

Markov Chain Monte Carlo and Bayesian Inference

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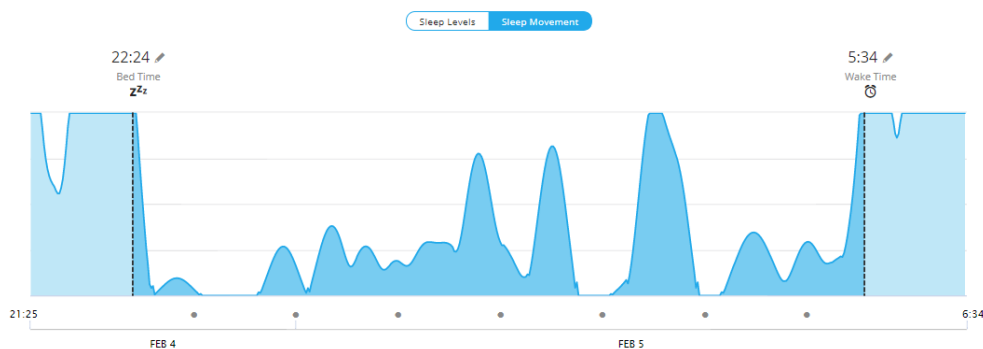
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Contents

1	Problem Description	2
1.1	Approach	2
2	Wake and Sleep Data Exploration	4
2.1	Falling Asleep Data	5
2.2	Waking Up Data	5
3	Logistic Function to Represent Transition	6
4	Prior Distribution for β and α	8
4.1	Markov Chain Monte Carlo	9
5	Posterior Probability of Sleep given Time	10
5.1	Bernoulli Variable	10
5.1.1	PyMC3 Model	11
5.2	Visualize Posteriors for β and α	12
5.3	Posterior for Sleep Visualization	13
5.3.1	Confidence Interval	15
5.4	Posterior Probability Distribution for Specific Time	16
5.5	Convergence in Markov Chain Monte Carlo	18
5.6	Trace Plots	18
6	Full Bayesian Model	19
6.1	Effect of Additional Information Visualization	21
7	Wake Model	22
7.1	Visualize Posterior for Wake given Time	23
7.2	Investigate the Wake Model	25
7.2.1	Confidence Interval	25

8	Model for Sleep Duration	26
8.1	Visualize Posterior Distribution for Sleep Duration	28
8.1.1	Query the Posterior Model	29
8.2	Visualize the Posterior and the Data	30
9	Conclusions	31

1 Problem Description



My Garmin Vivosmart watch tracks the time I fall asleep and wake up each day using motion sensing and heart rate monitoring. To augment this data, I have estimated likelihoods that I am asleep based on the condition of my bedroom light (on/off) and if my phone is charging (yes/no). My objective is to use this data to create a model that returns the probability I am asleep at a specific time for a given status of my bedroom light and phone. For a specific time, the probability of sleep given information about my bedroom light and phone is expressed as:

$$P(\text{sleep}|\text{bedroom light, phone charging})$$

In probability theory terms, this is the posterior probability at a specific time I am asleep given the status of my bedroom light and condition of my phone. The time is a continuous variable and the two additional pieces of information are discrete variables each with two states.

1.1 Approach

In order to solve this problem, I first need to express the final model in terms of Bayes Rule. At a specific time, the probability of sleep given my light and phone is:

$$P(s|L, C) = \frac{P(L, C|s) * P(s)}{P(L, C)}$$

Where $P(s)$ is the sleep probability t is the time, L is the condition of my bedroom light (on/off), and C is the charging status of my phone (yes/no).

$P(s)$ is the prior probability of sleep at the specified time. As time is a continuous variable, we will use an approximate Markov Chain Monte Carlo method to find the *most likely* probability distribution to find the prior.

One main assumption we make to simplify the inference is that the probability of my bedroom light condition and phone charging status are independent of one another with the knowledge of whether or not I am asleep. That is, these discrete variables are conditionally independent on sleep. Applying this assumption, the final model becomes at a specific time becomes:

$$P(s|L,C) = \frac{(P(L|s) * P(C|s) * P(s))}{P(L|s) * P(C|s) * P(s) + P(L|\bar{s}) * P(C|\bar{s}) * P(\bar{s})}$$

This is the final expanded expression we need to solve to find the posterior probability I am asleep at a specific time given the knowledge of my bedroom light and phone charging.

Time is a continuous variable, and specifying the entire joint probability for sleep given the time, $P(s)$ would be infeasible even with time discretized into one-minute intervals. Therefore, we will use an approximate method, Markov Chain Monte Carlo (MCMC), to find the posterior probability parameters. Markov Chain Monte Carlo samples based on maximizing the likelihood of the parameters under the data. To find the most likely posterior probability distribution of sleep given the time, we can use the average of all the samples of the parameters.

After the posterior probability of sleep as a function of time, $P(s)$, has been calculated, this becomes the prior in the final model. The other two variables, C and L , are discrete, which means we can specify the entire joint probability rather easily. The final model uses Bayes Rule to update the probability of sleep with the additional information. This represents the essence of Bayesian inference: attempting to become less wrong by updating predictions based on new evidence.

The general method is as follows, with additional details provided in the respective sections.

1. Format the data (done in separate notebook) and visualize
2. Choose function to represent probability of sleep given the time
3. Use Markov Chain Monte Carlo and the data to find most likely parameters for the selected posterior distribution
4. Use the posterior probability as the prior for applying Bayes Rule using additional data about light and phone status
5. Build a model for Bayesian Inference to find the probability of sleep given the time, light condition, and phone charging info
6. Interpret and visualize model

We can do this separately for both the sleep and waking data, although I will only build the complete model for the sleep data.

I make extensive use of the [PyMC3 library](#) for Markov Chain Monte Carlo and Bayesian Inference methods in this report.

2 Wake and Sleep Data Exploration

The wake and sleep data contains more than two months of information. The Garmin watch records when I fall asleep and wake up based on motion and heart rate. It is not 100% accurate as it often will think I'm sleeping if I turn off notifications and am quietly reading in bed. Sometimes we have to deal with imperfect data, and, because there are more truthful than false observations, we can expect the correct data to have a larger effect on the model.

First, we will import the required libraries, and visualize both the sleep data and the waking data.

```
In [1]: # pandas and numpy for data manipulation
import pandas as pd
import numpy as np

# scipy for algorithms
import scipy
from scipy import stats

# pymc3 for Bayesian Inference, pymc built on t
import pymc3 as pm
import theano.tensor as tt
import scipy

# matplotlib for plotting
import matplotlib.pyplot as plt
%matplotlib inline
from IPython.core.pylabtools import figsize
import matplotlib

import json
s = json.load(open('style/bmh_matplotlibrc.json'))
matplotlib.rcParams.update(s)
matplotlib.rcParams['figure.figsize'] = (10, 3)
matplotlib.rcParams['font.size'] = 14

# Number of samples for Markov Chain Monte Carlo
N_SAMPLES = 5000
```

WARNING (theano.tensor.blas): Using NumPy C-API based implementation for BLAS functions.

```
In [2]: # Data formatted in different notebook
sleep_data = pd.read_csv('data/sleep_data.csv')
wake_data = pd.read_csv('data/wake_data.csv')
```

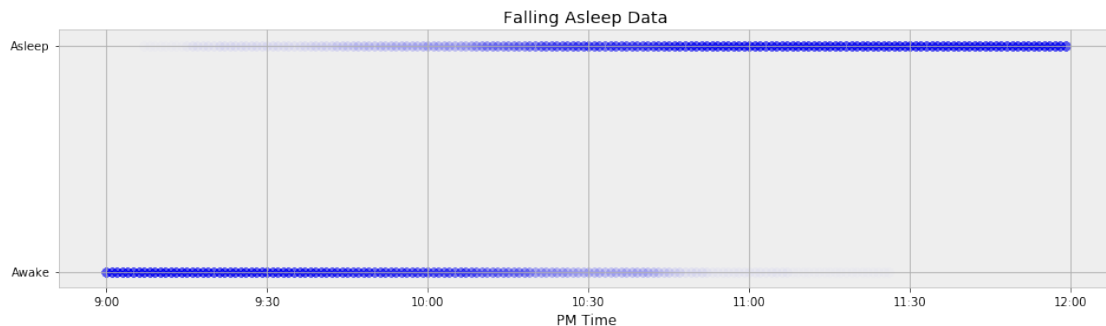
```
# Labels for plotting
sleep_labels = ['9:00', '9:30', '10:00', '10:30', '11:00', '11:30', '12:00']
wake_labels = ['5:00', '5:30', '6:00', '6:30', '7:00', '7:30', '8:00']
```

2.1 Falling Asleep Data

Each dot represents one observation at a specific time with the color intensity corresponding to the number of points at that time. We can see that I tend to fall asleep a little after 10:00 PM.

```
In [3]: figsize(16, 4)
```

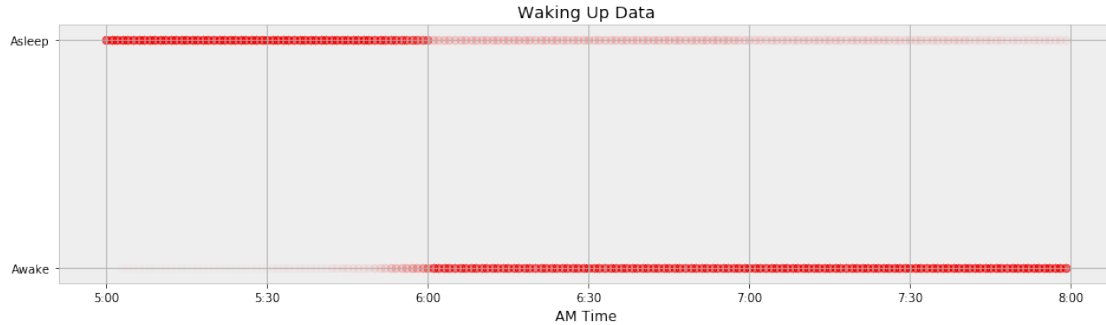
```
# Sleep data
plt.scatter(sleep_data['time_offset'], sleep_data['indicator'],
            s= 60, alpha=0.01, facecolor = 'b', edgecolors='b')
plt.yticks([0, 1], ['Awake', 'Asleep']); plt.xlabel('PM Time');
plt.title('Falling Asleep Data')
plt.xticks([-60, -30, 0, 30, 60, 90, 120], sleep_labels);
```



2.2 Waking Up Data

My alarm is set for 6:00 AM every day of the week, and the wake data is more consistent than the sleep data. I nearly always wake up within a 10 minute window around 6:00 AM.

```
In [4]: # Wake data
plt.scatter(wake_data['time_offset'], wake_data['indicator'],
            s= 50, alpha = 0.01, facecolor='r', edgecolors = 'r');
plt.yticks([0, 1], ['Awake', 'Asleep']); plt.xlabel('AM Time');
plt.title('Waking Up Data')
plt.xticks([-60, -30, 0, 30, 60, 90, 120], wake_labels);
```



3 Logistic Function to Represent Transition

We need to decide on a function to represent the transition from being awake to sleeping. There are a number of acceptable models, and here we will assume this transition can be modeled as a logistic function. A logistic function (also called a sigmoid) is a non-linear function bounded between 0 and 1. As $t \rightarrow -\infty$, $p(s) \rightarrow 0$ and, as $t \rightarrow +\infty$, $p(s) \rightarrow 1$. The expression for a logistic probability distribution for sleep as a function of time is:

$$p(s) = \frac{1}{1 + e^{\beta t}}$$

The β parameter is unknown and be esimated using Markov Chain Monte Carlo sampling. MCMC samples from the prior for each parameter, trying to maximize the proba-bilty of the parameter given the data.

