Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
- vector optimization

Optimization problem in standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal and locally optimal points

x is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

a feasible x is **optimal** if $f_0(x) = p^\star$; $X_{\rm opt}$ is the set of optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \mbox{minimize (over } z\mbox{)} & f_0(z) \\ \mbox{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$

examples (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
- $f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x = 1/e$ is optimal
- $f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x = 1$

Implicit constraints

the standard form optimization problem has an **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the explicit constraints
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\textstyle\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$
\begin{array}{ll}
\text{find} & x \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$
\begin{array}{ll}
\text{minimize} & 0 \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Convex optimization problem

standard form convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- f_0, f_1, . . . , f_m are convex; equality constraints are affine

often written as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

important property: feasible set of a convex optimization problem is convex

example

$$
\begin{array}{ll}
\text{minimize} & f_0(x) = x_1^2 + x_2^2 \\
\text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\
 & h_1(x) = (x_1 + x_2)^2 = 0
\end{array}
$$

example

$$\begin{array}{ll} \text{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \text{subject to} & f_1(x) = x_1/(1 + x_2^2) \leq 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

- f_0 is convex; feasible set $\{(x_1, x_2) \mid x_1 = -x_2 \leq 0\}$ is convex
- not a convex problem (according to our definition)
- equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \text{minimize} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 + x_2 = 0 \end{array}$$

Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \quad \|z - x\|_2 \le R \quad \Longrightarrow \quad f_0(z) \ge f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and

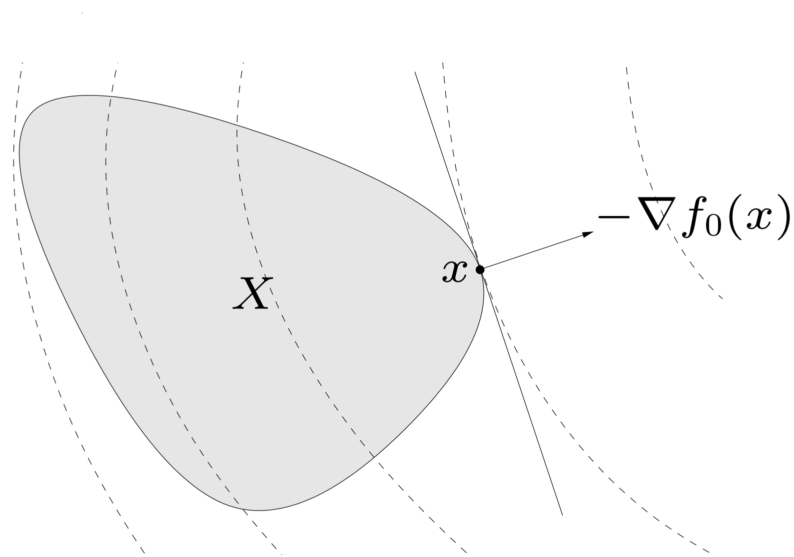
$$f_0(z) \le \theta f_0(y) + (1 - \theta) f_0(x) < f_0(x)$$

which contradicts our assumption that x is locally optimal

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$



if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

- **unconstrained problem**: x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad \nabla f_0(x) = 0$$

- **equality constrained problem**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \qquad Ax = b, \qquad \nabla f_0(x) + A^T\nu = 0$$

- **minimization over nonnegative orthant**

$$\text{minimize} \quad f_0(x) \quad \text{subject to} \quad x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \qquad x \succeq 0, \qquad \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **introducing equality constraints**

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\ y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \ldots, m \end{array}$$

- **minimizing over some variables**

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize} & \tilde{f}_0(x_1) \\ \text{subject to} & f_i(x_1) \le 0, \quad i = 1, \ldots, m \end{array}$$

where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

- **introducing slack variables for linear inequalities**

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \le b_i, \quad i = 1, \ldots, m \end{array}$$

