OPTIMISATION ALGORITHMS FOR DATA ANALYSIS

EXERCISES

Convex Sets

Exercise 1. Let $X := \{x_1, x_2, \ldots, x_m\}$ be a subset of points from $\mathbf{R}^n$, and let C and A denote, respectively, the convex and affine hulls of X. Show that $C \subset A$.

Exercise 2. Let C_1 and C_2 be two affine sets. When are we going to have $C_1 \cap C_2 \neq \emptyset$?

Exercise 3. Let $B(x_c, r) := \{x \in \mathbf{R}^n \mid \|x - x_c\|_2 \leq r\}$ be the Euclidean ball of radius $r > 0$ with centre $x_c \in \mathbf{R}^n$. Show that $B(x_c, r)$ is a convex set.

Exercise 4. Let C be a bounded convex set from $\mathbf{R}^n$. Show that convexity is retained under *scaling* and *translation*. That is, $\alpha C = \{\alpha x \mid x \in C\}$ and $C + a = \{x + a \mid x \in C\}$ are convex sets for any $\alpha \in \mathbf{R}$, $a \in \mathbf{R}^n$.

Exercise 5. Let C_1 and C_2 be two convex sets such that $C_1 \cap C_2 = \emptyset$. Show there exists a hyperplane $a^T x + b$ such that $a^T x_1 + b < 0$ for all $x_1 \in C_1$, and $a^T x_2 + b > 0$ for all $x_2 \in C_2$. (See Section 2.5 in Boyd's book).

Convex Functions

Exercise 6. Compute the gradient and hessian of the following functions

(a) $f(x) = \mathbf{1}^T A \mathbf{1}$	(b) $f(x) = \|x\|_2^2$	(c) $f(x) = x^T A x$
(d) $f(x) = \|x\|_1$	(e) $f(x) = \|Ax - b\|_2^2$	(f) $f(x) = \|x\|_2$

where $A \in \mathbf{R}^{n \times n}$, $b \in \mathbf{R}^n$, $x \in \mathbf{R}^n$ and $\mathbf{1}$ the all ones column vector.

Exercise 7. Show that the maximum entropy function $f(x) = \sum_{i=1}^n x_i \log(x_i)$ is *strictly* convex for $x \in \mathbf{R}_{++}$.

Exercise 8. Let g_i, $i \in \{1, \ldots, n\}$ be a collection of convex functions. Show that

(a) $f(x) = \max\{g_1(x), \ldots, g_n(x)\}$ is also convex.
(b) $f(x, \theta) = \sum_{i=1}^n \theta_i g_i(x)$ with $\theta_i \geq 0$ is
  (i) not necessarily convex in θ and x (jointly); but
  (ii) affine in θ for a fix x, and convex in x for a fix θ.

Exercise 9. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Show that $f(\mathbf{E}(x)) \leq \mathbf{E}(f(x))$, where $\mathbf{E}$ is the expectation w.r.t. variable x.

Exercise 10. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Prove, mathematically, that $\nabla f(x) = 0$ is a *necessary* and *sufficient* condition for (global) optimality. What happens to this condition when f is not convex?

Exercise 11. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function and $x^\star$ a minimiser of f, *i.e.* $f(x^\star) \leq f(x)$ for all $x \in \mathbf{R}^n$. Show that the following two inequalities must hold for all $x \in \mathbf{R}^n$:

(a) $\nabla f(x)^T (x^\star - x) \geq 0$.
(b) $\nabla f(x^\star)^T (x - x^\star) \geq 0$.

Exercise 12. Show that the normal distribution is log-convex.

Exercise 13. Let $f_1(x) = x^T P x$ and $f_2(x) = \sum_{i=1}^{n} \lambda_i x_i^2$, where λ_i is the i'th eigenvalue of $P \in \mathbf{R}^{n\times n}$. When will f_1 and f_2 be convex?
Hint: use the fact that $A = Q^T \Lambda Q$ where $Q \in \mathbf{R}^{n\times n}$ and Λ is a diagonal matrix containing the eigenvalues of A.

