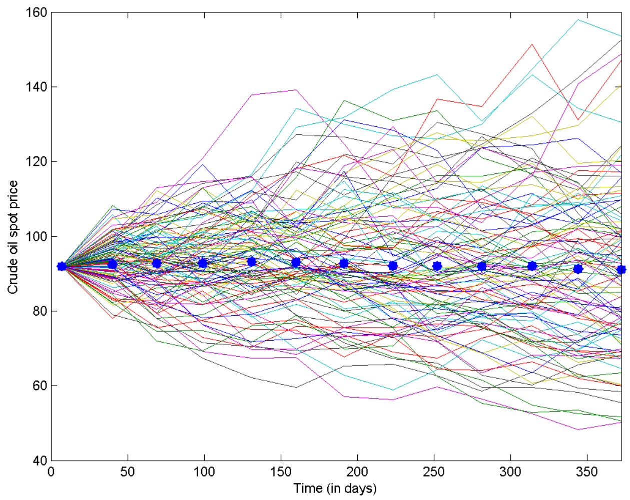

Figure 4.3: Simulated GBM paths fitting in average the term structure of futures prices (blue dotted points).

4.2. The Vasicek Mean-Reverting process

- An empirical property of several economic variables such as interest rates, inflation rates and even commodity prices, is the tendency toward lower levels (higher levels), when they are too high (low).

- This property is called **mean-reversion** and can be modeled using a so-called **Mean-Reverting (MR) process**. The effect of mean-revesion is described in figure (4.4).
- A Gaussian mean-reverting process is described by the following SDE

$$dX(t) = \alpha\,(\mu - X(t))\,dt + \sigma dW\,(t)\,, \alpha > 0.$$

- This model has been introduced in finance by Vasicek to model the instantaneous short rate. It is also named Ornstein-Uhlembeck process.
- We observe that

$$\mathbb{E}_t\,[dX(t)] = \alpha\,(\mu - X(t))\,dt,$$

  so that, assuming $\alpha > 0$, $\mathbb{E}_t\,[dX(t)] > 0$ when $X(t) < \mu$, i.e. we expect an increase (decrease) in the interest rate level when we are below (above) the level μ.
- Higher the value of α, faster the return toward the level μ. α is called speed of reversion, whilst μ determines the long-run mean-level.
- The distribution of X at any future time is Gaussian, so it allows for negative values.
- An extension, that guarantees positive interest rates, has been proposed by Cox, Ingersoll and Ross (CIR model).

4.2.1. A Note: The Ordinary Differential Equation $dx(t) = \alpha\,(\mu - x(t))\,dt$.

- We would like to solve the SDE in the Vasicek model. Let us start considering the deterministic version.

$$dx(t) = \alpha\,(\mu - x(t))\,dt.$$

- This is a first order ordinary differential equation. The procedure to solve is standard. We recall it here.

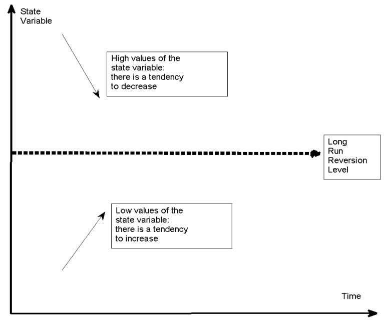

Figure 4.4: Mean reversion and expected change in the state variable (here an interest rate).

- We proceed trough the following steps:
  - We let $y(t) = g(t, x) = e^{\alpha t}x(t)$.
  - Then, $dy(t) = \alpha e^{\alpha t}x dt + e^{\alpha t}dx(t)$,
  - Therefore $dy(t) = \alpha e^{\alpha t}x(t)dt + e^{\alpha t}\alpha(\mu - x(t))\,dt$.
  - Therefore $dy(t)\,(t) = e^{\alpha t}\alpha\mu dt$
  - Finally $y\,(t) = y(0) + \alpha\mu\int_0^t e^{\alpha s}ds = y(0) + \mu\left(e^{\alpha t} - 1\right)$.

Very Important!!