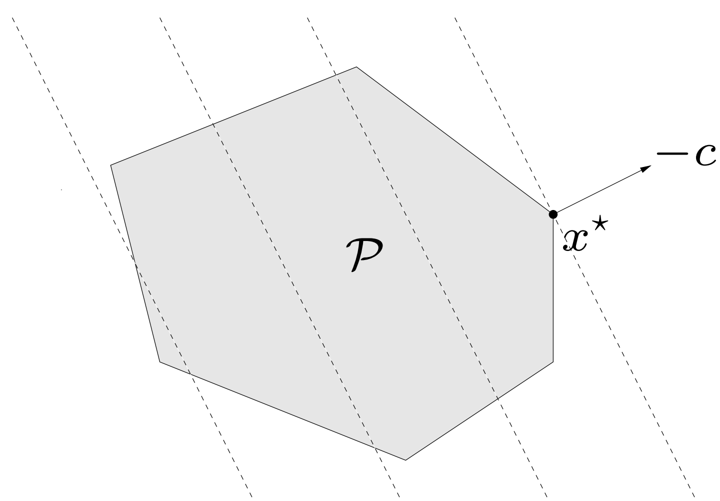
is equivalent to

$$\begin{array}{ll} \text{minimize (over } x,\ s) & f_0(x) \\ \text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\ & s_i \ge 0, \quad i = 1, \ldots m \end{array}$$

Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

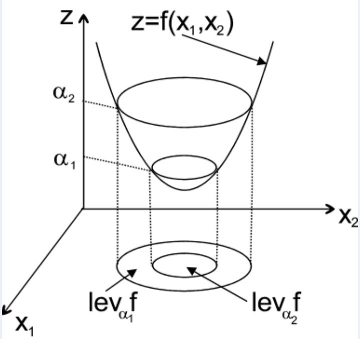
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Level Sets/Contour Lines

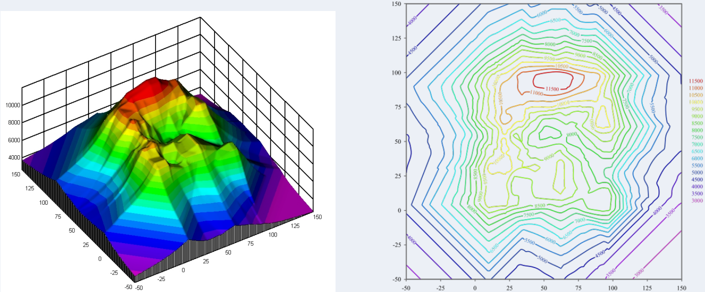
- The α-level set of a function $f(x) : R^n \to R$ is the set of points:

$$L_a = \{x \in R^n | f(x) = \alpha\}$$



Level Sets/Contour Lines

- When $f(x) : R^2 \rightarrow R$ then we also call them contour lines or isolines.



(a) (b)

Figure: See wikipedia for more and nicer examples!

btw, these are the plots we see in weather reports (winds, barometric, etc).

Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

- one unit of food j costs c_j, contains amount a_{ij} of nutrient i
- healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

piecewise-linear minimization

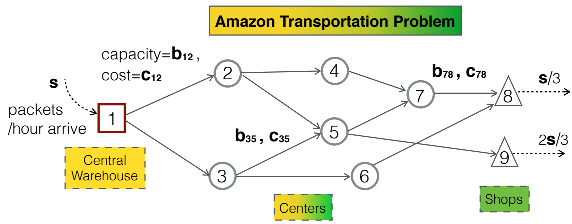
$$\text{minimize} \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & a_i^T x + b_i \leq t, \quad i = 1, \ldots, m \end{array}$$

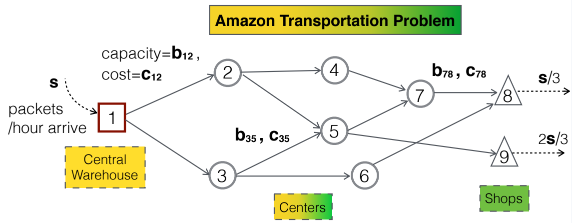
More LP Examples

- The transportation problem; or the min-cost/max-flow problem.
- The "Amazon" scenario:
  - Consider the product distribution network of Amazon where we need to transfer some packages (or, packets) from a central warehouse, to some local shops, through a set of intermediate distribution centers.
  - The system is described by a network graph $G = (\mathcal{N}, \mathcal{E})$.
  - c_{ij} is the per packet cost of using link $(i, j) \in \mathcal{E}$.
  - b_{ij} is the maximum number of packets that can be transferred over link $(i, j) \in \mathcal{E}$.
- We need to decide:
  1. Variables?
  2. Objective function?
  3. Constraints?



