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BOUNDS ON THE DENSITY OF STATES FOR SCHRÖDINGER OPERATORS

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ABSTRACT. We establish bounds on the density of states measure for Schrödinger operators. These are deterministic results that do not require the existence of the density of states measure, or, equivalently, of the integrated density of states. The results are stated in terms of a "density of states outer-measure" that always exists, and provides an upper bound for the density of states measure when it exists. We prove log-Hölder continuity for this density of states outer-measure in one, two, and three dimensions for Schrödinger operators, and in any dimension for discrete Schrödinger operators.

CONTENTS

1. INTRODUCTION

We study the density of states of the Schrödinger operator

$$H = -\Delta + V \quad \text{on} \quad L^2(\mathbb{R}^d), \tag{1.1}$$

where Δ is the Laplacian operator and V is a bounded potential. The density of states measure of an interval gives the "number of states per unit volume" with energy in the interval; its cumulative distribution function is the integrated density of states. Finite volume density of states measures, i.e., density of states measures for restrictions of the Schrödinger operator to finite volumes, are always well defined. The density of states measure is given by appropriate limits of finite volume density of states measures, when such limits exist. These limits are known to exist for Schrödinger operators where the potential V is in some sense uniform in space (e.g.,

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periodic potentials, ergodic Schrödinger operators), but not for general Schrödinger operators. The density of states measure and the corresponding integrated density of states cannot be defined for general Schrödinger operators. For this reason we introduce the density of states outer-measure, which always exists, and provides an upper bound for the density of states measure, when it exists. We prove upper bounds on the density of states outer-measure of small intervals, establishing log-Hölder continuity in one, two, and three dimensions for Schrödinger operators, and in any dimension for discrete Schrödinger operators.

We let

$$\Lambda_L(x) := x + \left]-\tfrac{L}{2}, \tfrac{L}{2}\right[^d = \left\{y \in \mathbb{R}^d;\ |y-x|_\infty < \tfrac{L}{2}\right\} \tag{1.2}$$

denote the (open) box of side L centered at $x \in \mathbb{R}^d$. By a box Λ_L we will mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. We write $\|\psi\| = \|\psi\|_2$ for $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ or $\psi \in \mathrm{L}^2(\Lambda)$. We set $V_\infty = \|V\|_\infty$, the norm of the bounded potential V. By χ_B we denote the characteristic function of the set B. Constants such as $C_{a,b,\ldots}$ will always be finite and depending only on the parameters or quantities $a, b, \ldots$; they will be independent of other parameters or quantities in the equation. Note that $C_{a,b,\ldots}$ may stand for different constants in different sides of the same inequality.

Given a finite box $\Lambda \subset \mathbb{R}^d$, we let $H_\Lambda^\sharp$ and $\Delta_\Lambda^\sharp$ be the restriction of H and Δ to $\mathrm{L}^2(\Lambda)$ with $\sharp$ boundary condition, where $\sharp = D$ (Dirichlet), N (Neumann), or P (periodic). We define finite volume density of states measures $\eta_{\Lambda,\sharp}$ on Borel subsets B of $\mathbb{R}^d$ by

$$\begin{aligned} \eta_{\Lambda,\sharp}(B) &:= \tfrac{1}{|\Lambda|} \operatorname{tr}\left\{\chi_B(H_\Lambda^\sharp)\right\} \quad \text{for} \quad \sharp = D, N, P, \\ \eta_{\Lambda,\infty}(B) &:= \tfrac{1}{|\Lambda|} \operatorname{tr}\left\{\chi_B(H)\chi_\Lambda\right\}. \end{aligned} \tag{1.3}$$

Note that for for all Borel subsets $B \subset]-\infty, E]$ we have

$$\eta_{\Lambda,\sharp}(B) \le C_{d,V_\infty,E} < \infty \quad \text{for} \quad \sharp = \infty, D, N, P. \tag{1.4}$$

Moreover, given $f \in C_c(\mathbb{R})$ and $\delta > 0$, there exists $L(d, V_\infty, \delta, f)$ such that for all $L \ge L(d, V_\infty, \delta, f)$ and $x_0 \in \mathbb{R}^d$ we have

$$\left|\eta_{\Lambda_L(x_0),\sharp_1}(f) - \eta_{\Lambda_L(x_0),\sharp_2}(f)\right| \le \delta \quad \text{for} \quad \sharp_1, \sharp_2 = \infty, D, N, P. \tag{1.5}$$

(This can be extracted from [DoIM, see Theorem 3.6, Theorem 6.2, and their proofs].) The finite volume integrated density of states are the corresponding cumulative distribution functions:

$$N_{\Lambda,\sharp}(E) := \eta_{\Lambda,\sharp}(]-\infty, E]). \tag{1.6}$$

For periodic and ergodic Schrödinger operators, density of states measures $\eta_\sharp$ can be defined as weak limits of the finite volume density of states measures $\eta_{\Lambda,\sharp}$ for sequences of boxes $\Lambda \to \mathbb{R}^d$ in an appropriate sense. In this case, the integrated density of states $N_\sharp(E) := \eta_\sharp(]-\infty, E])$ satisfies $N_\sharp(E) = \lim_{\Lambda\to\mathbb{R}^d} N_{\Lambda,\sharp}(E)$ except for a countable set of energies. Moreover, they all coincide, so we define the density of states measure η and the integrated density of states $N(E)$ by $\eta(B) := \eta_\sharp(B)$ and $N(E) := N_\sharp(E)$ for $\sharp = \infty, D, N, P$. (See [KM, PF, CL, DoIM, N].)

Since infinite volume density of states measures and integrated density of states cannot be defined for general Schrödinger operators, we define density of states outer-measures on Borel subsets B of $\mathbb{R}^d$ by

$$\begin{aligned} \eta^*_{L,\sharp}(B) &:= \sup_{x\in\mathbb{R}^d} \eta_{\Lambda_L(x),\sharp}(B) \\ \eta^*_{\sharp}(B) &:= \limsup_{L\to\infty} \eta^*_{L,\sharp}(B) \end{aligned}, \qquad \sharp = \infty, D, N, P. \tag{1.7}$$

These are always finite on bounded sets in view of (1.4). (They are indeed outer-measures, so we call them outer-measures for lack of a better name.) Moreover, it follows from (1.5) that for all $E_1, E_2 \in \mathbb{R}$, $E_1 \le E_2$, and $\delta > 0$ we have

$$\eta^*_{\sharp_1}([E_1, E_2]) \le \eta^*_{\sharp_2}([E_1 - \delta, E_2 + \delta]) \quad \text{for all} \quad \sharp_1, \sharp_2 = \infty, D, N, P. \tag{1.8}$$

We will say that we have continuity of the density of states outer-measure $\eta^*_\sharp$ if

$$\lim_{\varepsilon\to 0} \eta^*_\sharp([E-\varepsilon, E+\varepsilon]) = 0 \quad \text{for all} \quad E \in \mathbb{R}. \tag{1.9}$$

In view of (1.8), continuity of $\eta^*_\sharp$ for some value of $\sharp$ implies continuity of $\eta^*_\sharp$ for all values of $\sharp$, and we have

$$\eta^*_\infty([E_1, E_2]) = \eta^*_D([E_1, E_2]) = \eta^*_N([E_1, E_2]) = \eta^*_P([E_1, E_2]) \tag{1.10}$$

for all $E_1, E_2 \in \mathbb{R}$, $E_1 \le E_2$. In this case we set

$$\eta^*([E_1, E_2]) := \eta^*_\sharp([E_1, E_2]) \quad \text{for} \quad \sharp = \infty, D, N, P, \tag{1.11}$$

and say that we have continuity of the density of states outer-measure.

We are ready to state our main result. Note that if the density of states measure $\eta_\sharp$ exists, we always have

$$\eta_\sharp(B) \le \eta^*_\sharp(B) \quad \text{for all Borel sets} \quad B \subset \mathbb{R}^d, \tag{1.12}$$

and hence continuity of the density of states outer-measure implies continuity of the integrated density of states

Theorem 1.1. *Let H be a Schrödinger operator as in* (1.1)*, where $d = 1, 2, 3$. Then we have continuity of the density of states outer-measure. Moreover, given $E_0 \in \mathbb{R}$, for all $E \le E_0$ and $\varepsilon \le \frac{1}{2}$ we have*

$$\eta^*\left([E, E+\varepsilon]\right) \le \frac{C_{d,V_\infty,E_0}}{\left(\log \frac{1}{\varepsilon}\right)^{\kappa_d}}, \quad \textit{where} \quad \kappa_1 = 1,\ \kappa_2 = \tfrac{1}{4},\ \kappa_3 = \tfrac{1}{8}. \tag{1.13}$$

We also prove a similar result for discrete Schrödinger operators, i.e., for

$$H = -\Delta + V \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{1.14}$$

where V is a bounded potential and Δ is the centered discrete Laplacian,

$$\Delta\psi(x) = \sum_{y\in\mathbb{Z}^d;\ |x-y|=1} \psi(y) \quad \text{for} \quad x \in \mathbb{Z}^d. \tag{1.15}$$

(Our results are still valid if we take Δ to be any translation invariant finite range self-adjoint operator on $\ell^2(\mathbb{Z}^d)$.) In $\mathbb{Z}^d$ we define the box of side L centered at $x \in \mathbb{Z}^d$ by

$$\Lambda = \Lambda_L(x) = \left\{ y \in \mathbb{Z}^d;\ |y - x|_\infty \le \tfrac{L}{2} \right\}, \tag{1.16}$$

and define finite volume operators $H^\sharp_\Lambda$ and $\Delta^\sharp_\Lambda$ as the restriction of H and Δ to $\ell^2(\Lambda)$ with $\sharp$ boundary condition, where $\sharp = D$ (Dirichlet, i.e., simple boundary condition) or P (periodic). We define finite volume density of states measures $\eta_{\Lambda,\sharp}$ as in (1.3) and density of states outer-measures $\eta^*_{L,\sharp}, \eta^*_\sharp$ as in (1.7) for $\sharp = \infty, D, P$.

In the discrete case it is easy to see that we also have (1.8), and hence continuity of $\eta^*_\sharp$ for some value of $\sharp$ implies (1.10), in which case we define η^* as in (1.11).

Theorem 1.2. *Let H be a discrete Schrödinger operator as in* (1.14)*. Then for all $d = 1, 2, \ldots$ we have continuity of the density of states outer-measure, and for all $E \in \mathbb{R}$ and $\varepsilon \leq \frac{1}{2}$ we have*

$$\eta^* \left([E, E + \varepsilon]\right) \leq \frac{C_{d,V_\infty}}{\log \frac{1}{\varepsilon}}. \tag{1.17}$$

We are not aware of previous results in the generality of Theorems 1.1 and 1.2. Published results appear to be restricted to cases where we have existence of the integrated density of states. For periodic potentials, continuity of the the integrated density of states is equivalent to the nonexistence of eigenvalues, a nontrivial result proved by Thomas [T]. For ergodic Schrödinger operators, continuity of the the integrated density of states is equivalent to the nonexistence of energies that are eigenvalues of infinite multiplicity with probability one (see [CL, Lemma V.2.1]). Although Schrödinger operators can have eigenvalues of infinite multiplicity (see [ThE]), it is hard to imagine how a fixed energy can be an eigenvalue of infinite multiplicity for almost all realizations of an ergodic Schrödinger operator.

Craig and Simon proved log-Hölder continuity (with exponent 1) of the integrated density of states for one-dimensional ergodic Schrödinger operators [CrS1] and for ergodic discrete Schrödinger operators in any dimension [CrS2]. Delyon and Souillard [DS] provided a simple proof of continuity of the integrated density of states in the discrete case. But continuity of the the integrated density of states for multi-dimensional (continuous) ergodic Schrödinger operators, albeit expected, has been hard to prove in full generality. It is Problem 14 in [Si2], where it was called (in 2000) a 15 year old open problem.

For random Schrödinger operators continuity of the integrated density of states follows from a suitable Wegner estimate. The most general result is due to Combes, Hislop and Klopp [CoHK] that proved that for the Anderson model, both continuous and discrete, we always have continuity of the integrated density of states if the single-site probability distribution has no atoms. (They show that the integrated density of states has as much regularity as the concentration function of the single-site probability distribution.) Germinet and Klein [GK2] proved log-Hölder continuity of the integrated density of states for the continuous Anderson model with arbitrary single-site probability distribution (e.g., Bernouilli) in the region of localization. (More precisely, in the region of applicability of the multiscale analysis; the log-Hölder continuity of the integrated density of states is derived from the conclusions of the multiscale analysis.)

