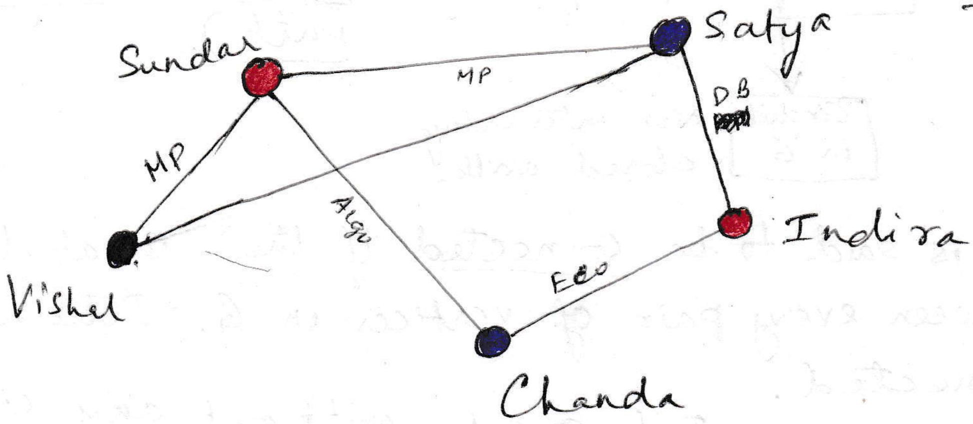
Graph Coloring

Student Name	Courses Opted
Sundar	DM – MP – Algo
Satya	MP – Networks – DB – PM
Indira	Algo – DB – Economics
Chanda	Economics – Cloud
Vishal	MP – AI

Need to slot out the exams for the different courses. How can the exams be slotted and how many slots are needed to avoid conflict.



Time Slots needed:

1 – Red
2 – Blue
3 – Black

Example for students:

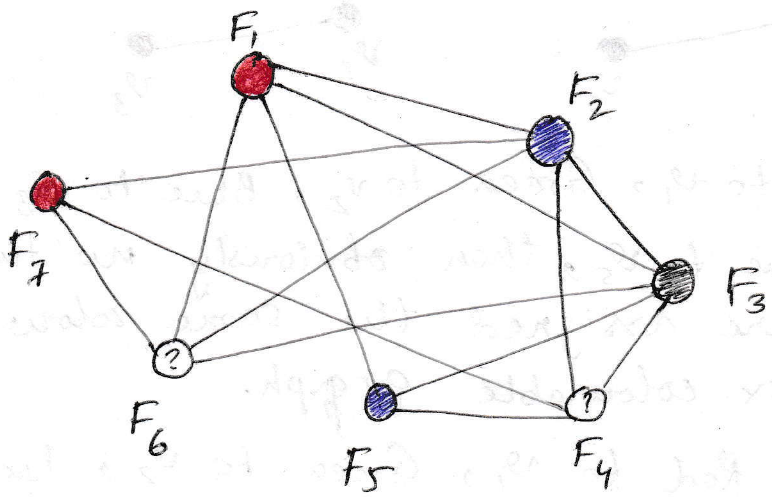
Suppose Indian airlines has seven flights $F_1, F_2, \ldots F_7$ all starting from Delhi. Following table shows the towns touched down by each flight.

Flight	Touched Towns
F_1	Chandigar → Bangalore → Mumbai → Kochi
F_2	Lucknow → Patna → Kolkata → Kochi
F_3	Ahmedabad → Bangalore → Kochi — Chennai
F_4	Ahmedabad → Nagpur → Patna
F_5	Ahmedabad → Panaji → Bangalore
F_6	Jaipur → Kolkata → Kochi
F_7	Jaipur → Bhopal → Patna

For servicing the planes, the company decides the flights to fly on Monday, ~~Tues~~ Wednesday and Friday. It is also ~~to~~ decided that no more than one flight per day will touch down any of the towns.

Indian airlines wants to know whether this is possible. If so, how can the flights be allocated in these 3 days.

Soln. To solve this problem, we first construct a graph with ~~9~~ 7 vertices. Each vertex represents a flight. If two flights have a common town to touch down, then they are joined by an edge. The 3 days are represented by 3 colors, say day 1 (red), day 2 (blue) and day 3 (black).



