Chapter 5. Stochastic Processes with Jumps

- The Brownian motion is a process which is continuous in time and space.
- As a consequence, it cannot capture extreme movements.
- The Brownian motion is, in fact, Gaussian, i.e. it has symmetric distribution with zero excess kurtosis.
- Extreme movements, i.e. skewness and excess kurtosis, can be captured by allowing, for example, discontinuity in space, i.e. introducing jumps.
- Possible examples of such processes are
  - Jump Diffusion processes, like the Merton JD or the Kou JD;
  - Time Changed Brownian motions, like the VG process.
- The construction of these processes requires some preliminary facts listed in the following section.

In the remaining of this chapter, we assume the following.

- The dynamics of the log-price is

$$s\left(t\right)=\log S\left(t\right),$$

  where

$$s(t)=at+X(t).$$

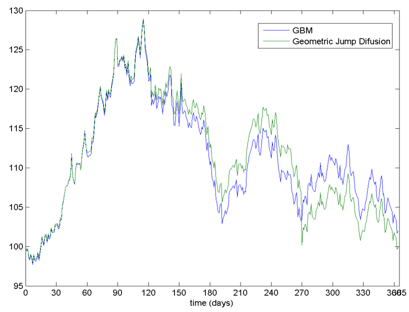

Figure 5.1: Sample trajectories of a stock price $S(t)$ in the cases in which $X(t)$ is either an Arithmetic Brownian motion or a jump-diffusion process of the form $X_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Z_k$. In this example, the jump size Z is Gaussian (Merton, 1976). The process X is obtained from the Arithmetic Brownian motion (the continuous parts are identical) by superimposing the compound Poisson process.

- $X(t)$ is the stochastic process of interest.

5.1. Preliminaries

5.1.1. The Poisson process

Fact 44 *A Poisson process is an increasing, positive stochastic process $N(t)$ on $\mathbb{N}$ with independent and stationary increments which are Poisson distributed with instantaneous rate of arrival $\lambda > 0$. In other words, for any $0 < s < t$ the following hold.*

1. $N(0) = 0$;

2. *$N(t) - N(s)$ is independent of the information set $\mathcal{F}(s)$ generated up to time s;*
3. *$N(t) - N(s) \sim N(t-s) \sim Poi(\lambda(t-s))$.*

Moreover, the characteristic function of $N(t)$ is

$$\phi_N(u;t) = e^{\lambda t(e^{iu}-1)},$$

where $i = \sqrt{-1}$ is the imaginary unit. Further,

$$\mathbb{E}\left(N(t)\right) = \lambda t$$

and

$$\mathbb{V}ar\left(N(t)\right) = \lambda t.$$

Hence, we note the following.

- It follows from properties (1) and (3) above that $N(t) \sim Poi(\lambda t)$.
- It follows from the definition of the Poisson distribution (see Appendix), that the increments can only take values 1 or 0 according to whether an arrival occurs or not.
- Hence, the Poisson process counts the arrivals in a system, like calls at a call center or shocks in the market.
- By definition of Poisson distribution, there cannot be more than one jump per time period.
- Hence, the Poisson process can only generate a finite number of jumps over a finite time horizon.
- For this reason, the Poisson process is said to have finite activity.

5.1.2. The Compound Poisson process

In order to gain some additional flexibility in modelling the size (severity) of the jumps, the Poisson process can be used to construct a more flexible process by assigning a specific distribution to the severities.

Fact 45 *A compound Poisson process is a stochastic process $Y(t)$ of the form*

$$Y(t) = \sum_{k=1}^{N(t)} Z_k,$$

where $\{Z_k\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables which are assumed independent from the Poisson process $N(t)$.

Moreover, the compound Poisson process has characteristic function

$$\phi_Y(u;t) = e^{\lambda t(\phi_Z(u)-1)},$$

where $\phi_Z(u)$ denotes the characteristic function of the random variable Z. It follows that

$$\mathbb{E}\left(Y(t)\right) = \lambda \mathbb{E}\left(Z\right) t,$$

and

$$\mathbb{V}ar\left(Y(t)\right) = \lambda \mathbb{E}\left(Z^2\right) t.$$

We can think of the compound Poisson process as follows.