The cases $d = 1$ and $d = 2, 3$ of Theorem 1.1 have separate proofs, the proof for $d = 1$ being similar to the proof of Theorem 1.2. Note that it suffices to establish (1.13) and (1.17) with Dirichlet boundary condition ($\sharp = D$), since we would then have (1.11). Thus in the following sections we assume Dirichlet boundary condition and drop it from the notation.

Theorem 1.2 and the $d = 1$ case of Theorem 1.1 are proved in Section 2; they are immediate consequences of Theorems 2.2 and 2.3, respectively.

Section 3 is devoted to multi-dimensional Schrödinger operators. We start by studying the local behavior of approximate solutions of the stationary Schrödinger equation in Subsection 3.1; see Theorem 3.1. Solutions of the stationary Schrödinger equation admit a local decomposition into a homogeneous harmonic polynomial and a lower order term [HW, B]; in Lemma 3.2 we establish a quantitative version of this decomposition with explicit estimates of the lower order term. This result is extended to approximate solutions in Lemma 3.3, implying Theorem 3.1. We then state and prove Theorem 3.4, a version of Bourgain and Kenig's quantitative unique continuation principle [BoK, Lemma 3.10], in which we make explicit the dependence on the parameters relevant to this article. Finally, in Subsection 3.3 we prove Theorem 3.7, which implies the $d = 2, 3$ cases of Theorem 1.1.

The restriction to $d = 1, 2, 3$ in Theorem 1.1 is due to the present form of the quantitative unique continuation principle (Theorem 3.4), where there is a term $Q^{\frac{4}{3}}$ in the exponent on the left hand side of (3.62). If we had Q^{β} in (3.62), we would be able to prove Theorem 3.7, and hence Theorem 1.1, for dimensions $d < \frac{\beta}{\beta-1}$. Since $\beta = \frac{4}{3}$, we get $d < 4$. It is reasonable to expect that something like Theorem 3.4 holds with $\beta = 1+$ (there are no counterexamples for real potentials), in which case Theorem 1.1 would hold for all d, with $\kappa_d = \frac{\beta - d(\beta-1)}{2\beta} = \frac{1}{2}-$ for $d \geq 2$ in (1.13).

2. Discrete and one-dimensional Schrödinger operators

To prove Theorem 1.2 and the $d = 1$ case of Theorem 1.1, we will select a class of approximate eigenfunctions for which we establish a global upper bound, and use Lemma 2.1 to pick an approximate eigenfunction for which we have a lower bound for the global upper bound. In more detail: Given an energy E, $0 < \varepsilon \leq \frac{1}{2}$, and a box Λ, we set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$ and consider the linear space $\operatorname{Ran} P$. (Note that $\psi \in \operatorname{Ran} P$ is an approximate eigenfunction for H_Λ in the sense that $\|(H_{\Lambda_L} - E)\,\psi\| \leq \varepsilon \|\psi\|$.) We select a linear subspace of $\mathcal{F}$ of $\operatorname{Ran} P$ for which the L^∞-norms are uniformly bounded in terms of the L^2-norms (a global upper bound). We then use Lemma 2.1 to pick $\psi_0 \in \mathcal{F}$ for which we have a lower bound for $\|\psi_0\|_\infty$. Comparing this lower bound with the global upper bound yields the bound on $\eta_\Lambda([E, E+\varepsilon])$.

2.1. A lower bound for the maximal L^∞ norm.

Lemma 2.1. *Let $\mathcal{V}$ be a finite dimensional linear subspace of $\mathrm{L}^\infty(\Omega, \mathbb{P})$, where $(\Omega, \mathbb{P})$ is a probability space. Then there exists $\psi \in \mathcal{V}$ with $\|\psi\|_2 = 1$ such that*

$$\|\psi\|_\infty \geq \sqrt{\dim \mathcal{V}}. \tag{2.1}$$

This lemma is known to follow immediately from the theory of absolutely summing operators, but can also be proved by a direct argument. We present both proofs for completeness.

Proof of Lemma 2.1 using absolutely summing operators. Let $\mathcal{V}_p$ denote the linear space $\mathcal{V}$ viewed as subspace of $\mathrm{L}^p(\Omega, \mathbb{P})$ and let $I^{p,q}$ be the identity map from $\mathcal{V}_p$ to $\mathcal{V}_q$, with $\pi_2(I^{p,q})$ being its 2-summing norm. (We refer to [DiJA] for the definition and properties of the 2-summing norm.) Then $\pi_2(I^{2,2}) = \sqrt{\dim \mathcal{V}}$, since it is the same as the Hilbert-Schmidt norm of $I^{2,2}$. Factor $I^{2,2} = I^{2,\infty} I^{\infty,2}$, so $\pi_2(I^{2,2}) \leq \left\|I^{2,\infty}\right\| \pi_2(I^{\infty,2})$ by the the ideal property [DiJA, item 2.4]. Since $\pi_2(I^{\infty,2}) \leq 1$ [DiJA, Example 2.9(d)], we have $\left\|I^{2,\infty}\right\| \geq \sqrt{\dim \mathcal{V}}$, and the lemma follows. □

Proof of Lemma 2.1 (direct proof). Using the the Gelfand-Neumark Theorem (e.g., [S, Section 73]) we can assume, without loss of generality, that Ω is a compact

Hausdorff space and $\mathrm{L}^\infty(\Omega,\mathbb{P}) = C(\Omega)$. Thus $\mathcal{V}$ is a finite dimensional linear subspace of $C(\Omega) \subset \mathrm{L}^2(\Omega,\mathbb{P})$. Let $N = \dim \mathcal{V}$, and pick an orthonormal basis $\{\phi_j\}_{j=1}^N$ for $\mathcal{V}$. In particular,

$$\phi(x,y) := \sum_{j=1}^N \overline{\phi_j(x)}\phi_j(y) \in C(\Omega^2), \tag{2.2}$$

and we have

$$N = \int_\Omega \phi(x,x)\,\mathbb{P}(\mathrm{d}x) = \int_\Omega \left\{ \frac{\phi(x,x)}{\sqrt{\phi(x,x)}} \right\}^2 \mathbb{P}(\mathrm{d}x) \le \int_\Omega \max_{y\in\Omega} \left\{ \frac{|\phi(x,y)|}{\sqrt{\phi(x,x)}} \right\}^2 \mathbb{P}(\mathrm{d}x). \tag{2.3}$$

Since $\mathbb{P}$ is a probability measure, there exists $x_0 \in \Omega$ such that

$$\max_{y\in\Omega} \frac{|\phi(x_0,y)|}{\sqrt{\phi(x_0,x_0)}} \ge \sqrt{N}. \tag{2.4}$$

Setting

$$\psi = \frac{\phi(x_0,\cdot)}{\sqrt{\phi(x_0,x_0)}} = \frac{1}{\sqrt{\phi(x_0,x_0)}} \sum_{j=1}^N \overline{\phi_j(x_0)}\phi_j, \tag{2.5}$$

we have $\psi \in \mathcal{V}$, $\|\psi\|_2 = 1$, and $\|\psi\|_\infty \ge \sqrt{N}$. □

2.2. **Discrete Schrödinger operators.** Theorem 1.2 is an immediate consequence of the following theorem.

Theorem 2.2. *Let H be a discrete Schrödinger operator as in* (1.14). *Then for all $0 < \varepsilon \le \frac{1}{2}$ and boxes $\Lambda = \Lambda_L$ with $L \ge L_{d,V_\infty} \log \frac{1}{\varepsilon}$ we have*

$$\eta_\Lambda\left([E, E+\varepsilon]\right) \le \frac{C_{d,V_\infty}}{\log \frac{1}{\varepsilon}} \quad \text{for all} \quad E \in \mathbb{R}. \tag{2.6}$$

Proof. Let $\Lambda_L = \Lambda_L(x_0)$ with $x_0 \in \mathbb{Z}^d$, $E \in \mathbb{R}$, and $\varepsilon \in]0, \frac{1}{2}]$. We set $P = \chi_{[E,E+\varepsilon]}(H_{\Lambda_L})$, and note that

$$\|(H_{\Lambda_L} - E)\,\psi\|_\infty \le \|(H_{\Lambda_L} - E)\,\psi\| \le \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P, \tag{2.7}$$

since we have $\|\psi\|_\infty \le \|\psi\|$ for all $\psi \in \ell^2(\Lambda)$.

We assume

$$\rho := \eta_{\Lambda_L}\left([E, E+\varepsilon]\right) = \tfrac{1}{|\Lambda_L|} \operatorname{tr} P > 0, \tag{2.8}$$

since otherwise there is nothing to prove.

We fix $R \in 2\mathbb{N}$, $R < L$, to be selected later, and pick $\mathcal{G} \subset \Lambda_L$ such that

$$\Lambda_L = \bigcup_{y\in\mathcal{G}} \Lambda_R(y) \quad \text{and} \quad \tfrac{|\Lambda_L|}{|\Lambda_R|} \le \#\mathcal{G} \le 2^d \tfrac{|\Lambda_L|}{|\Lambda_R|}. \tag{2.9}$$

Note that $(L-1)^d < |\Lambda_L| = \left(2\left\lfloor \frac{L}{2} \right\rfloor + 1\right)^d \le (L+1)^d$, and $|\Lambda_R| = (R+1)^d$. We set

$$\partial_2 \Lambda_R(y) = \left\{ x \in \Lambda_R(y);\ |x-y|_\infty \in \left\{ \tfrac{R}{2}, \tfrac{R}{2} - 1 \right\} \right\}, \tag{2.10}$$

and let

$$\partial_R \Lambda_L = \bigcup_{y\in\mathcal{G}} \partial_2 \Lambda_R(y). \tag{2.11}$$

We have

$$|\partial_2 \Lambda_R(y)| \leq c_d R^{d-1}, \quad \text{so} \quad |\partial_R \Lambda_L| \leq 2^d c_d R^{d-1} \tfrac{|\Lambda_L|}{|\Lambda_R|} \leq 2^d c_d \tfrac{|\Lambda_L|}{R}. \tag{2.12}$$

We take

$$L > \tfrac{2^{d+1} c_d}{\rho} + 2, \tag{2.13}$$

since otherwise there is nothing to prove for large L, and pick

$$R \in \left[\tfrac{2^{d+1} c_d}{\rho}, \tfrac{2^{d+1} c_d}{\rho} + 2\right) \cap 2\mathbb{N}. \tag{2.14}$$

We now consider the vector space

$$\mathcal{F} = \{\psi \in \operatorname{Ran} P;\ \psi(x) = 0 \quad \text{for all} \quad x \in \partial_R \Lambda_L\}. \tag{2.15}$$

Since $\mathcal{F}$ is the vector subspace of $\operatorname{Ran} P$ defined by $|\partial_R \Lambda_L|$ linear conditions, it follows from (2.8), (2.12), and (2.14) that

$$\dim \mathcal{F} \geq \rho\, |\Lambda_L| - |\partial_R \Lambda_L| \geq \tfrac{1}{2} \rho\, |\Lambda_L| \geq 1. \tag{2.16}$$

Let $\psi \in \mathcal{F}$ with $\|\psi\| = 1$. If $y \in \mathcal{G}$, it follows from (1.14), (1.15), and (2.7) that if we know that $|\psi(x)| \leq C$ for all x with $|x - y|_\infty = k+1, k+2$, then we must have $|\psi(x)| \leq CA + \varepsilon$ for $|x - y|_\infty = k$, where $A = 2d - 1 + \|V - E\|_\infty$. Since $\psi(x) = 0$ if $|x - y|_\infty = \frac{R}{2}, \frac{R}{2} - 1$, we get

$$|\psi(x)| \leq \varepsilon \sum_{j=0}^{\frac{R}{2} - 2 - |x-y|_\infty} A^j \leq \varepsilon \left(\tfrac{R}{2} - 1\right) A^{\frac{R}{2} - 2} \quad \text{for all} \quad x \in \Lambda_{R-4}(y), \tag{2.17}$$

so $|\psi(x)| \leq \varepsilon \left(\frac{R}{2} - 1\right) A^{\frac{R}{2} - 2}$ for all $x \in \Lambda_R(y)$. We conclude, using (2.9), that

$$\|\psi\|_\infty \leq \varepsilon \left(\tfrac{R}{2} - 1\right) A^{\frac{R}{2} - 2}. \tag{2.18}$$

We now use Lemma 2.1, obtaining $\psi_0 \in \mathcal{F}$, $\|\psi_0\| = 1$, such that

$$\|\psi_0\|_\infty \geq \sqrt{\frac{\dim \mathcal{F}}{|\Lambda_L|}} \geq \sqrt{\tfrac{1}{2}\rho}. \tag{2.19}$$

(The volume $|\Lambda_L|$ appears because the measure in Lemma 2.1 is normalized.) Combining (2.18), (2.19), and (2.14) we get

$$\sqrt{\tfrac{1}{2}\rho} \leq \varepsilon \left(\tfrac{R}{2} - 1\right) A^{\frac{R}{2} - 2} \leq \varepsilon \tfrac{2^d c_d}{\rho} A^{\frac{2^d c_d}{\rho} - 1} \leq \varepsilon \tfrac{2^d c_d}{\rho} A^{\frac{2^d c_d}{\rho}}, \tag{2.20}$$

which implies

$$\rho \leq \frac{C_{d, \|V - E\|_\infty}}{\log \frac{1}{\varepsilon}}, \tag{2.21}$$

which is valid when (2.14) holds, i.e., $L \geq C'_{d, \|V-E\|_\infty} \log \frac{1}{\varepsilon}$.

