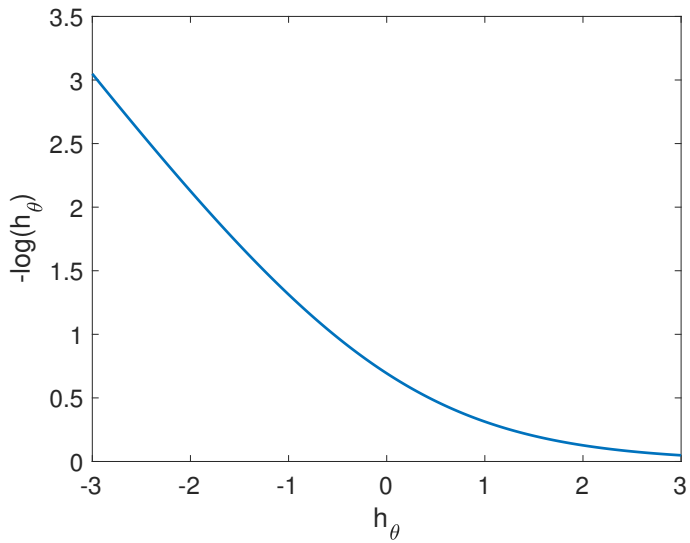
Overview

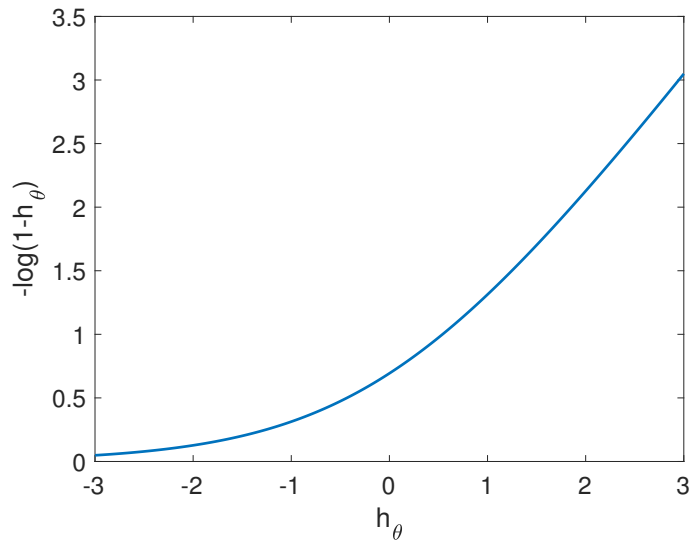
- Support Vector Machines (SVMs)
- Maximising Margin
- Gradient Descent for SVMs

Logistic Regression: Choice of Cost Function

- Hypothesis: $h_\theta(x) = sign(\theta^T x)$
- Parameters: θ
- Cost Function: $J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \log(1 + e^{-y^{(i)} \theta^T x^{(i)}})$
- Goal: Select θ that minimise $J(\theta)$

Loss function: $\log(1 + e^{-y\theta^T x})/log(2)$ (the $log(2)$ is an optional scaling factor):