- At time t a jump occurs.
- When this happens, the Poisson process increases of 1 unit.
- At the same time, a random draw Z is taken from a given distribution to quantify the jump size and it is summed up to the value of the process at the previous time point.
- The compound Poisson process has finite activity, like the Poisson process.

5.1.3. The Gamma process

Alternative processes which captures jump arrivals and size simultaneously like the compound Poisson process are available. One example is given by the Gamma process.

Fact 46 *A Gamma process is a positive, non decreasing stochastic process $Y(t)$ with independent and stationary increments which follow a Gamma distribution (see Appendix), i.e.*

- $Y(0) = 0$;
- $Y(t) - Y(s)$ *is independent of the information set up to time* $s < t$;
- $Y(t) - Y(s) \sim Y(t-s) \sim \Gamma(\alpha(t-s), \lambda)$.

The characteristic function of the Gamma process is

$$\phi_Y(u;t) = \left(\frac{\lambda}{\lambda - iu}\right)^{\alpha t};$$

therefore

$$\mathbb{E}\left(Y(t)\right) = \frac{\alpha}{\lambda}t,$$

and

$$\mathbb{V}ar\left(Y(t)\right) = \frac{\alpha}{\lambda^2}t.$$

Remark 47 *The Gamma process differs from the compound Poisson process in two aspects.*

1. *The Gamma process has infinite activity, as there can be an infinite number of jumps of very small size in a finite time period.*
2. *In the case of the Gamma process it is not possible to separare the rate of arrival of the jumps from their distribution.*