Convex Problems

Exercise 14. Variation of Problem 4.1 in Boyd's book [BV04]. Consider the optimisation problem

$$\begin{array}{ll} \text{maximise} & -f(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 0 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, \quad x_2 \geq 0. \end{array}$$

Write the optimisation problem in standard (convex) form and make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value:

(a) $f(x_1, x_2) = x_1 + x_2$ (b) $f(x_1, x_2) = -x_1 - x_2$.
(c) $f(x_1, x_2) = x_1$ (d) $f(x_1, x_2) = \max\{x_1, x_2\}$
(e) $f(x_1, x_2) = x_1^2 + 9x_2^2$.

Exercise 15. Consider the optimisation problem

$$\begin{array}{ll} \text{minimise} & \|s - x\|_1 \\ \text{subject to} & \mathbf{1}^T x = 1 \\ & x \succeq 0 \end{array}$$

with $s, x \in \mathbf{R}^n$.

(a) Sketch the feasible set for $n = 2$.
(b) Show that the problem can be reformulated as an LP (linear programme).

Exercise 16. Consider the optimisation problem

$$\text{minimise} \qquad \|Ax - b\|_p + \gamma \|x\|_p$$

with $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $\gamma > 0$. Answer the following questions:

(a) does a solution to the problem exists for all $p \in \{1, 2, \infty\}$? When will we have $f^\star = \infty$?
(b) for which $p \in \{1, 2, \infty\}$ is there a unique solution?
(c) can we ever have $f^\star = 0$ if $\gamma > 0$?
(d) what happens to the problem if $\gamma < 0$?
(e) what's the role of parameter γ? Namely, how does it affect the set of solutions?
(f) rewrite the problem so that $-\mathbf{1} \preceq x^\star \preceq \mathbf{1}$.

Exercise 17. Consider the optimisation problem

$$\text{minimise} \qquad \|Ax - b\|_\infty$$

with $x \in \mathbf{R}^n$, $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$. Rewrite the problem as an LP.

Exercise 18. Consider the *non-convex* optimisation problem

$$\begin{array}{ll} \text{minimise} & -\|x\|_\infty \\ \text{subject to} & \|x\|_2 \leq 1 \end{array}$$

where $x \in \mathbf{R}^n$. Use the results from Exercise 17 to show that it can be reformulated as a convex problem. Hint: draw sets $C_2 = \{x \in \mathbf{R}^2 \mid \|x\|_2 \leq 1\}$ and $C_\infty = \{x \in \mathbf{R}^2 \mid \|x\|_\infty \leq 1\}$.

Exercise 19. The Hamilton restaurant is back and you've been asked to help with its special recipe: an Irish Paella. As always, the restaurant values more efficiency and money than taste. Your task is to minimise the cost of a ration of Paella subject to some nutrition requirements:

The list of products you're allowed to use is the following:

Protein	Fat	Carbohydrates	Sugar	Sodium	Fiber
12	20	120	30	5	10

TABLE 1. Recommended nutrients intake per meal (in grams).

Product	Price (€)	Protein	Fat	Carbohydrates	Sugar	Sodium	Fiber
Rice	1	3	0.5	57.8	0	0	1.7
Stock	0.5	1	1	1	0	4	0
Pepper	1.5	0.9	0.1	1.8	0	0	0.4
Onion	0.5	1	0	10	0	0	0
Prawns	3	14	0	1	0	2	0
Salmon	2	22	9	0	0	1	0
Olive Oil	3	0	100	0	0	0	0
Magic ingredient	4	1.9	0.3	18	24	0	5

TABLE 2. Price and nutrient intake per 100 g.

Write your task as a (convex) optimisation problem in standard form. Try to write the problem in a compact form; *i.e.* using matrices and vectors. Solve it with CVX (http://cvxr.com/cvx/).