$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \sum_{(i,j)\in\mathcal{E}} c_{ij}x_{ij} = \boldsymbol{c}^T\boldsymbol{x} && (1)\\
\text{s.t.} \quad & 0 \le x_{ij} \le b_{ij}, \qquad \forall\,(i,j)\in\mathcal{E} && (2)\\
& \sum_{i:(i,j)\in\mathcal{E}} = \sum_{k:(j,k)\in\mathcal{E}} x_{jk}, \qquad \forall\, j \in \mathcal{N}\ \{1,8,9\} && (3)\\
& x_{12} + x_{13} = s && (4)\\
& x_{78} + x_{68} = s/3 && (5)\\
& x_{58} = 2s/3 && (6)\\
, \quad & x_{ij} \ge 0, \quad \forall\,(i,j)\in\mathcal{E} && (7)
\end{aligned}
$$

- The solution $\boldsymbol{x}^*$ is the optimal transportation policy (packets over each link).
  - Note that we assume continuous splitting of packets is possible;
  - The policy is applied for, say, every day during the next 3 months.



- A twist in the problem:
  - Assume that the transportation cost might change every day, c_d, $d = 1, 2, \ldots, 7$.
  - We want to find the transportation policy (**the same across all days**) that will minimize the maximum cost (i.e., the worst case) that we will pay in each day.
- The new problem can be written:

$$\begin{aligned} \min_{\boldsymbol{x}} \; & \{ \max_{d=1\ldots7} \; \boldsymbol{c}_d^T \boldsymbol{x} \, \} \\ \text{s.t.} \; & (2) - (7) \end{aligned}$$

- Is this problem a convex one? and LP?

Epigraph Form

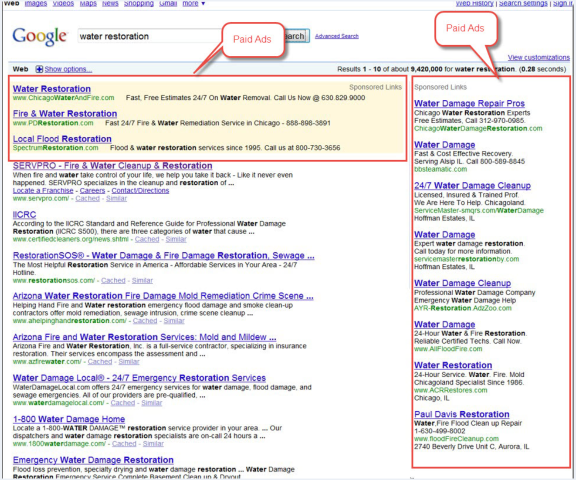
- We used a standard trick:
  - Use the epigraph form of a problem, transforming it this way to an LP.
- All convex problems can be transformed to problems with linear objectives!
- Convex problem:

$$\begin{array}{ll} \min_x & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- Problem in epigraph form:

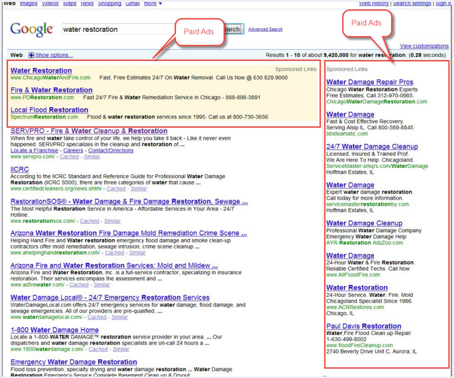
$$\begin{array}{ll} \min_{x,t} & t \\ \text{subject to} & f_0(\boldsymbol{x}) - t \leq 0, \\ & f_i(\boldsymbol{x}) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

More LP Examples - Sponsored Search



- A search engine (SE) wishes to allocate its ad slots, for certain keywords, to businesses ("clients") that want to be advertised.

Sponsored Search



- Some context:
  - the higher an ad is displayed, more likely to be "clicked";
  - the SE is being paid only if the client’s link is "clicked" by a user.
  - clients indicate to which searches (which keywords) they want to be displayed, and how much they are willing to pay per click to the SE;
  - clients also determine the maximum budget they are willing to pay (e.g. for each day).