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% SIMULATING JUMP PROCESSES       %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Assigning the number of simulated paths
%(nsimul), time to maturity (expiry), number of steps
%(nsteps), time step (dt) and observation times (timestep):
clear all; nsimul=5, expiry=1, nsteps=250;
dt=expiry/nsteps; timestep=[0:dt:expiry]';
%Assigning parameters
lambdaP=5; muZ=-0.05; sigmaZ=0.1; alpha=5; lambdaG=10;
%Simulate increments of the Poisson process
dN=poissrnd(lambdaP*dt,[nsteps,nsimul]);
%Simulate Poisson process (use cumulative sum of the increments)
cdN=[zeros(1,nsimul); cumsum(dN)];
%1. Simulate increments of the CPP for Gaussian jump sizes
dJ=muZ*dN+sigmaZ*sqrt(dN).*randn(nsteps,nsimul);
%2. Simulate CPP process (use cumulative sum of the increments):
cdJ=[zeros(1,nsimul); cumsum(dJ)];
%3. Simulate increments of the Gamma process:
dG=gamrnd(dt*alpha,1/lambdaG,[nsteps,nsimul]);
%4. Simulate Gamma process (use cumulative sum of the increments
cdG=[zeros(1,nsimul); cumsum(dG)];
%Plot simulated paths:
h=figure('Color', [ 1 1 1])
subplot(3,1,1); plot(timestep, cdN);xlabel('Time (years)')
title('Simulated Paths of the Poisson Process')
subplot(3,1,2); plot(timestep, cdJ);xlabel('Time (years)')
title('Simulated Paths of the Compound Poisson Process')
subplot(3,1,3); plot(timestep, cdG); xlabel('Time (years)')
title('Simulated Paths of the Gamma Process')
```

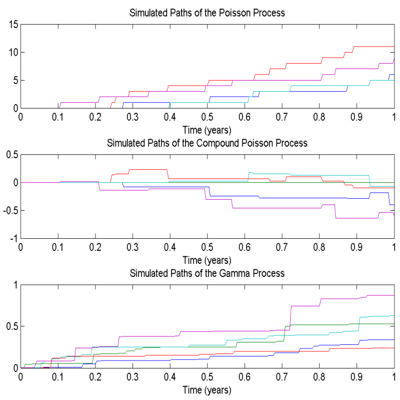

Figure 5.2: Simulated paths of the Poisson process $N(t) \sim Poi(\lambda t)$ for $\lambda = 5$ (top panel); the Compound Poisson process with Gaussian jump severities with parameters $\lambda = 5$, $\mu_Z = -0.05$, $\sigma_Z = 0.1$ (middle panel); the Gamma process $G(t) \sim \Gamma(\alpha t, \lambda)$, with parameters $\alpha = 5$, $\lambda = 10$ (bottom panel).

5.2. Jump Diffusion processes

Fact 48 (Jump Diffusion process) *A Jump Diffusion process is a stochastic process $X(t)$ with independent and stationary increments which is obtained as the sum of an Arithmetic Brownian Motion and an independent compound Poisson process, i.e.*

$$X(t) = \mu t + \sigma W(t) + \sum_{j=1}^{N(t)} Z_k,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$ and

- *$W(t)$ is a Brownian motion,*
- *$N(t)$ is a Poisson process with instantaneous rate of arrival $\lambda > 0$ and independent of $W(t)$,*
- *$\{Z_k\}_{k\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, which are independent of both the Brownian motion and the Poisson process.*

We can further 'specialize' the compound Poisson part of the JD process by specifying the distribution of the jump severities. Common choices for this distribution in financial applications are the Gaussian distribution and the exponential distribution.

5.2.1. The Merton Jump Diffusion Process

Fact 49 *Let us assume that the jump severities follow a Gaussian distribution, i.e. $Z \sim \mathcal{N}\left(\mu_Z, \sigma_Z^2\right)$. Then, the JD process $X(t)$ is a Merton JD process.*

- The process takes its name from Robert Merton who first used it for financial applications.
- The choice of modelling dynamics using a JD process is quite common in financial applications due to the following observation:
  - stock prices appear to have small continuous movements most of the time (due, for example, to a temporary imbalance between demand and supply);

  - sometimes though they experience large jumps upon the arrival of important information with more than just a marginal impact.
- By its very nature, important information arrives only at discrete points in time and the jumps it causes have finite activity.

Properties of the Merton JD process

- $\mathbb{E}X\left(t\right) = \left(\mu + \lambda\mu_Z\right)t$
- $\mathbb{V}ar\left(X\left(t\right)\right) = \left(\sigma^2 + \lambda\left(\mu_Z^2 + \sigma_Z^2\right)\right)t$
- The indices of skewness and excess kurtosis are respectively

$$\begin{aligned} Skew\left(t\right) &= \frac{\lambda\mu_Z\left(\mu_Z^2 + 3\sigma_Z^2\right)}{\left(\sigma^2 + \lambda\left(\mu_Z^2 + \sigma_Z^2\right)\right)^{3/2}\sqrt{t}}, \\ \mathbb{E}\mathbb{K}urt\left(t\right) &= \frac{\lambda\left(\mu_Z^4 + 6\mu_Z^2\sigma_X^2 + 3\sigma_Z^4\right)}{\left(\sigma^2 + \lambda\left(\mu_Z^2 + \sigma_Z^2\right)\right)^2 t}. \end{aligned}$$

Interpretation of the parameters

- $\mu =$ drift of the process.
- $\sigma =$ volatility of the Brownian motion.
- $\lambda =$ rate of arrival of the jumps; it controls the level of excess kurtosis.
- $\mu_Z =$ mean of the jump sizes; it controls the sign of the skewness index. Hence, the Merton jump diffusion has a distribution which is skewed to the left if $\mu_Z < 0$ and skewed to the right if $\mu_Z > 0$.
- $\sigma_Z =$ volatility of the jump sizes.

Matlab Code

```
function m=get_moments_JD(mu, sg, lambda, muZ, sgZ, t)