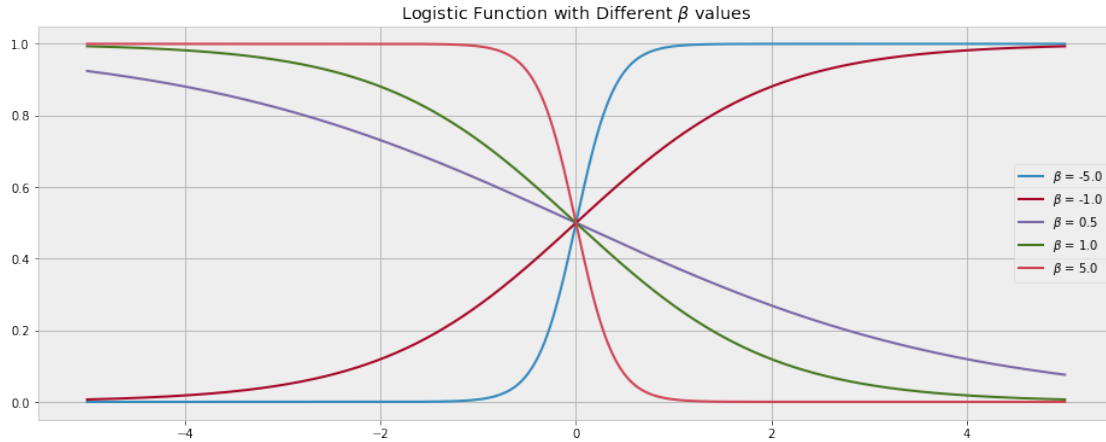
Several logistic functions with various β parameters are shown below:

In [5]: `figsize(16, 6)`

```
# Logistic function with only beta
def logistic(x, beta):
    return 1. / (1. + np.exp(beta * x))

# Plot examples with different betas
x = np.linspace(-5, 5, 1000)
for beta in [-5, -1, 0.5, 1, 5]:
    plt.plot(x, logistic(x, beta), label = r"$\beta$ = %.1f" % beta)

plt.legend();
plt.title(r'Logistic Function with Different $\beta$ values');
```



There is one problem with the basic logistic function as shown above: the transition is centered at 0. However, in my sleeping data, the transition is around 10:00 pm for sleeping and 6:00 am for waking. We address this by adding an offset, called a bias, to adjust the location of the logistic function. The logistic function now is:

$$P(s) = \frac{1}{1 + e^{\beta t + \alpha}}$$

This introduces another unknown parameter, α , which we will also find from Markov Chain Monte Carlo.

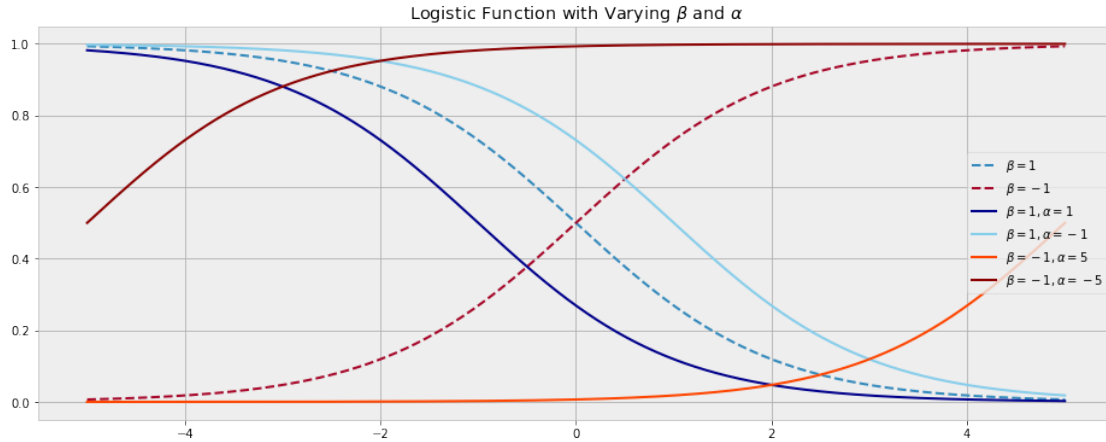
The logistic function with various α and β parameters is shown below.

```
In [6]: # Logistic function with both beta and alpha
def logistic(x, beta, alpha=0):
    return 1.0 / (1.0 + np.exp(np.dot(beta, x) + alpha))

x = np.linspace(-5, 5, 1000)

plt.plot(x, logistic(x, beta=1), label=r"$\beta = 1$", ls="--", lw=2)
plt.plot(x, logistic(x, beta=-1), label=r"$\beta = -1$", ls="--", lw=2)

plt.plot(x, logistic(x, 1, 1),
         label=r"$\beta = 1, \alpha = 1$", color="darkblue")
plt.plot(x, logistic(x, 1, -1),
         label=r"$\beta = 1, \alpha = -1$", color="skyblue")
plt.plot(x, logistic(x, -1, 5),
         label=r"$\beta = -1, \alpha = 5$", color="orangered")
plt.plot(x, logistic(x, -1, -5),
         label=r"$\beta = -1, \alpha = -5$", color="darkred")
plt.legend();
plt.title(r'Logistic Function with Varying $\beta$ and $\alpha$');
```



β shifts the direction and steepness of the curve, while α changes the location. We will use MCMC to find the most likely value of these parameters under the data.

4 Prior Distribution for β and α

We have no evidence to suggest what the prior distributions for the model parameters β and α are ahead of time. Therefore, we can model them as if they came from a normal distribution. The normal, or Gaussian, distribution is defined by the mean, μ , and the precision, τ . The precision is the reciprocal of the standard deviation, σ . The mean defines the location of the distribution and the precision shows the spread. A larger value of τ indicates the data is less spread out (it is more precise) and hence the variation is smaller. The mean can be either positive or negative, but the precision will always be positive. A normal distribution as defined here is represented as:

$$f(x|\mu, \tau) = \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{\tau}{2}(x - \mu)^2\right)$$

Probability density functions for three normal distributions are shown below.

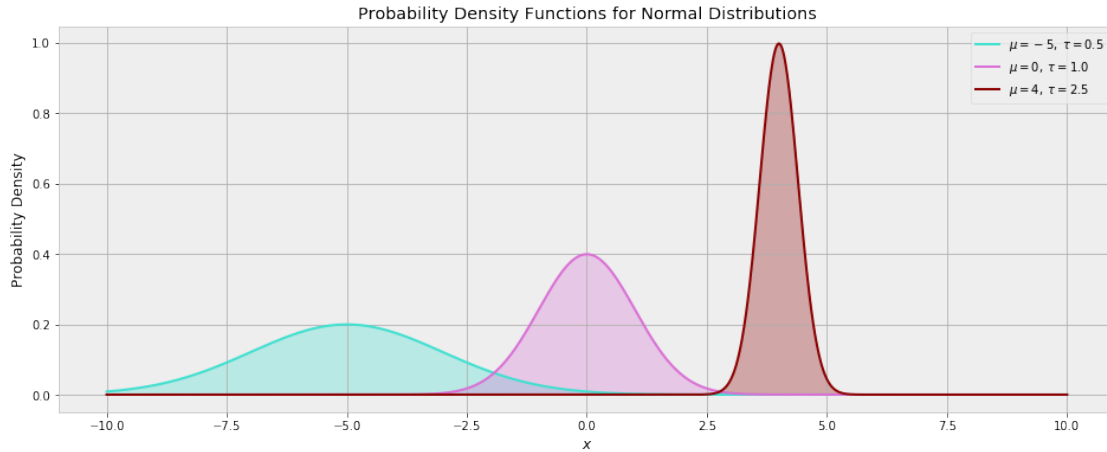
```
In [7]: # Set up the plotting parameters
        nor = stats.norm
        x = np.linspace(-10, 10, 1000)
        mu = (-5, 0, 4)
        tau = (0.5, 1, 2.5)
        colors = ("turquoise", "orchid", "darkred")

        # Plot 3 pdfs for different normal distributions
        params = zip(mu, tau, colors)
        for param in params:
            y = nor.pdf(x, loc = param[0], scale = 1 / param[1])
```



```
plt.plot(x, y,
        label="$\mu = %d, \tau = %.1f$" % (param[0], param[1]),
        color = param[2])
plt.fill_between(x, y, color = param[2], alpha = 0.3)

plt.legend();
plt.xlabel("$x$")
plt.ylabel("Probability Density")
plt.title("Probability Density Functions for Normal Distributions");
```



The expected value of a normal distribution is the mean.

$$E[X|\mu, \tau] = \mu$$

The variance of a normal distribution is equal to:

$$Var[X|\mu, \tau] = \frac{1}{\tau}$$

Again, we have no assumptions about the value for either μ or τ in the prior distributions for α and β . When we initialize the model, we can use $\mu = 0$ and a relatively large variance such as $\tau = 0.05$. Markov Chain Monte Carlo will sample values of μ and τ that try to maximize the likelihood of α and β under the data.

4.1 Markov Chain Monte Carlo

Markov Chain Monte Carlo will sample both β and α from two normal distributions to find the parameters. Each iteration (state), an estimate for both β and α are drawn from the prior. If the parameters increase the probability of the data, the state is accepted, but if the parameters are not in agreement with the data, the state is rejected. Monte Carlo refers to the sampling part of the algorithm. Markov Chain means that the next state is

only dependent on the current state in a first order process (second order depends on the current and 1 previous step, third order on the current and 2 previous steps and so on). MCMC will return every sample of the parameters for the number of specified steps. This is known as the model trace. To find the most likely parameters, we can take the average of the samples in the trace. MCMC does not give an exact answer, but rather tries to find the maximum likelihood states under the data.