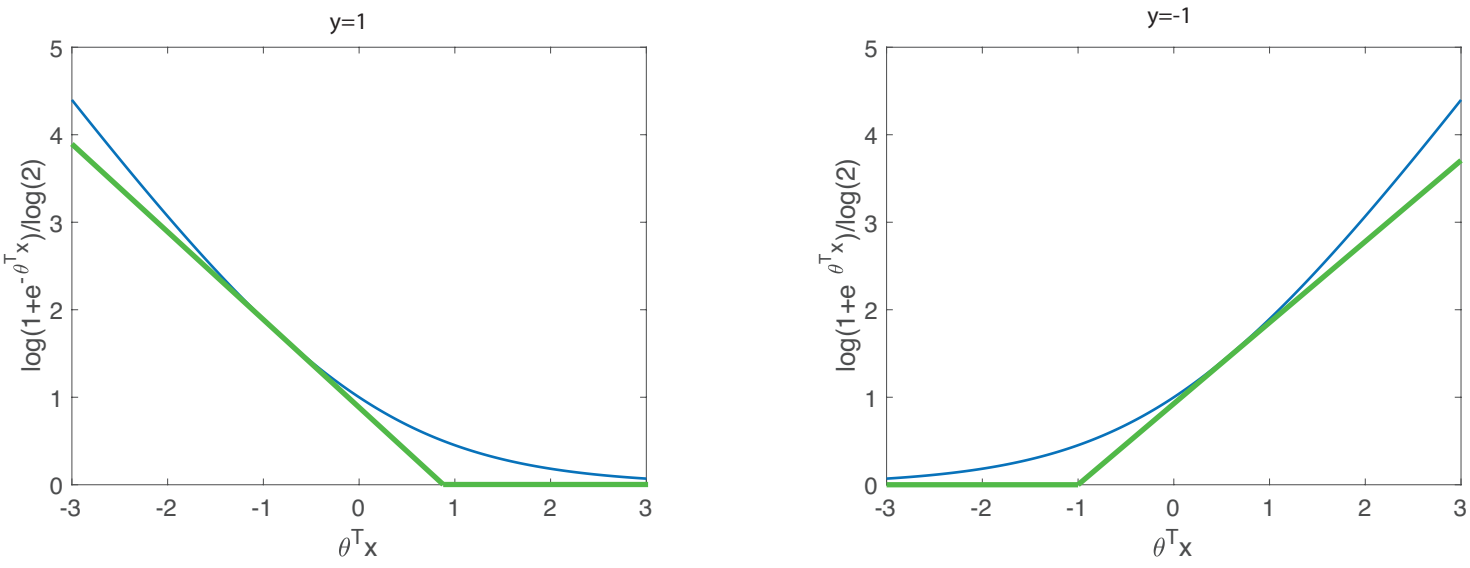




gives a small penalty when $\theta^T x \gg 0$ and $y = 1$, and when $\theta^T x \ll 0$ and $y = -1$.

SVM: Choice of Cost Function

In an SVM use the "hinge" loss function $\max(0, 1 - y\theta^T x)$:



Main differences from logistic loss function:

- hinge-loss is not differentiable ("non-smooth")
- hinge loss assigns zero penalty to all values of θ which ensure $\theta^T x \geq 1$ when $y = 1$, and $\theta^T x \leq -1$ when $y = -1$

SVM: Choice of Cost Function

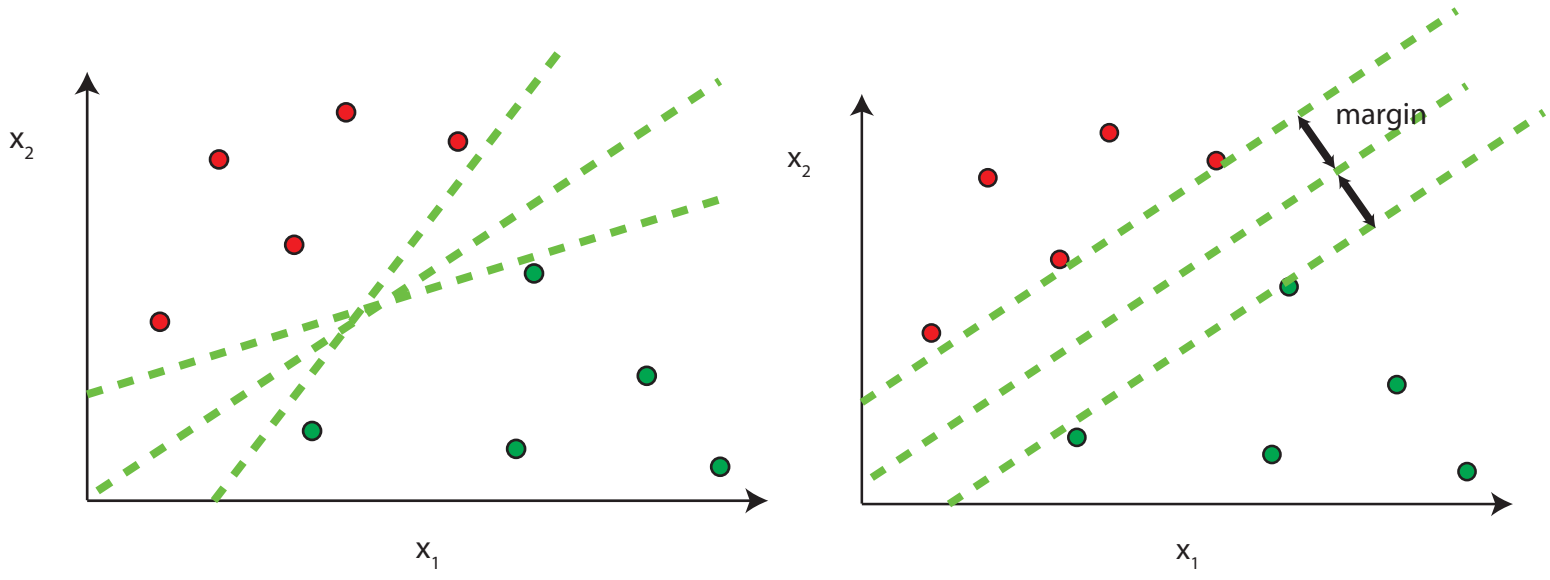
In an SVM use the "hinge" loss function $\max(0, 1 - y\theta^T x)$:

- So long as $y\theta^T x > 0$ then by scaling up θ sufficiently, e.g. to 10θ or 100θ, then we can always force $y\theta^T x > 1$ i.e. $\max(0, 1 - y\theta^T x) = 0$
- To get sensible behaviour we have to penalise large values of θ. We do this by adding penalty $\theta^T\theta = \sum_{j=1}^{n} \theta_j^2$
- Final SVM cost function is:

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \max(0, 1 - y^{(i)}\theta^T x^{(i)}) + \lambda\theta^T\theta$$

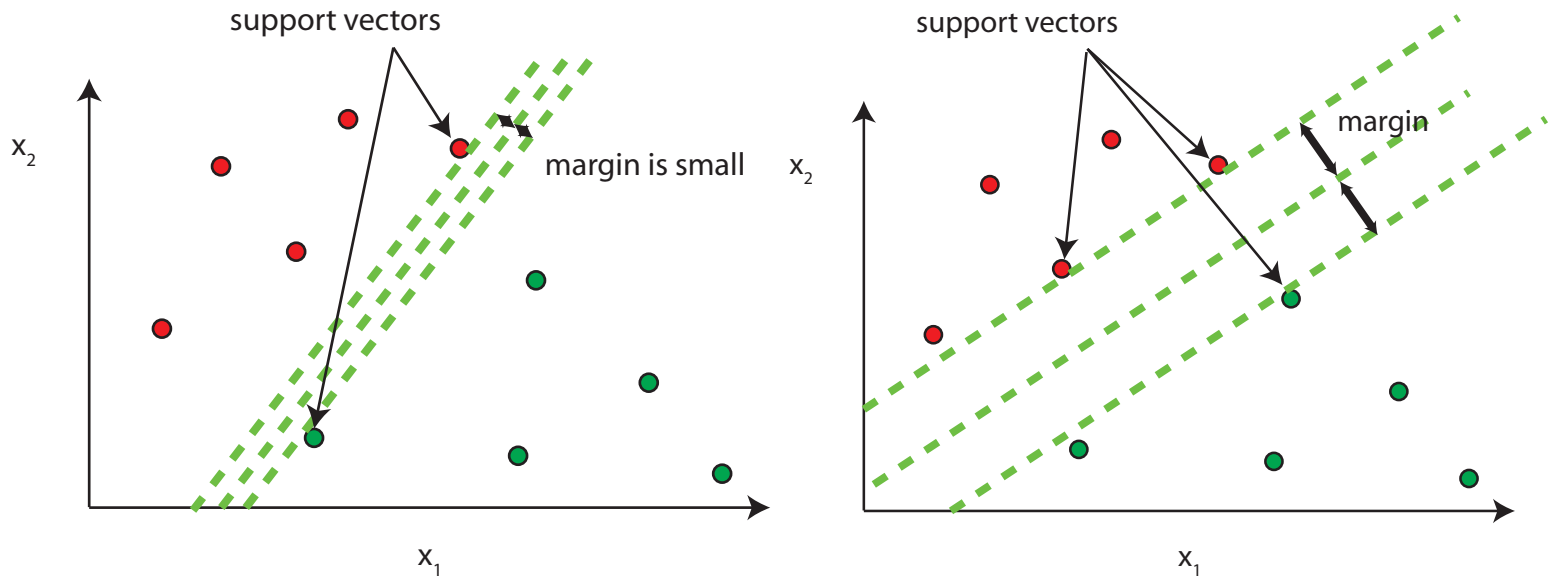
where $\lambda > 0$ is a weighting parameter that we have to choose.