m(1,:)= (mu+lambda*muZ)*t; %mean
m(2,:)= (sg*sg+lambda*(muZ.^2+sgZ.^2))*t; %variance
numsk= lambda.*muZ.*(muZ.^2+3*sgZ.^2);
densk=(sg.^2+lambda.*(muZ.^2+sgZ.^2))^1.5*t.^0.5;
m(3,:)=numsk./densk;%skewness
numk= lambda.*(muZ.^4+6*muZ.^2.*sg.^2+3*sgZ.^4);
denk=(sg.^2+lambda.*(muZ.^2+sgZ.^2))^2*t;
m(4,:)=numk./denk;%excess kurtosis
```

Parameters fitting

- The simplest method (straigthforward but not very accurate) to fit the parameters is to use the method of moments procedure.
- It consists in minimizing the distance between sample moments (such as sample mean, sample variance, sample skewness and sample kurtosis) with theoretical ones.
- For example, over year 2012, the log-return series of crude oil price were characterized by the sample moments in Table (5.1):

Mean	Variance	Skewness	Excess Kurtosis
-0.0003	0.0144	0.1417	4.3605

Table 5.1: Sample moments of daily log-price changes in oil price in year 2012.

- We can solve for the MJD parameters such that theoretical moments fit the ones in Table (5.1). This can be done trough the following commands in the Matlab command window.


```
%fitting parameters
>>x0(1)=0; x0(2)=0.05; x0(3)=0.5; x0(4)=0.01; x0(5)=0.17
>>[xopt fval]= fminsearch(@(x) sum(((get_moments_JD(x(1), x(2), x(3), x(4), x(5),1)-ms').^2)),x0)
>>mJD=get_moments_JD(xopt(1), xopt(2), xopt(3), xopt(4), xopt(5),1)
```

We obtain the parameter estimates as in Table (5.2).

μ	σ	λ	μ_Z	σ_Z
-0.0037	0.0407	0.5373	0.0064	0.1541

Table 5.2: Calibrated parameters of the MJD model to sample moments of daily log-price changes in oil price for year 2012.

Simulating the Merton JD process

- The simulation procedure for the trajectories of the Merton Jump Diffusion process is based on the following two observations.
  1. The increments of the Poisson process are independent and follow a Poisson distribution with rate $\lambda(t_{j+1} - t_j)$;
  2. conditioned on the number of jumps occurred from t_j to t_{j+1}, the sum of the jump severities is Gaussian with given mean and variance.
- Hence, the simulation algorithm can be organized as follows.

  Step 1. Simulate the continuous part of the JD diffusion process, i.e. the ABM, on the given time partition.

  Step 2. Simulate the number of jumps occurring from t_j to t_{j+1}, i.e. $N \sim Poi(\lambda(t_j, t_{j+1}))$.

  Step 3. Generate $Z \sim \mathcal{N}(0,1)$; set $J = \mu_Z N + \sigma_Z \sqrt{N} Z$.

  Step 4. Sum the ABM and J.
- Simulated paths are illustrated in Figure (5.3). A comparison between the density of the MJD model and the Gaussian with same mean and variance is given in Figure (5.4).

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% SIMULATING THE MERTON JD Process %%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Assigning the number of simulated paths
%(nsimul), time to maturity (expiry), number of steps
%(nsteps), time step (dt) and observation times (timestep):
clear all; nsimul=50, expiry=1, nsteps=250;
dt=expiry/nsteps; timestep=[0:dt:expiry]';
%Assigning parameters
mu=-0.0003; sigma=0.0425; lambda=0.5175;
muZ=0.0064; sigmaZ=0.1520;
%Simulate increments of the ABM
dW=mu*dt+sigma*sqrt(dt).*randn(nsteps,nsimul);
%Simulate increments of the CPP
dN=poissrnd(lambda*dt,[nsteps,nsimul]);
dJ=muZ*dN+sigmaZ*sqrt(dN).*randn(nsteps,nsimul);
dX=dW+dJ;
%Simulate MJD process (use cumulative sum of the increments):
cdX=[zeros(1,nsimul); cumsum(dX)];
%Plot simulated paths:
h=figure('Color', [ 1 1 1])
plot(timestep, cdX);xlabel('Time (years)')
title('Simulated Paths of the Merton JD Process')
```

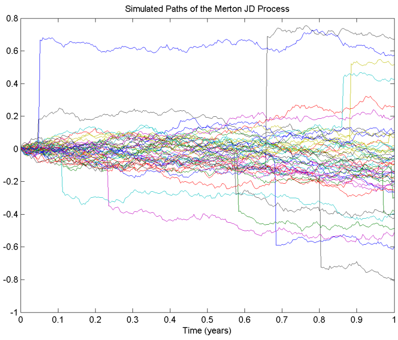

Figure 5.3: Simulated paths of the Merton jump diffusion process. Parameters: $\mu = -0.0003$, $\sigma = 0.0425$, $\lambda = 0.5175$, $\mu_Z = 0.0064$, $\sigma_Z = 0.1520$.