Typical problems that can be solved using graph coloring are —

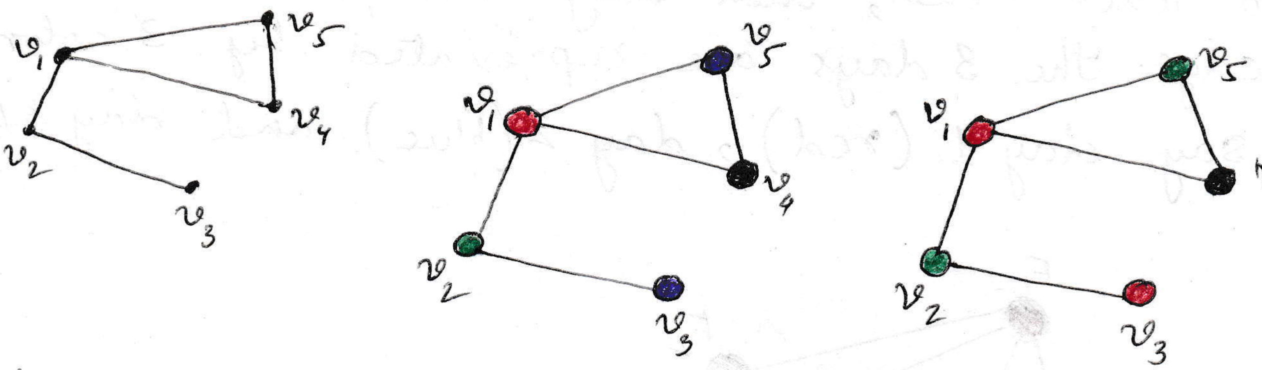
① How many colours do we need to colour the countries of a map in such a way that adjacent countries are coloured differently?

② How many days have to be scheduled for committee meetings in a parliament if every committee intends to meet one day and some members of parliament serve different committees?

A vertex colouring of a graph $G = (V, E)$ is a map $c : V \to S$, such that $c(v) \neq c(w)$, where v and w are adjacent vertices. The elements of the set S are called available colours. The smallest number of elements of S, say 'k', is what we try to find for any problem. This integer 'k' is the (vertex-) chromatic number of graph G and is denoted by $\chi(G)$.

A graph G with $\chi(G) = k$ is called k-chromatic. If $\chi(G) \leq k$, we call G k-colourable.

Let's consider the following graph G. Let's consider set $S = \{$Red, Green, Black, Blue$\}$.



If we assign, Red to v_1, Green to v_2, Blue to v_3, Black to v_4 and Blue to v_5, then obviously no two adjacent vertices are assigned the same colour. So G is a 4-vertex colourable graph.

Also, we can assign Red to v_1, Green to v_2, Red to v_3, Black to v_4 and Green to v_5. This way graph G is 3-colourable.

It is not possible to colour the graph with 2 colours in a way that no two adjacent vertices have the same colour. So, G is a 3-chromatic graph, ie. $\chi(G) = 3$.

A k-chromatic graph is a graph that needs at least k-colours for its colouring.

→ A single-vertex graph is 1-chromatic. A null graph is also 1-chromatic.

→ If a graph has a loop at vertex v_i, then v_i is adjacent to itself. So no colouring is possible for G. To avoid this, loops are removed from a graph.

→ If there are parallel edges joining two vertices, all of them except one can be removed.

For the above two points, graph-coloring is applicable for simple graphs.

→ If H is a sub-graph of G, $\chi(H) \leq \chi(G)$.

Theorems:

Theorem 1: A graph is k-vertex colourable if and only if each block in it is k-vertex colourable.

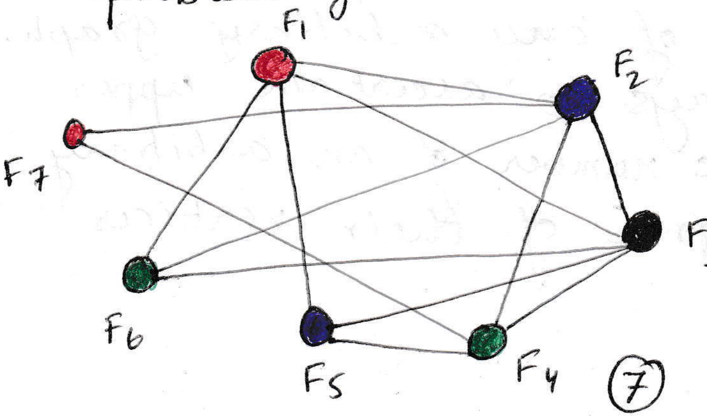
Theorem 2: The chromatic number of a graph is k if the chromatic number of each block of it is k.