Since $\sigma(H_{\Lambda_L}) \subset [-2d - V_\infty, 2d + V_\infty]$, we have $\eta_{\Lambda_L}([E, E+\varepsilon]) = 0$ unless $|E| \leq 2d + V_\infty + \frac{1}{2}$, so we get (2.6) if $L \geq L_{d, V_\infty} \log \frac{1}{\varepsilon}$. □

2.3. **One-dimensional Schrödinger operators.** The case $d = 1$ of Theorem 1.1 is an immediate consequence of the following theorem. Note that one dimensional boxes are intervals.

Theorem 2.3. *Let H be a Schrödinger operator as in* (1.1) *with $d = 1$. Given $E_0 \in \mathbb{R}$, there exists L_{V_∞,E_0} such that for all $0 < \varepsilon \le \frac{1}{2}$, open intervals $\Lambda = \Lambda_L$ with $L \ge L_{V_\infty,E_0} \log \frac{1}{\varepsilon}$, and energies $E \le E_0$, we have*

$$\eta_\Lambda\left([E, E+\varepsilon]\right) \le \frac{C_{V_\infty,E_0}}{\log \frac{1}{\varepsilon}}. \tag{2.22}$$

Proof. Let $\Lambda = \Lambda_L =]a_0, a_0 + L[$, $E \in \mathbb{R}$, $\varepsilon \in]0, \frac{1}{2}]$. We set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$. Recall that $\operatorname{Ran} P\chi_\Lambda \subset \mathcal{D}(\Delta_\Lambda) \subset C^1(\Lambda)$ since $d = 1$, and note that

$$\|(H_\Lambda - E)\,\psi\| \le \varepsilon\,\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{2.23}$$

Given $0 < R < L$, set $a_j = a_0 + jR$ for $j = 1, 2, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1$. We introduce the vector space

$$\mathcal{F}_R := \left\{ \psi \in \operatorname{Ran} P;\;\; \psi(a_j) = \psi'(a_j) = 0 \text{ for } j = 1, 2, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1 \right\}. \tag{2.24}$$

Given $\psi \in \mathcal{F}_R$ and $j = 1, \ldots, \left\lceil \frac{L}{R} \right\rceil - 1$, it follows from Gronwall's inequality (see [Ho]), $\psi(a_j) = \psi'(a_j) = 0$, and (2.23) that for all $x \in]a_j - R, a_j + R[\cap \Lambda$ we have

$$|\psi(x)| \le \mathrm{e}^{K|x-a_j|} \left| \int_{a_j}^{x} \mathrm{e}^{-K|y-a_j|} \left|(H_\Lambda - E)\,\psi(y)\right| \,\mathrm{d}y \right| \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon\, \|\psi\|\,, \tag{2.25}$$

where $K = 1 + \|V - E\|_\infty$. Since Λ is the union of these intervals, we conclude that

$$\|\psi\|_\infty \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon\, \|\psi\| \quad \text{for all} \quad \psi \in \mathcal{F}_R. \tag{2.26}$$

We now assume that

$$\rho := \eta_{\Lambda_L}\left([E, E+\varepsilon]\right) = \tfrac{1}{L} \operatorname{tr} P > \tfrac{4}{L}, \tag{2.27}$$

since otherwise there is nothing to prove for large L.. Taking $R = \frac{4}{\rho}$, it follows from (2.27) that

$$\dim \mathcal{F}_R \ge \rho L - 2\left(\left\lceil \tfrac{L}{R} \right\rceil - 1\right) \ge \rho L - 2\tfrac{L}{R} = \tfrac{1}{2}\rho L > 1. \tag{2.28}$$

Applying Lemma 2.1, we obtain $\psi_0 \in \mathcal{F}_R$, $\psi_0 \neq 0$, such that

$$\|\psi_0\|_\infty \ge \sqrt{\frac{\dim \mathcal{F}_R}{L}}\, \|\psi_0\| \ge \sqrt{\tfrac{1}{2}\rho}\, \|\psi_0\|\,. \tag{2.29}$$

It follows from (2.26) and (2.29) that

$$\sqrt{\tfrac{1}{2}\rho} \le (2K)^{-\frac{1}{2}}\, \mathrm{e}^{KR} \varepsilon = (2K)^{-\frac{1}{2}}\, \mathrm{e}^{\frac{4K}{\rho}}\, \varepsilon. \tag{2.30}$$

Thus, we get

$$\rho \le \frac{8K}{\log \frac{1}{\varepsilon}}, \tag{2.31}$$

if $L > \frac{4}{\rho} \ge \frac{\log \frac{1}{\varepsilon}}{2K}$.

Since $\sigma(H_\Lambda) \subset [-V_\infty, \infty[$, we have $\eta_\Lambda([E, E+\varepsilon]) = 0$ unless $E \ge -V_\infty - \frac{1}{2}$. Thus, given $E_0 \in \mathbb{R}$, there exists L_{V_∞,E_0} such that, for all $0 < \varepsilon \le \frac{1}{2}$, open intervals $\Lambda = \Lambda_L$ with $L \ge L_{V_\infty,E_0} \log \frac{1}{\varepsilon}$, and energies $E \le E_0$, we have (2.22). $\square$

3. Multi-dimensional Schrödinger operators

To prove Theorem 1.1 for $d = 2, 3$, we will select a class of approximate eigenfunctions for which we can establish uniform local upper bounds, and pick an approximate eigenfunction for which we have a global lower bound for the global upper bound. The local upper bounds will come from the local behavior of approximate solutions of the stationary Schrödinger equation (Theorem 3.1); the global upper bound will come from the quantitative unique continuation principle (Theorem 3.4).

Given $x \in \mathbb{R}^d$ and $\delta > 0$, we set $B(x, \delta) := \{y \in \mathbb{R}^d;\ |y - x| < \delta\}$.

3.1. **Local behavior of approximate solutions of the stationary Schrödinger equation.**

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^d$ be an open subset, where $d = 2, 3, \ldots$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$, Let $B(x_0, r_0) \subset \Omega$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$. Suppose $\mathcal{F}$ is a linear subspace of $H^2(\Omega)$ such that*

$$\|(-\Delta + W)\, \psi\|_{\mathrm{L}^\infty(B(x_0,r_0))} \leq C_{\mathcal{F}}\, \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \textit{for all} \quad \psi \in \mathcal{F}. \tag{3.1}$$

Then there exist constants $\gamma_d > 0$ and $0 < r_1 = r_1(d, W_\infty) < r_0$, where $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$, with the property that for all $N \in \mathbb{N}$ there is a linear subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$\dim \mathcal{F}_N \geq \dim \mathcal{F} - \gamma_d N^{d-1}, \tag{3.2}$$

such that for all $\psi \in \mathcal{F}_N$ we have

$$|\psi(x)| \leq \left(C_{d,W_\infty,r_1}^{N^2} |x - x_0|^{N+1} + C_{\mathcal{F}} \right) \|\psi\|_{\mathrm{L}^2(\Omega)} \quad \textit{for all} \quad x \in B(x_0, r_1). \tag{3.3}$$

We take $d = 2, 3, \ldots$, and set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We consider sites $x \in \mathbb{R}^d$, partial derivatives $\partial_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, d$, multi-indices $\alpha \in \mathbb{N}_0^d$, and set

$$x^\alpha = \prod_{j=1}^d x_j^{\alpha_j}, \quad D^\alpha = \prod_{j=1}^d \partial_j^{\alpha_j}, \quad |\alpha| = \sum_{j=1}^d |\alpha_j|, \quad \alpha! = \prod_{j=1}^d \alpha_j!. \tag{3.4}$$

We let $\mathcal{H}_m^{(d)} = \mathcal{H}_m(\mathbb{R}^d)$ denote the vector space of homogenous harmonic polynomials on $\mathbb{R}^d$ of degree $m \in \mathbb{N}_0$, and recall that [ABR, Proposition 5.8 and exercises] we have $\dim \mathcal{H}_0^{(d)} = 1$, $\dim \mathcal{H}_1^{(d)} = d$, and, for $m = 2, 3, \ldots$,

$$\dim \mathcal{H}_m^{(d)} = \binom{d + m - 1}{d - 1} - \binom{d + m - 3}{d - 1}. \tag{3.5}$$

In particular, we have

$$\dim \mathcal{H}_m^{(2)} = 2 \quad \text{and} \quad \dim \mathcal{H}_m^{(3)} = 2m + 1 \quad \text{for} \quad m = 2, 3, \ldots, \tag{3.6}$$

and $\dim \mathcal{H}_m^{(d)} < \dim \mathcal{H}_{m+1}^{(d)}$ for $d > 2$. Moreover

$$\lim_{m \to \infty} \frac{\dim \mathcal{H}_m^{(d)}}{m^{d-2}} = \frac{2}{(d-2)!} \quad \text{for} \quad d \geq 2. \tag{3.7}$$

We also define $\mathcal{H}_{\leq N}^{(d)} = \bigoplus_{m=0}^N \mathcal{H}_m^{(d)}$, the vector space of harmonic polynomials on $\mathbb{R}^d$ of degree $\leq N$. It follows from (3.7) that for $d = 2, 3, \ldots$ there exists a constant

$\gamma_d > 0$ such that

$$\dim \mathcal{H}^{(d)}_{\le N} = \sum_{m=0}^{N} \dim \mathcal{H}^{(d)}_m \le \gamma_d N^{d-1} \quad \text{for all} \quad N \in \mathbb{N}. \tag{3.8}$$

Let

$$\mathbf{\Phi}(x) = \mathbf{\Phi}_d(x) := \begin{cases} (d(d-2)\omega_d)^{-1} |x|^{-d+2} & \text{if} \quad d = 3, 4, \dots \\ -\frac{1}{2\pi} \log |x| & \text{if} \quad d = 2 \end{cases}. \tag{3.9}$$

$\mathbf{\Phi}(x)$ is the fundamental solution to Laplace's equation; ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. In particular,

$$-\Delta \mathbf{\Phi}(x) = \delta(x) \quad \text{on} \quad \mathbb{R}^d, \tag{3.10}$$

and

$$|D^\alpha \mathbf{\Phi}(x)| \le C_{d,|\alpha|} |x|^{-d+2-|\alpha|}. \tag{3.11}$$

Given $\Omega = B(x, r)$ for some $x \in \mathbb{R}^d$ and $r > 0$ and $W \in \mathrm{L}^\infty(\Omega)$ real valued, we set $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$, and consider the the stationary Schrödinger equation

$$-\Delta \phi + W\phi = 0 \quad \text{a.e. on} \quad \Omega. \tag{3.12}$$

We let $\mathcal{E}_0(\Omega) = \mathcal{E}_0(\Omega, W)$ denote the vector subspace formed by solutions $\phi \in H^2(\Omega)$. We define linear subspaces

$$\mathcal{E}_N(\Omega) = \left\{ \phi \in \mathcal{E}_0(\Omega);\ \limsup_{x \to x_0} \tfrac{|\phi(x)|}{|x - x_0|^N} < \infty \right\} \quad \text{for} \quad N \in \mathbb{N}. \tag{3.13}$$

Note that $\mathcal{E}_1(\Omega) = \{\phi \in \mathcal{E}_0(\Omega);\ \phi(x_0) = 0\}$, $\mathcal{E}_N(\Omega) \supset \mathcal{E}_{N+1}(\Omega)$ for all $N \in \mathbb{N}_0$, and $\cap_{N=0}^\infty \mathcal{E}_N(\Omega) = \{0\}$ by the unique continuation principle.