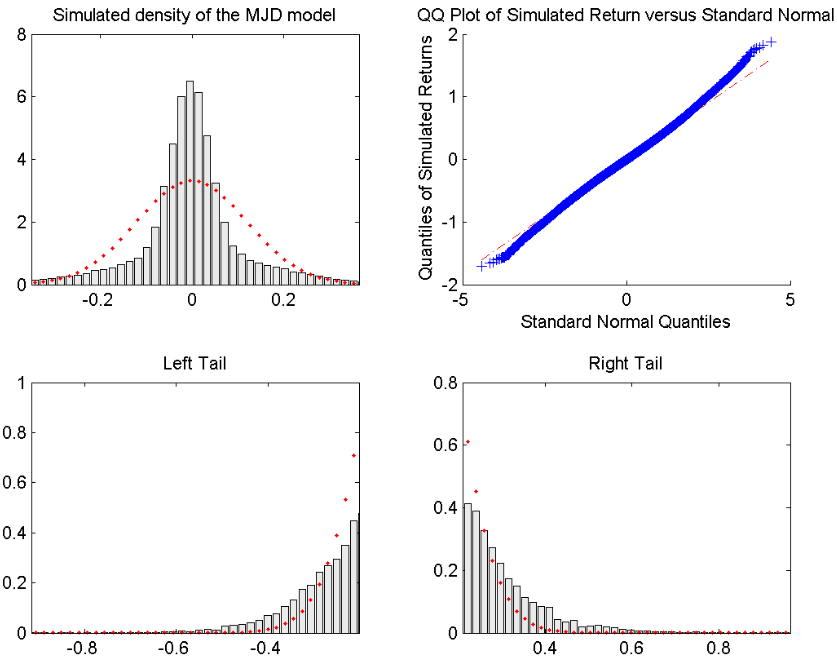

Figure 5.4: Top left: Simulated density of the Merton jump diffusion process at 1 day horizon and superimposed Gaussian density with same mean and variance. Top right: qqplot of simulated returns. Bottom left: Left tail of the simulated returns versus Gaussian tail. Bottom right: right tail of the simulated returns versus Gaussian tail. Parameters as in Table (5.2).

5.2.2. The Kou process

In the case of the Kou process, the jump sizes follow a double exponential distribution with parameters (p, η_1, η_2), i.e. their density function is given by

$$p\eta_1 e^{-\eta_1 y}1_{(y\geq 0)} + (1-p)\,\eta_2 e^{\eta_2 y}1_{(y<0)}, \quad \eta_1, \eta_2 > 0, p \in [0,1].$$

Properties of the Kou JD process

- $\mathbb{E}(X(t)) = (\mu + \lambda\,(p/\eta_1 - (1-p)/\eta_2))\,t$
- $\mathbb{V}ar\,(X(t)) = \left(\sigma^2 + 2\lambda\left(p/\eta_1^2 + (1-p)/\eta_2^2\right)\right)t$
- The indices of skewness and excess kurtosis are respectively

$$\begin{aligned}
\mathbb{S}kew\,(t) &= \frac{6\lambda\left(p/\eta_1^3 - (1-p)/\eta_2^3\right)}{\left(\sigma^2 + 2\lambda\left(p/\eta_1^2 + (1-p)/\eta_2^2\right)\right)^{3/2}\sqrt{t}},\\
\mathbb{E}\mathbb{K}urt\,(t) &= \frac{24\lambda\left(p/\eta_1^4 + (1-p)/\eta_2^4\right)}{\left(\sigma^2 + 2\lambda\left(p/\eta_1^2 + (1-p)/\eta_2^2\right)\right)^{2} t}.
\end{aligned}$$

Interpretation of the parameters

- $\mu =$ drift of the process.
- $\sigma =$ volatility of the Brownian motion.
- $\lambda =$ rate of arrival of the jumps; it controls the level of excess kurtosis.
- $p =$ probability of an upward jump.
- $\eta_1 =$ parameter of the exponential distribution controlling the upward jumps; therefore, the upward jumps have mean $1/\eta_1$.