Theorem 3: The chromatic number of a complete graph with 'n' vertices K_n is 'n'.

The chromatic number of the $(K_n - v_i)$ graph obtained by deletion of a vertex v_i is $(n-1)$.

Corollary: If a complete graph K_n is the sub-graph of a graph G, then $\chi(G) \geq n$.

This can be used to solve the Flight scheduling problem given earlier. In the graph, K_4 is a sub-graph induced by F_1, F_2, F_3 and F_6. So, by the above corollary, the chromatic number of the overall graph is ≥ 4. It is actually 4. Hence, the seven flights can be scheduled in 4 days, not 3 days.



Theorem 4: The chromatic number of a cycle with 'n' vertices, C_n, is 2 if even and 3 if odd.

Corollary: If C_n is a sub-graph of G, then
(i) $\chi(G) \geq 2$, if n is even
(ii) $\chi(G) \geq 3$, if n is odd.

Theorem 5: The chromatic number of a non-null graph is 2 if and only if the graph is bipartite.

Corollary 1: The chromatic number of a complete bigraph is 2, that is $\chi(K_{m,n}) = 2$.

Corollary 2: The chromatic number of a tree with 2 or more vertices is 2.

Konig's Theorem: The chromatic number of a graph with at least one edge is 2, if and only if the graph contains no ~~of~~ cycles of odd length.

Combining above two theorems –
For a non-null graph G, the following statements are equivalent –
(i) $\chi(G) = 2$
(ii) G is bipartite
(iii) G contains no cycle of odd length.

Upper bound of chromatic number –
Though we are able to find the exact chromatic number of K_n, C_n and $K_{m,n}$, no efficient and convenient procedure is known for finding the chromatic number of any arbitrary graph. However, there are ways to ascertain upper bounds for the chromatic number of an arbitrary graph, provided the degree of their vertices are known.

Brooks Theorem (1941):

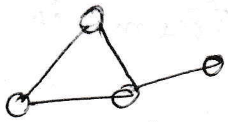
Generalized theorem:

For any graph G, the chromatic number is at most one greater than the maximum degree, ie. $\chi(G) \leq \Delta(G)+1$.

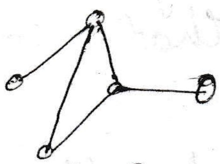
However, Brooks showed that is upper-bound can be improved by 1, if G has no complete graph of $(\Delta(G)+1)$ vertices.

So, by Brooks theorem, $\chi(G) \leq \Delta(G)$, except when G is a complete graph with $(\Delta(G)+1)$ vertices.

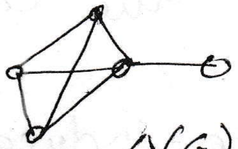
Note: One more exclusion is cycles with odd number of vertices.



$\Delta(G) = 2$
$\chi(G) \leq 2+1$
i.e. $\chi(G) \leq 3$



$\Delta(G) = 3$
$\chi(G) \leq 3$ (K_3 present as subgraph)
i.e. $\chi(G) \leq 3$



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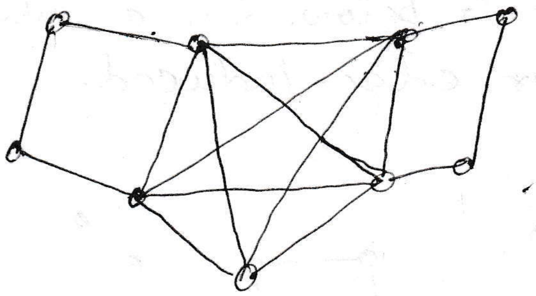
$\Delta(G) = 3$
$\chi(G) \leq 3+1$ ($\because K_4$ is present and $\Delta(G) = 3$)
or, $\chi(G) \leq 4$



$\Delta(G) = 5$
K_5 present
$\therefore \chi(G) \leq 5$, as Brook's theorem applies here because the complete graph which exists is not having $\Delta(G)+1$ ie. 6 vertices

$\chi(G) \geq 5$, by corollary of Theorem 3.