A solution of the equation (3.12) admits a local decomposition into a homogeneous harmonic polynomial and a lower order term [HW, B]. The following lemma is a quantitative version of this decomposition; it gives an explicit estimate of the lower order term.

Lemma 3.2. *Let $\Omega = B(x_0, 3r_0)$ for some $x_0 \in \mathbb{R}^d$ and $r_0 > 0$, $d = 2, 3, \dots$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$. For all $N \in \mathbb{N}_0$ there exists a linear map $Y^{(\Omega)}_N \colon \mathcal{E}_N(\Omega) \to \mathcal{H}^{(d)}_N$ such that for all $\phi \in \mathcal{E}_N(\Omega)$ we have*

$$\begin{aligned} &\left| \phi(x) - \left(Y^{(\Omega)}_N \phi \right)(x - x_0) \right| \\ &\quad \le r_0^{-\frac{d}{2}} \left(r_0^{-1} \widetilde{C}_{d, r_0^2 W_\infty} \right)^{N+1} \left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}} ((N+1)!)^{d-2} |x - x_0|^{N+1} \|\phi\|_{\mathrm{L}^2(\Omega)} \end{aligned} \tag{3.14}$$

for all $x \in \overline{B}(x_0, \frac{r_0}{2})$. As a consequence, for all $N \in \mathbb{N}_0$ we have

$$\mathcal{E}_{N+1}(\Omega) = \ker Y^{(\Omega)}_N \quad \text{and} \quad \dim \mathcal{E}_{N+1}(\Omega) \ge \dim \mathcal{E}_N(\Omega) - \dim \mathcal{H}^{(d)}_N. \tag{3.15}$$

In particular, if $\mathcal{J}$ is a vector subspace of $\mathcal{E}_0(\Omega)$ we have

$$\dim \mathcal{J} \cap \mathcal{E}_{N+1}(\Omega) \ge \dim \mathcal{J} - \gamma_d N^{d-1} \quad \text{for all} \quad N \in \mathbb{N}, \tag{3.16}$$

where γ_d is the constant in (3.8).

Proof. We prove the lemma for $\Omega = B(0, 3)$; the general case then follows by translating and dilating. We set $\Omega' = B\left(0, \frac{3}{2}\right)$, and note that $\phi \in \mathcal{E}_0$ ($\mathcal{E}_n = \mathcal{E}_n(\Omega)$) satisfies elliptic regularity estimates:

$$\|\phi\|_{\mathrm{L}^\infty(\Omega')} \le C_{d, W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)}, \tag{3.17}$$

$$\|\nabla\phi\|_{\mathrm{L}^\infty(B(0,1))} \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^\infty(\Omega')} . \tag{3.18}$$

The estimate (3.17) follows immediately from [GiT, Theorem 8.17]. If we knew $\phi \in C^2(\Omega)\cap\mathcal{E}_0$, the estimate (3.18) would follow directly from [GiT, Theorem 8.32]. To prove (3.18) for arbitrary $\phi \in \mathcal{E}_0$, we fix a mollifier $\alpha \in C^\infty(\mathbb{R}^d)$ (i.e., $\alpha \ge 0$, $\int \alpha(x)\,\mathrm{d}x = 1$, $\operatorname{supp}\alpha \subset B(0,1)$), let $\alpha_n(x) = n^d\alpha(nx)$ for $n \in \mathbb{N}$, and define $\phi_n = \alpha_n * \phi$ on $\mathbb{R}^d$. (We extend ϕ to $\mathbb{R}^d$ by $\phi(x) = 0$ for $x \notin \Omega$.) We have $\phi_n \in C^\infty(\mathbb{R}^d)$ and $\phi_n \to \phi$ in $H^2(\Omega')$. (See [GiT, Chapter 7].) Since $\phi \in \mathcal{E}_0$, we have

$$-\Delta\phi_n = \alpha_n * (-\Delta)\phi = \alpha_n * ((-\Delta + W)\phi) - \alpha_n * (W\phi) = -\alpha_n * (W\phi) \quad \text{on} \quad \Omega'. \tag{3.19}$$

In addition, setting $\Omega'' = B(0, \frac{5}{4})$, taking $n \ge 4$, and using Young's inequality for convolutions, we have

$$\begin{aligned} \|\phi_n\|_{\mathrm{L}^\infty(\Omega'')} &\le \|\phi\|_{\mathrm{L}^\infty(\Omega')} , \\ \|(-\Delta + W)\phi_n\|_{\mathrm{L}^\infty(\Omega'')} &\le \|\alpha_n * (W\phi)\|_{\mathrm{L}^\infty(\Omega'')} + \|W\phi_n\|_{\mathrm{L}^\infty(\Omega'')} \le 2W_\infty \|\phi\|_{\mathrm{L}^\infty(\Omega')} . \end{aligned} \tag{3.20}$$

Appealing to [GiT, Theorem 8.32], and using (3.20), we get

$$\begin{aligned} \|\nabla\phi_n\|_{\mathrm{L}^\infty(B(0,1))} &\le C_{d,W_\infty} \Big(\|\phi_n\|_{\mathrm{L}^\infty(\Omega'')} + \|(-\Delta + W)\phi_n\|_{\mathrm{L}^\infty(\Omega'')}\Big) \\ &\le C'_{d,W_\infty} \|\phi\|_{\mathrm{L}^\infty(\Omega')} \quad \text{for} \quad n \ge 4. \end{aligned} \tag{3.21}$$

Since we can find a subsequence ϕ_{n_k} such that $\nabla\phi_{n_k} \to \nabla\phi$ a.e. on Ω', (3.18) follows from (3.21).

Given $\phi \in \mathcal{E}_0$ we consider its Newtonian potential given by

$$\psi(x) = -\int_{\Omega'} W(y)\phi(y)\mathbf{\Phi}(x-y)\,\mathrm{d}y \quad \text{for} \quad x \in \mathbb{R}^d. \tag{3.22}$$

In view of (3.17), we have

$$|\psi(x)| \le W_\infty \|\phi\|_{\mathrm{L}^\infty(\Omega')} \|\mathbf{\Phi}\|_{\mathrm{L}^1(\Omega)} \le C_{d,W_\infty} W_\infty \|\phi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad x \in \Omega'. \tag{3.23}$$

It follows from (3.10) that $\Delta\psi = W\phi$ weakly in Ω'. Thus, letting $h = \phi - \psi$ we have $\Delta h = 0$ weakly in Ω', so we conclude that h is a harmonic function in $\Omega' \supset \overline{B}(0,1)$. In particular (see [ABR, Corollary 5.34 and its proof]), h is real analytic in Ω' and

$$h(x) = \sum_{m=0}^{\infty} p_m(x) \quad \text{for all} \quad x \in B(0,1), \tag{3.24}$$

where $p_m \in \mathcal{H}_m^{(d)}$ for all $m = 0, 1, \ldots$, and for $m = 1, 2, \ldots$ we have

$$|p_m(x)| \le C_d\, m^{d-2} \, |x|^m \sup_{y\in\partial B(0,1)} |h(y)| \quad \text{for all} \quad x \in B(0,1). \tag{3.25}$$

In addition, it follows from the mean value property that for all $y \in \partial B(0,1)$ we have

$$|h(y)| \le \tfrac{1}{\left|B\left(y,\frac{1}{2}\right)\right|} \int_{B\left(y,\frac{1}{2}\right)} |h(y')| \,\mathrm{d}y' \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)} , \tag{3.26}$$

using (3.17) and (3.23). Thus, for all $m = 1, 2, \ldots$ it follows from (3.25) that

$$|p_m(x)| \le C_{d,W_\infty} m^{d-2} \|\phi\|_{\mathrm{L}^2(\Omega)} \, |x|^m \quad \text{for all} \quad x \in B(0,1). \tag{3.27}$$

Setting $h_N = \sum_{m=0}^{N} p_m(x) \in \mathcal{H}^{(d)}_{\le N}$, it follows that

$$|h(x) - h_N(x)| \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)} (N+1)^{d-2} |x|^{N+1} \quad \text{for all} \quad x \in \overline{B}\left(0, \tfrac{1}{2}\right). \tag{3.28}$$

For each $y \in \mathbb{R}^d \setminus \{0\}$ we consider $\mathbf{\Phi}_y(x) = \mathbf{\Phi}(x-y)$, a harmonic function on $\mathbb{R}^d \setminus \{y\}$. In particular, $\mathbf{\Phi}_y(x)$ is real analytic in $B(0,|y|)$, so, defining

$$J_m(x,y) = \sum_{\alpha \in \mathbb{N}_0^d,\, |\alpha| = m} \tfrac{1}{\alpha!} D^\alpha \mathbf{\Phi}(y) x^\alpha \quad \text{for} \quad x \in \mathbb{R}^d, \tag{3.29}$$

we have (see [ABR])

$$\mathbf{\Phi}(x-y) = \mathbf{\Phi}_y(x) = \sum_{m=0}^{\infty} J_m(x,y) \quad \text{for all} \quad x \in B(0,|y|), \tag{3.30}$$

the series converging absolutely and uniformly on compact subsets of $B(0,|y|)$. Moreover, $J_m(\cdot,y) \in \mathcal{H}^{(d)}_m$, and for all $y \in \mathbb{Z}^d$ and $m = 1, 2, \ldots$ we have (see [ABR, Corollary 5.34 and its proof]) that

$$|J_m(x,y)| \le C_d\, m^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^m \sup_{x' \in \partial B\left(0, \frac{3}{4}|y|\right)} |\mathbf{\Phi}_y(x')| \le C_d\, m^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^m \mathbf{\Phi}(\tfrac{y}{4}), \tag{3.31}$$

for all $x \in B\left(0, \frac{3}{4}|y|\right)$. Setting $\mathbf{\Phi}_{y,N}(x) = \sum_{m=0}^{N} J_m(x,y) \in \mathcal{H}^{(d)}_{\le N}$, it follows that for $x \in \overline{B}(0, \frac{1}{2}|y|)$ we have

$$|\mathbf{\Phi}_y(x) - \mathbf{\Phi}_{y,N}(x)| \le C_d (N+1)^{d-2} \left(\tfrac{4|x|}{3|y|}\right)^{N+1} \mathbf{\Phi}(\tfrac{y}{4}). \tag{3.32}$$

We now proceed by induction. We define $Y_0 \colon \mathcal{E}_0 \to \mathcal{H}^{(d)}_0$ by $Y_0\phi = \phi(0)$. Given $\phi \in \mathcal{E}_0$, it follows from the mean value theorem and the elliptic regularity estimates (3.17) and (3.18) that

$$|\phi(x) - \phi(0)| \le \sup_{y \in B(0,1)} |\nabla \phi(y)|\, |x| \le C_{d,W_\infty} \|\phi\|_{\mathrm{L}^2(\Omega)} |x| \quad \text{for} \quad x \in \overline{B}(0,1). \tag{3.33}$$

Thus the lemma holds for $N = 0$.