- $\eta_2 =$ parameter of the exponential distribution controlling the downward jumps; therefore, the downward jumps have mean $1/\eta_2$.

References on Jump Diffusion processes

- Kou, S. G. 2002. A jump-diffusion model for option pricing, Management Science, 48, 8, 1086-1101.
- Merton, R. 1976. Option pricing when underlying stock returns are discontinuous, Journal of Financial Economics, 125-144.

5.3. Time Changed Brownian motions

An alternative way of constructing stochastic processes with jumps is to consider an Arithmetic Brownian Motion on a time scale which is not governed by the standard calendar time, but by a random clock. These processes are called Time Changed Brownian Motions.

Fact 50 *A Time Changed Brownian motion is a process of the form*

$$X(t) = \theta G(t) + \sigma W(G(t)), \qquad \theta \in \mathbb{R}, \sigma > 0,$$

where $W(t)$ is a Brownian motion and $G(t)$ is a positive, increasing stochastic process independent of W. The law of the increments of the process G is what allows us to characterize the resulting process X.

Constructing Time Changed Brownian Motions has particular economic appeal as

- this construction finds its rationale in the following: uncertainty in prices changes is originated by the time at which the next investor enters the market with a transaction altering the current price values, and the amount by which this current price is changed by. The random clock models the time at which the next transaction will take place; the 'size' of the price change is instead captured by the Brownian motion component.
- Empirical evidence shows that stock log-returns are Gaussian but only under trade time, rather than standard calendar time.

- Further, the time change construction recognizes that stock prices are largely driven by news, and the time between one piece of news and the next is random as is its impact.
- Finally, this construction offers a high degree of mathematical tractability as, once we operate under business time, log-returns are once again Gaussian and therefore the results derived for the Black-Scholes model still hold.

A Time Changed Brownian motion commonly used in finance is the Variance Gamma process.

5.3.1. The Variance Gamma process

Fact 51 *Let us assume that G is a Gamma process with parameters $\alpha = \lambda = k^{-1}$, for any positive constant k, so that $\mathbb{E}G(t) = t$ and $\mathbb{V}ar(G(t)) = kt$. Then, $X(t)$ is a VG process.*

We note the following

- The parameters of the Gamma process are chosen so that $\mathbb{E}(G(t)) = t$, i.e. the process chosen as random clock is an unbiased representation of calendar time.
- The VG process has infinite activity; specifically it is characterized by an infinite number of jumps of small size in a finite time period.
- The VG process has finite variation, i.e. it is characterized by a finite number of jumps of big size in a finite time period.

Other examples of time changed Brownian motions used for financial applications are the Normal Inverse Gaussian and the CGMY process.

Properties of the VG process

- The probability density function is

$$2\frac{e^{\theta x/\sigma^2}}{k^{t/k}\sigma\sqrt{2\pi}\Gamma(t/k)}\left(\frac{x^2}{\theta^2+2\sigma^2/k}\right)^{\frac{t}{2k}-\frac{1}{4}}K_{\frac{t}{k}-\frac{1}{2}}\left(\frac{|x|}{\sigma^2}\sqrt{\theta^2+2\sigma^2/k}\right). \tag{5.1}$$

- The characteristic function of the VG process is

$$\phi_X(u;t) = \left(1 - iu\theta k + u^2\frac{\sigma^2}{2}k\right)^{-\frac{t}{k}}, \tag{5.2}$$