Maximising Margin



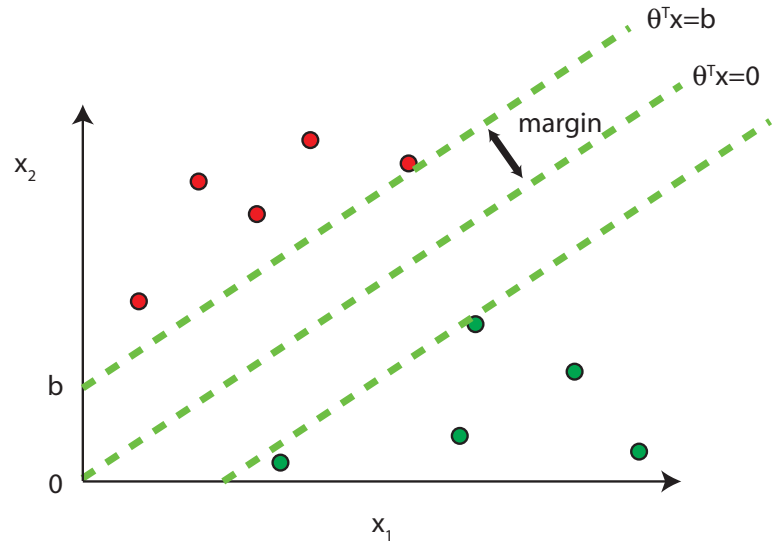
- We have some freedom in the choice of line used to separate two classes
- Idea: select separating line that maximises the **margin**

Maximising Margin



- Margin is determined by points that touch (or "support") the upper and lower boundaries, and not affected by other points
- Hinge loss function $\max(0, 1 - y\theta^T x)$ assigns zero penalty to points where $y\theta^T x \geq 1$, i.e. value of cost function $J(\theta)$ is determined only by those points for which $y\theta^T x < 1$.