Fact 34 (Solving the ode $dx(t) = \alpha(\mu - x(t))\,dt$) *The ode $dx(t) = \alpha(\mu - x(t))\,dt$ admits solution*

$$x\,(t) = e^{-\alpha t}y\,(t) \Rightarrow x\,(t) = e^{-\alpha t}x(0) + \mu\left(1 - e^{-\alpha t}\right).$$

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%MEAN REVERSION%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;expiry=10;
timestep=linspace(0,expiry,100)';
mu=100; alpha=1;
X0=100; sol1=mu+(X0-mu)*exp(-alpha*timestep);
X1=120; sol2=mu+(X1-mu)*exp(-alpha*timestep);
X2=80; sol3=mu+(X2-mu)*exp(-alpha*timestep);
%Plot solutions:
h=figure('Color', [ 1 1 1])
plot(timestep', [sol1, sol2, sol3],'.')
xlabel('Time (years)')
legend('Model 1: x_0=100, \mu = 100, \alpha = 1', ...
        'Model 2: x_0=120, \mu= 100, \alpha = 1 ',...
        'Model 3: x_0=80, \mu = 100, \alpha = 1')
print(h,'-dpng','LecBMFigMeanReversion.png')
```

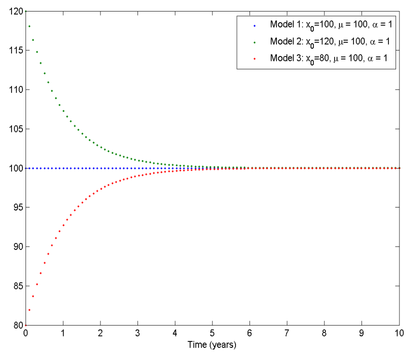

Figure 4.5: Solution of the equation $dx = \alpha(\mu - x)\,dt$ changing the initial condition x_0.

4.2.2. Solving the SDE $dX(t) = \alpha(\mu - X(t))\,dt + \sigma dW(t)$

- We need to solve

$$dX(t) = \alpha(\mu - X(t))\,dt + \sigma dW(t). \tag{4.1}$$

- By analogy with the previous ode, let us define

$$Y(t) = g\,(t, Xt) = e^{\alpha t}X(t),$$

  and apply Itô's Lemma.
- Then

$$
\begin{aligned}
& dY(t) \\
= & \left(\underbrace{\frac{\partial g(t,X)}{\partial t}}_{\alpha e^{\alpha t} X} + \alpha(\mu - X)\underbrace{\frac{\partial g(t,X)}{\partial X}}_{e^{\alpha t}} + \frac{1}{2}\sigma^2 \underbrace{\frac{\partial^2 g(t,X)}{\partial X^2}}_{0}\right) dt + \sigma \underbrace{\frac{\partial g(t,X)}{\partial X}}_{e^{\alpha t}} dW(t). \\
= & \left(\alpha e^{\alpha t} X + \alpha(\mu - X) e^{\alpha t}\right) dt + \sigma e^{\alpha t} dW(t) \\
= & \alpha \mu e^{\alpha t} dt + \sigma e^{\alpha t} dW(t).
\end{aligned}
$$

- Therefore

$$
\begin{aligned}
Y(t) &= Y(0) + \int_0^t \alpha \mu e^{\alpha s} ds + \int_0^t \sigma e^{\alpha s} dW(s) \\
&= Y(0) + \mu\left(e^{\alpha t} - 1\right) + \int_0^t \sigma e^{\alpha s} dW(s).
\end{aligned}
$$

Very Important!!

Fact 35 *The solution of the SDE 4.1 is*

$$
X(t) = e^{-\alpha t} Y(t) = e^{-\alpha t} X(0) + \mu\left(1 - e^{-\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-s)} dW(s).
$$

Very Important!!

In addition, we also have that

$$
\begin{aligned}
X(t) &\sim \mathcal{N}\left(\mathbb{E}_0(X(t)), \mathbb{V}ar_0(X(t))\right), \\
\mathbb{E}_0(X(t)) &= e^{-\alpha t} X(0) + \mu\left(1 - e^{-\alpha t}\right), \\