- The expected value is
$$\mathbb{E}(X(t)) = \theta t.$$
- The variance is
$$\mathbb{V}ar(X(t)) = \left(\sigma^2+\theta^2 k\right)t.$$
- The indices of skewness and excess kurtosis are respectively

$$\begin{aligned} \mathbb{S}kew(t) &= \frac{\left(3\sigma^2+2\theta^2 k\right)\theta k}{\left(\sigma^2+\theta^2 k\right)^{3/2}\sqrt{t}}, \\ \mathbb{E}\mathbb{K}urt(t) &= \frac{\left(3\sigma^4+12\sigma^2\theta^2 k+6\theta^4 k^2\right)k}{\left(\sigma^2+\theta^2 k\right)^2 t}. \end{aligned}$$

Interpretation of the parameters

- $\theta \in \mathbb{R}$: mean of the VG process; it also controls the sign on the skewness index. Hence, the VG process has distribution skewed to the left if $\theta < 0$, and skewed to the right if $\theta > 0$. If $\theta = 0$, the process has symmetric distribution.
- $\sigma > 0$: it controls the variance of the VG process. If $\sigma = 0$, the VG process reduces to the Gamma process.
- $k > 0$: variance rate of the Gamma process. It controls the level of excess kurtosis.

Matlab Code

```
function m=get_moments_VG(theta, sg, kappa,t)

m(1,:)= theta*t; %mean
m(2,:)= sg*sg*t+theta*theta*kappa*t; %variance
numsk= (3*sg^2+2*theta^2*kappa)*theta*kappa;
densk=(sg^2+theta^2*kappa)^(3/2)*t.^0.5;
m(3,:)=numsk./densk;%skewness
numk=(3*sg^4+12*sg^2*theta^2*kappa+6*theta^4*kappa^2)*kappa;
denk=(sg*sg+theta*theta*kappa)^2*t;
m(4,:)=numk./denk;%kurtosis
```

Simulation of the VG process

The simulation procedure of the Variance Gamma process is based on the following two observations.

1. The increments of the Gamma process are independent and follow a Gamma distribution $\Gamma((t_{j+1} - t_j)/k, 1/k)$;
2. conditioned on the increments of the Gamma clock, the increments of the VG process are Gaussian with given mean and variance.

Hence, the simulation algorithm can be organized as follows.

Step 1. Simulate the increments from t_j to t_{j+1} of the Gamma clock, i.e. $G \sim \Gamma((t_{j+1} - t_j)/k, 1/k)$.