$\therefore 5 \leq \chi(G) \leq 5$ ie. $\underline{\chi(G) = 5}$

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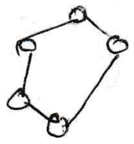


$\Delta(G) = 2$
$\chi(G) \leq 2+1$
ie. $\chi(G) \leq 3$

$\Delta(G) = 2$
$\chi(G) \leq 2$

Theorem: For any graph G, $\chi(G) \leq 1 + \max[\delta(G')]$, where the maximum is taken over all minimum degree induced subgraphs G' of G. This theorem is also called Szekeres and Wilf theorem. (1968) $\delta(G')$ represents the minimum degree for each of the induced subgraphs, G'.

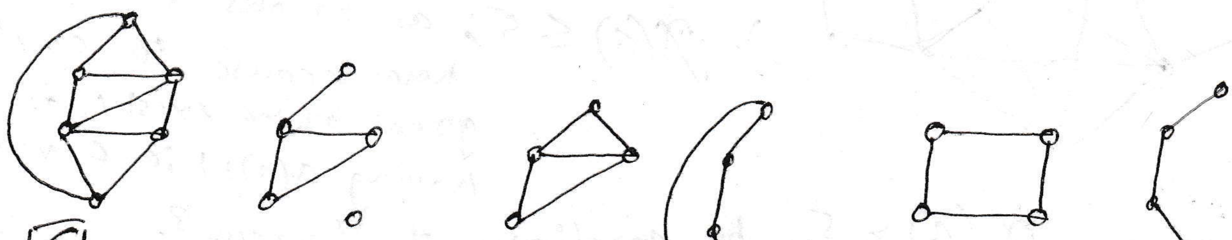
Definition of sub-graph & induced subgraph:

Subgraph 's' of a graph 'G' is a graph whose set of vertices and edges are all subsets of G. Since every set is a subset of itself, every graph is a subgraph of itself.

A vertex-induced subgraph is one that consists of some of the vertices of the original graph and all of the edges that connect them in the original.

An edge-induced subgraph consists of some of the edges of the original graph and the vertices that are at their endpoint.

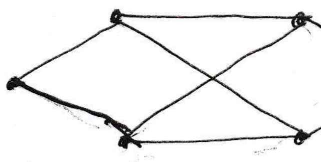
Let's consider the graph G below. G_1 is a sub-graph, but neither vertex-induced nor edge-induced.



$\boxed{G}$ G_1 G_2' G_2'' G_3' G_3''

G_2' and G_2'' are vertex-induced subgraphs of G.
G_3' and G_3'' are edge-induced subgraphs of G.

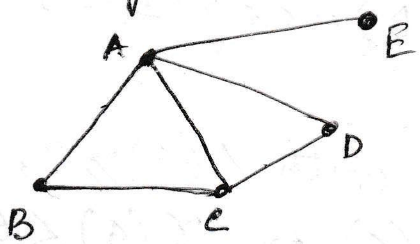
Q. Find chromatic number of -



or, $\chi(G) \leq 3$.

Lower Bounds of Chromatic number:

Independent vertex set – Let G be a graph and A be a subset of its vertex. If no two vertices in A are adjacent, then A is called independent set of vertices.



For example, in the graph in the figure, set of vertices $A = \{B, D, E\}$ are independent as none of the pairs formed by the three vertices namely B-D, D-E or B-E are adjacent. However, $\{A, D, E\}$ is not independent as both pairs A-D and A-E are adjacent.

Independence number – The cardinal number of the largest independent set in a graph G is called the independence number or vertex-independence number of G. It is denoted by $\alpha(G)$.
In the previous example, $\alpha(G) = 3$.

Clique: Let G be a graph and Z be a subset of its vertex set. If every pair of vertices in Z are adjacent, then Z is called a clique of G.
For example, in the previous figure $\{A, B, C\}$ and $\{A, C, D\}$ are cliques while $\{A, D, E\}$ is not.

Clique Number: The cardinal number of the largest clique in a graph G is called the clique number of graph G. It is denoted by $\omega(G)$.
In the previous graph, $\{A, B, C\}$ and $\{A, C, D\}$ are largest cliques. So the clique number $\omega(G) = 3$.

