Solutions for Mid Term Exams, CS7DS2/CS4405

Exercise 1

1. **dom** f is a convex set. $f'(x) = -4x,\ f''(x) = -4 < 0$. So, f is strictly concave.

2. **dom** f is not convex because $x_1 \in \mathcal{N}$, which is not a convex set. Thus, f is neither.

Exercise 2

dom f is a convex set.

$$\frac{\partial f(x)}{\partial x_1} = 2x_1 - \alpha x_2, \quad \frac{\partial f(x)}{\partial x_2} = 2x_2 - \alpha x_1,$$

$$\frac{\partial f^2(x)}{\partial x_1 \partial x_1} = 2, \quad \frac{\partial f^2(x)}{\partial x_1 \partial x_2} = -\alpha, \quad \frac{\partial f^2(x)}{\partial x_2 \partial x_1} = -\alpha, \quad \frac{\partial f^2(x)}{\partial x_2 \partial x_2} = 2,$$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & -a \\ -a & 2 \end{bmatrix}$$

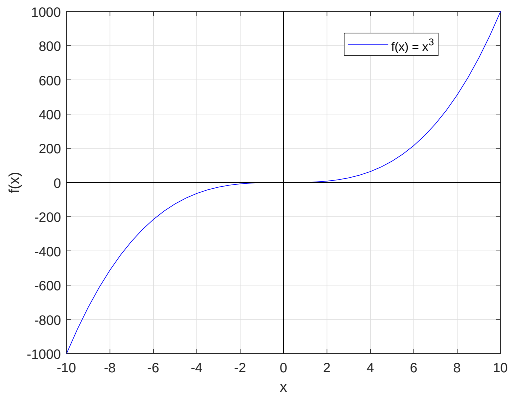
We examine whether the Hessian is positive semi-definite. According to Sylvester's criterion, this holds when all principal minors of the matrix are non-negative. For this 2×2 matrix, the principal minors are:

$$2 > 0, \qquad 2 \cdot 2 - (-a) \cdot (-a) = 4 - a^2\,, \tag{1}$$

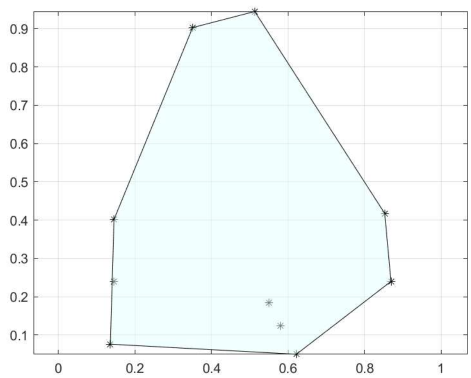
where the last term is non-negative (≥ 0) only when $\alpha \in [-2, 2]$. Hence, f is convex for these values of α.

Exercise 3

1. $f(x) = x^3,\ x \in \boldsymbol{R}$ in (a). A random set of points in $\boldsymbol{R}$ and its respective convex hull is the marked area in (b).



(a)

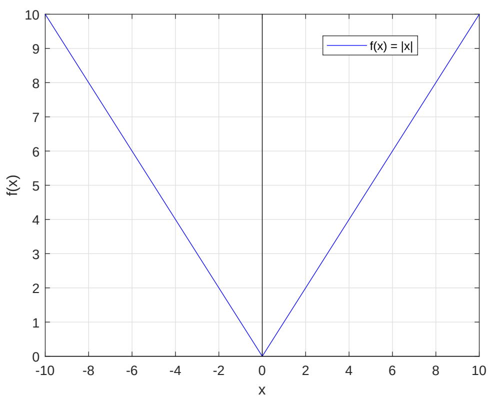


(b)

2. $f(x) = |x|,\ x \in \boldsymbol{R}$, is convex but not-differentiable in $x = 0$. The convexity can be proved by inspecting its graph, or by using the definition of a convex function:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y), \tag{2}$$

where $x, y \in \boldsymbol{R}$, and $\theta \in [0,1]$. Finally, recall that we can prove the convexity of a function by checking if its epigraph is a convex set.



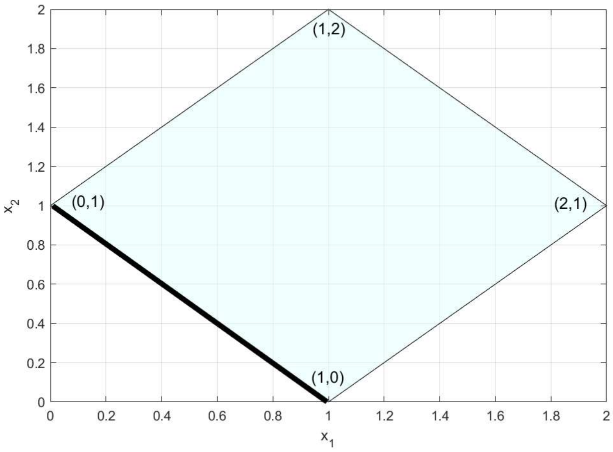
(a)

3. The shaded rectangle formed by points: (0,1), (1,0), (2,1), (1,2). All points in the line segment between (0,1) and (1,0) are minimal points of the polyhedron with respect to the cone $\preceq_{\boldsymbol{R}^2_+}$.