Step 2. Generate $Z \sim \mathcal{N}(0, 1)$; set $X = \theta G + \sigma\sqrt{G}Z$.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%% SIMULATING THE VARIANCE GAMMA %%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Assigning the number of simulated paths
%(nsimul), time to maturity (expiry), number of steps
%(nsteps), time step (dt) and observation times (timestep):
clear all;
nsimul=50, expiry=1, nsteps=250; dt=expiry/nsteps;
timestep=[0:dt:expiry]';
%Assigning parameters
theta=-0.4; sigma=0.3; kappa=0.25;
%Simulate increments of the Gamma process:
dG=gamrnd(dt/kappa,kappa,[nsteps,nsimul]);
%Simulate increments of the ABM on the Gamma clock scale
dX=theta*dG+sigma*sqrt(dG).*randn(nsteps,nsimul);
%Simulate VG process (use cumulative sum of the increments):
cdX=[zeros(1,nsimul); cumsum(dX)];
%Plot simulated paths:
h=figure('Color', [ 1 1 1])
plot(timestep, cdX)
title('Simulated Paths of the VG Process')
xlabel('Time (years)')
```

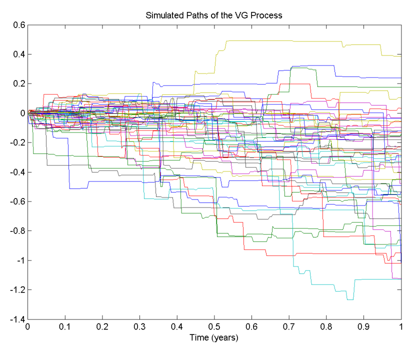

Figure 5.5: Simulated paths of the VG process. Parameters: $\theta = -0.4$, $\sigma = 0.3$, $\kappa = 0.25$.

References on Time Changed Brownian motions

- Barndorff-Nielsen, O. E. 1995. Normal inverse Gaussian distributions and the modeling of stock returns. Research report N. 300, Department of Theoretical Statistics, Aarhus University.
- Carr, P., Geman, H., Madan, D. B. and Yor,M. 2002. The fine structure of asset returns: an empirical investigation, Journal of Business, 75, 305-32.

- Carr, P. and Wu, L. 2004. Time-changed Lévy processes and option pricing. Journal of Financial Economics, 71, 113-141.
- Madan, D. B., Carr, P., and Chang, E. 1998. The variance gamma process and option pricing, European Finance Review, 2, 79-105.
- Madan, D. B. and Milne,F. 1991. Option pricing with VG martingale components, Mathematical Finance, 1, 39-45.
- Madan, D. B. and Seneta, E. 1990. The variance gamma (VG) model for share market returns. Journal of Business, 63, 511-24.

5.4. Final Remark: Lévy processes

- All the processes presented in this section share the feature of independent and stationary increments.
- The Brownian motion share the same feature as well.
- They differ in the distribution chosen to model these increments.
- A process with independent and stationary increments is called Lévy process.
- Lévy processes are widely used in financial applications.
- All processes can be made more rich in terms of features they can capture by assuming, for example, time dependent parameters, or by using more complex processes as stochastic clocks.
- For example: the instantaneous volatility of any of the processes presented above is constant. This assumption can be relaxed by assuming time dependent parameters. However, the resulting more general process will no longer have independent and stationary increments.
- The main problem with Lévy processes is that they cannot capture the volatility clustering effects, which can be captured by other models such as stochastic volatility models.

- Lévy processes and the stochastic volatility model complement each other: jump processes have a relative advantage in analytical tractability and they better capture short-term behavior of financial time series, whilst stochastic volatility models have a richer time dependency structure and are more useful to model long term behavior.
- Figure (24) compares the MJD, the VG and the Gaussian pdf at different horizons (1, 10, 20 and 90 days). We can see that the three densities approach one to the other as time horizon lengths. This is due to the nature of independency of the increments of the three processes. This allows the central limit theorem to operate: in practice the skewness and the kurtosis fade away very quickly.

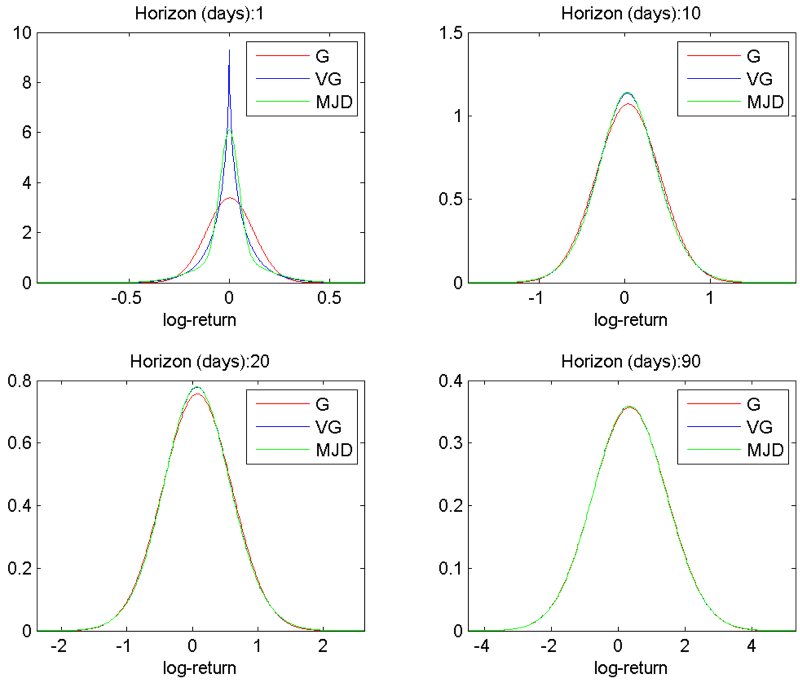

Figure 5.6: PDF of Gaussian (G), Variance-Gamma (VG) and Merton jump-diffusion (MJD) processes at different time horizons. Parameters are chosen to fit sample moments of daily log-returns of oil prices in 2012.