Note:
(1) A single vertex set is an independent set.
(2) A complete subgraph of a graph G is a clique of G.
(3) A null graph has ~~not~~ no clique.
(4) $\omega(G) \geq 2$ for any graph.
(5) A clique in a graph G is independent set of vertices in the complement of G.

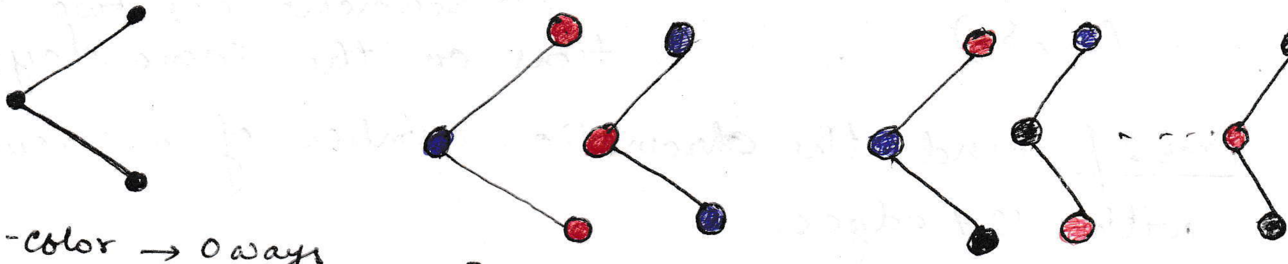
Theorem 1: For a graph G with 'n' vertices, and $\alpha(G)$ independence number, $\chi(G) \geq \dfrac{n}{\alpha(G)}$.

Theorem 2: For a graph G with 'n' vertices and $\omega(G)$ clique number, $\chi \geq \omega(G)$

Chromatic Polynomials:

Chromatic polynomial is a polynomial which gives the number of different ways that the vertices of a graph can be coloured with a certain number of available or choosable colours.

~~Let's~~ take the example of a graph G with 3 vertices, as below.



1-color → 0 ways   2 colors → 2 ways   3 colors → 12 ways

$P(x) = x^3 - 2x^2 + x$

$P(1) = 1^3 - 2.1^2 + 1 = 0$

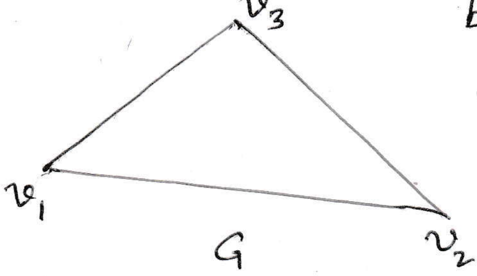
$P(2) = 2^3 - 2.2^2 + 2 = 2$

$P(3) = 3^3 - 2.3^2 + 3 = 12$

$P(4) = 4^3 - 2.4^2 + 4 = 36$

So, chromatic polynomials give us how many different ways we can colour a graph based on the available colours.

Again, let's consider the graph below, which is a triangle or complete graph with 3 vertices. Let λ be the number of available colours for G. We want to find the chromatic polynomial $P(G, \lambda)$, for this graph.



If v_1, v_2 and v_3 are the vertices of G, then v_1 can be assigned λ number of colours, v_2 can be assigned $(\lambda - 1)$ number of colours, and v_3 can be assigned $(\lambda - 2)$ number of colours.

Therefore, G may be coloured in $\lambda(\lambda-1)(\lambda-2)$ ways. Thus $P(G,\lambda) = \lambda^3 - 3\lambda^2 + 2\lambda$, is the chromatic polynomial of G.

But what does it mean?

In a case where $\lambda = 2$, ie. the number of the available colours is 2, the value of the Chromatic polynomial is 0. $(2^3 - 3.2^2 + 2.2 = 8 - 12 + 4 = 0)$

This means it's not possible to colour the graph with 2 colours. This also matches with the fact that for a complete graph K_3, chromatic number is 3.

For $\lambda = 3$, $P(G,\lambda) = 3^3 - 3.3^2 + 2.3 = 6$. So there are 6 ways to colour the graph G with 3 colours.

Like this, we can find out the possible ways to colour a graph, given the Chromatic polynomial $P(G,\lambda)$ and the number of colours λ.