We now let $N \in \mathbb{N}$ and suppose that the lemma is valid for $N-1$. If $\phi \in \mathcal{E}_N$, it follows that $\phi \in \mathcal{E}_{N-1}$ with $Y_{N-1}\phi = 0$, so by the induction hypothesis

$$|\phi(x)| \le C_N \|\phi\|_{\mathrm{L}^2(\Omega)} |x|^N \quad \text{for all} \quad x \in \overline{B}\left(0, \tfrac{1}{2}\right), \tag{3.34}$$

where

$$C_N = \widetilde{C}^N_{d,W_\infty} \left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}} (N!)^{d-2}. \tag{3.35}$$

Using (3.31) and (3.34), we define

$$\psi_N(x) = -\int_{\Omega'} W(y)\phi(y) \mathbf{\Phi}_{y,N}(x)\, \mathrm{d}y \in \mathcal{H}^{(d)}_{\le N}. \tag{3.36}$$

We fix $x \in \overline{B}\left(0, \frac{1}{2}\right)$ and estimate

$$|\psi(x) - \psi_N(x)| \le W_\infty \int_{\Omega'} |\phi(y)|\, |\mathbf{\Phi}_{y,>N}(x)|\, \mathrm{d}y, \tag{3.37}$$

where $\mathbf{\Phi}_{y,>N}(x) = \mathbf{\Phi}_y(x) - \mathbf{\Phi}_{y,N}(x)$. Appealing to (3.32) and (3.34), we get

$$\int_{\overline{B}(0,\frac{1}{2})\setminus B(0,2|x|)} |\phi(y)| \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y \le C_d C_N \, \|\phi\|_{\mathrm{L}^2(\Omega)} \, (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1} . \tag{3.38}$$

If $y \notin B(0, 2\,|x|)$ we have $|y| \ge 2\,|x| \ge 1$, and hence, using (3.32),

$$\begin{aligned} \int_{\Omega'\setminus\left(B(0,2|x|)\cup\overline{B}(0,\frac{1}{2})\right)} |\phi(y)| \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y \\ \le C_d (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} \mathbf{\Phi}(\tfrac{1}{4}) \, |x|^{N+1} \int_{\Omega'} |\phi(y)| \, \mathrm{d}y \\ \le C_d (N+1)^{d-2} \left(\tfrac{4}{3}\right)^{N+1} |x|^{N+1} \, \|\phi\|_{\mathrm{L}^2(\Omega)} . \end{aligned} \tag{3.39}$$

Using (3.31) and (3.34), we get

$$\begin{aligned} &\int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |\phi(y)| \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y \\ &\le C_N \|\phi\|_{\mathrm{L}^2(\Omega)} \int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^N \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y \\ &\le C_N \|\phi\|_{\mathrm{L}^2(\Omega)} \int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^N \, |\mathbf{\Phi}(x-y)| \, \mathrm{d}y \\ &\quad + C_d C_N \|\phi\|_{\mathrm{L}^2(\Omega)} \sum_{m=0}^{N} m^{d-2} \left(\tfrac{4}{3}\,|x|\right)^m \int_{\overline{B}(0,\frac{1}{2})\cap B(0,2|x|)} |y|^{N-m} \left|\mathbf{\Phi}(\tfrac{y}{4})\right| \mathrm{d}y \\ &\le C_d C_N \|\phi\|_{\mathrm{L}^2(\Omega)} \left(1 + N^{d-2} \left(\tfrac{4}{3}\right)^{N+1}\right) |x|^{N+1} , \end{aligned} \tag{3.40}$$

where we used $|x-y| \le 3\,|x|$ for $y \in B(0, 2\,|x|)$. (Note that we get $|x|^{N+2}$ if $d \ge 3$ and $|x|^{(N+2)-}$ if $d = 2$.) Also using (3.31), we get

$$\begin{aligned} \int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)| \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y &\le \int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)| \, |\mathbf{\Phi}(x-y)| \, \mathrm{d}y \\ &\quad + C_d \sum_{m=0}^{N} m^{d-2} \left(\tfrac{4}{3}\,|x|\right)^m \int_{\Omega'\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)| \, |y|^{-m} \left|\mathbf{\Phi}(\tfrac{y}{4})\right| \mathrm{d}y \\ &\le C_d \|\phi\|_{\mathrm{L}^2(\Omega)} \left(1 + N^{d-2} \left(\tfrac{4}{3}\right)^{N+1}\right) , \end{aligned} \tag{3.41}$$

where we used $|x| \le \frac{1}{2}$. Since $|x| > \frac{1}{4}$ if $y \in B(0, 2\,|x|) \setminus \overline{B}(0, \frac{1}{2})$, we obtain

$$\begin{aligned} \int_{(\Omega'\cap B(0,2|x|))\setminus\overline{B}(0,\frac{1}{2})} |\phi(y)| \, |\mathbf{\Phi}_{y,>N}(x)| \, \mathrm{d}y \\ \le C_d \|\phi\|_{\mathrm{L}^2(\Omega)} \left(1 + N^{d-2} \left(\tfrac{16}{3}\right)^{N+1}\right) |x|^{N+1} . \end{aligned} \tag{3.42}$$

Putting together (3.37), (3.38), (3.39), (3.40), and (3.42), we conclude that for all $x \in \overline{B}(0, \frac{1}{2})$ we have ($C_N \ge 1$)

$$|\psi(x) - \psi_N(x)| \le C_d C_N W_\infty (N+1)^{d-2} \left(\tfrac{16}{3}\right)^{N+1} |x|^{N+1} \, \|\phi\|_{\mathrm{L}^2(\Omega)} . \tag{3.43}$$

We now define $Y_N\phi = h_N + \psi_N \in \mathcal{H}_N^{(d)}$. Since $\phi = h + \psi$, for all $x \in \overline{B}\left(0, \frac{1}{2}\right)$ it follows from (3.28), (3.43), and (3.35), that

$$\begin{aligned}
|\phi(x) - (Y_N\phi)(x)| &\leq |h(x) - h_N(x)| + |\psi(x) - \psi_N(x)| \qquad (3.44)\\
&\leq (C_{d,W_\infty} + C_d W_\infty C_N)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\leq \widetilde{C}_{d,W_\infty} C_N (N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\leq \widetilde{C}_{d,W_\infty}\left(\widetilde{C}_{d,W_\infty}^N\left(\tfrac{16}{3}\right)^{\frac{N(N+1)}{2}}(N!)^{d-2}\right)(N+1)^{d-2}\left(\tfrac{16}{3}\right)^{N+1}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)}\\
&\leq \widetilde{C}_{d,W_\infty}^{N+1}\left(\tfrac{16}{3}\right)^{\frac{(N+1)(N+2)}{2}}((N+1)!)^{d-2}|x|^{N+1}\|\phi\|_{\mathrm{L}^2(\Omega)},
\end{aligned}$$

by choosing the constant $\widetilde{C}_{d,W_\infty}$ in (3.35) large enough. This completes the induction.

The lemma is proven, as (3.15) is an immediate consequence of (3.14), and (3.16) follows from (3.15) and (3.8). □

Theorem 3.1 is an immediate consequence from the following lemma.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an open subset, where $d = 2, 3, \ldots$, and fix a real valued function $W \in \mathrm{L}^\infty(\Omega)$. Let $B(x_0, r_1) \subset \Omega$ for some $x_0 \in \mathbb{R}^d$ and $r_1 > 0$. Suppose $\mathcal{F}$ is a linear subspace of $H^2(\Omega)$ such that*

$$\|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B(x_0,r_1))} \leq C_{\mathcal{F}}\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{for all} \quad \psi \in \mathcal{F}. \tag{3.45}$$

Then there exists $0 < r_2 = r_2(d, W_\infty) < r_1$, where $W_\infty = \|W\|_{\mathrm{L}^\infty(\Omega)}$, with the property that for all $r \in]0, r_2]$ there is a linear map $Z_r \colon \mathcal{F} \to \mathcal{E}_0(B(x_0, r))$ such that

$$\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B(x_0,r))} \leq C_{d,r}C_{\mathcal{F}}\|\psi\|_{\mathrm{L}^2(\Omega)} \quad \text{where} \quad \lim_{r\to 0} C_{d,r} = 0. \tag{3.46}$$

As a consequence, for all $N \in \mathbb{N}$ there is a vector subspace $\mathcal{F}_N$ of $\mathcal{F}$, with

$$\dim \mathcal{F}_N \geq \dim \mathcal{F} - \gamma_d N^{d-1}, \tag{3.47}$$

where γ_d is the constant in (3.8), such that for all $\psi \in \mathcal{F}_N$ we have

$$\begin{aligned}
|\psi(x)| &\leq \left(\widehat{C}_{d,W_\infty,r_1}^{N+1}((N+1)!)^{d-2}\,3^{N^2}|x - x_0|^{N+1} + C_{\mathcal{F}}\right)\|\psi\|_{\mathrm{L}^2(\Omega)} \qquad (3.48)\\
&\leq \left(C_{d,W_\infty,r_1}^{N^2}|x - x_0|^{N+1} + C_{\mathcal{F}}\right)\|\psi\|_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

for all $x \in \overline{B}(x_0, \frac{r_2}{4})$.

Proof. It suffices to consider $x_0 = 0$. We set $B_r = B(0, r)$. Given $0 < r < r_1$ and $\psi \in \mathcal{F}$, we define $Z_r\psi \in \mathcal{E}_0(B_r)$ as the unique solution $\phi \in H^2(B_r)$ to the Dirichlet problem on B_r given by

$$\begin{cases} -\Delta\phi + W\phi = 0 & \text{on} \quad B_r \\ \phi = \psi \quad \text{on} \quad \partial B_r \end{cases}. \tag{3.49}$$

This map is well defined in view of [GiT, Theorem 8.3] and is clearly a linear map.

To prove (3.46) we will use the Green's function $G_r(x, y)$ for the ball B_r. We recall that, abusing the notation by writing $\mathbf{\Phi}(|x|)$ instead of $\mathbf{\Phi}(x)$ (see [GiT, Section 2.5]; note that with our definition $\mathbf{\Phi}(x) = -\Gamma(|x|)$),

$$G_r(x, y) = \begin{cases} \mathbf{\Phi}(|x - y|) - \mathbf{\Phi}\left(\frac{|y|}{r}\left|x - \frac{r^2}{|y|^2}y\right|\right) & \text{if} \quad y \neq 0 \\ \mathbf{\Phi}(|x|) - \mathbf{\Phi}(r) & \text{if} \quad y = 0 \end{cases}. \tag{3.50}$$

Using Green's representation formula [GiT, Eq. (2.21)] for ψ and $Z_r\psi$, for all $x \in B_r$ we have

$$\psi(x) = -\int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x,\zeta)\mathrm{d}S(\zeta) - \int_{B_r} W(y)\psi(y)G_r(x,y)\mathrm{d}y \qquad (3.51)$$
$$+ \int_{B_r} (-\Delta + W(y))\psi(y)G_r(x,y)\mathrm{d}y,$$
$$Z_r\psi(x) = -\int_{\partial B_r} \psi(\zeta)\partial_\nu G_r(x,\zeta)\mathrm{d}S(\zeta) - \int_{B_r} W(y)Z_r\psi(y)G_r(x,y)\mathrm{d}y, \qquad (3.52)$$

where $\mathrm{d}S$ denotes the surface measure and ∂_ν is the normal derivative. Since by an explicit calculation we have, with $p_2 = 2$ and $p_d = \frac{d-1}{d-2}$for $d \geq 3$, that for all $x \in B_r$

$$\|G_r(x,\cdot)\|_{\mathrm{L}^1(B_r)} \leq C_d' r^{\frac{d(p_d-1)}{p_d}} \; \|G_r(x,\cdot)\|_{\mathrm{L}^{p_d}(B_r)} \leq C_d r^{\frac{d(p_d-1)}{p_d}}, \qquad (3.53)$$

it follows that

$$\|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} \qquad (3.54)$$
$$\leq C_d r^{\frac{d(p_d-1)}{p_d}} \left(W_\infty \|\psi - Z_r\psi\|_{\mathrm{L}^\infty(B_r)} + \|(-\Delta + W)\psi\|_{\mathrm{L}^\infty(B_r)} \right).$$

Selecting $r_2 \in]0, r_1[$ such that $C_d r_2^{\frac{d(p_d-1)}{p_d}} (1 + W_\infty) \leq \frac{1}{2}$, and using (3.45), we get (3.46).