\mathbb{V}ar_0(X(t)) &= \sigma^2 \int_0^t e^{-2\alpha(t-s)}(s)\, ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right),
\end{aligned}
$$

where, in order to compute the variance, we have exploited the Ito isometry.

4.2.3. The (auto)-covariance function

- Let us consider two time instants, t and s, $t < s$. We have for $t < s$ (but similarly for $s < t$) that $c_X(t,s)$ is given by

$$
\begin{aligned}
c_X(t,s) &= cov\left(\sigma \int_0^t e^{-\alpha(t-u)} dW(u), \sigma \int_0^s e^{-\alpha(t-u)} dW(u)\right) \\
&= \sigma^2 cov\left(\int_0^{min(t,s)} e^{-\alpha(t-u)} dW(u), \int_0^{min(t,s)} e^{-\alpha(s-u)} dW(u)\right) \\
&= \sigma^2 e^{-\alpha(t+s)} cov\left(\int_0^t e^{\alpha u} dW(u), \int_0^t e^{\alpha u} dW(u)\right) \\
& \quad \text{by the isometry property} \\
&= \sigma^2 e^{-\alpha(t+s)} \int_0^t e^{2\alpha u} d(u) \\
&= \sigma^2 \frac{e^{-\alpha(s-t)}}{2\alpha}.
\end{aligned}
$$

- With a similar reasoning, if we take t and s with $s < t$, we have

$$
c_X(t,s) = \sigma^2 \frac{e^{-\alpha(t-s)}}{2\alpha}.
$$

Fact 36 *The auto-covariance function of the Vasicek model is given by*

$$
c_X(t,s) = \sigma^2 \frac{e^{-\alpha|t-s|}}{2\alpha}.
$$

This result can be exploited to generate simultaneously the entire trajectory of the Vasicek model: we can simulate the full path by drawing samples from a multivariate normal distribution with the above covariance matrix.

4.2.4. Matlab: Simulation of the Vasicek model

Here we simulate the Vasicek model exploiting the solution in (35).

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%SIMULATING THE VASICEK MODEL%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;close all
%Model: dr = a * (b - r ) * dt + sg * dW
%Assign Inputs
r0=0.05; a=10; b=0.07; sg=0.1; nstep=200; horizon=1;
nsimul=1000; dt=horizon/nstep;
%Compute the variance of the increments
vol2=(1-exp(-2*a*dt))/(2*a);
rall=[];
for j=1:nsimul
    %Initialize the interest rate vector
    r=zeros(nstep+1,1); r(1)=r0;
    %Simulate the increments
    dW=randn(nstep,1)*vol2^0.5;
    %Start iteration
    for i=1:nstep
        r(i+1)=b+exp(-a*dt)*(r(i)-b)+sg*dW(i);
    end
    %store the simulated path
    rall=[rall, r]
end
%Plot the sample path
h=figure('Color',[1 1 1])
plot([0:nstep]*dt,[rall, b+(r0-b)*exp(-a*[0:nstep]*dt)']);
xlabel('Time')
legend('Simulated path','Expected path')
title('Simulated path of the Vasicek model dr=a(b-r)dt+sg*dW')
print(h,'-djpg','LecBM_SimVasicek_new.jpg')
```