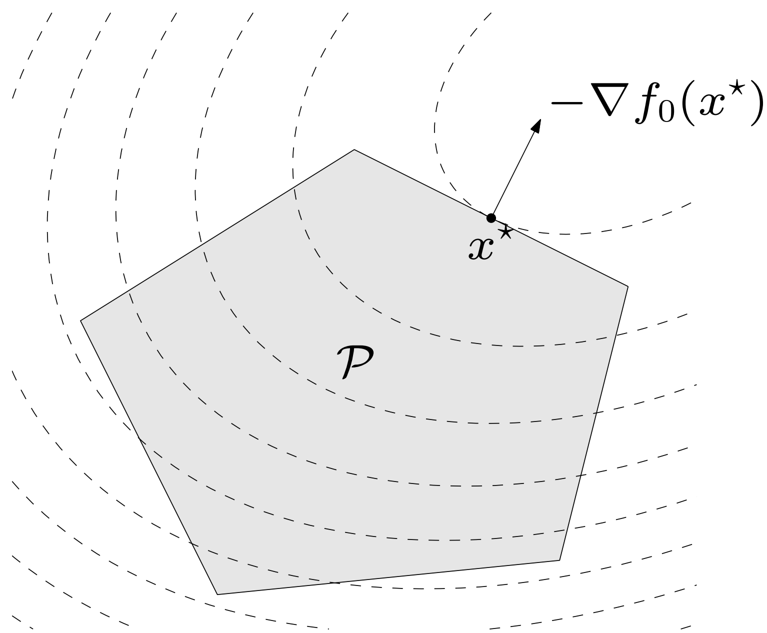
Sponsored Search

- The problem of sponsored search ads (Google, Yahoo!, etc):
  - A search engine (SE) wishes to allocate its ad slots, for certain keywords, to a set of clients/buyers/bidders.
  - There is a set $\mathcal{N}$ of $N = |\mathcal{N}$ such clients, where clients $i \in \mathcal{N}$ has submitted a budget of B_i Euros (e.g., for 1 day), and is willing to pay p_i Euros to the SE per click it receives.
  - There is a set $\mathcal{K}$ of $1, 2, \ldots, K$ available slots.
  - $c_{ij} \in [0, 1]$ is the probability that client i's ad will be clicked if it is displayed in the jth search slot; the higher the better, i.e., $c_{ij} \geq c_{i(j+1)}$, for every i, j.
  - SE needs to decide, for each keyword, whether it will display the ad of each company (buyer), and if so, in which slot. Each ad is displayed only in one slot at most.
- What is the ad slot allocation policy that maximizes the revenue of the SE?

Quadratic program (QP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P x + q^T x + r \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Examples

least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

- analytical solution $x^\star = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, *e.g.*, $l \preceq x \preceq u$

linear program with random cost

$$\begin{array}{ll} \text{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma\, \mathbf{var}(c^T x) \\ \text{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

- c is random vector with mean $\bar{c}$ and covariance Σ
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Robust linear programming

the parameters in optimization problems are often uncertain, *e.g.*, in an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m, \end{array}$$

there can be uncertainty in c, a_i, b_i

two common approaches to handling uncertainty (in a_i, for simplicity)

- deterministic model: constraints must hold for all $a_i \in \mathcal{E}_i$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m, \end{array}$$

- stochastic model: a_i is random variable; constraints must hold with probability η

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & \mathbf{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m \end{array}$$

Quadratically constrained quadratic program (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Generalized inequalities

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$

generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad\qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \mathbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \dots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

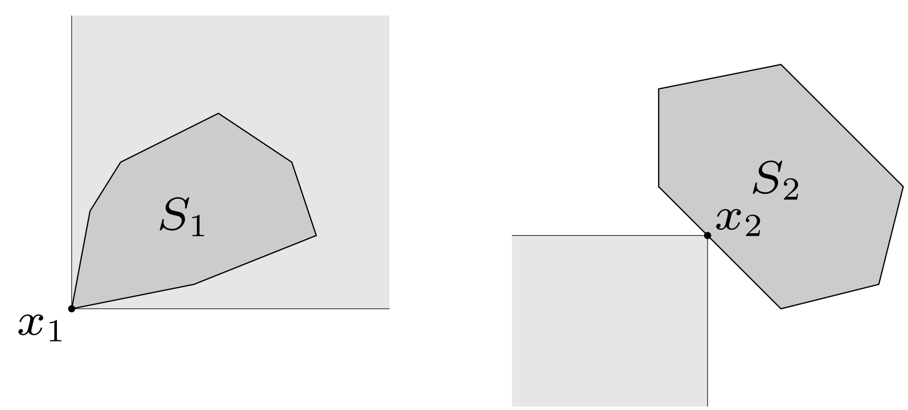
$$y \in S \quad \Longrightarrow \quad x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \quad \Longrightarrow \quad y = x$$