Now let $\mathcal{J} = \operatorname{Ran} Z_{r_2}$, a linear subspace of $\mathcal{E}_0(B_{r_2})$; note that

$$\dim \mathcal{J} + \dim \ker Z_{r_2} = \dim \mathcal{F}. \qquad (3.55)$$

We set $\mathcal{J}_N = \mathcal{J} \cap \mathcal{E}_{N+1}(B_{r_2})$ and $\mathcal{F}_N = Z_{r_2}^{-1}(\mathcal{J}_N)$. It follows from (3.16) and (3.55) that

$$\dim \mathcal{F}_N = \dim \ker Z_{r_2} + \dim \mathcal{J}_N \geq \dim \mathcal{F} - \gamma_d N^{d-1}. \qquad (3.56)$$

If $\psi \in \mathcal{F}_N$, we have $Z_{r_2}\psi \in \mathcal{E}_{N+1}(B_{r_2})$ and

$$\|\psi\|_{\mathrm{L}^\infty(B_{r_2})} \leq \|\psi - Z_{r_2}\psi\|_{\mathrm{L}^\infty(B_{r_2})} + \|Z_{r_2}\psi\|_{\mathrm{L}^\infty(B_{r_2})}, \qquad (3.57)$$

so (3.48) follows from (3.46) and (3.14). □

3.2. **A quantitative unique continuation principle for approximate solutions of the stationary Schrödinger equation.** We state and prove a a version of Bourgain and Kenig's quantitative unique continuation principle [BoK, Lemma 3.10], in which we make explicit the dependence on the parameters relevant to this article. We give a proof following [GK2, Theorem A.1].

Given subsets A and B of $\mathbb{R}^d$, and a function φ on set B, we set $\varphi_A := \varphi\chi_{A\cap B}$. In particular, given $x \in \mathbb{R}^d$ and $\delta > 0$ we write $\varphi_{x,\delta} := \varphi_{B(x,\delta)}$.

Theorem 3.4. *Let Ω be an open subset of $\mathbb{R}^d$ and consider a real measurable function V on Ω with $\|V\|_\infty \leq K < \infty$. Let $\psi \in \mathrm{H}^2(\Omega)$ be real valued and let $\zeta \in \mathrm{L}^2(\Omega)$ be defined by*

$$-\Delta\psi + V\psi = \zeta \quad \text{a.e. on} \quad \Omega. \qquad (3.58)$$

Let $\Theta \subset \Omega$ be a bounded measurable set where $\|\psi_\Theta\|_2 > 0$. Set

$$Q(x,\Theta) := \sup_{y\in\Theta} |y - x| \quad \text{for} \quad x \in \Omega. \qquad (3.59)$$

Consider $x_0 \in \Omega \setminus \overline{\Theta}$ *such that*

$$Q = Q(x_0, \Theta) \geq 1 \quad \text{and} \quad B(x_0, 6Q+2) \subset \Omega. \tag{3.60}$$

Then, given

$$0 < \delta \leq \min \left\{ \operatorname{dist}(x_0, \Theta), \tfrac{1}{24} \right\}, \tag{3.61}$$

we have

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log \frac{\|\psi_\Omega\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|\zeta_\Omega\|_2^2, \tag{3.62}$$

where $m > 0$ *is a constant depending only on* d.

We we will apply this theorem with $\delta \ll 1 \ll Q$.

The proof of this theorem is based on the the Carleman-type inequality estimate given in [BoK, Lemma 3.15], [EV, Theorem 2], We state it as in [GK2, Lemma A.5].

Lemma 3.5. *Given* $\varrho > 0$, *the function* $w_\varrho(x) = \varphi(\frac{1}{\varrho}|x|)$ *on* $\mathbb{R}^d$, *where* $\varphi(s) := s\, \mathrm{e}^{-\int_0^s \frac{1-\mathrm{e}^{-t}}{t}\, \mathrm{d}t}$, *is a strictly increasing continuous function on* $[0,\infty[$, C^∞ *on* $]0,\infty[$, *satisfying*

$$\tfrac{1}{C_1\varrho}|x| \leq w_\varrho(x) \leq \tfrac{1}{\varrho}|x| \quad \text{for} \quad x \in B(0,\varrho), \quad \text{where} \quad C_1 = \varphi(1)^{-1} \in]2,3[. \tag{3.63}$$

Moreover, there exist positive constants C_2 *and* C_3, *depending only on* d, *such that for all* $\alpha \geq C_2$ *and all real valued functions* $f \in \mathrm{H}^2(B(0,\varrho))$ *with* $\operatorname{supp} f \subset B(0,\varrho) \setminus \{0\}$ *we have*

$$\alpha^3 \int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha} f^2 \, \mathrm{d}x \leq C_3\, \varrho^4 \int_{\mathbb{R}^d} w_\varrho^{2-2\alpha} (\Delta f)^2 \, \mathrm{d}x. \tag{3.64}$$

Proof of Theorem 3.4. Let $x_0 \in \Omega \setminus \overline{\Theta}$ satisfy (3.60), where C_1 is defined in (3.63). For convenience we may assume $x_0 = 0$, in which case $\Theta \subset B(0, 2C_1 Q)$, and take $\Omega = B(0,\varrho)$, where $\varrho = 2C_1Q + 2$.

Let δ be as in (3.61), and fix a function $\eta \in C_c^\infty(\mathbb{R}^d)$ given by $\eta(x) = \xi(|x|)$, where ξ is an even C^∞ function on $\mathbb{R}$, $0 \leq \xi \leq 1$, such that

$$\begin{gathered} \xi(s) = 1 \text{ if } \tfrac{3\delta}{4} \leq |s| \leq 2C_1Q, \quad \xi(s) = 0 \text{ if } |s| \leq \tfrac{\delta}{4} \text{ or } |s| \geq 2C_1Q+1, \\ \left|\xi^{(j)}(s)\right| \leq \left(\tfrac{4}{\delta}\right)^j \text{ if } |s| \leq \tfrac{3\delta}{4}, \quad \left|\xi^{(j)}(s)\right| \leq 2^j \text{ if } |s| \geq 2C_1Q, \quad j = 1,2. \end{gathered} \tag{3.65}$$

Note that $|\nabla\eta(x)| \leq \sqrt{d}\,|\xi'(|x|)|$ and $|\Delta\eta(x)| \leq d\,|\xi''(|x|)|$.

We will now apply Lemma 3.5 to the function $\eta\psi$. In what follows C_1, C_2, C_3 are the constants of Lemma 3.5, which depend only on d. By C_j, $j = 4,5,\ldots$, we will always denote an appropriate nonzero constant depending only on d.

Given $\alpha \geq C_2 > 1$ (without loss of generality we take $C_2 > 1$), it follows from (3.64) that

$$\begin{aligned} \frac{\alpha^3}{3C_3\varrho^4} \int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha} \eta^2\psi^2 \, \mathrm{d}x \leq \tfrac{1}{3} \int_{\mathbb{R}^d} w_\varrho^{2-2\alpha} (\Delta(\eta\psi))^2 \, \mathrm{d}x \leq \int_{\mathbb{R}^d} w_\varrho^{2-2\alpha} \eta^2 (\Delta\psi)^2 \, \mathrm{d}x \\ + 4 \int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha} |\nabla\eta|^2 |\nabla\psi|^2 \, \mathrm{d}x + \int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha} (\Delta\eta)^2 \psi^2 \, \mathrm{d}x, \end{aligned} \tag{3.66}$$

where $\operatorname{supp}\nabla\eta \subset \left\{\frac{\delta}{4} \leq |x| \leq \frac{3\delta}{4}\right\} \cup \{2C_1Q \leq |x| \leq 2C_1Q+1\}$.

Using (3.58), recalling $\|V\|_\infty \le K$, and noting that $w_\varrho \le 1$ on $\operatorname{supp}\eta$, we get

$$\int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}\eta^2(\Delta\psi)^2\,\mathrm{d}x \le 2K^2\int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha}\eta^2\psi^2\,\mathrm{d}x + 2\int_{\mathbb{R}^d} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x. \tag{3.67}$$

We take

$$\alpha_0 := \alpha\rho^{-\frac{4}{3}} \ge C_4\left(1+K^{\frac{2}{3}}\right), \tag{3.68}$$

ensuring $\alpha > C_2$ and

$$\frac{\alpha^3}{3C_3\varrho^4} = \frac{\alpha_0^3}{3C_3} \ge 6K^2. \tag{3.69}$$

As a consequence, using (3.63) and recalling (3.59), we obtain

$$\int_{\mathbb{R}^d} w_\varrho^{-1-2\alpha}\eta^2\psi^2\,\mathrm{d}x \ge \left(\frac{\varrho}{Q}\right)^{1+2\alpha}\|\psi_\Theta\|_2^2 \ge (2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2. \tag{3.70}$$

Combining (3.66), (3.67), (3.69), and (3.70), we conclude that

$$\begin{aligned}\frac{2\alpha_0^3}{9C_3}(2C_1)^{1+2\alpha}\|\psi_\Theta\|_2^2 \le{}& 4\int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}|\nabla\eta|^2|\nabla\psi|^2\,\mathrm{d}x \\ &+ \int_{\operatorname{supp}\nabla\eta} w_\varrho^{2-2\alpha}(\Delta\eta)^2\psi^2\,\mathrm{d}x + 2\int_{\operatorname{supp}\eta} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x.\end{aligned} \tag{3.71}$$

We have

$$\begin{aligned}&\int_{\{2C_1Q\le|x|\le 2C_1Q+1\}} w_\varrho^{2-2\alpha}\left(4|\nabla\eta|^2|\nabla\psi|^2 + (\Delta\eta)^2\psi^2\right)\mathrm{d}x \\ &\quad\le 4d^2\left(\frac{C_1\varrho}{2C_1Q}\right)^{2\alpha-2}\int_{\{2C_1Q\le|x|\le 2C_1Q+1\}}\left(4|\nabla\psi|^2+\psi^2\right)\mathrm{d}x \\ &\quad\le C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}\int_{\{2C_1Q-1\le|x|\le 2C_1Q+2\}}\left(\zeta^2+(1+K)\psi^2\right)\mathrm{d}x \\ &\quad\le C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}\left(\|\zeta_\Omega\|_2^2 + (1+K)\|\psi_\Omega\|_2^2\right),\end{aligned} \tag{3.72}$$

where we used an interior estimate (e.g., [GK1, Lemma A.2]). Similarly,

$$\begin{aligned}&\int_{\left\{\frac{\delta}{4}\le|x|\le\frac{3\delta}{4}\right\}} w_\varrho^{2-2\alpha}\left(4|\nabla\eta|^2|\nabla\psi|^2 + (\Delta\eta)^2\psi^2\right)\mathrm{d}x \\ &\quad\le 16d^2\delta^{-2}\left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\int_{\left\{\frac{\delta}{4}\le|x|\le\frac{3\delta}{4}\right\}}\left(4|\nabla\psi|^2+\psi^2\right)\mathrm{d}x \\ &\quad\le C_6\delta^{-2}\left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\int_{\{|x|\le\delta\}}\left(\zeta^2+(K+\delta^{-2})\psi^2\right)\mathrm{d}x \\ &\quad\le C_6\delta^{-2}\left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\left(\|\zeta_\Omega\|_2^2 + (K+\delta^{-2})\|\psi_{0,\delta}\|_2^2\right).\end{aligned} \tag{3.73}$$

In addition,

$$\int_{\operatorname{supp}\eta} w_\varrho^{2-2\alpha}\eta^2\zeta^2\,\mathrm{d}x \le \left(4\delta^{-1}C_1\varrho\right)^{2\alpha-2}\|\zeta_\Omega\|_2^2 \le \left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\|\zeta_\Omega\|_2^2. \tag{3.74}$$

Thus, if we have

$$\alpha_0^3\left(\tfrac{8}{5}\right)^{2\alpha}\|\psi_\Theta\|_2^2 \ge C_7(1+K)\|\psi_\Omega\|_2^2, \tag{3.75}$$

we obtain

$$C_5\left(\tfrac{5}{4}C_1\right)^{2\alpha-2}(1+K)\left\|\psi_\Omega\right\|_2^2 \le \tfrac{1}{2}\frac{2\alpha_0^3}{9C_3}\left(2C_1\right)^{1+2\alpha}\left\|\psi_\Theta\right\|_2^2, \tag{3.76}$$

so we conclude that

$$\frac{\alpha_0^3}{9C_3}\left(2C_1\right)^{1+2\alpha}\left\|\psi_\Theta\right\|_2^2 \le C_8\delta^{-2}\left(16\delta^{-1}C_1^2Q\right)^{2\alpha-2}\left(\left(K+\delta^{-2}\right)\left\|\psi_{0,\delta}\right\|_2^2+\left\|\zeta_\Omega\right\|_2^2\right), \tag{3.77}$$

where we used (3.61). Thus,

$$\alpha_0^3Q^2\left(\left(8C_1Q\right)^{-1}\delta\right)^{2\alpha}\left\|\psi_\Theta\right\|_2^2 \le C_9\left(\left(K+\delta^{-2}\right)\left\|\psi_{0,\delta}\right\|_2^2+\left\|\zeta_\Omega\right\|_2^2\right), \tag{3.78}$$

which implies

$$\alpha_0^3Q^4\left(\frac{\delta}{Q}\right)^{4\alpha+4}\left\|\psi_\Theta\right\|_2^2 \le C_{10}\left((1+K)\left\|\psi_{0,\delta}\right\|_2^2+\delta^2\left\|\zeta_\Omega\right\|_2^2\right), \tag{3.79}$$

since $\frac{\delta}{Q} \le \frac{1}{24} < \frac{1}{8C_1}$ by (3.61).