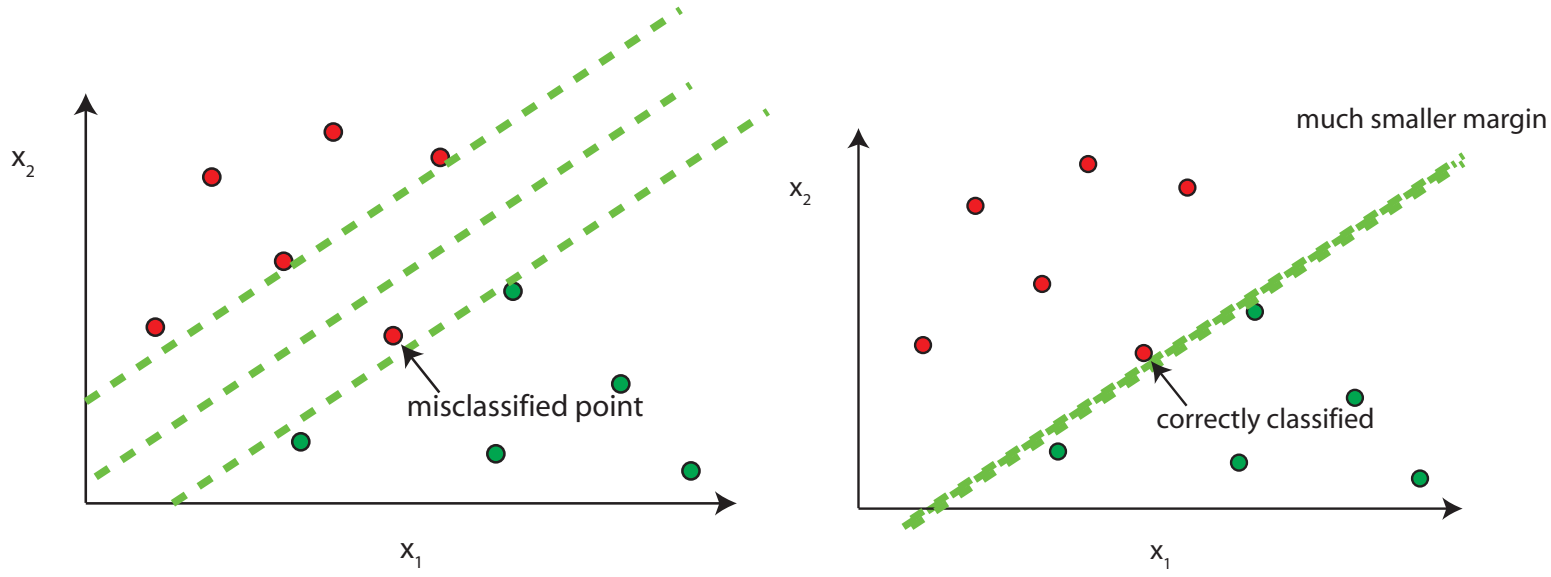
Maximising Margin



- $\theta^T x = 0$ is the decision boundary. $\theta^T x = b$ is a parallel line shifted up by b, and line passing through point $x^{(i)}$ has $b = \theta^T x^{(i)}$
- $|b| = y^{(i)}\theta^T x^{(i)}$ since $y^{(i)} = 1$ or $y^{(i)} = -1$
- Margin[1] equals $\frac{|b|}{\sqrt{\theta^T\theta}} = \frac{y^{(i)}\theta^T x^{(i)}}{\sqrt{\theta^T\theta}}$. Maximising $\frac{y^{(i)}\theta^T x^{(i)}}{\sqrt{\theta^T\theta}}$ is the same as minimising $-\frac{y^{(i)}\theta^T x^{(i)}}{\sqrt{\theta^T\theta}}$.
- Hinge loss $\max(0, 1 - y\theta^T x)$ assigns zero penalty to points where $y\theta^T x \geq 1$ i.e. value of cost function $J(\theta)$ is determined only by those points for which $y\theta^T x < 1$

[1] https://en.wikipedia.org/wiki/Distance_from_a_point_to_a_plane

Maximising Margin



- There is a trade-off between maximising the margin and classification accuracy
- Recall cost function $J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \max(0, 1 - y^{(i)} \theta^T x^{(i)}) + \lambda \theta^T \theta$
- Decreasing λ tends to reduce misclassification but leads to smaller margin
- Increasing λ tends to increase misclassification but leads to larger margin

SVM Summary

- Hypothesis: $h_\theta(x) = sign(\theta^T x)$
- Parameters: θ
- Cost Function: $J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \max(0, 1 - y^{(i)} \theta^T x^{(i)}) + \lambda \theta^T \theta$
- Goal: Select θ that minimise $J(\theta)$

Gradient Descent for SVMs

As before, can find θ using:

- Start with some θ
- Repeat:

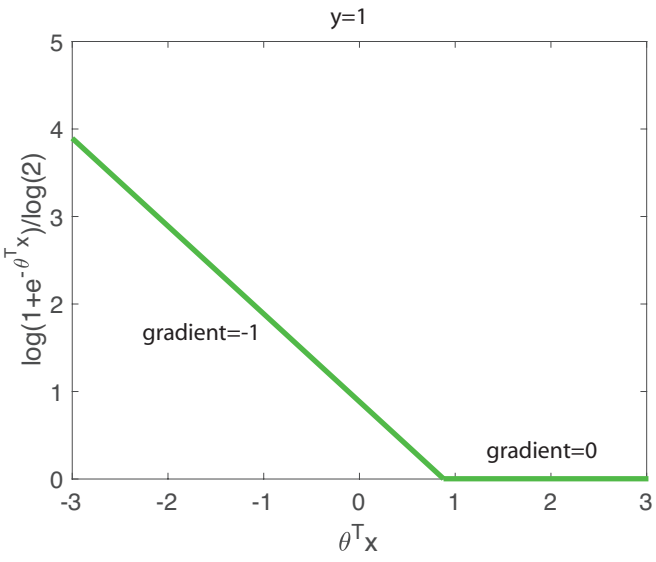
  Update vector θ to new value which makes $J(\theta)$ smaller