example $(K = \mathbf{R}^2_+)$

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2



Convexity with respect to generalized inequalities

$f : \mathbf{R}^n \to \mathbf{R}^m$ is K-convex if $\mathbf{dom}\, f$ is convex and

$$f(\theta x + (1-\theta)y) \preceq_K \theta f(x) + (1-\theta)f(y)$$

for x, $y \in \mathbf{dom}\, f$, $0 \leq \theta \leq 1$

example $f : \mathbf{S}^m \to \mathbf{S}^m$, $f(X) = X^2$ is $\mathbf{S}^m_+$-convex

proof: for fixed $z \in \mathbf{R}^m$, $z^T X^2 z = \|Xz\|_2^2$ is convex in X, *i.e.*,

$$z^T(\theta X + (1-\theta)Y)^2 z \leq \theta z^T X^2 z + (1-\theta) z^T Y^2 z$$

for $X, Y \in \mathbf{S}^m$, $0 \leq \theta \leq 1$

therefore $(\theta X + (1-\theta)Y)^2 \preceq \theta X^2 + (1-\theta)Y^2$

Vector optimization

general vector optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& h_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

vector objective $f_0 : \mathbf{R}^n \to \mathbf{R}^q$, minimized w.r.t. proper cone $K \in \mathbf{R}^q$

convex vector optimization problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

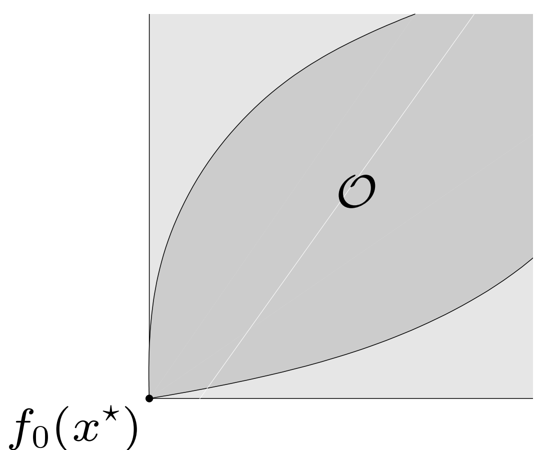
with f_0 K-convex, f_1, . . . , f_m convex

Optimal and Pareto optimal points

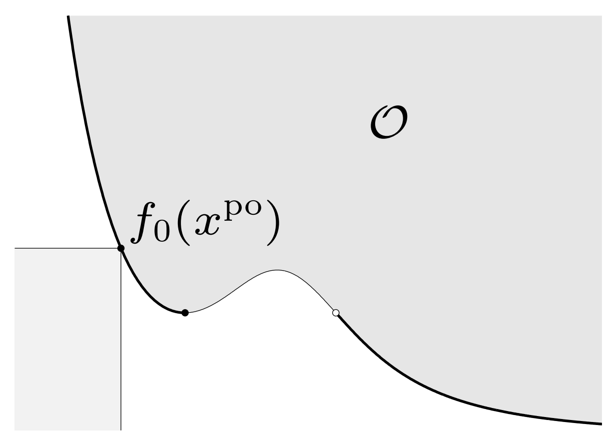
set of achievable objective values

$$\mathcal{O} = \{f_0(x) \mid x \text{ feasible}\}$$

- feasible x is **optimal** if $f_0(x)$ is the minimum value of $\mathcal{O}$
- feasible x is **Pareto optimal** if $f_0(x)$ is a minimal value of $\mathcal{O}$



$x^\star$ is optimal



$x^{\rm po}$ is Pareto optimal

Multicriterion optimization

vector optimization problem with $K = \mathbf{R}^q_+$

$$f_0(x) = (F_1(x), \dots, F_q(x))$$

- q different objectives F_i; roughly speaking we want all F_i's to be small
- feasible $x^\star$ is optimal if