Note: The smallest λ for which $P(G,\lambda) \neq 0$ is the chromatic number, $\chi(G)$. In other words, $P(G,\lambda) = 0$ for $\lambda < \chi(G)$. This is observed in the above example as $P(G,2) = 0$, $P(G,3) = 6$ and $\chi(G) = 3$.

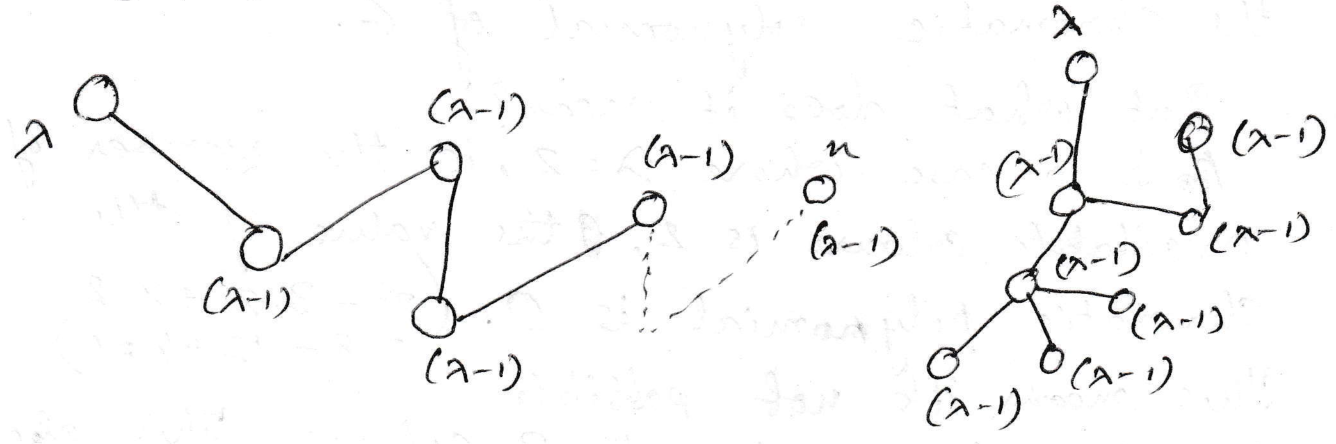
Note: For a null graph with 'n' vertices, ie. $|V| = n$ and $E = \phi$, $P(G,\lambda) = \lambda^n$.

Theorem: For a complete graph $G = K_n$, chromatic polynomial $P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)\ldots(\lambda-n+1)$

$$= \frac{\lambda!}{(\lambda-n)!}$$

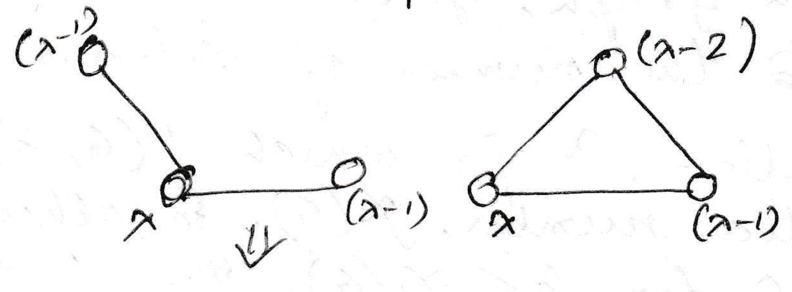
→ For paths, say with 'n' vertices,

$$P(G, \lambda) = \lambda(\lambda-1)^{n-1}$$



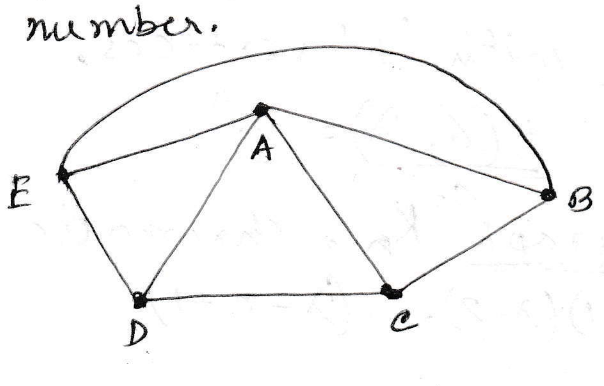
→ For trees too, $P(G, \lambda) = \lambda(\lambda-1)^{n-1}$

→ If graph G has multiple components, the chromatic polynomial of individual components have to be multiplied ~~together~~ using product rule to derive the chromatic polynomial of the whole graph.



