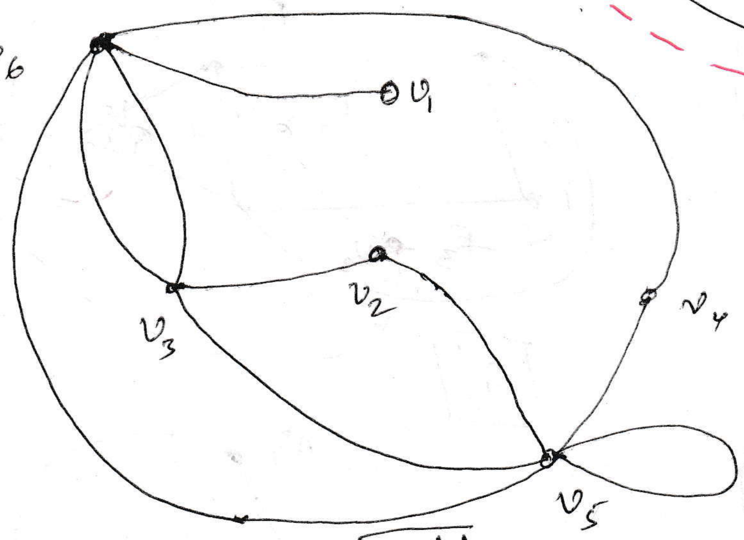
Soln.:



$\boxed{G}$

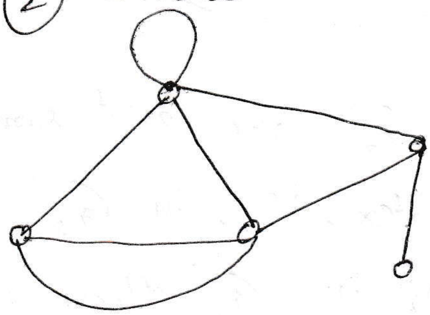
$\Rightarrow$



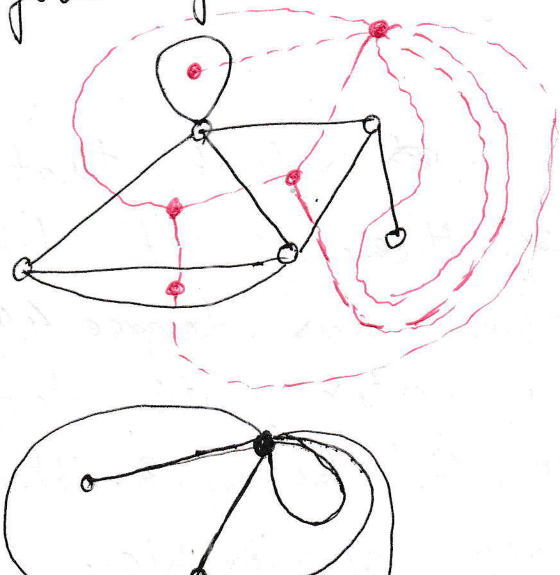


Dual graph - $\boxed{G'}$

② Draw the dual of following graph.