$$y \text{ feasible} \quad \Longrightarrow \quad f_0(x^\star) \preceq f_0(y)$$

  if there exists an optimal point, the objectives are noncompeting

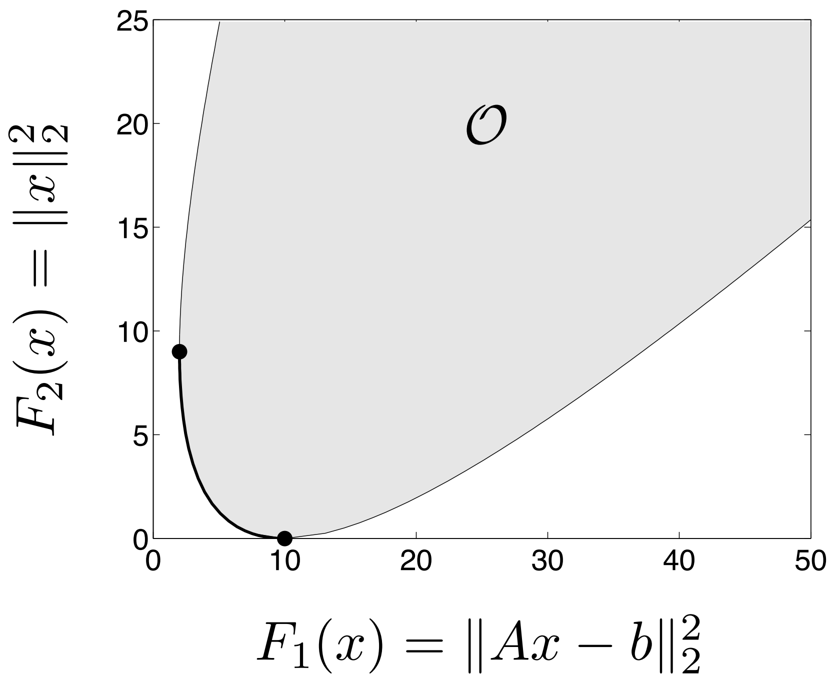
- feasible x^{po} is Pareto optimal if

$$y \text{ feasible}, \quad f_0(y) \preceq f_0(x^{\text{po}}) \quad \Longrightarrow \quad f_0(x^{\text{po}}) = f_0(y)$$

  if there are multiple Pareto optimal values, there is a trade-off between the objectives

Regularized least-squares

minimize (w.r.t. $\mathbf{R}^2_+$) $\quad (\|Ax - b\|_2^2, \|x\|_2^2)$



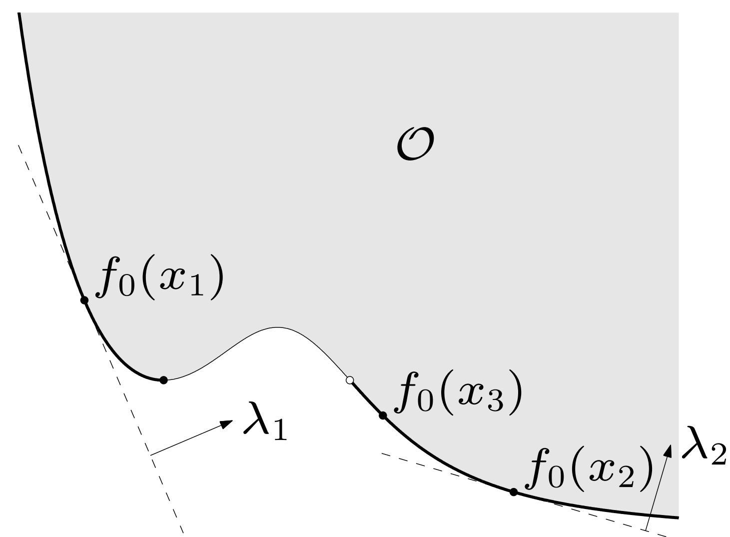
example for $A \in \mathbf{R}^{100\times 10}$; heavy line is formed by Pareto optimal points

Scalarization

to find Pareto optimal points: choose $\lambda \succ_{K^*} 0$ and solve scalar problem

$$\begin{array}{ll} \text{minimize} & \lambda^T f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

if x is optimal for scalar problem, then it is Pareto-optimal for vector optimization problem



for convex vector optimization problems, can find (almost) all Pareto optimal points by varying $\lambda \succ_{K^*} 0$

Scalarization for multicriterion problems

to find Pareto optimal points, minimize positive weighted sum

$$\lambda^T f_0(x) = \lambda_1 F_1(x) + \cdots + \lambda_q F_q(x)$$

examples

- regularized least-squares problem of page 4–43

take $\lambda = (1, \gamma)$ with $\gamma > 0$

$$\text{minimize} \quad \|Ax - b\|_2^2 + \gamma \|x\|_2^2$$

for fixed γ, a LS problem