We can't use gradient descent directly since

$$J(\theta) = \frac{1}{m} \sum_{i=1}^{m} \max(0, 1 - y^{(i)} \theta^T x^{(i)}) + \lambda \theta^T \theta$$

is not differentiable due to max(). But can still use a **subgradient** approach.

Gradient Descent for SVMs



- Subgradient of $\max(0, 1 - z)$ is -1 when $z \leq 1$ and 0 when $z > 1$.
- Derivative of $1 - y\theta^T x$ with respect to θ_j is yx_j.
- Putting these together, subgradient of $\max(0, 1 - y\theta^T x)$ is

$$\begin{cases} -yx_j & \text{when } y\theta^T x \leq 1 \\ 0 & \text{when } y\theta^T x > 1 \end{cases}$$

Gradient Descent for SVMs

For $J(\theta) = \frac{1}{m}\sum_{i=1}^{m} \max(0, 1 - y^{(i)}\theta^T x^{(i)}) + \lambda\theta^T\theta$, subgradient with respect to θ_j is:

- $2\lambda\theta_j - \frac{1}{m}\sum_{i=1}^{m} y^{(i)}x_j^{(i)}\mathbb{1}(y^{(i)}\theta^T x^{(i)} \leq 1)$ where $\mathbb{1}(y^{(i)}\theta^T x^{(i)} \leq 1) = 1$ when $y^{(i)}\theta^T x^{(i)} \leq 1$ and zero otherwise.

So subgradient descent algorithm for SVMs is:

- Start with some θ
- Repeat:

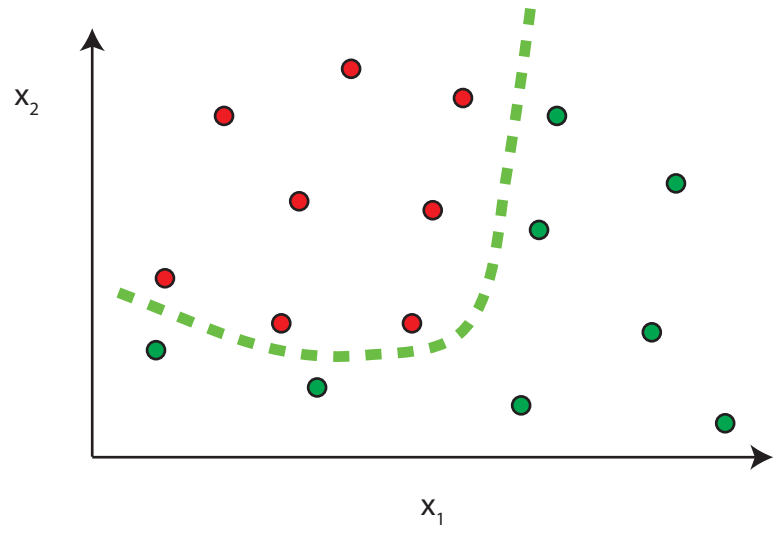
  for j=0 to n

  $\{tempj := \theta_j - \alpha(2\lambda\theta_j - \frac{1}{m}\sum_{i=1}^{m} y^{(i)}x_j^{(i)}\mathbb{1}(y^{(i)}\theta^T x^{(i)} \leq 1))$

  for j=0 to n $\{\theta_j := tempj\}$

$J(\theta)$ is convex, has a single minimum. Iteration moves downhill until it reaches the minimum

Nonlinear Decision Boundary



- As with linear and logistic regression we can add extra "features" e.g. use $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2$, so predict $y = 1$ when $\theta_0 + \theta_1 x_1 + \theta_2 x_2 + \theta_3 x_1^2 + \theta_4 x_2^2 > 0$