$P(G, \lambda) = \{\lambda(\lambda-1)^2\}\left\{\frac{\lambda!}{(\lambda-3)!}\right\}$

<u>Example</u>: Find the chromatic polynomial of the following graph. After that, find its chromatic number.



<u>Theorem</u>: For a graph G with 'n' vertices, chromatic polynomial $P(G, \lambda) = c_1 \, {}^{\lambda}C_1 + c_2 \, {}^{\lambda}C_2 + c_3 \, {}^{\lambda}C_3 + \dots + c_n \, {}^{\lambda}C_n$.

Each c_i has to evaluated individually for each given graph.

Soln Since the given graph has 5 vertices,

$$P(G,\lambda) = c_1\, {}^{\lambda}C_1 + c_2\, {}^{\lambda}C_2 + c_3\, {}^{\lambda}C_3 + c_4\, {}^{\lambda}C_4 + c_5\, {}^{\lambda}C_5$$

Now, since there is a complete sub-graph with 3 vertices in G (ABC/ACD/ADE),

$$\chi(G) \geq 3.$$

So, G cannot be coloured with one colour i.e. $c_1 = 0$ and G cannot also be coloured with two colours i.e. $c_2 = 0$.

Now, say there are 3 available colours or $\lambda = 3$. These 3 colours can be assigned to A, B, C in $\lfloor 3 = 6$ different ways. But since we will not be left with any colour, the colour which has been assigned to B must be assigned to E too and the colour assigned to C must be assigned to E too. Also, this can be done only in 1 way. So, $c_3 = 6$.

Next we assume number of available colours as 4 or $\lambda = 4$. These 4 colours can be assigned to 3 vertices A, B and C in ${}^4P_3 = 4 \times 3 \times 2$
$= 24$ ways.

We still have one colour left. This can be assigned to D or E, so there are 2 choices.

$$\therefore\ c_4 = 24 \times 2 = 48.$$

If there are 5 available colours, it can be assigned to 5 vertices in $\lfloor 5 = 120$ ways.

$$\therefore\ c_5 = 120$$

So, the chromatic polynomial

$$P(G,\lambda) = 6 \cdot {}^{\lambda}C_3 + 48 \cdot {}^{\lambda}C_4 + 120 \cdot {}^{\lambda}C_5$$
$$= \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$
$$+ \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)$$

We already know $\chi(G) \geq 3$.

For $\lambda = 3$, $P(G,\lambda) = 3(3-1)(3-2)$ [∵ Second and third terms of the Chromatic poly. will become 0, $(\lambda-3)$]

$= 6$

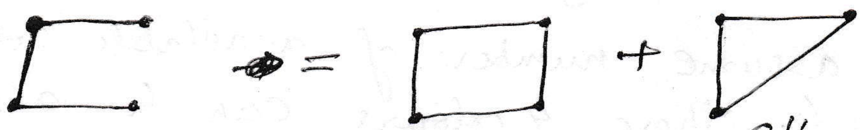
∴ For, $\lambda = 3$, $P(G,\lambda) \neq 0$

∴ The chromatic number of the given graph is 3.

Decomposition Theorem:

Let v_1 and v_2 be the two non-adjacent vertices of a simple graph G. Let G' be a graph generated from G adding an edge between v_1 and v_2. Let G'' be a graph obtained from G by merging the vertices v_1 and v_2 together.

Then $\boxed{P(G,\lambda) = P(G',\lambda) + P(G'',\lambda)}$



G G' G''

⇓

$P(G,\lambda) = \lambda(\lambda-1)^3$ as G is a tree with 4 vertices

⇓

$P(G,\lambda) = \lambda(\lambda-1)(\lambda-2)$, as G'' is a complete graph with 3 vertices

$$\therefore P(G',\lambda) = P(G,\lambda) - P(G'',\lambda)$$
$$= \lambda(\lambda-1)^3 - \lambda(\lambda-1)(\lambda-2)$$
$$= \lambda(\lambda-1)\{\lambda^2 - 2\lambda + 1 - \lambda + 2\}$$
$$= (\lambda^2-\lambda)(\lambda^2 - 3\lambda + 3)$$
$$\therefore P(G',\lambda) = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda$$