When modeling with MCMC up to 50% of the initial steps, referred to as the burn-in part of the trace, are discarded because the algorithm returns more likely parameters as the number of samples increases. The initial samples are less likely than the latter samples on average. There are a number of methods to test for convergence of MCMC, including visually inspecting the trace, and calculating the auto-correlation of the trace (a lower auto-correlation is an indicator of convergence). We will look at the trace in this example, but will not take rigorous steps to address convergence. There are also a number of methods to choose a smart starting value for the Markov Chain such as Maximum A Posterior estimation. Choosing an intelligent initial value can speed up convergence.

5 Posterior Probability of Sleep given Time

We have all the pieces for the posterior probability and can now put them together. The logistic function describes the transition from awake to asleep, but we do not know the parameters β and α . The aim is to find the parameters of the logistic function which maximize the likelihood of the observed data. The parameters are assumed to come from a normal distribution defined by a mean, μ and a variance, τ . The MCMC algorithm will sample values of μ and τ for both α and β to try and maximize the parameters of the logistic function given the data.

The data is connected to the parameters through a Bernoulli Variable.

5.1 Bernoulli Variable

A Bernoulli variable is a discrete random variable that is either 0 or 1. In our example, we can model asleep or awake as a Bernoulli variable where awake is 0 and asleep is 1. The Bernoulli variable for sleep depends on the time, in a manner defined by the logistic function.

$$\text{Sleep Probability, } S_i \sim \text{Ber}(p(t_i)), \quad i = 1..N$$

$p(t_i)$ is the logistic function with the independent variable time, so this becomes:

$$P(\text{sleep}) = \text{Ber}\left(\frac{1}{1 + e^{(\beta t_i + \alpha)}}\right)$$

The goal of MCMC is to find the α and β parameters using the data and assuming normal priors.

5.1.1 PyMC3 Model

We are using a powerful Bayesian Inference library in Python called PyMC3. This library has features for running Markov Chain Monte Carlo and other inference algorithms. This report does not detail PyMC3, but a great book for getting started is *Probabilistic Programming and Bayesian Methods for Hackers* by Cameron Davidson-Pilon which is available for free on [GitHub](#)

The following code creates the model and performs MCMC, drawing `N_SAMPLES` number of samples for β and α . The specific sampling algorithm is [Metropolis Hastings](#). We feed in the data and tell the model it is observations of the Bernoulli variable. The model then tries to maximize the parameters under the data.

```
In [8]: # Sort the values by time offset
sleep_data.sort_values('time_offset', inplace=True)

# Time is the time offset
time = np.array(sleep_data.loc[:, 'time_offset'])

# Observations are the indicator
sleep_obs = np.array(sleep_data.loc[:, 'indicator'])

In [9]: with pm.Model() as sleep_model:
    # Create the alpha and beta parameters
    alpha = pm.Normal('alpha', mu=0.0, tau=0.05, testval=0.0)
    beta = pm.Normal('beta', mu=0.0, tau=0.05, testval=0.0)

    # Create the probability from the logistic function
    p = pm.Deterministic('p', 1. / (1. + tt.exp(beta * time + alpha)))

    # Create the bernoulli parameter which uses the observed data
    observed = pm.Bernoulli('obs', p, observed=sleep_obs)

    # Starting values are found through Maximum A Posterior estimation
    # start = pm.find_MAP()

    # Using Metropolis Hastings Sampling
    step = pm.Metropolis()

    # Sample from the posterior using the sampling method
    sleep_trace = pm.sample(N_SAMPLES, step=step);
```

Multiprocess sampling (2 chains in 2 jobs)

CompoundStep

>Metropolis: [beta]

>Metropolis: [alpha]

The number of effective samples is smaller than 10% for some parameters.

The trace variable contains all of the samples drawn from the posterior for β and α . We can graph these samples to explore how they change over the course of sampling. The idea of MCMC is that the samples get more likely given the data as the algorithm continues. In other words, the MCMC algorithm converges on the most likely values as the samples increase. We expect the latter values drawn from the posterior to be more accurate than the earlier values. In Markov Chain Monte Carlo, it is common practice to discard a portion of the samples, usually about 50%, which are known as the burn-in samples. For this report I am not discarding any samples, but in a real application, we would run the model for many more steps and discard the initial samples.

5.2 Visualize Posteriors for β and α

The values returned in the trace are all the samples drawn for the parameters. We can visually inspect these values in histograms.

```
In [10]: # Extract the alpha and beta samples
         # Currently using all, including the burn-in period
         alpha_samples = sleep_trace["alpha"][:, None]
         beta_samples = sleep_trace["beta"][:, None]

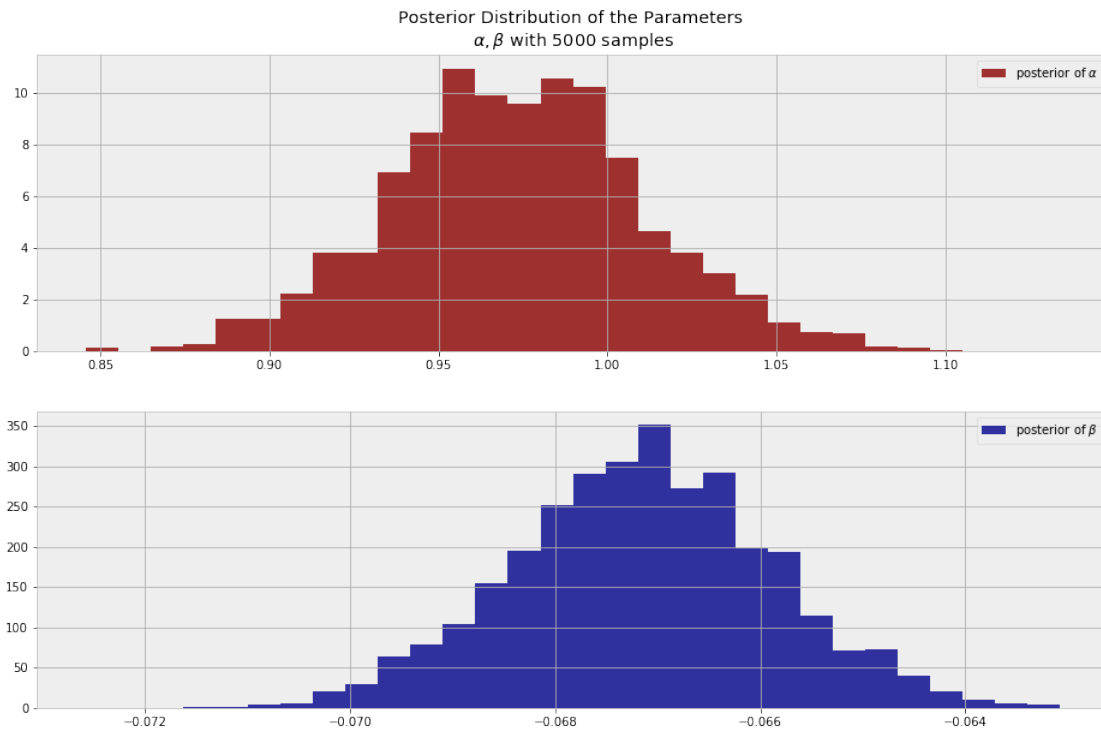
In [11]: figsize(16, 10)

plt.subplot(211)
plt.title(r"""Posterior Distribution of the Parameters
           $\alpha$, $\beta$ with %d samples""" % N_SAMPLES)

plt.hist(alpha_samples, histtype='stepfilled',
         color = 'darkred', bins=30, alpha=0.8,
         label=r"posterior of $\alpha$", normed=True);
plt.legend()

plt.subplot(212)

plt.hist(beta_samples, histtype='stepfilled',
         color = 'darkblue', bins=30, alpha=0.8,
         label=r"posterior of $\beta$", normed=True)
plt.legend();
```



If the β values were centered around 0 that would indicate time has no effect on the probability of being asleep. The α values also are not at 0, indicating that there is an offset from 10:00 pm in terms of being asleep. I choose to represent the times as an offset from 10:00 PM to avoid dealing with data times as much as possible.

The spread of the data gives us a measure of uncertainty about the data. A larger spread indicates more uncertainty. As there is considerable overlap in the observations for awake and asleep, the uncertainty is expected to be large. To find the most likely posterior distribution for sleep given the time, we take the average of the α and β samples.

5.3 Posterior for Sleep Visualization

```
In [12]: # Time values for probability prediction
time_est = np.linspace(time.min()- 15, time.max() + 15, 1e3)[: , None]

# Take most likely parameters to be mean values
alpha_est = alpha_samples.mean()
beta_est = beta_samples.mean()

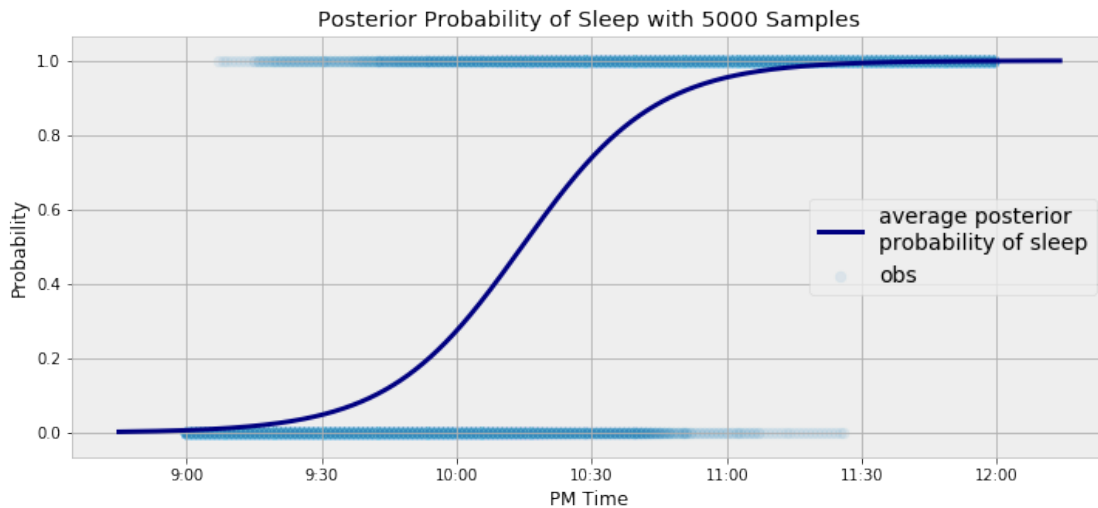
# Probability at each time using mean values of alpha and beta
sleep_est = logistic(time_est, beta=beta_est, alpha=alpha_est)

In [13]: figsize(12, 5)
```

```

plt.plot(time_est, sleep_est, color = 'navy',
         lw=3, label="average posterior \nprobability of sleep")
plt.scatter(time, sleep_obs, edgecolor = 'skyblue',
            s=50, alpha=0.1, label='obs')
plt.title('Posterior Probability of Sleep with %d Samples' % N_SAMPLES);
plt.legend(prop={'size':14})
plt.ylabel('Probability')
plt.xlabel('PM Time');
plt.xticks([-60, -30, 0, 30, 60, 90, 120], sleep_labels);