We now choose α. Requiring (3.68), to satisfy (3.75) it suffices to also require

$$\alpha \ge C_{11}\left(1+\log\frac{\left\|\psi_\Omega\right\|_2}{\left\|\psi_\Theta\right\|_2}\right). \tag{3.80}$$

Thus we can satisfy (3.68) and (3.75) by taking

$$\alpha = C_{12}\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\left\|\psi_\Omega\right\|_2}{\left\|\psi_\Theta\right\|_2}\right). \tag{3.81}$$

Combining with (3.79), and recalling $Q \ge 1$, we get

$$\begin{aligned}\left(1+K^{\frac{2}{3}}\right)^3\left(\frac{\delta}{Q}\right)^{C_{13}\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\left\|\psi_\Omega\right\|_2}{\left\|\psi_\Theta\right\|_2}\right)}&\left\|\psi_\Theta\right\|_2^2 \\ &\le C_{14}\left((1+K)\left\|\psi_{0,\delta}\right\|_2^2+\delta^2\left\|\zeta_\Omega\right\|_2^2\right),\end{aligned} \tag{3.82}$$

and hence,

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\left\|\psi_\Omega\right\|_2}{\left\|\psi_\Theta\right\|_2}\right)}\left\|\psi_\Theta\right\|_2^2 \le \left\|\psi_{0,\delta}\right\|_2^2+\delta^2\left\|\zeta_\Omega\right\|_2^2, \tag{3.83}$$

where $m > 0$ is a constant depending only on d. □

We will apply Theorem 3.4 to approximate eigenfunctions of Schrödinger operators defined on a box Λ with Dirichlet boundary condition. In this case the second condition in (3.60) seems to restrict the application of Theorem 3.4 to sites $x_0 \in \Lambda$ sufficiently far away from the boundary of Λ. But, as noted in [GK2, Corollary A.2], in this case Theorem 3.4 can be extended to sites near the boundary of Λ as in the following corollary.

Corollary 3.6. *Consider the Schrödinger operator $H_\Lambda := -\Delta_\Lambda + V$ on $\mathrm{L}^2(\Lambda)$, where $\Lambda = \Lambda_L(x_0)$ is the open box of side $L > 0$ centered at $x_0 \in \mathbb{R}^d$, Δ_Λ is the Laplacian with either Dirichlet or periodic boundary condition on Λ, and V a is bounded potential on Λ with $\|V\|_\infty \le K < \infty$. Let $\psi \in \mathcal{D}(\Delta_\Lambda)$ and fix a bounded measurable set $\Theta \subset \Lambda$ where $\left\|\psi_\Theta\right\|_2 > 0$. Set $Q(x,\Theta) := \sup_{y\in\Theta}|y-x|$*

for $x \in \Lambda$, and consider $x_0 \in \Lambda \setminus \overline{\Theta}$ such that $Q = Q(x_0, \Theta) \geq 1$. Then, given $0 < \delta \leq \min\left\{\operatorname{dist}(x_0,\Theta), \frac{1}{24}\right\}$ such that $B(x_0,\delta) \subset \Lambda$, we have

$$\left(\frac{\delta}{Q}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(Q^{\frac{4}{3}}+\log\frac{\|\psi\|_2}{\|\psi_\Theta\|_2}\right)} \|\psi_\Theta\|_2^2 \leq \|\psi_{x_0,\delta}\|_2^2 + \delta^2 \|H_\Lambda \psi\|_2^2, \tag{3.84}$$

where $m > 0$ is a constant depending only on d.

This corollary is proved exactly as [GK2, Corollary A.2].

3.3. **Two and three dimensional Schrödinger operators.**

Theorem 3.7. *Let H be a Schrödinger operator as in (1.1), where $d = 2, 3$. Given $E_0 \in \mathbb{R}$, there exists L_{d,V_∞,E_0} such that for all $0 < \varepsilon \leq \frac{1}{2}$, open boxes $\Lambda = \Lambda_L$ with $L \geq L_{d,V_\infty,E_0}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{8}}$, and energies $E \leq E_0$, we have*

$$\eta_\Lambda([E, E+\varepsilon]) \leq \frac{C_{d,V_\infty,E_0}}{\left(\log\frac{1}{\varepsilon}\right)^{\frac{4-d}{8}}}. \tag{3.85}$$

Proof. We fix $\varepsilon \in]0, \frac{1}{2}]$, let $L \geq L_0(\varepsilon)$, where $L_0(\varepsilon) > 0$ will be specified later, and take a box $\Lambda = \Lambda_L$. Since $\sigma(H_\Lambda) \subset [-V_\infty, \infty[$, it suffices to consider $E_0 \geq -V_\infty - 1$ and $E \in [-V_\infty - 1, E_0]$. We set $P = \chi_{[E,E+\varepsilon]}(H_\Lambda)$; note that $\operatorname{Ran} P \subset \mathcal{D}(\Delta_\Lambda) \subset H^2(\Lambda)$ and

$$\|(H_\Lambda - E)\psi\| \leq \varepsilon \|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{3.86}$$

Moreover, for $\psi \in \operatorname{Ran} P$ we have

$$\begin{aligned}\|\psi\|_\infty &= \left\| \mathrm{e}^{-(H_\Lambda+V_\infty)}\mathrm{e}^{(H_\Lambda+V_\infty)}\psi \right\|_\infty \\ &\leq \left\| \mathrm{e}^{-(H_\Lambda+V_\infty)} \right\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} \left\| \mathrm{e}^{(H_\Lambda+V_\infty)}\psi \right\| \leq C_d \mathrm{e}^{E_0+V_\infty+1}\|\psi\|,\end{aligned} \tag{3.87}$$

where we used that for $t > 0$

$$\left\| \mathrm{e}^{-t(H_\Lambda+V_\infty)} \right\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} \leq \left\| \mathrm{e}^{t\Delta_\Lambda} \right\|_{\mathrm{L}^2(\Lambda)\to\mathrm{L}^\infty(\Lambda)} \leq \left\| \mathrm{e}^{t\Delta} \right\|_{\mathrm{L}^2(\mathbb{R}^d)\to\mathrm{L}^\infty(\mathbb{R}^d)} < \infty. \tag{3.88}$$

Since $P(H_\Lambda - E)\psi = (H_\Lambda - E)P\psi = (H_\Lambda - E)\psi$ for $\psi \in \operatorname{Ran} P$, we conclude that

$$\|(H_\Lambda - E)\psi\|_\infty \leq \varepsilon\, C_{d,V_\infty,E_0}\|\psi\| \quad \text{for all} \quad \psi \in \operatorname{Ran} P. \tag{3.89}$$

Let

$$\rho := \eta_\Lambda([E, E+\varepsilon]) = \tfrac{1}{L^d}\operatorname{tr} P. \tag{3.90}$$

Recalling the estimate $\operatorname{tr} P \leq C_{d,V_\infty,E_0} L^d$ (e.g., [GK1, Eq. (A.7)]), where $C_{d,V_\infty,E_0} \geq 1$, we obtain the uniform upper bound

$$\rho \leq \rho_{\mathrm{ub}} := C_{d,V_\infty,E_0} \quad \text{with} \quad \rho_{\mathrm{ub}} \geq 1. \tag{3.91}$$

We assume that

$$L^d > 2^{3d+1}\gamma_d \tfrac{\rho_{\mathrm{ub}}}{\rho}, \tag{3.92}$$

since otherwise there is nothing to prove for L sufficiently large. Here γ_d is the constant in Theorem 3.1; we assume $2^d\gamma_d \geq 1$ without loss of generality. We take R such that

$$2^{d+1}\gamma_d \tfrac{\rho_{\mathrm{ub}}}{\rho} \leq R^d < \left(\tfrac{L}{4}\right)^d; \tag{3.93}$$

note that

$$2 \leq \rho R^d \quad \text{and} \quad 2 \leq R^d. \tag{3.94}$$

In particular, it follows from (3.91) and (3.93) that

$$N := \left\lfloor \left(\tfrac{\rho}{2^{d+1}\gamma_d}\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \right\rfloor \geq \left\lfloor \rho_{\mathrm{ub}}^{\frac{1}{d-1}} \right\rfloor \geq 1. \tag{3.95}$$

We pick $\mathcal{G} \subset \Lambda$ such that

$$\overline{\Lambda} = \bigcup_{y \in \mathcal{G}} \overline{\Lambda}_R(y) \quad \text{and} \quad \#\mathcal{G} = \left(\lceil \tfrac{L}{R} \rceil\right)^d \in \left[\left(\tfrac{L}{R}\right)^d, \left(\tfrac{2L}{R}\right)^d\right] \cap \mathbb{N}. \tag{3.96}$$

Given $y_1 \in \mathcal{G}$, we apply Theorem 3.1 with $\Omega = \Lambda \supset B(y_1, 1)$, $W = V - E$, and $\mathcal{F} = \operatorname{Ran} P$. The hypothesis (3.1) follows from (3.89). We conclude that there exists a vector subspace $\mathcal{F}_{y_1,N}$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, V_\infty, E_0) \in (0,1)$, such that, using also (3.95) and (3.93), we have

$$\dim \mathcal{F}_{y_1,N} \geq \rho L^d - \gamma_d N^{d-1} \geq 1, \tag{3.97}$$

and for all $\psi \in \mathcal{F}_{y_1,N}$ we have

$$|\psi(y_1 + x)| \leq \left(C_{d,V_\infty,E_0}^{N^2} |x|^{N+1} + \varepsilon C_{d,V_\infty,E_0}\right) \|\psi\| \quad \text{if} \quad |x| < r_0. \tag{3.98}$$

Picking $y_2 \in \mathcal{G}$, $y_2 \neq y_1$, and applying Theorem 3.1 with $\Omega = \Lambda \supset B(y_2, 1)$, $W = V - E$, and $\mathcal{F} = \mathcal{F}_{y_1,N}$, we obtain a vector vector subspace $\mathcal{F}_{y_1,y_2,N}$ of $\mathcal{F}_{y_1,N}$, and hence of $\operatorname{Ran} P$, such that

$$\dim \mathcal{F}_{y_1,y_2,N} \geq \dim \mathcal{F}_{y_1,N} - \gamma_d N^{d-1} \geq \rho L^d - 2\gamma_d N^{d-1} \geq 1, \tag{3.99}$$

and (3.98) holds for all $\psi \in \mathcal{F}_{y_1,y_2,N}$ also with y_2 substituted for y_1. Repeating this procedure until we exhaust the sites in $\mathcal{G}$, we conclude that there exists a vector subspace $\mathcal{F}_R$ of $\operatorname{Ran} P$ and $r_0 = r_0(d, V_\infty, E_0) \in (0,1)$, such that

$$\dim \mathcal{F}_R \geq \rho L^d - \left(\tfrac{2L}{R}\right)^d \gamma_d N^{d-1} \geq \tfrac{1}{2}\rho L^d \geq 2^{3d}\gamma_d \rho_{\mathrm{ub}} \geq 1, \tag{3.100}$$

where we used the assumption (3.92), and for all $\psi \in \mathcal{F}_R$ and $y \in \mathcal{G}$ we have

$$|\psi(y + x)| \leq \left(C_{d,V_\infty,E_0}^{N^2} |x|^{N+1} + \varepsilon C_{d,V_\infty,E_0}\right) \|\psi\| \quad \text{if} \quad |x| < r_0. \tag{3.101}$$

