Optimisation for Data Analysis **CS7DS2/CS4405**

Mid Term Exams - March 2018

Duration: 55 minutes

Exercise 1 (10 Points). Prove whether the following functions are strictly convex, convex, strictly concave, concave or neither of these.

1. $f(x) = -2x^2$, with $f : \mathbf{R} \longrightarrow \mathbf{R}$, $\mathbf{dom}\ f = \mathbf{R}$, and $x \in \mathbf{R}$.
2. $f(x) = 4x_1^2 + 3x_2^2 + 5x_3^2 + 6x_1x_2 + x_1x_3 - 3x_1 - 2x_2 + 15$, with $f : \mathbf{R}^3 \to \mathbf{R}$ and $\mathbf{dom}\ f = \{x_1, x_2, x_3 \in \mathbf{R} \mid x_3 \leq 10, x_1 \in \mathcal{N}, x_2 \geq 0\}$ where $\mathcal{N} = \{0, 1, 2, 3, 4, 5\}$

Exercise 2 (10 Points). For which values of parameter α is the following function convex?

$$f(x_1, x_2) = x_1^2 - \alpha x_1 x_2 + x_2^2 \tag{1.1}$$

with $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$, $\mathbf{dom}\ f = \mathbf{R}^2$, $x_1, x_2 \in \mathbf{R}$.

Exercise 3 (15 Points). Draw (sketch) the following:

1. a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ that is not convex. Also, draw a set that is not convex and its convex hull.
2. a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ that is convex but not-differentiable at least in one point. Explain how you can prove the convexity of this function, i.e., which formula, rule, etc. you can use.
3. draw a polyhedron in $\mathbf{R}^2$ that has only minimal elements (not minimum) with respect to the cone $\mathbf{R}^2_+$ (i.e., use the generalized inequality $\preceq_{\mathbf{R}^2_+}$); and write the mathematical expression for this set.

Exercise 4 (10 Points). Consider the function:

$$f(x) = \min_x \{f_1(x), f_2(x)\} \tag{1.2}$$

where $x \in \mathbf{R}^n$, $f_1 : \mathbf{R}^n \to \mathbf{R}$ and $f_2 : \mathbf{R}^n \to \mathbf{R}$ are concave functions with $\mathbf{dom}\ f = \mathbf{dom}\ f_1 \cap \mathbf{dom}\ f_2$. Prove that $f(x)$ is also a concave function.

Exercise 5 (10 Points). Consider the following optimization problem:

$$\begin{array}{ll} \underset{x_1, x_2}{\text{minimize}} & 1 \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 2 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array} \tag{1.3}$$

with $x_1, x_2 \in \mathbf{R}$.

1. Is this a convex optimization problem? Justify your answer.
2. Draw (sketch) the feasible set of the problem. What type of set is this? (e.g., cone, hyperplane, hyperspace, polyhedron, Euclidean ball, polytope, etc.). Justify your answer.
3. Find the optimal value $p^\star$ of this problem and a solution $(x_1^\star, x_2^\star)$.

Exercise 6 (15 Points). Consider the following optimization problem:

$$\begin{array}{ll} \underset{x}{\text{maximize}} & f_0(x) \\ \text{subject to} & f_1(x) = 0 \\ & f_2(x) \geq 0 \\ & Ax = b \end{array} \tag{1.4}$$

where $x \in \mathbf{R}^n$, $f_0 : \mathbf{R}^n \to \mathbf{R}$ is concave, $f_1 : \mathbf{R}^n \to \mathbf{R}$, $f_2 : \mathbf{R}^n \to \mathbf{R}$, $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$. You are asked to:

1. Write the problem in *standard optimization* form;
2. Under what conditions is it a convex optimization problem?
3. Assume now that these conditions are satisfied, and write the problem in its *epigraph form*. Is this new problem a *linear program* (LP)? Justify your answer.

Exercise 7 (30 Points). You are the master-chef of a famous restaurant in Dublin. You are given a list of ingredients (shown in Table 1.1) and you are asked to design a recipe. You know that a healthy meal should contain different nutrients in quantities at least equal to the values shown in Table 1.2.

1. **Version 1**: Formulate the optimisation problem that minimizes the cost of your recipe while ensuring that it is a healthy one. Write the necessary variables and parameters in detail (matrices, vectors, etc.), and their constraints. Write this optimization problem as a *linear program* in *standard* form.
2. **Version 2:** Now consider the following slightly different problem: not only you are interested in minimizing the cost of the recipe, but now you additionally wish your recipe to be as similar as possible to the following "delicious" recipe: Rice 300 g; Pepper 100 g; Onion 200 g; Salmon 100 g. You are asked to formulate this new optimization problem taking into account that your boss is more interested (say, 2 times more) in minimizing the cost than offering a "delicious" meal.

Product	Price (€)	Protein	Fat	Sugar
Rice	1	3	0.5	2
Pepper	1.5	2	1	0
Onion	0.5	1	0	0
Salmon	2	4	5	4

Table 1.1: Price and nutrient intake per 100 g.

Protein	Fat	Sugar
12	10	30

Table 1.2: Minimum quantities per nutrient, for a healthy meal (in grams).

Notes

- The mathematical expression $x \in \mathcal{S}$ means that x is an element of ("belongs to") the set $\mathcal{S}$. $\mathbf{R}$ is the set of real numbers. The term **convex** means just"convex", i.e., not "convex or concave". If the question is related to concavity ("concave"), this will be explicitly stated.
- All answers should be justified/proved. The available time for this test is 55 minutes. Please write clearly your answers, your name, student number, and number all the pages.