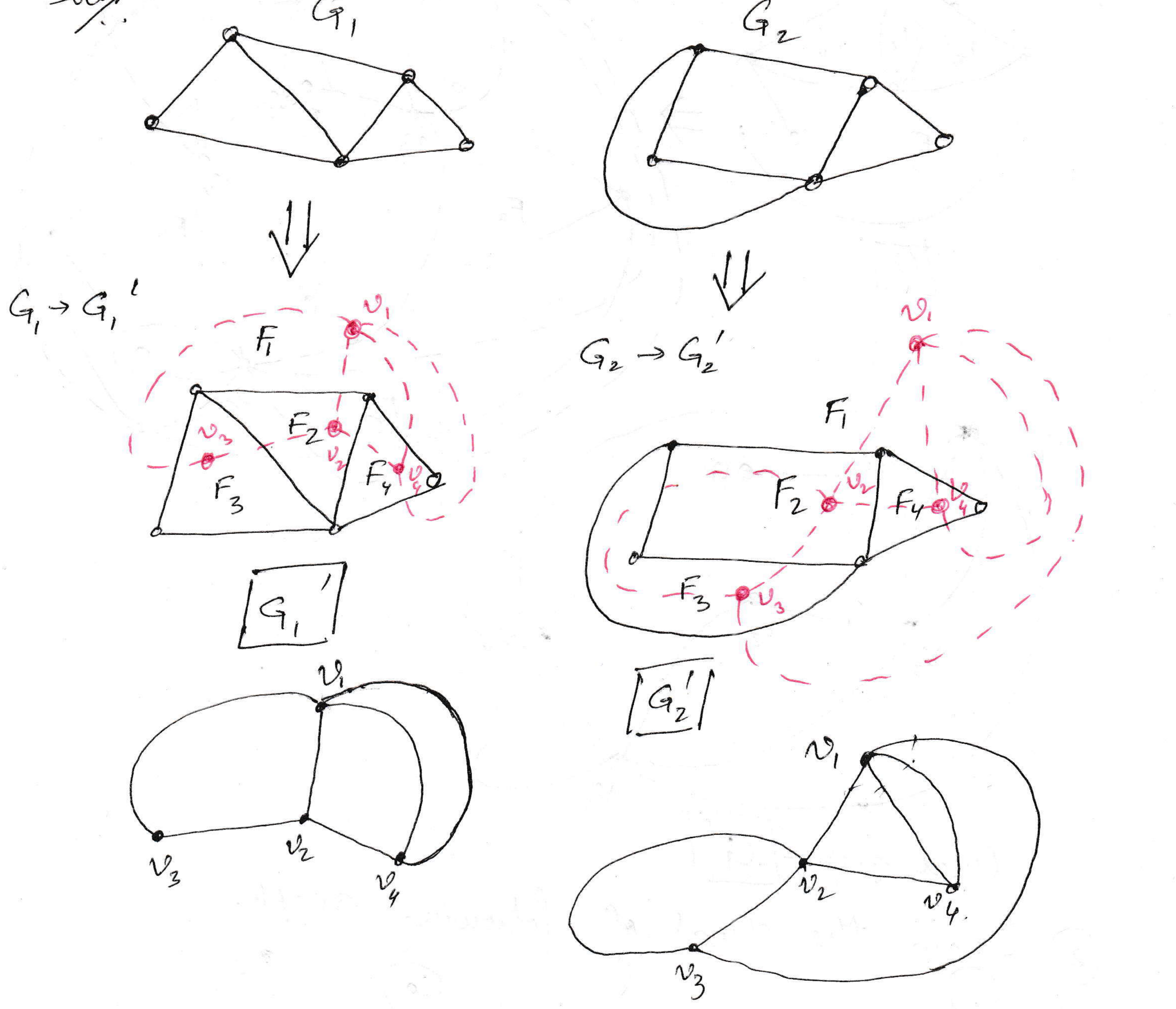
$\Rightarrow$



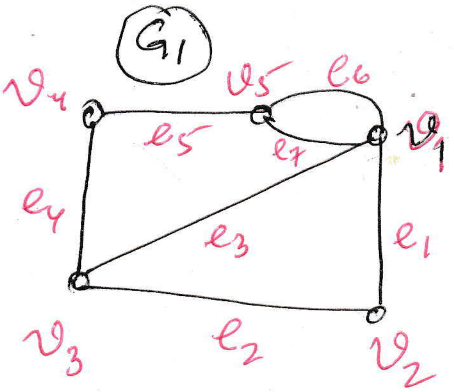
Dual graph G'

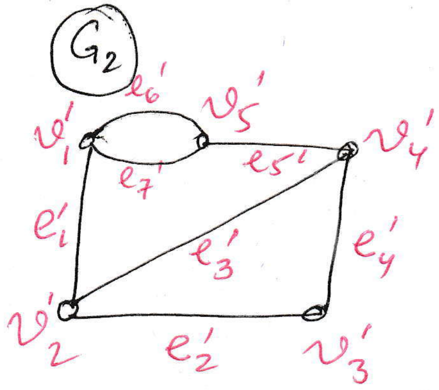
Problem/ Give an example to show a graph that is drawn in two different ways as planar graph. Show that their duals are different. (WBUT 2014)

Soln.



It is evident that G_1' and G_2' are not similar. Also, there is at least one vertex, v_1, in G_2' which has degree 4. However, in G_1' there is no vertex with degree 4, the highest degree in G_1' being 3. Hence, G_1' and G_2' are not even isomorphic.





Vertex-to-vertex one-to-one correspondence:

$v_1 \to v_1'$, $v_2 \to v_2'$, $v_3 \to v_3'$, $v_4 \to v_4'$, $v_5 - v_5'$

Edge-wise one-to-one correspondence:

$e_1 \to e_1'$, $e_2 \to e_2'$, $e_3 \to e_3'$, $e_4 \to e_4'$, $e_5 \to e_5'$
$e_6 \to e_6'$, $e_7 \to e_7'$

Incidence relationship of vertices and edges:

$e_1 \to v_1 v_2 \mid e_1' \to v_1' v_2'$, $e_2 \to v_2 v_3 \mid e_2' \to v_2' v_3'$,

$e_3 \to v_1 v_3 \mid e_3' \to v_2' v_4'$, $e_4 \to v_3 v_4 \mid e_4' \to v_3' v_4'$,

$e_5 \to v_4 v_5 \mid e_5' \to v_4' v_5'$, $e_6 \to v_1 v_5 \mid e_6' \to v_1' v_5'$

$e_7 \to v_1 v_5 \mid e_7' \to v_1' v_5'$

So there is at least one incidence relationship in G_1 which is not matching with G_2.

Hence the graphs are not isomorphic.