The polyhedron is described by the following set of inequalities:

$$\begin{aligned} x_1 + x_2 &\geq 1, \\ x_1 - x_2 &\geq 1, \\ -x_1 + x_2 &\leq 1, \\ x_1 + x_2 &\leq 3 \end{aligned}$$

Another possible answer for this question is to use a polyhedron that is described only with the first inequality (doesn't have to be closed).



(a)

Exercise 4

For concavity, the following should hold:

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta) f(y)$$

So we get that:

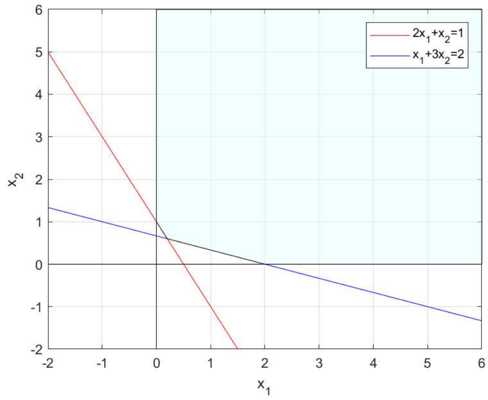
$$\begin{aligned} f(\theta x + (1-\theta)y) &= \min\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\ &\geq \min\{\theta f_1(x) + (1-\theta) f_1(y), \theta f_2(x) + (1-\theta) f_2(y)\}, \text{ because } f_1, f_2 \text{ are concave} \\ &\geq \theta \min\{f_1(x), f_2(x)\} + (1-\theta)\min\{f_1(y), f_2(y)\}, \\ &= \theta f(x) + (1-\theta) f(y) \end{aligned}$$

Exercise 5

1. This is a convex optimization problem. $x_1, x_2 \in \boldsymbol{R}$, which is a convex set. The objective function $f_0(x) = 1$ is affine, so it is convex. The same holds for the inequality constraints too.

2. The shaded area in the figure below is the problem's feasible set. It is a polyhedron by definition: *the solution set of a finite number of linear equalities and inequalities.*

3. The objective function does not depend on (x_1, x_2) so this is actually a feasibility problem. Since the constraint set is non-empty (i.e., there exist feasible solutions), the problem's optimal solution is $p^* = 1$ (the objective function). Any feasible point is a solution, e.g., point (2,2).



(a)

Exercise 6

1. The standard form of the problem is:

$$\begin{aligned}\min_{x} \quad & -f_0(x) \\ \text{subject to:} \quad & f_1(x) = 0 \\ & -f_2(x) \leq 0 \\ & Ax - b = 0\end{aligned}$$

2. The above is a convex optimization problem only if f_1 is affine and $-f_2$ is convex.

3. The epigraph form of the problem is:

$$\begin{aligned}\min_{x,t} \quad & t \\ \text{subject to:} \quad & -f_0(x) - t \leq 0 \\ & f_1(x) = 0 \\ & -f_2(x) \leq 0 \\ & Ax - b = 0\end{aligned}$$

This problem is a linear program only if f_0, f_1, f_2 are affine functions.

Exercise 7

1. The variable vector $x \in \boldsymbol{R}_+^4$ denotes the quantity for each of the 4 ingredients in chunks of 100 grams, i.e., if $x_1 = 2$ we get 200 grams of rice.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vector $c \in \mathbf{R}^4$ denotes the cost per 100 grams for each of the ingredients:

$$c = \begin{bmatrix} 1 \\ 1.5 \\ 0.5 \\ 2 \end{bmatrix}$$

Vector $b \in \mathbf{R}^3$ denotes the minimum quantities per nutrient constraint:

$$b = \begin{bmatrix} 12 \\ 10 \\ 30 \end{bmatrix}$$

The nutrient intake matrix $A \in \mathbf{R}^{3\times4}$ is given as:

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 0.5 & 1 & 0 & 5 \\ 2 & 0 & 0 & 4 \end{bmatrix}$$

The problem is formulated as:

$$\begin{aligned} \min_x \; & c^T x \\ \text{subject to:} \; & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Converting to standard form by subtracting slack variables:

$$\begin{aligned} \min_{x,s} \; & c^T x \\ \text{subject to:} \; & Ax - s = b \\ & x, s \geq 0 \end{aligned}$$

where:

$$s = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

contains the slack variables for each constraint.

2. The new optimization problem is formulated as follows:

$$\begin{aligned} \min_x \; & \lambda_1 c^T x + \lambda_2 ||x - x_d||_2^2 \\ \text{subject to:} \; & Ax \geq b \\ & x \geq 0 \end{aligned}$$

where

$$x_d = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

corresponds to the amount of ingredients required for the delicious recipe. By setting, for example, $\lambda_1 = 2$ and $\lambda_2 = 1$ we can prioritize cost minimization over obtaining a delicious recipe.