We let Q_R denote the orthogonal projection onto $\mathcal{F}_R$. Since $\operatorname{tr} Q_R = \dim \mathcal{F}_R$, it follows from (3.100) that that we can find a box $\Lambda_1 = \Lambda_1(x_1) \subset \Lambda$ such that

$$\operatorname{tr} \chi_{\Lambda_1} Q_R \chi_{\Lambda_1} \geq (\lceil L \rceil)^{-d} \tfrac{1}{2}\rho L^d \geq 2^{-(d+1)}\rho. \tag{3.102}$$

But $Q_R = Q_R P = P Q_R$ since $\mathcal{F}_R \subset \operatorname{Ran} P$, and hence

$$2^{-(d+1)}\rho \leq \operatorname{tr} \chi_{\Lambda_1} Q_R \chi_{\Lambda_1} = \operatorname{tr} \chi_{\Lambda_1} P Q_R \chi_{\Lambda_1} \leq \|\chi_{\Lambda_1} P\|_1 \|Q_R \chi_{\Lambda_1}\|. \tag{3.103}$$

Recall that

$$\|\chi_{\Lambda_1} P\|_1 = \|P \chi_{\Lambda_1}\|_1 \leq C_{d,V_\infty,E_0}. \tag{3.104}$$

(This is [Si1, Theorem B.9.2] when $\Lambda = \mathbb{R}^d$. But by an argument similar to (3.88) the crucial estimate [Si1, Eq. (B11)] holds on finite boxes Λ with constants uniform in Λ, so a careful reading of the proof of [Si1, Theorem B.9.2] shows that the result holds on finite boxes Λ with constants uniform in Λ.). We thus conclude that

$$\|Q_R \chi_{\Lambda_1}\| \geq C'_{d,V_\infty,E_0}\rho \quad \text{with} \quad C'_{d,V_\infty,E_0} > 0, \tag{3.105}$$

so there exists $\psi_0 = Q_R \psi_0$ with $\|\psi_0\| = 1$ such that

$$\|\chi_{\Lambda_1}\psi_0\| \geq \gamma\rho, \quad \text{where} \quad \gamma = \tfrac{1}{2}C'_{d,V_\infty,E_0} > 0. \tag{3.106}$$

(Note that $\gamma\rho < \|\psi_0\| = 1$.)

We pick $y_0 \in \mathcal{G}$ such that

$$\tfrac{1}{2} \leq \tfrac{1}{4}R \leq \operatorname{dist}(y_0, \Lambda_1) \leq \tfrac{5\sqrt{d}}{2}R, \tag{3.107}$$

which can done by our construction, and apply Corollary 3.6 with $x_0 = y_0$, $\Theta = \Lambda_1$, and potential $V - E$; note that (recall (3.59))

$$\tfrac{R}{4} + \sqrt{d} \leq Q = Q(y_0, \Lambda_1) \leq \tfrac{5\sqrt{d}}{2}R + \sqrt{d} \leq 3\sqrt{d}R. \tag{3.108}$$

Let $0 < \delta < \delta_0 := \min\left\{\frac{1}{24}, r_0\right\}$, where r_0 is as in (3.101). It follows from Corollary 3.6, using (3.86), that

$$\left(\tfrac{\delta}{3\sqrt{d}R}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}} - \log\|\psi_0\chi_{\Lambda_1}\|_2\right)} \|\psi_0\chi_{\Lambda_1}\|_2^2 \leq \left\|\psi_0\chi_{B(y_0,\delta)}\right\|_2^2 + \varepsilon^2, \tag{3.109}$$

with a constant $m = m_d > 0$ and $K = \|V - E\|_\infty$. Using (3.101) and (3.106), we get

$$\left(\tfrac{\delta}{3\sqrt{d}R}\right)^{m\left(1+K^{\frac{2}{3}}\right)\left(R^{\frac{4}{3}} - \log(\gamma\rho)\right)} (\gamma\rho)^2 \leq C_d C_{d,V_\infty,E_0}^{N^2} \delta^{2(N+1)+d} + C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.110}$$

Since $\rho \geq 2R^{-d}$ and $\frac{\delta}{3\sqrt{d}R} < \frac{\delta}{3\sqrt{d}} < 1$ by (3.94), the inequality (3.110) implies the existence of strictly positive constants $\widetilde{R} = \widetilde{R}_{d,V_\infty,E_0}$ and $M = M_{d,V_\infty,E_0}$ such that

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq C_{d,V_\infty,E_0}^{N^2} \delta^{2N} + C_{d,V_\infty,E_0}\,\varepsilon^2 \quad \text{for} \quad R \geq \widetilde{R}. \tag{3.111}$$

We require

$$R > \widehat{R} = \max\left\{\widetilde{R}, \delta_0^{-1}\right\}, \tag{3.112}$$

and choose δ by (note $C_{d,V_\infty,E_0}^N \geq 1$)

$$\delta = \left(C_{d,V_\infty,E_0}^N R\right)^{-1} < \delta_0, \quad \text{so} \quad \tfrac{\delta}{R} = C_{d,V_\infty,E_0}^N \delta^2 = \left(C_{d,V_\infty,E_0}^N R^2\right)^{-1}, \tag{3.113}$$

obtaining

$$\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq \left(\tfrac{\delta}{R}\right)^N + C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.114}$$

We now take $d = 2, 3$ and take R large enough so that

$$\left(\tfrac{\delta}{R}\right)^N \leq \tfrac{1}{2}\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}}, \quad \text{i.e.,} \quad \left(C_{d,V_\infty,E_0}^N R^2\right)^{N - MR^{\frac{4}{3}}} \geq 2. \tag{3.115}$$

To see this, note that $\frac{4}{3} < \frac{d}{d-1}$ for $d = 2, 3$, so

$$MR^{\frac{4}{3}} < N = \left\lfloor \left(\tfrac{\rho}{2^{d+1}\gamma_d}\right)^{\frac{1}{d-1}} R^{\frac{d}{d-1}} \right\rfloor \quad \text{if} \quad \rho > C'_{d,V_\infty,E_0} R^{\frac{d-4}{3}}, \tag{3.116}$$

and hence

$$\left(C_{d,V_\infty,E_0}^N R^2\right)^{N - MR^{\frac{4}{3}}} \geq 4^{N - MR^{\frac{4}{3}}} \geq 2 \quad \text{if} \quad \rho > C''_{d,V_\infty,E_0} R^{\frac{d-4}{3}}. \tag{3.117}$$

We now choose R by

$$\rho = c_{d,V_\infty,E_0} R^{\frac{d-4}{3}}, \tag{3.118}$$

where the constant c_{d,V_∞,E_0} is chosen large enough to ensure that all the conditions (3.93), (3.112), and (3.117) (and hence (3.115)) are satisfied. (This can be done using (3.91).) It then follows from (3.114) and (3.115) that

$$\tfrac{1}{2}\left(\tfrac{\delta}{R}\right)^{MR^{\frac{4}{3}}} \leq C_{d,V_\infty,E_0}\,\varepsilon^2, \quad \text{i.e.,} \quad \left(C^N_{d,V_\infty,E_0} R^2\right)^{-MR^{\frac{4}{3}}} \leq 2C_{d,V_\infty,E_0}\,\varepsilon^2. \tag{3.119}$$

Recalling (3.95), and using (3.118) with a sufficiently large constant c_{d,V_∞,E_0}, we get from (3.119) that

$$\mathrm{e}^{-M'R^{\frac{8}{3}}} = \mathrm{e}^{-M'R^{\frac{d-4}{3(d-1)}+\frac{d}{d-1}+\frac{4}{3}}} \leq C_{d,V_\infty,E_0}\,\varepsilon^2, \tag{3.120}$$

where $M' = M'_{d,V_\infty,E_0}$. Thus

$$\log\tfrac{1}{\varepsilon} \leq C_{d,V_\infty,E_0} R^{\frac{8}{3}} = \frac{C'_{d,V_\infty,E_0}}{\rho^{\frac{8}{4-d}}}, \tag{3.121}$$

and hence

$$\rho \leq C_{d,V_\infty,E_0}\left(\log\tfrac{1}{\varepsilon}\right)^{-\frac{4-d}{8}}, \tag{3.122}$$

as long as L is large enough to satisfy (3.93) with the choice of R in (3.118), namely $L \geq L_{d,V_\infty,E_0}\left(\log\frac{1}{\varepsilon}\right)^{\frac{3}{8}}$. □

References

[ABR] Axler, S., Bourdon, P., Ramey, W.: *Harmonic function theory.* Second edition. Graduate Texts in Mathematics **137**. Springer-Verlag, New York, 2001

[B] Bers, L.: Local behavior of solutions of general linear elliptic equations. Comm. Pure Appl. Math. **8**, 473-496 (1955)

[BoK] Bourgain, J., Kenig, C.: On localization in the continuous Anderson-Bernoulli model in higher dimension, Invent. Math. **161**, 389-426 (2005)

[CL] Carmona, R, Lacroix, J.: Spectral Theory of Random Schrödinger Operators. Boston: Birkhaüser, 1990

[CoHK] Combes, J.M., Hislop, P.D., Klopp, F.: Optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. Duke Math. J. **140**, 469-498 (2007)

[CrS1] Craig, W., Simon, B.: Subharmonicity of the Lyaponov index. Duke Math. J. **50**, 551-560 (1983)

[CrS2] Craig, W., Simon, B.: Log Hölder continuity of the integrated density of states for stochastic Jacobi matrices. Comm. Math. Phys. **90**, 207-218 (1983)

[DS] Delyon, F., Souillard, B.: Remark on the continuity of the density of states of ergodic finite difference operators. Comm. Math. Phys. **94**, 289-291(1984)

[DiJA] Diestel, J., Jarchow, H., Tonge, A.: *Absolutely Summing Operators.* Cambridge Studies in Advanced Mathematics **43**. Cambridge University Press, Cambridge, 1995

[DoIM] Doi, S., Iwatsuka, A., Mine, T.: The uniqueness of the integrated density of states for the Schrödinger operators with magnetic fields. Math. Z. **237**, 335-371 (2001)

[EV] Escauriaza, L., Vessella, S.: Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients. Inverse problems: theory and applications (Cortona/Pisa, 2002), 79-87, Contemp. Math. **333**, Amer. Math. Soc., Providence, RI, 2003

[GK1] Germinet, F., Klein, A.: A characterization of the Anderson metal-insulator transport transition. Duke Math. J. **124**, 309-351 (2004)

[GK2] Germinet, F., Klein, A.: A comprehensive proof of localization for continuous Anderson models with singular random potentials. J. Europ. Math. Soc. To appear

[GiT] Gilbarg, D.. Trudinger, N.: *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001

[HW] Hartman, P., Wintner, A.: On the local behavior of solutions of non-parabolic partial differential equations. Amer. J. Math. **75**, 449-476. (1953)

[Ho] Howard, R.: The Gronwall Inequality [Online].
Available: http://www. math.sc.edu/ howard/Notes/gronwall.pdf

[KM] Kirsch, W., Martinelli, F.: On the density of states of Schrdinger operators with a random potential. J. Phys. A **15**, 2139-2156 (1982)

[N] Nakamura, S.: A remark on the Dirichlet-Neumann decoupling and the integrated density of states. J. Funct. Anal. **179**, 136-152 (2001)

[PF] Pastur, L., Figotin, A.: Spectra of Random and Almost-Periodic Operators. Heidelberg: Springer-Verlag, 1992

[S] Simmons, G. F.: *Introduction to topology and modern analysis.* McGraw-Hill Book Co., Inc., New York-San Francisco, Calif.-Toronto-London, 1963

[Si1] Simon, B.: Schrödinger semi-groups. Bull. Amer. Math. Soc. **7**, 447-526 (1982)

[Si2] Simon, B.: Schrödinger operators in the twenty-first century. Mathematical Physics 2000, 283-288, Imp. Coll. Press, London, 2000

[T] Thomas, L.: Time dependent approach to scattering from impurities in a crystal. Comm. Math. Phys. **33**, 335-343 (1973)

[ThE] Thurlow, C., Eastham, M.: The existence of eigenvalues of infinite multiplicity for the Schrödinger operator. Proc. Roy. Soc. Edinburgh Sect. **A 86**, 61-64. (1980)

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