```



The posterior probability increases from 0 to 1 as the time gets later. The model is not perfect because of the noise in the data, but it is an adequate approximation based on the observations and can provide useful estimates.

```
In [36]: print('The probability of sleep increases to above 50% at 10:{} PM.'.format(int
```

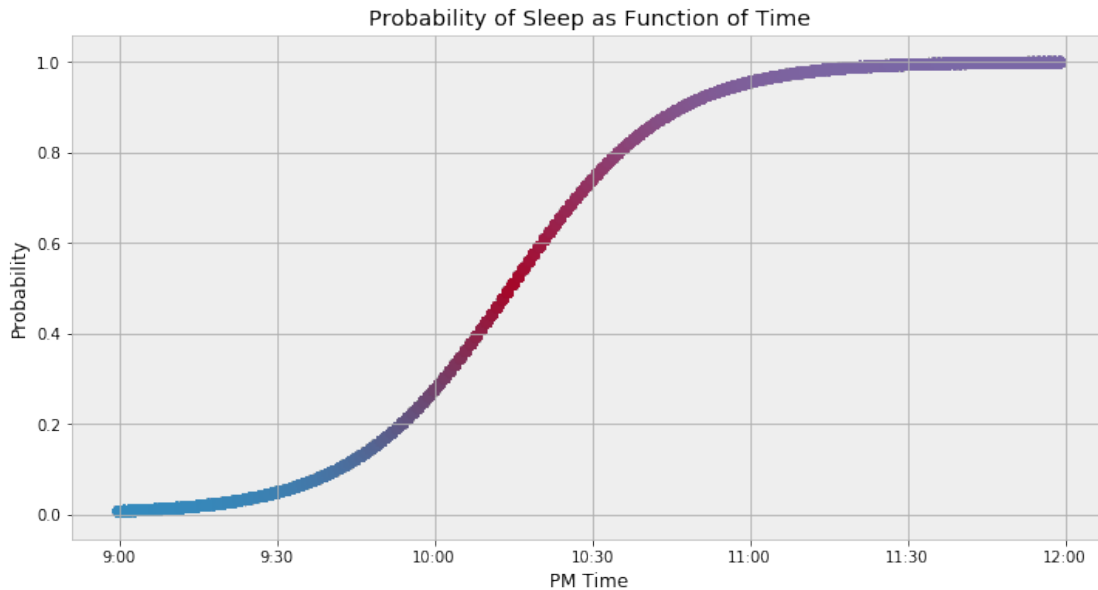
The probability of sleep increases to above 50% at 10:14 PM.

```

In [14]: colors = ["#348ABD", "#A60628", "#7A68A6"]
         cmap = matplotlib.colors.LinearSegmentedColormap.from_list("BMH", colors)
         figsize(12, 6)
         probs = sleep_trace['p']

         plt.scatter(time, probs.mean(axis=0), cmap = cmap,
                     c = probs.mean(axis=0), s = 50);
         plt.title('Probability of Sleep as Function of Time')
         plt.xlabel('PM Time');
         plt.ylabel('Probability');
         plt.xticks([-60, -30, 0, 30, 60, 90, 120], sleep_labels);

```



The posterior can be queried at any time (as an offset from 10:00 PM) to find the probability I am asleep.

```
In [15]: print('10:00 PM probability of being asleep: {:.2f}%'.
            format(100 * logistic(0, beta_est, alpha_est)))
          print('9:30 PM probability of being asleep: {:.2f}%'.
                format(100 * logistic(-30, beta_est, alpha_est)))
          print('10:30 PM probability of being asleep: {:.2f}%'.
                format(100 * logistic(30, beta_est, alpha_est)))
```

```
10:00 PM probability of being asleep: 27.42%.
9:30 PM probability of being asleep: 4.80%.
10:30 PM probability of being asleep: 73.90%.
```

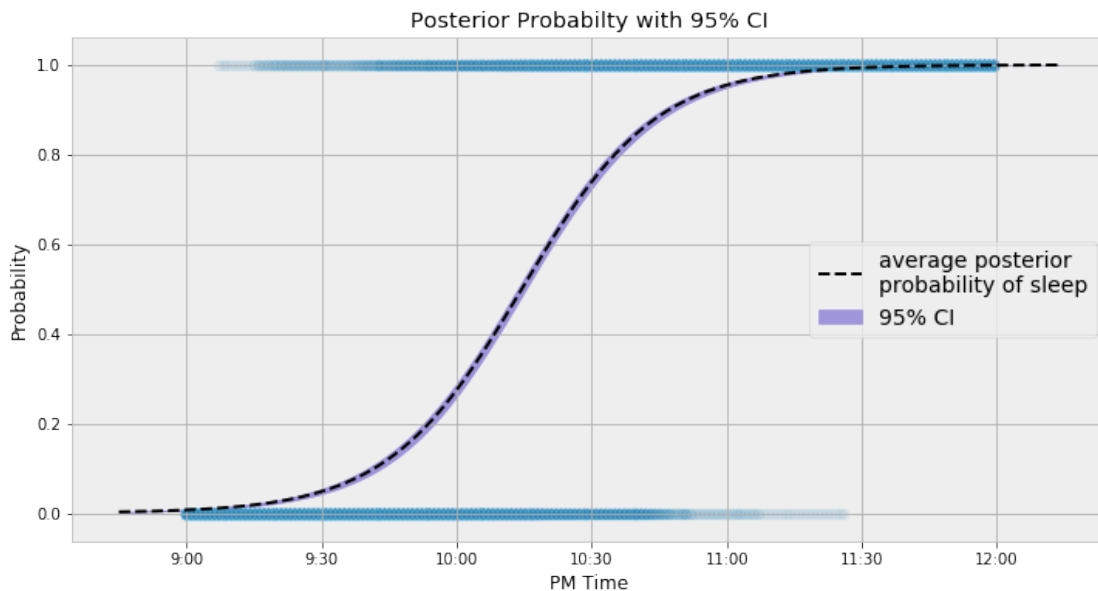
5.3.1 Confidence Interval

There are many other diagnostics of the model that we can perform. For example, we know there is a considerable amount of uncertainty in our estimates for α and β . To reflect this in the graph, we can include the 95% confidence interval at each time based on all of the samples.

```
In [16]: sleep_all_est = logistic(time_est.T, beta_samples, alpha_samples)
          quantiles = stats.mstats.mquantiles(sleep_all_est, [0.025, 0.975], axis=0)

In [17]: plt.fill_between(time_est[:, 0], *quantiles, alpha=0.6,
                          color='slateblue', label = '95% CI')
```

```
plt.plot(time_est, sleep_est, lw=2, ls='--',
        color='black', label="average posterior \nprobability of sleep")
plt.xticks([-60, -30, 0, 30, 60, 90, 120], sleep_labels);
plt.scatter(time, sleep_obs, edgecolor = 'skyblue', s=50, alpha=0.1);
plt.legend(prop={'size':14})
plt.xlabel('PM Time'); plt.ylabel('Probability');
plt.title('Posterior Probabilty with 95% CI');
```



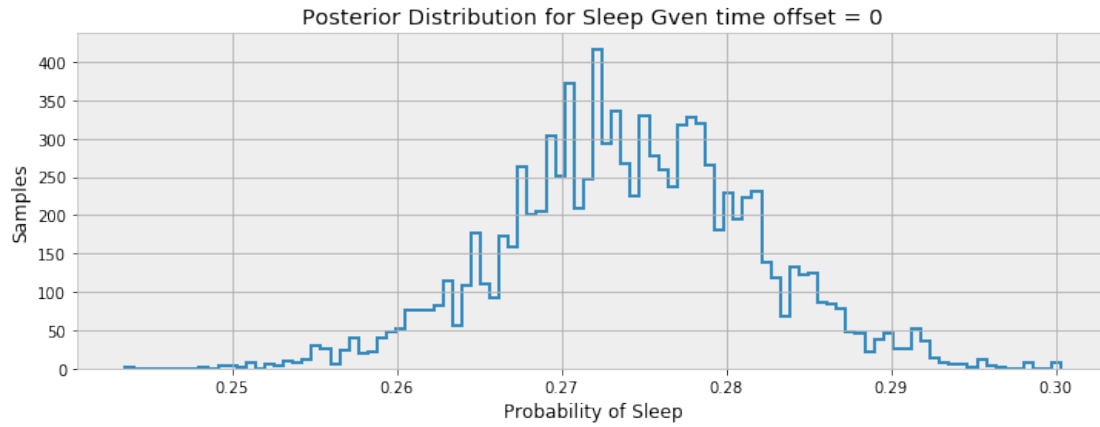
At each time, there is a measure of uncertainty as to whether or not I am asleep. This represents the fact that MCMC does not return the True parameters.

5.4 Posterior Probability Distribution for Specific Time

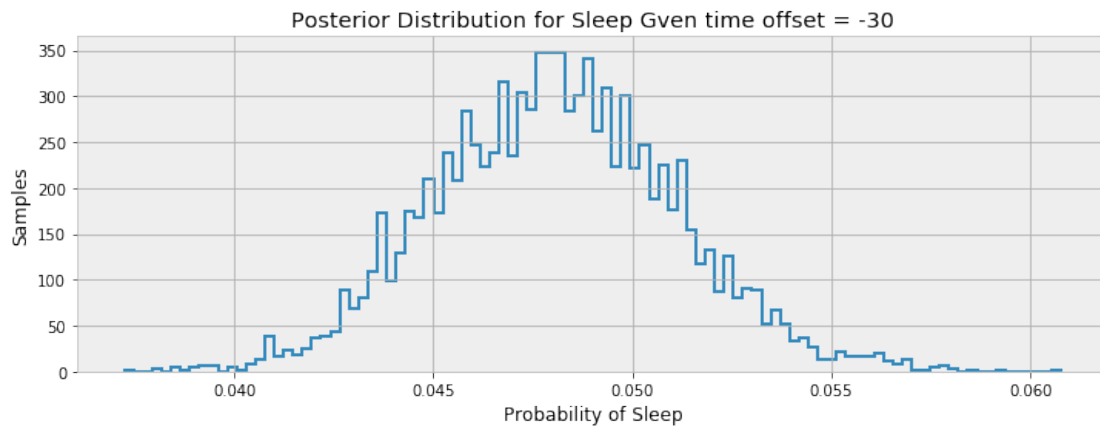
We can also plot the posterior distribution of sleep at a time as a histogram based on all of the samples for the paramters. This gives us another look at the uncertainty in the model.

```
In [18]: def sleep_posterior(time_offset):
    figsize(12, 4)
    prob = logistic(time_offset, beta_samples, alpha_samples)
    plt.hist(prob, bins=100, histtype='step', lw=2)
    plt.title('Posterior Distribution for Sleep Gven time offset = %d' %
              time_offset)
    plt.xlabel('Probability of Sleep'); plt.ylabel('Samples')
    plt.show();
```

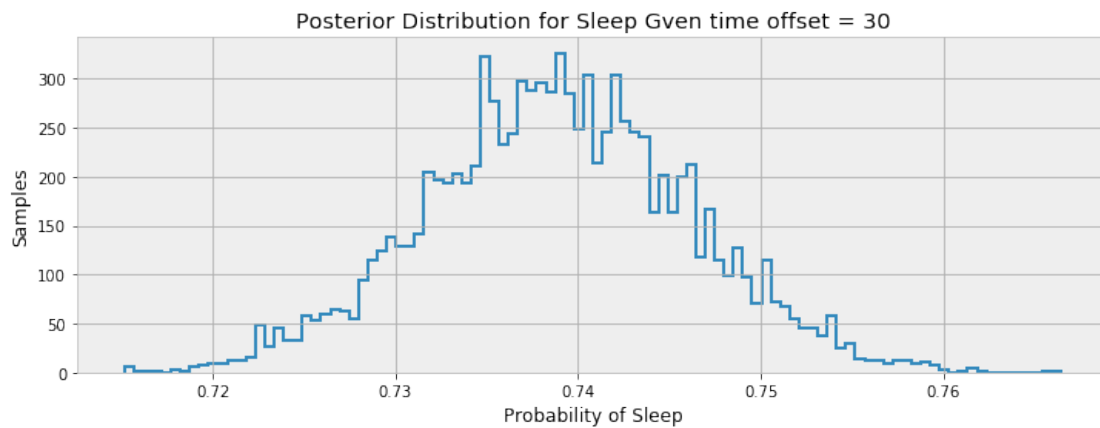
```
In [19]: sleep_posterior(0)
```

```
In [20]: sleep_posterior(-30)
```



```
In [21]: sleep_posterior(30)
```



5.5 Convergence in Markov Chain Monte Carlo

How can we know if the model converged? We can look at the trace, or the path of the values over sampling. Another option is to look at the auto-correlation of the samples. In Markov Chain modeling, the samples are correlated with themselves because the next value depends on the current state (or the current state and past states based on the order). Initially, the algorithm tends to wander about the search space and will have a high auto-correlation. As the algorithm converges, the samples will settle down around a value and one measure of convergence is a low auto-correlation. We will not do a rigorous study of convergence in this report, but we can plot the traces of all the samples.

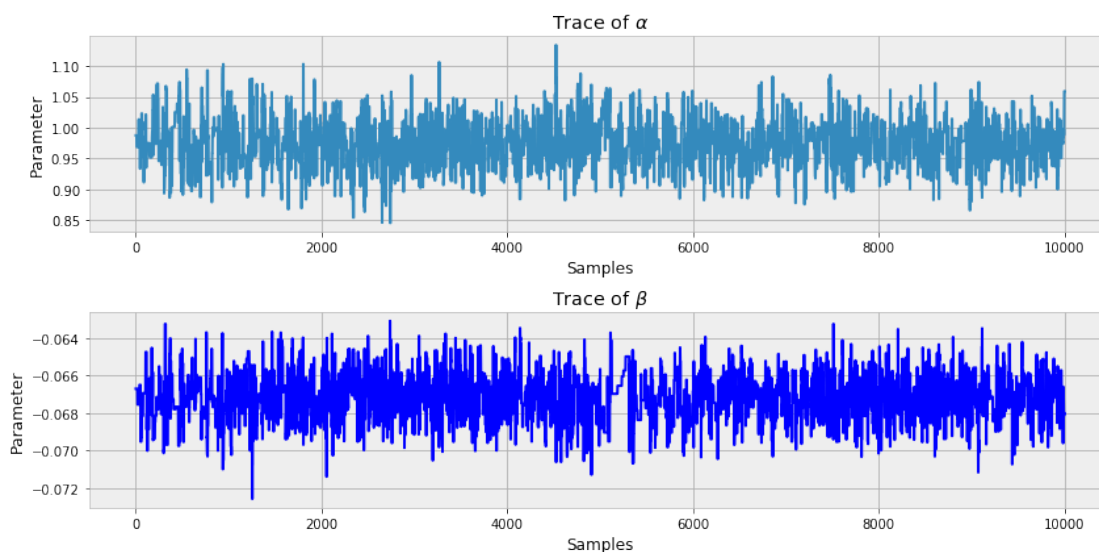
5.6 Trace Plots

The plots below show all the samples of α and β from the algorithm.

```
In [22]: figsize(12, 6)
```

```
# Plot alpha trace
plt.subplot(211)
plt.title(r'Trace of  $\alpha$ ')
plt.plot(alpha_samples)
plt.xlabel('Samples'); plt.ylabel('Parameter');

# Plot beta trace
plt.subplot(212)
plt.title(r'Trace of  $\beta$ ')
plt.plot(beta_samples, color='b')
plt.xlabel('Samples'); plt.ylabel('Parameter');
plt.tight_layout(h_pad=0.8)
```



6 Full Bayesian Model

The model we have currently returns the probability I am asleep given only the time $p(s|t)$. There are additional pieces of information such as if my bedroom light is on, or if my phone is charging, that would improve the model. The way to incorporate additional evidence in Bayesian inference is by using Bayes Rule. We can use the posterior probability for if I am asleep as the prior in the new model. I don't have data on the likelihoods for my light and phone, but I can provide estimates based on habits. Following are the likelihoods for these pieces of evidence:

$$P(L = 1|s) = 0.01$$

$$P(L = 1|\bar{s}) = 0.90$$

and

$$P(C = 1|s) = 0.95$$

$$P(C = 1|\bar{s}) = 0.40$$

Where $L = 1$ indicates my bedroom light is on and $C = 1$ indicates my phone is charging. $P(C = 1|\bar{s}) = 0.40$ means given that I am not asleep, there is a 40% probability my phone is charging.

Applying Bayes' Equation, the final posterior probability for sleep at a given time knowing the condition of my light and phone is:

$$P(s|L, C) = \frac{(P(L|s) * P(C|s) * P(s))}{P(L|s) * P(C|s) * P(s) + P(L|\bar{s}) * P(C|\bar{s}) * P(\bar{s})}$$

We have all the parts we need as the prior $P(s)$ was calculated from MCMC. We simply apply Bayes Rule with all the evidence.

```
In [23]: alpha_sleep_est = alpha_est
        beta_sleep_est = beta_est
```

```
In [24]: # Returns the prior based on the average parameters from MCMC
        def sleep_prior(time_offset):
            return logistic(time_offset,
                           beta=beta_sleep_est,
                           alpha=alpha_sleep_est)
```

```
In [25]: # Returns the posterior probability of sleep given the time and condition of th
        def sleep_posterior(time_offset, light, charge,
                           light_likelihood_sleep=0.01,
                           light_likelihood_awake=0.90,
```

```

        charge_likelihood_sleep=0.95,
        charge_likelihood_awake=0.40):
prior_sleep = sleep_prior(time_offset)
prior_awake = 1 - prior_sleep

# Change the priors based on the evidence
if not light:
    light_likelihood_sleep = 1 - light_likelihood_sleep
    light_likelihood_awake = 1 - light_likelihood_awake
if not charge:
    charge_likelihood_sleep = 1 - charge_likelihood_sleep
    charge_likelihood_awake = 1 - charge_likelihood_awake

# Calculate the numerator and denominator in Baye's Equation
numerator = light_likelihood_sleep * charge_likelihood_sleep * prior_sleep
denominator = (numerator +
               (light_likelihood_awake * charge_likelihood_awake * prior_awa

# Return the posterior probability
posterior_sleep = numerator / denominator
return posterior_sleep

```

We can now use this model to make predictions with all the information about time, light condition, and phone charging status.

Let's look at how the extra evidence affects the probability. First up, what is the probability I am asleep at 10:00 pm if my light is off and my phone charging?

```

In [26]: print('The prior for this case is      {:.2f}%.'.
              format(100 * sleep_prior(0)))
          print('The posterior for this case is {:.2f}%.'.
              format(100 * sleep_posterior(0, 0, 1)))

```

The prior for this case is 27.42%.

The posterior for this case is 89.88%.

How about if it is 10:30 pm, my light is on, and my phone is charging?

```

In [27]: print('The prior for this case is      {:.2f}%.'.
              format(100 * sleep_prior(30)))
          print('The posterior for this case is {:.2f}%.'.
              format(100 * sleep_posterior(30, 1, 1)))

```

The prior for this case is 73.90%.

The posterior for this case is 6.95%.

One more situation. It is 9:45 pm, my light is off, and my phone is not charging.

```
In [28]: print('The prior for this case is      {:.2f}%'.  
             format(100 * sleep_prior(-15)))  
         print('The posterior for this case is {:.2f}%'.  
             format(100 * sleep_posterior(-15, 0, 0)))
```

The prior for this case is 12.12%.

The posterior for this case is 10.22%.

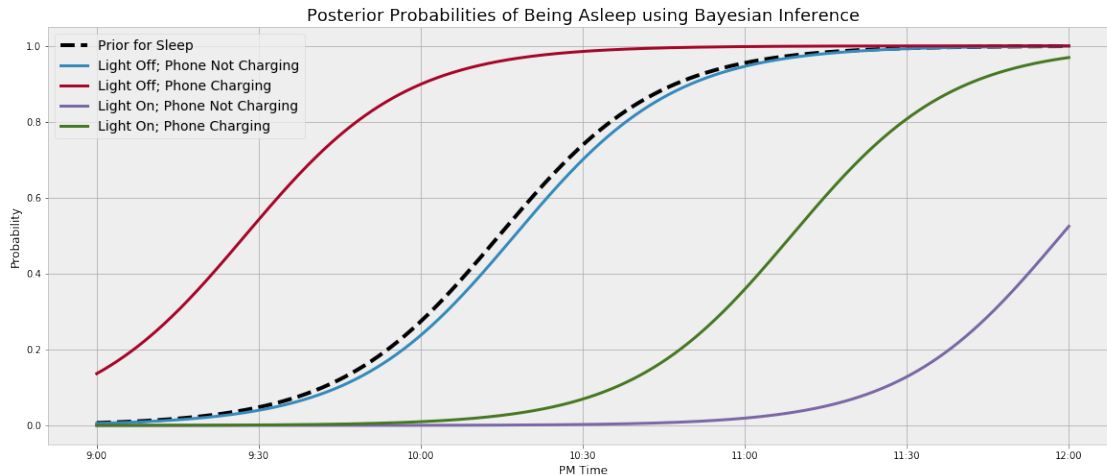
We can see how incorporating more information changes the estimate. As we gather more data, the model becomes more accurate because we use the observations and Bayes Rule. This is a great example of Bayesian Inference, where we update our beliefs to incorporate more information. Bayesian Inference means becoming less wrong about the world with each new piece of evidence!

6.1 Effect of Additional Information Visualization

To show how this information changes the probability, we can make the same plots showing probability of sleep vs time incorporating the new evidence. There are four additional situations that shift the curve.

```
In [29]: figsize(20, 8)
```

```
x = np.linspace(-60, 120, 1e3)  
priors = logistic(x, beta=beta_sleep_est, alpha=alpha_sleep_est )  
  
evidence = [[0, 0], [0, 1], [1, 0], [1, 1]]  
labels = ['Light Off; Phone Not Charging', 'Light Off; Phone Charging',  
          'Light On; Phone Not Charging', 'Light On; Phone Charging']  
  
plt.plot(x, priors, label='Prior for Sleep', ls = '--', color = 'k', lw = 4)  
  
for i, obs in enumerate(evidence):  
    posteriors = []  
    for time_offset in x:  
        posterior = sleep_posterior(time_offset, obs[0], obs[1])  
        posteriors.append(posterior)  
  
    plt.plot(x, posteriors, label=labels[i], lw = 3)  
  
plt.legend(prop={'size':14}); plt.xticks([-60, -30, 0, 30, 60, 90, 120], sleep_  
plt.xlabel('PM Time'); plt.ylabel('Probability')  
plt.title("Posterior Probabilities of Being Asleep using Bayesian Inference",  
          size = 18);
```



The graph shows the impact of incorporating more knowledge. If we know the light is on, the probability I am asleep at a given time decreases, and if the light is off and the phone charging, the probability I am sleeping at a given time increases! Bayesian Inference is extremely useful!

7 Wake Model

We can repeat the same procedure to find a model for the wake data. The process is basically the same although I will only calculate the posterior for sleep given the time. After we have this, we can incorporate more information, but I will not do those calculations in this report.

```
In [30]: # Sort the values by time offset
wake_data.sort_values('time_offset', inplace=True)

# Time is the time offset
time = np.array(wake_data.loc[:, 'time_offset'])

# Observations are the indicator
wake_obs = np.array(wake_data.loc[:, 'indicator'])

with pm.Model() as wake_model:
    # Create the alpha and beta parameters
    alpha = pm.Normal('alpha', mu=0.0, tau=0.05, testval=0.0)
    beta = pm.Normal('beta', mu=0.0, tau=0.05, testval=0.0)

    # Create the probability from the logistic function
    p = pm.Deterministic('p', 1. / (1. + tt.exp(beta * time + alpha)))
```

```

# Create the bernoulli parameter which uses the observed data
observed = pm.Bernoulli('obs', p, observed=wake_obs)

# Starting values are found through Maximum A Posterior estimation
# start = pm.find_MAP()

# Using Metropolis Hastings Sampling
step = pm.Metropolis()

# Sample from the posterior using the sampling method
wake_trace = pm.sample(N_SAMPLES, step=step);

```

Multiprocess sampling (2 chains in 2 jobs)

CompoundStep

>Metropolis: [beta]

>Metropolis: [alpha]

The number of effective samples is smaller than 10% for some parameters.

7.1 Visualize Posterior for Wake given Time

```

In [31]: # Extract the alpha and beta samples
# Currently using all, including the burn-in period
alpha_samples = wake_trace["alpha"][:, None]
beta_samples = wake_trace["beta"][:, None]

# Time values for probability prediction
time_est = np.linspace(time.min()- 15, time.max() + 15, 1e3)[:, None]

# Take most likely parameters to be mean values
alpha_est = alpha_samples.mean()
beta_est = beta_samples.mean()

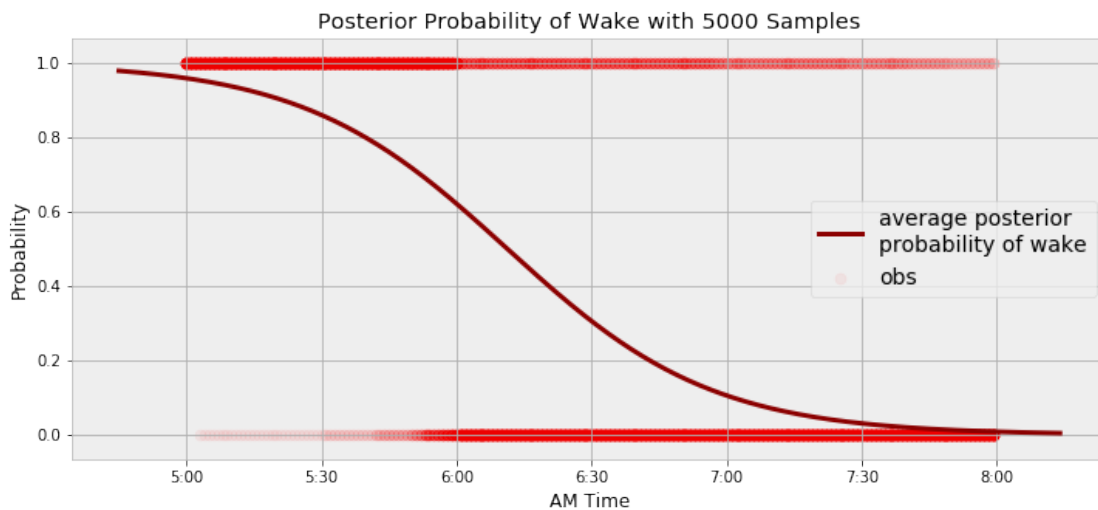
# Probability at each time using mean values of alpha and beta
wake_est = logistic(time_est, beta=beta_est, alpha=alpha_est)

figsize(12, 5)

plt.plot(time_est, wake_est, color = 'darkred',
         lw=3, label="average posterior \nprobability of wake")
plt.scatter(time, wake_obs, edgecolor = 'r', facecolor = 'r',
           s=50, alpha=0.05, label='obs')
plt.title('Posterior Probability of Wake with %d Samples' % N_SAMPLES);
plt.legend(prop={'size':14})
plt.ylabel('Probability')

```

```
plt.xlabel('AM Time');
plt.xticks([-60, -30, 0, 30, 60, 90, 120], wake_labels);
```



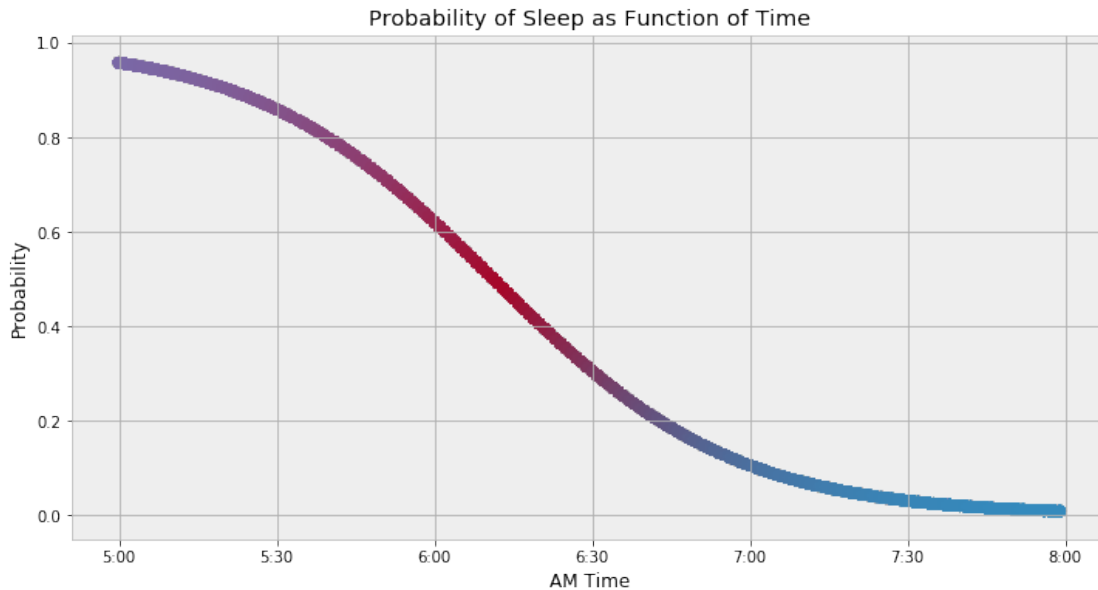
The model does not have a steep transition, which is what I expected because I tend to wake up right around 6:00 AM. There were several times when I woke up several hours later which shifted the curve to the right.

```
In [43]: print('The probability of being awake passes 50% at 6:{} AM.'.format(int(time_e
```

The probability of being awake passes 50% at 6:11 AM.

```
In [37]: colors = ["#348ABD", "#A60628", "#7A68A6"]
cmap = matplotlib.colors.LinearSegmentedColormap.from_list("BMH", colors)
figsize(12, 6)
probs = wake_trace['p']

plt.scatter(time, probs.mean(axis=0), cmap = cmap,
            c = probs.mean(axis=0), s = 50);
plt.title('Probability of Sleep as Function of Time')
plt.xlabel('AM Time');
plt.ylabel('Probability');
plt.xticks([-60, -30, 0, 30, 60, 90, 120], wake_labels);
```

7.2 Investigate the Wake Model

```
In [44]: print('Probability of being awake at 5:30 AM: {:.2f}%.'.
              format(100 - (100 * logistic(-30, beta=beta_est, alpha=alpha_est))))
print('Probability of being awake at 6:00 AM: {:.2f}%.'.
      format(100 - (100 * logistic(0, beta=beta_est, alpha=alpha_est))))
print('Probability of being awake at 6:30 AM: {:.2f}%.'.
      format(100 - (100 * logistic(30, beta=beta_est, alpha=alpha_est))))
```

```
Probability of being awake at 5:30 AM: 14.09%.
Probability of being awake at 6:00 AM: 37.93%.
Probability of being awake at 6:30 AM: 69.47%.
```

7.2.1 Confidence Interval

To represent the uncertainty, we can calculate the 95% confidence interval for the probability of being asleep at each time. MCMC does not return the True distribution, but an approximation that we hope represents the observations.

```
In [ ]: wake_all_est = logistic(time_est.T, beta_samples, alpha_samples)
        quantiles = mquantiles(wake_all_est, [0.025, 0.975], axis=0)

In [ ]: plt.fill_between(time_est[:, 0], *quantiles,
                        alpha=0.6, color='salmon', label = '95% CI')
        plt.plot(time_est, wake_est, lw=2, ls='--',
```

```

        color='black', label="average posterior \nprobability of sleep")
plt.xticks([-60, -30, 0, 30, 60, 90, 120], wake_labels);
plt.scatter(time, sleep_obs, edgecolor = 'red',
            facecolor = 'red', s=50, alpha=0.1);
plt.legend(prop={'size':14})
plt.xlabel('AM Time'); plt.ylabel('Probability');
plt.title('Posterior Probabilty with 95% CI');

```

8 Model for Sleep Duration

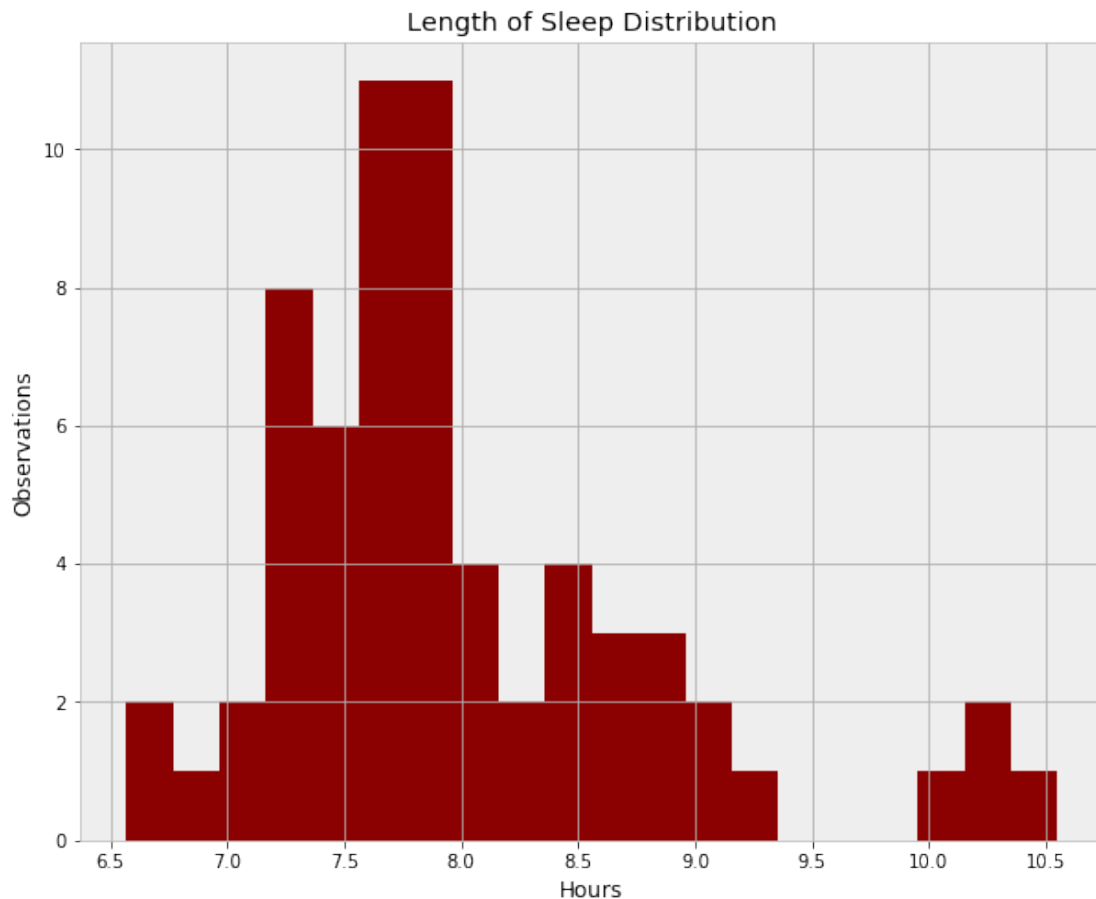
We can also form a model to estimate the most likely length of time I am asleep. We can first look at the data and then determine which function fits the probability distribution.

```

In [45]: raw_data = pd.read_csv('data/sleep_wake.csv')
        raw_data['length'] = 8 - (raw_data['Sleep'] / 60) + (raw_data['Wake'] / 60)
        duration = raw_data['length']

In [46]: figsize(10, 8)
        plt.hist(duration, bins = 20, color = 'darkred')
        plt.xlabel('Hours'); plt.title('Length of Sleep Distribution');
        plt.ylabel('Observations');

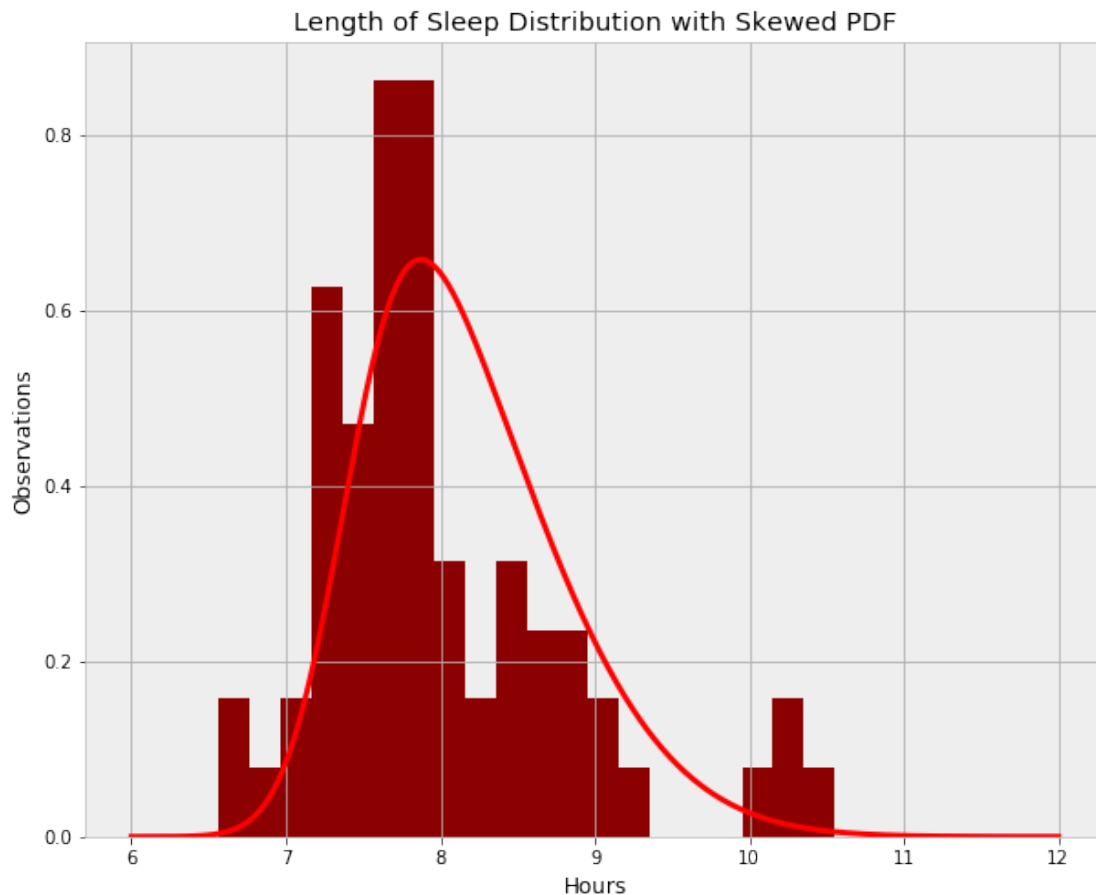
```



The distribution is skewed to the right. Therefore, we can use a skewed distribution to model the length of sleep. We might also want to use a bi-modal distribution because there appear to be two modes. For now, I will stick to representing the length of sleep distribution as a skewed normal. A skewed normal distribution is shown below.

```
In [47]: a = 3
fig, ax = plt.subplots(1, 1)
x = np.linspace(6, 12, 1e3)

figsize(10, 8)
plt.hist(duration, bins = 20, color = 'darkred', normed=True)
plt.xlabel('Hours'); plt.title('Length of Sleep Distribution with Skewed PDF');
plt.ylabel('Observations');
plt.plot(x, stats.skewnorm.pdf(x, a, loc = 7.4, scale=1), 'r-',
        lw=3, label='skewnorm pdf');
```



```
In [48]: with pm.Model() as duration_model:
        # Three parameters to sample
        alpha_skew = pm.Normal('alpha_skew', mu=0, tau=0.5, testval=3.0)
        mu_ = pm.Normal('mu', mu=0, tau=0.5, testval=7.4)
        tau_ = pm.Normal('tau', mu=0, tau=0.5, testval=1.0)

        # Duration is a deterministic variable
        duration_ = pm.SkewNormal('duration', alpha = alpha_skew, mu = mu_,
                                   sd = 1/tau_, observed = duration)

        # Metropolis Hastings for sampling
        step = pm.Metropolis()
        duration_trace = pm.sample(N_SAMPLES, step=step)
```

Multiprocess sampling (2 chains in 2 jobs)

CompoundStep

>Metropolis: [tau]

>Metropolis: [mu]

>Metropolis: [alpha_skew]

The number of effective samples is smaller than 10% for some parameters.

```
In [49]: # Extract the most likely estimates from the sampling
        alpha_skew_samples = duration_trace['alpha_skew']
        mu_samples = duration_trace['mu']
        tau_samples = duration_trace['tau']

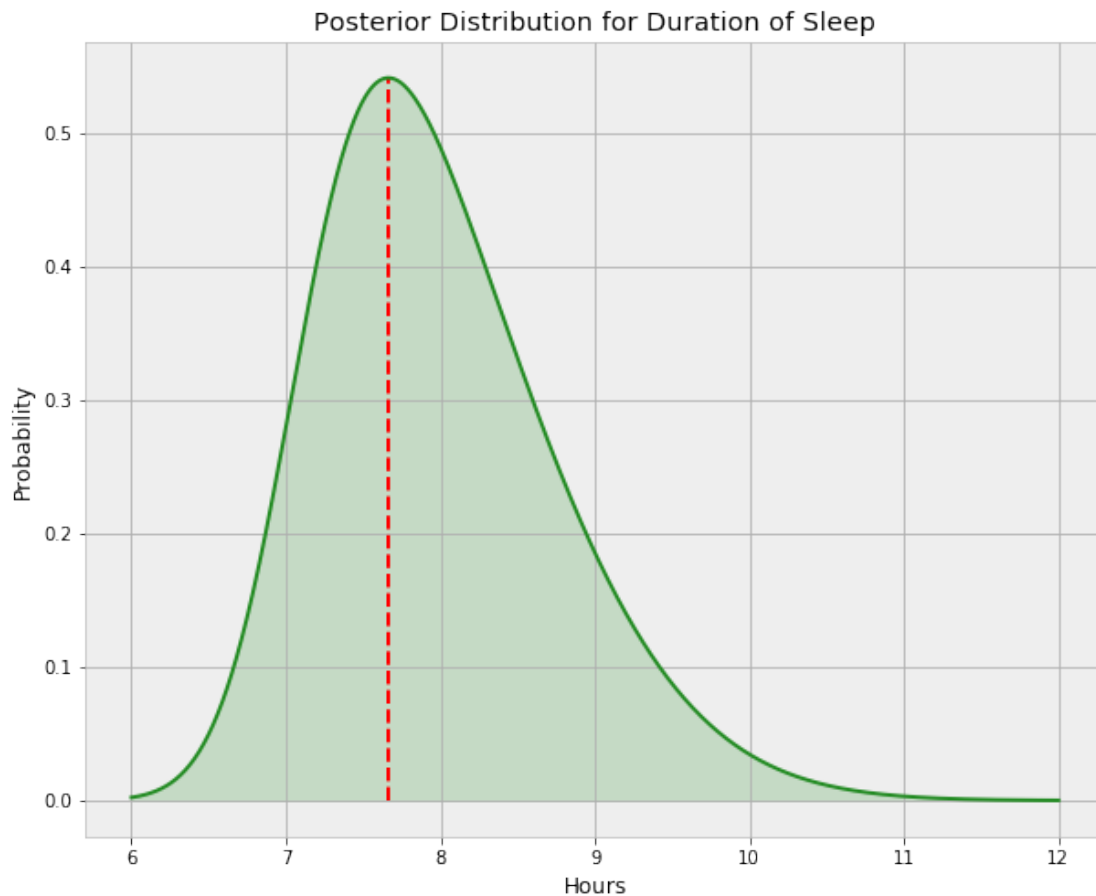
        alpha_skew_est = alpha_skew_samples.mean()
        mu_est = mu_samples.mean()
        tau_est = tau_samples.mean()
```

8.1 Visualize Posterior Distribution for Sleep Duration

```
In [50]: x = np.linspace(6, 12, 1000)
        y = stats.skewnorm.pdf(x, a = alpha_skew_est, loc=mu_est, scale=1/tau_est)
        plt.plot(x, y, color = 'forestgreen')
        plt.fill_between(x, y, color = 'forestgreen', alpha = 0.2);
        plt.xlabel('Hours'); plt.ylabel('Probability');
        plt.title('Posterior Distribution for Duration of Sleep');
        plt.vlines(x = x[np.argmax(y)], ymin=0, ymax=y.max(),
                    linestyle='--', linewidth=2, color='red',
                    label = 'Most Likely Duration');

        print('The most likely duration of sleep is {:.2f} hours.'.format(x[np.argmax(y)]))
```

The most likely duration of sleep is 7.66 hours.



8.1.1 Query the Posterior Model

```
In [51]: print('Probability of at least 6.5 hours of sleep = {:.2f}%'.  
            format(100 * (1 - stats.skewnorm.cdf(6.5, a = alpha_skew_est, loc = mu_es  
            print('Probability of at least 8.0 hours of sleep = {:.2f}%'.  
            format(100 * (1 - stats.skewnorm.cdf(8.0, a = alpha_skew_est, loc = mu_es  
            print('Probability of at least 9.0 hours of sleep = {:.2f}%'.  
            format(100 * (1 - stats.skewnorm.cdf(9.0, a = alpha_skew_est, loc = mu_es
```

Probability of at least 6.5 hours of sleep = 99.12%.

Probability of at least 8.0 hours of sleep = 44.38%.

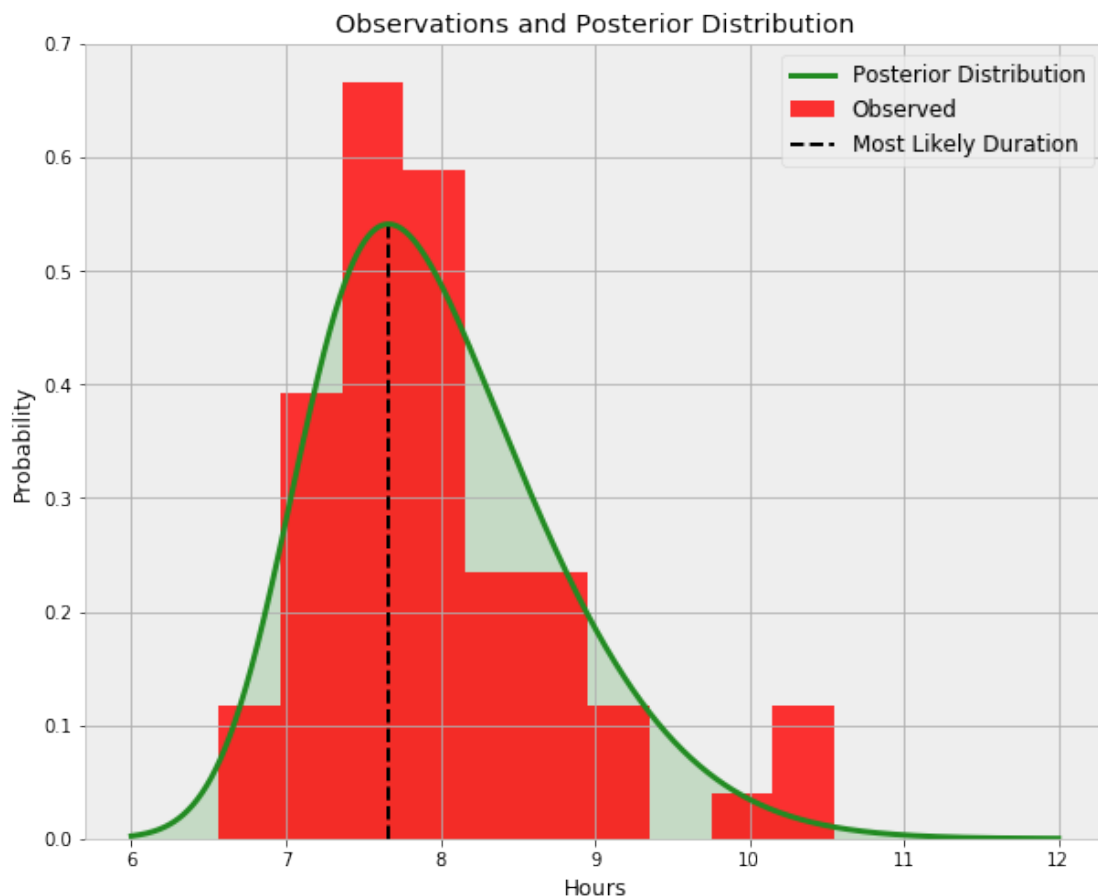
Probability of at least 9.0 hours of sleep = 11.01%.

8.2 Visualize the Posterior and the Data

```
In [52]: x = np.linspace(6, 12, 1000)
        y = stats.skewnorm.pdf(x, a = alpha_skew_est, loc=mu_est, scale=1/tau_est)

        # Plot the posterior distribution
        plt.plot(x, y, color = 'forestgreen',
                 label = 'Posterior Distribution', lw = 3)
        plt.fill_between(x, y, color = 'forestgreen', alpha = 0.2);

        # Plot the observed values
        plt.hist(duration, bins=10, color = 'red', alpha=0.8,
                 label='Observed', normed=True)
        plt.xlabel('Hours'); plt.ylabel('Probability');
        plt.title('Observations and Posterior Distribution');
        plt.vlines(x = x[np.argmax(y)], ymin=0, ymax=y.max(),
                  linestyle='--', linewidth=2, color='k',
                  label = 'Most Likely Duration');
        plt.legend(prop={'size':12});
```



The posterior skewed normal distribution looks to fit the data well. However, the data may be better modeled as two separate distributions given the second mode to the right. The second mode is not captured in a single skewed normal distribution. Overall, this model still provides a reasonable estimate for the probability of duration of my sleep.

9 Conclusions

Based on the observations, we can state the following:

- **On average I fall asleep by 10:14 PM**
- **On average I wake up by 6:11 AM**
- **My average duration of sleep is 7.66 hours**

In this report, I used real-world data and Bayesian Inference to form models about my sleep patterns. I first used Markov Chain Monte Carlo to find the parameters for the posterior probability distribution of sleep given the time. Then, I updated the model using Baye's Rule and additional evidence. The final model for the probability of sleep at a specific time knowing my phone status, and light condition has the form:

$$P(s|L,C) = \frac{(P(L|s) * P(C|s) * P(s))}{P(L|s) * P(C|s) * P(s) + P(L|\bar{s}) * P(C|\bar{s}) * P(\bar{s})}$$

where $P(s)$ was the posterior probability of sleep as a function of time found from MCMC. The final model can also be supplemented by three additional models showing:

- * Posterior probability I am asleep in the evening at a given time
- * Posterior probability I am asleep in the morning at a given time
- * Posterior distribution of the duration of my sleep

The models have already provided me with knowledge about my sleeping patterns, and more data would only improve the applicability. I could incorporate additional information such as day of the week or daily activities to see how they affect my sleep and then make adjustments as required. Although this report made several assumptions and did not fully investigate the models, it was a great start towards analyzing actual data with Bayesian methods. Projects, especially with real applications, are the best way to learn, and I look forward to applying Bayesian methods to additional problems!