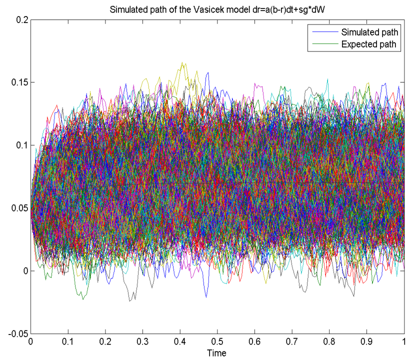

Figure 4.6: Parameters: $r_0 = 0.05$; α=10; $\mu = 0.07$; $\sigma = 0.1$; nstep=200; horizon=1.

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%VASICEK densities at different time horizons%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all
%Assign parameters
mu=0.09; sg=0.05; alpha=0.8;rt=0.04;
horizon=[0.25 0.5 0.75 1 5];
%Compute Exp. Value and variance
meanVas=mu+exp(-alpha*horizon).*(rt-mu);
varVas=sg*sg*(1-exp(-2*alpha*horizon))/(2*alpha);
range=linspace(mu-3*sg/(2*alpha)^0.5,...
               mu+3*sg/(2*alpha)^0.5,200);
pdfV=[];
for i=1:length(horizon)
meanV=meanVas(i);
stdV=varVas(i).^0.5;
pdfVas=pdf('norm', range,meanV,stdV);
pdfV=[pdfV;pdfVas];
end
h=figure('Color',[1 1 1])
plot(range,pdfV)
title('Pdf of the short rate at different times')
legend('0.25 yrs','0.5 yrs','0.75 yrs','1 years','5 yrs')
print(h,'-dpng','LecBMFigpdfVasicek.jpg')
```

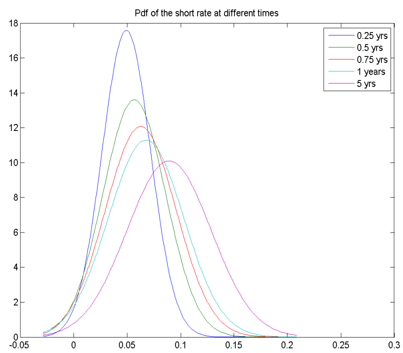

Figure 4.7: Density of the MR(0.09,0.8, 0.05) at different horizons.

MEAN-REVERTING PROCESS MR(α, μ, σ): FACTS	
The SDE	$dX(t) = \alpha(\mu - X(t))\,dt + \sigma dW(t), X(0) = x_0.$
The solution	$X(t) = e^{-\alpha t}X(0) + \mu\left(1 - e^{-\alpha t}\right) + \sigma \int_0^t e^{-\alpha(t-s)}dW(s).$
The distribution of $X(t)$	$X(t) \sim \mathcal{N}\left(\mathbb{E}_0(X(t)), \mathbb{V}ar_0(X(t))\right).$
The mean of $X(t)$	$\mathbb{E}_0(X(t)) = e^{-\alpha t}X(0) + \mu\left(1 - e^{-\alpha t}\right).$
The variance of $X(t)$	$\mathbb{V}ar_0(X(t)) = \sigma^2 \int_0^t e^{-2\alpha(t-s)}(s)\,ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$
The stationary distribution of $X(t), (t \to \infty)$	$X(t) \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{2\alpha}\right)\ if \alpha > 0.$
The auto-covariance of $X(t)$	$c_X(t,s) = \frac{\sigma^2}{2\alpha}e^{-\alpha\|t-s\|}.$

4.2.5. EXTENSION: MR WITH DETERMINISTIC VOLATILITY

- We can generalize the Vasicek model to a deterministic time-varying volatility.
- The SDE becomes
$$dX(t) = \alpha(\mu - X(t))\,dt + \sigma(t)\,dW(t).$$
- It has solution
$$X(t) = X(0) + \mu\left(1 - e^{-\alpha t}\right) + \int_0^t \sigma(s)\,e^{-\alpha(t-s)}dW(s).$$

- The solution has the following properties (variance and covariance are computed using the isometry property):
$$\begin{aligned}
X(t) &\sim \mathcal{N}\left(\mathbb{E}_0(X(t)), \mathbb{V}ar_0(X(t))\right), \\
\mathbb{E}_0(X(t)) &= X(0) + \mu\left(1 - e^{-\alpha t}\right), \\
\mathbb{V}ar_0(X(t)) &= \int_0^t \sigma^2(s)\,e^{-2\alpha(t-s)}ds; \\
\mathbb{C}ov_0(X(t), X(s)) &= \int_0^{min(t,s)} \sigma^2(u)\,e^{-2\alpha(t+s-2u)}du.
\end{aligned}$$

4.3. THE COX-INGERSOLL-ROSS MODEL (CIR) MODEL

The sde is given by
$$dX(t) = \alpha(\mu - X(t))\,dt + \sigma\sqrt{X(t)}dW(t).$$
This model has been introduced by Cox, Ingersoll and Ross to model the dynamics of the instantaneous interest rate. The peculiar form of the diffusion coefficient has been chosen to ensure that the process does not achieve negative values, still preserving the analytical tractability.

4.3.1. SOLVING THE SDE $dX(t) = \alpha(\mu - X(t))\,dt + \sigma\sqrt{X(t)}dW(t)$

- The SDE for the short rate is given by:
$$dX(t) = \alpha(\mu - X(t))\,dt + \sigma\sqrt{X}dW(t).$$
- This model shares with the Vasicek one the form of the drift term, so that it allows for mean reversion and interest rates cannot explode.
  - The mean-reversion property implies also that interest rate displays a steady state distribution.
- The difference with respect to the Vasicek model is the appearance of the square root term $\sqrt{X}$ in the diffusion term: