Advances in Mathematics **133**, 163–184 (1998)
Article No. AI971688

Extended States in the Anderson Model on the Bethe Lattice

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Received January 18, 1995; accepted May 29, 1995

We prove that the Anderson Hamiltonian $H_\lambda = -\Delta + \lambda V$ on the Bethe lattice has "extended states" for small disorder. More precisely, given any closed interval I contained in the interior of the spectrum of the Laplacian on the Bethe lattice, we prove that for small disorder H_λ has purely absolutely continuous spectrum in I with probability one (i.e., $\sigma_{ac}(H_\lambda) \cap I = I$ and $\sigma_{pp}(H_\lambda) \cap I = \sigma_{sc}(H_\lambda) \cap I = \varnothing$ with probability one), and its integrated density of states is continuously differentiable on the interval I.

Key Words: random Schrödinger operators; Anderson model; extended states; absolutely continuous spectrum; localization.

1. INTRODUCTION

The Bethe lattice (or Cayley tree), $\mathbb{B}$, is an infinite connected graph with no closed loops and a fixed degree (number of nearest neighbors) at each vertex (site or point). The degree is called the coordination number and the connectivity, K, is one less the coordination number. The distance between two sites x and y will be denoted by $d(x, y)$ and is equal to the length of the shortest path connecting x and y.

The Anderson model on the Bethe lattice is given by the random Hamiltonian

$$H_\lambda = \tfrac{1}{2}\Delta + \lambda V \tag{1.1}$$

on

$$\ell^2(\mathbb{B}) = \left\{ u\colon \mathbb{B} \to \mathbb{C};\ \sum_{x \in \mathbb{B}} |u(x)|^2 < \infty \right\}.$$

* Partially supported by the NSF under Grant DMS-9208029.

0001-8708/98 $25.00

The (centered) Laplacian Δ is defined by

$$(\Delta u)(x) = \sum_{y:\, d(x,\, y) = 1} u(y); \tag{1.2}$$

it has spectrum $\sigma(\Delta) = [-2\sqrt{K}, 2\sqrt{K}]$ (e.g., [3]); we use $\frac{1}{2}\Delta$ in the definition of H_λ to simplify some formulas (notice that in the Bethe lattice $-\Delta$ and Δ are unitarily equivalent). V is a random potential, with $V(x)$, $x \in \mathbb{B}$, being independent, identically distributed random variables with common probability distribution μ. The characteristic function of μ will be denoted by h, i.e., $h(t) = \int e^{-itv}\, d\mu(v)$. The real parameter λ is called the *disorder*.

This model was introduced by Anderson [7] to describe the motion of a quantum-mechanical electron in a crystal with impurities. In one and two dimensions it was argued that, as long as the potential was random (i.e., $\lambda \neq 0$), the model exhibits exponential localization (i.e., pure point spectrum with exponentially decaying eigenfunctions). In three and more dimensions both localized and extended states (i.e., absolutely continuous spectrum) are expected for small disorder, with the energies of extended and localized states being separated by the “mobility edge.” A new approach to the study of such questions was given by Abou-Chacra, Anderson, and Thouless [1], who developed a self-consistent approximation for the study of localization which becomes exact in the Bethe lattice. The resulting equations were further studied by Abou-Chacra and Thouless [2], who showed that on the Bethe lattice there should be a mobility edge for small disorder. They calculated that the energy at which localization breaks down converges to $(K+1)/2$ in the zero disorder limit.

The physics literature contains many papers which study the Anderson model in the Bethe lattice; the most recent ones being the work of Mirlin and Fyodorov [24] and of Miller and Derrida [23]; we refer to their list of references for other related work. Miller and Derrida performed a weak disorder expansion inside the spectrum of the zero disorder Hamiltonian and computed perturbatively the density of states and conducting properties corresponding to extended states. They also found the existence of an energy which converges to the edge $\sqrt{K}$ of the spectrum of $\frac{1}{2}\Delta$ in the zero disorder limit, above which the density of states and the conducting properties vanish to all orders in perturbation theory.

It follows from ergodicity (the ergodic theorem in the Bethe lattice is discussed in the Appendix of [3]) that the spectrum of the Hamiltonian H_λ is given by

$$\sigma(H_\lambda) = \sigma(\tfrac{1}{2}\Delta) + \lambda \operatorname{supp} \mu = [-\sqrt{K}, \sqrt{K}] + \lambda \operatorname{supp} \mu$$

with probability one [10, 26]. For each choice of V the spectrum of H_λ can be decomposed into pure point spectrum, $\sigma_{pp}(H_\lambda)$, absolutely continuous spectrum, $\sigma_{ac}(H_\lambda)$, and singular continuous spectrum, $\sigma_{sc}(H_\lambda)$. Ergodicity gives the existence of sets $\Sigma_{\lambda,\,pp}, \Sigma_{\lambda,\,ac}, \Sigma_{\lambda,\,sc} \subset \mathbb{R}$ such that $\sigma_{pp}(H_\lambda) = \Sigma_{\lambda,\,pp}$, $\sigma_{ac}(H_\lambda) = \Sigma_{\lambda,\,ac}$, and $\sigma_{sc}(H_\lambda) = \Sigma_{\lambda,\,sc}$ with probability one [10, 21].

Localization for the Anderson Hamiltonian is by now well understood. In one dimension there are mathematical proofs of exponential localization for any disorder (e.g., [9, 12, 15, 21] and others). In the multidimensional case exponential localization is proved for large disorder or low energy (e.g., [4, 6, 11–14, 18, 28] and others). Only localization in two dimensions for small disorder is still an open problem.

But extended states are another matter. Up to now there was no proof of the occurrence of absolutely continuous spectrum (i.e., extended states) in the Anderson model.

For the Bethe lattice there were no rigorous results up to recently. Kunz and Souillard [22] gave an outline of what should be proven: analyticity of the density of states for distributions close to the Cauchy distribution, localization for large disorder or low energies, and existence of extended states for small disorder. The first was proved by Acosta and Klein [3], the second by Aizenman and Molchanov [6], and the third we prove in this article.

Aizenman [4] has proved localization in the Bethe lattice for energies beyond $(K+1)/2$ at weak disorder, confirming half of Abou-Chacra and Thouless' prediction [2]. For the case when the potential at a single site has a Cauchy distribution, Aizenman [5] has announced a proof of the existence of extended states (absolutely continuous spectrum), inside the spectrum of $\frac{1}{2}\Delta$, for small disorder.

In this article we always assume that $K \geqslant 2$ (*so* $\mathbb{B}$ *is not the line* $\mathbb{R}$) *and that* $h(t)$ *is differentiable on* $(0, \infty)$, *with* $h'(t)$ *absolutely continuous and bounded on* $(0, \infty)$, *and* $h''(t)$ *also bounded. These conditions are satisfied by any probability distribution* μ *with a finite second moment* (*e.g., uniform, Gaussian or Bernoulli distributions*) *and by the Cauchy distribution.*

We will prove that the Anderson Hamiltonian on the Bethe lattice has "extended states" for small disorder. More precisely, given any closed interval I contained in the interior of the spectrum of $\frac{1}{2}\Delta$ on the Bethe lattice, we will prove that for small disorder H_λ has purely absolutely continuous spectrum in I with probability one, and its integrated density of states is continuously differentiable on the interval. These results agree with Miller and Derrida's conclusions [23].

The main result of this article is

THEOREM 1.1. *For any* E, $0 < E < \sqrt{K}$, *there exists* $\lambda(E) > 0$, *such that for any* λ *with* $|\lambda| < \lambda(E)$ *the spectrum of* H_λ *in* $[-E, E]$ *is purely absolutely* *continuous with probability one; i.e., we have* $\Sigma_{\lambda,\, ac} \cap [-E, E] = [-E, E]$ *and* $\Sigma_{\lambda,\, pp} \cap [-E, E] = \Sigma_{\lambda,\, sc} \cap [-E, E] = \varnothing$.

The Green's function of H_λ is given by

$$G_\lambda(x, y; z) = \langle x|\, (H_\lambda - z)^{-1}\, |y\rangle \tag{1.3}$$

for $x, y \in \mathbb{B}$ and $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta > 0$. The integrated density of states $N_\lambda(E)$ (see [3] for a discussion of the integrated density of states in the Bethe lattice) is defined by

$$N_\lambda(E) = \mathbb{E}(\langle x|\, \chi_{(-\infty,\, E]}(H_\lambda)\, |x\rangle) \qquad \text{for any} \quad x \in \mathbb{B}. \tag{1.4}$$

Our result for the integrated density of states is

THEOREM 1.2. *For any* E, $0 < E < \sqrt{K}$, *there exists* $\lambda(E) > 0$, *such that for any* λ *with* $|\lambda| < \lambda(E)$ *the integrated density of states* $N_\lambda(E')$ *is continuously differentiable on the interval* $(-E, E)$ *with* $N'_\lambda(E') = \lim_{\eta \downarrow 0} (1/\pi) \operatorname{Im} \mathbb{E}(G_\lambda(x, x; E' + i\eta))$ *for any* $x \in \mathbb{B}$.

We will show that Theorem 1.1 follows from

THEOREM 1.3. *For any* E, $0 < E < \sqrt{K}$, *there exists* $\lambda(E) > 0$, *such that for all* $x \in \mathbb{B}$ *we have*

$$\sup_{\lambda;\, |\lambda| < \lambda(E)} \ \sup_{E';\, |E'| \leqslant E} \ \sup_{\eta;\, 0 < \eta} \mathbb{E}(|G_\lambda(x, x; E' + i\eta)|^2) < \infty. \tag{1.5}$$

We will actually prove more. For any $x \in \mathbb{B}$ and any potential V, $G_\lambda(x, x; E + i\eta)$ is a continuous function of $(\lambda, E, \eta) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$; to prove it one uses the resolvent identity plus the fact that, as long as $\eta > 0$, we have $\lambda V/(\lambda V - i\eta) \to 0$ strongly as $\lambda \to 0$. It then follows from the dominated convergence theorem that $\mathbb{E}(G_\lambda(x, x; E + i\eta))$ and $\mathbb{E}(|G_\lambda(x, x; E + i\eta)|^2)$ are also continuous functions of $(\lambda, E, \eta) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$. In the next theorem we prove that we can let $\eta \downarrow 0$ inside the spectrum of $\frac{1}{2}\Delta$.

THEOREM 1.4. *For any* E, $0 < E < \sqrt{K}$, *there exists* $\lambda(E) > 0$, *such that for all* $x \in \mathbb{B}$ *the continuous functions*

$$(\lambda, E', \eta) \in (-\lambda(E), \lambda(E)) \times [-E, E] \times (0, \infty) \to \mathbb{E}(|G_\lambda(x, x; E' + i\eta)|^2) \tag{1.6}$$

and

$$(\lambda, E', \eta) \in (-\lambda(E), \lambda(E)) \times [-E, E] \times (0, \infty) \to \mathbb{E}(G_\lambda(x, x; E' + i\eta)) \tag{1.7}$$

have continuous extensions to $(-\lambda(E), \lambda(E)) \times [-E, E] \times [0, \infty)$.

Theorem 1.3 is clearly an immediate consequence of the first statement in Theorem 1.4. Theorem 1.2 follows from the second statement in Theorem 1.4, since $\mathbb{E}(G_\lambda(x, x; E + i\eta))$ is the Stieltjes transform of the integrated density of states.

This article is organized as follows: In Section 2 we derive Eqs. (2.12) and (2.18), which give $\mathbb{E}(G_\lambda(x, x; E + i\eta))$ and $\mathbb{E}(|G_\lambda(x, x; E + i\eta)|^2)$ in terms of functions $\zeta_{\lambda, z}(\varphi^2)$ and $\xi_{\lambda, z}(\varphi_+^2, \varphi_-^2)$, defined in (2.11) and (2.16), which are fixed points for the nonlinear Eqs. (2.13) and (2.19). In Section 3 we introduce the appropriate Banach spaces and operators for rewriting the solutions of (2.13) and (2.19) as fixed points for certain nonlinear operators ((3.14) and (3.16)). We then use the implicit function theorem to perform a fixed point analysis and prove Theorem 1.4. Section 4 contains a criterion for the absolute continuity of a measure in terms of its Stieltjes transform and its application to the proof of Theorem 1.1 from Theorem 1.3. Appendix A contains an alternative (more intuitive) derivation of the equations of Section 2, using the "supersymmetric replica trick." The precise version of the implicit function theorem we use in Section 3 is given in Appendix B.

An announcement of some of the results in this article appeared in [19].

2. THE MAIN EQUATIONS

We fix an arbitrary site in $\mathbb{B}$ which we will call the origin and denote by 0. Given two nearest neighbors sites $x, y \in \mathbb{B}$, we will denote by $\mathbb{B}^{(x \mid y)}$ the lattice obtained by removing from $\mathbb{B}$ the branch emanating from x that passes through y; if we do not specify which branch was removed we will simply write $\mathbb{B}^{(x)}$. Each vertex in $\mathbb{B}^{(x)}$ has degree $K+1$, with the single exception of x which has degree K.

Given $\Lambda \cap \mathbb{B}$, we will use $H_{\lambda, \Lambda}$ to denote the operator H_λ restricted to $\ell^2(\Lambda)$ with Dirichlet boundary conditions. The Green's function corresponding to $H_{\lambda, \Lambda}$ will be denoted by

$$G_{\lambda, \Lambda}(x, y; z) = \langle x| \, (H_{\lambda, \Lambda} - z)^{-1} \, |y\rangle \tag{2.1}$$

for $x, y \in \Lambda$ and $z = E + i\eta$ with $E \in \mathbb{R}$, $\eta > 0$. We will write H_λ, $H_\lambda^{(x \mid y)}$, and $H_\lambda^{(x)}$ for $H_{\lambda, \mathbb{B}}$, $H_{\lambda, \mathbb{B}^{(x \mid y)}}$, and $H_{\lambda, \mathbb{B}^{(x)}}$, respectively. Similarly, we will use

$G_\lambda(x, y; z)$ for $G_{\lambda, \mathbb{B}}(x, y; z)$ and $G_\lambda(z)$, $G_\lambda^{(x\,|\,y)}(z)$, $G_\lambda^{(x)}(z)$ for $G_\lambda(0, 0; z)$, $G_{\lambda, \mathbb{B}^{(x\,|\,y)}}(x, x; z)$, $G_{\lambda, \mathbb{B}^{(x)}}(x, x; z)$, respectively.

PROPOSITION 2.1. *For any $\lambda \in \mathbb{R}$, $E \in \mathbb{R}$, and $\eta > 0$ we have*

$$G_\lambda(z) = -\left(z - \lambda V(0) + \tfrac{1}{4} \sum_{x:\, d(x,0)=1} G_\lambda^{(x\,|\,0)}(z)\right)^{-1} \tag{2.2}$$

and for any two nearest neighbors sites $x, y \in \mathbb{B}$

$$G_\lambda^{(x\,|\,y)}(z) = -\left(z - \lambda V(x) + \tfrac{1}{4} \sum_{x':\, d(x',x)=1,\, x' \neq y} G_\lambda^{(x'\,|\,x)}(z)\right)^{-1}. \tag{2.3}$$

Proof. We will prove (2.2); (2.3) is proven in exactly the same way. Let us write

$$H_\lambda = \tilde{H}_\lambda + \Gamma, \tag{2.4}$$

where

$$\tilde{H}_\lambda = \lambda V(0) \oplus \left(\bigoplus_{x:\, d(x,0)=1} H_\lambda^{(x\,|\,0)}\right), \tag{2.5}$$

the direct sum corresponding to the decomposition $\mathbb{B} = \{0\} \cup (\bigcup_{x: d(x,0)=1} \mathbb{B}^{(x\,|\,0)})$. The operator Γ has matrix elements $\langle x|\, \Gamma\, |0\rangle = \langle 0|\, \Gamma\, |x\rangle = \frac{1}{2}$ if $d(x, 0) = 1$, with all other matrix elements being 0. The resolvent identity gives

$$(H_\lambda - z)^{-1} = (\tilde{H}_\lambda - z)^{-1} + (\tilde{H}_\lambda - z)^{-1}\, \Gamma (H_\lambda - z)^{-1}. \tag{2.6}$$

Hence, taking matrix elements we get

$$G_\lambda(z) = (\lambda V(0) - z)^{-1} + \tfrac{1}{2} (\lambda V(0) - z)^{-1} \sum_{x:\, d(x,0)=1} G_\lambda(x, 0; z), \tag{2.7}$$

and for each x with $d(x, 0) = 1$

$$G_\lambda(x, 0; z) = \tfrac{1}{2} G_\lambda^{(x\,|\,0)}(z)\, G_\lambda(z); \tag{2.8}$$

(2.2) follows from (2.7) and (2.8). ∎

PROPOSITION 2.2. *For any $\lambda \in \mathbb{R}$, $E \in \mathbb{R}$, and $\eta > 0$ we have*

$$G_\lambda(z) = \frac{i}{\pi} \int_{\mathbb{R}^2} e^{i(z - \lambda V(0))\, \varphi^2} \exp\left\{\frac{i}{4} \sum_{x:\, d(x,0)=1} G_\lambda^{(x\,|\,0)}(z)\, \varphi^2\right\} d^2\varphi, \tag{2.9}$$

and for any two nearest neighbors sites $x, y \in \mathbb{B}$

$$e^{(i/4)\, G_\lambda^{(x \mid y)}(z)\, \varphi^2} = -\frac{1}{\pi} \int_{\mathbb{R}^2} e^{-i\varphi \cdot \varphi'}\, \partial \left\{ e^{i(z - \lambda V(x))\, \varphi'^2} \times \exp\left\{ \frac{i}{4} \sum_{x' : d(x', x) = 1,\, x' \neq y} G_\lambda^{(x' \mid x)}(z)\, \varphi'^2 \right\} \right\} d^2\varphi', \quad (2.10)$$

where $\varphi^2 = \varphi \cdot \varphi$ and $\partial f(\varphi^2) = f'(\varphi^2)$.

Proof. If we perform the integration in (2.9) and (2.10) we obtain (2.2) and (2.3). ∎

Remark. The above derivation for (2.9) and (2.10) is not very intuitive. These equations appear naturally in the "supersymmetric formalism." In Appendix A we give an intuitive derivation using the "supersymmetric replica trick."

THEOREM 2.3. *For any $\lambda \in \mathbb{R}$, $E \in \mathbb{R}$, and $\eta > 0$ let*

$$\zeta_{\lambda, z}(\varphi^2) = \mathbb{E}(e^{(i/4)\, G_\lambda^{(0)}(z)\, \varphi^2}). \quad (2.11)$$

Then

$$\mathbb{E}(G_\lambda(z)) = \frac{i}{\pi} \int_{\mathbb{R}^2} e^{iz\varphi^2} h(\lambda\varphi^2) [\zeta_{\lambda, z}(\varphi^2)]^{K+1}\, d^2\varphi \quad (2.12)$$

and

$$\zeta_{\lambda, z}(\varphi^2) = -\frac{1}{\pi} \int_{\mathbb{R}^2} e^{-i\varphi \cdot \varphi'}\, \partial \{ e^{iz\varphi'^2} h(\lambda\varphi'^2) [\zeta_{\lambda, z}(\varphi'^2)]^K \}\, d^2\varphi'. \quad (2.13)$$

Proof. If we take expectations in (2.9) and (2.10), with respect to the potential's probability distribution, and recall that the $V(x)$, $x \in \mathbb{B}$, are independent, identically distributed random variables, we get (2.12) and (2.13). ∎

If $\lambda = 0$ we can calculate $G_0^{(0)}(z)$ obtaining [3]

$$\zeta_{0, z}(\varphi^2) = e^{(i/2K)\{ -z + \sqrt{z^2 - K} \}\, \varphi^2}, \quad (2.14)$$

where we always make the choice $\mathrm{Im} \sqrt{\ } > 0$. If $|E| < \sqrt{K}$, we have the pointwise limit

$$\zeta_{0, E}(\varphi^2) \equiv \lim_{n \downarrow 0} \zeta_{0, z}(y^2) = e^{(1/2K)(-iE - \sqrt{K - E^2})\, \varphi^2}. \quad (2.15)$$

THEOREM 2.4. *For any $\lambda \in \mathbb{R}$, $E \in \mathbb{R}$, and $\eta > 0$ let*

$$\xi_{\lambda,z}(\varphi_+^2, \varphi_-^2) = \mathbb{E}\left(\exp\left\{\frac{i}{4}\left(G_\lambda^{(0)}(z)\,\varphi_+^2 - \overline{G_\lambda^{(0)}(z)}\,\varphi_-^2\right)\right\}\right) \tag{2.16}$$

$$= \mathbb{E}\left(\exp\left\{\frac{1}{4}\left[i\mathscr{R}_\lambda^{(0)}(z)(\varphi_+^2 - \varphi_-^2) - \mathscr{I}_\lambda^{(0)}(z)(\varphi_+^2 + \varphi_-^2)\right]\right\}\right), \tag{2.17}$$

where $\mathscr{R}_\lambda^{(x\,|\,0)}(z) + i\mathscr{I}_\lambda^{(x\,|\,0)}(z)$ is the decomposition of $G_\lambda^{(x\,|\,0)}(z)$ into its real and imaginary parts. Then

$$\mathbb{E}(|G_\lambda(z)|^2) = \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{iE(\varphi_+^2 - \varphi_-^2) - \eta(\varphi_+^2 + \varphi_-^2)} \times h(\lambda(\varphi_+^2 - \varphi_-^2))[\xi_{\lambda,z}(\varphi_+^2, \varphi_-^2)]^{K+1}\, d^2\varphi_+\, d^2\varphi_- \tag{2.18}$$

and

$$\xi_{\lambda,z}(\varphi_+^2, \varphi_-^2) = \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{-i(\varphi_+\cdot\varphi'_+ - i\varphi_-\cdot\varphi'_-)}\, \partial_+\partial_-\{e^{iE(\varphi'^2_+ - \varphi'^2_-) - \eta(\varphi'^2_+ + \varphi'^2_-)} \times h(\lambda(\varphi'^2_+ - \varphi'^2_-))[\xi_{\lambda,z}(\varphi'^2_+, \varphi'^2_-)]^K\}\, d^2\varphi'_+\, d^2\varphi'_-, \tag{2.19}$$

with

$$\partial_{\pm} g(\varphi_+^2, \varphi_-^2) = \frac{\partial}{\partial\varphi_\pm^2}\, g(\varphi_+^2, \varphi_-^2). \tag{2.20}$$

Proof. From (2.9) we get

$$|G_\lambda(z)|^2 = \frac{1}{\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2} e^{i(E - V(0))(\varphi_+^2 - \varphi_-^2) - \eta(\varphi_+^2 + \varphi_-^2)} \times e^{\{(1/4)\sum_{x:\, d(x,0)=1}[i\mathscr{R}_\lambda^{(x\,|\,0)}(z)(\varphi_+^2 - \varphi_-^2) - \mathscr{I}_\lambda^{(x\,|\,0)}(z)(\varphi_+^2 + \varphi_-^2)]\}}\, d^2\varphi_+\, d^2\varphi_-. \tag{2.21}$$

Taking expectations we get (2.18). To prove (2.19), we use (2.10), (2.16), and take expectations. ∎

For $\lambda = 0$ we have

$$\xi_{0,z}(\varphi_+^2, \varphi_-^2) = \zeta_{0,z}(\varphi_+^2)\,\overline{\zeta_{0,z}(\varphi_-^2)}. \tag{2.22}$$

Again, as in (2.15), when $|E| < \sqrt{K}$ we have the pointwise limit

$$\xi_{0,E}(\varphi_+^2, \varphi_-^2) \equiv \lim_{\eta \downarrow 0} \xi_{0,z}(\varphi_+^2, \varphi_-^2) = e^{(1/2K)\{-iE(\varphi_+^2 - \varphi_-^2) - \sqrt{K-E^2}(\varphi_+^2 + \varphi_-^2)\}}. \tag{2.23}$$

3. A FIXED POINT ANALYSIS

Following Campanino and Klein [8, 17, 20], we introduce the Hilbert space $\mathscr{H}$ given as the completion of

$$\{f\colon [0, \infty) \to \mathbb{C} \text{ continuously differentiable}; \, \|f\|_{\mathscr{H}} \equiv |||f|||_2 < \infty\}, \tag{3.1}$$

where for $1 \leqslant p \leqslant \infty$

$$|||f|||_p^2 = \|f(\varphi^2)\|^2_{L^p(\mathbb{R}^2, d^2\varphi)} + \|2\partial f(\varphi^2)\|^2_{L^p(\mathbb{R}^2, d^2\varphi)} \tag{3.2}$$

with $\partial f(\varphi^2) = f'(\varphi^2)$, and the operators

$$(Tf)(\varphi^2) = -\frac{1}{\pi} \int_{\mathbb{R}^2} e^{-i\varphi \cdot \varphi'} \, \partial f(\varphi'^2) \, d^2\varphi' \tag{3.3}$$

and $B_{\lambda,z} = M(e^{iz\varphi^2} h(\lambda\varphi^2))$, where for a given function $g = g(\varphi^2)$ we use $M(g)$, or $M(g(\varphi^2))$, to denote the operator given by multiplication by $g(\varphi^2)$:

$$(M(g)\, f)(\varphi^2) = g(\varphi^2)\, f(\varphi^2). \tag{3.4}$$

If $\mathscr{F}$ denotes the Fourier transform in $\mathbb{R}^2$,

$$(\mathscr{F} g)(\varphi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i\varphi \cdot \varphi'} g(\varphi') \, d^2\varphi', \tag{3.5}$$

we have [8, 20]

$$(Tf)(\varphi^2) = -2(\mathscr{F}[\partial f(\varphi'^2)])(\varphi) \tag{3.6}$$

and

$$\partial(Tf)(\varphi^2) = -\tfrac{1}{2}(\mathscr{F}[f(\varphi'^2)])(\varphi). \tag{3.7}$$

It follows that T is unitary on $\mathscr{H}$. It is also easy to see that $B_{\lambda,z}$ is a bounded operator on $\mathscr{H}$.

Let me now introduce the Hilbert space $\mathscr{K}=\mathscr{H}\otimes\mathscr{H}$, which is the completion of

$$\{g\colon [0,\infty)\times[0,\infty)\to\mathbb{C} \text{ of class } C^2;\ \|g\|_{\mathscr{K}}\equiv |||g|||_2<\infty\}, \tag{3.8}$$

where for $1\leqslant p\leqslant\infty$

$$\begin{aligned}|||g|||_p^2 &= \|g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2,\,d^2\varphi_+\,d^2\varphi_-)}\\ &\quad+\|2\partial_+ g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2,\,d^2\varphi_+\,d^2\varphi_-)}\\ &\quad+\|2\partial_- g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2,\,d^2\varphi_+\,d^2\varphi_-)}\\ &\quad+\|4\partial_+\partial_- g(\varphi_+^2,\varphi_-^2)\|^2_{L^p(\mathbb{R}^2\times\mathbb{R}^2,\,d^2\varphi_+\,d^2\varphi_-)},\end{aligned} \tag{3.9}$$

we set $\mathscr{T}=T\otimes T$, so $\mathscr{T}$ is unitary on $\mathscr{K}$. We also define

$$\mathscr{B}_{\lambda,z}=M(e^{iE(\varphi_+^2-\varphi_-^2)-\eta(\varphi_+^2+\varphi_-^2)}h(\lambda(\varphi_+^2-\varphi_-^2))), \tag{3.10}$$

where as before $M(g(\varphi_+^2,\varphi_-^2))$ denotes multiplication by the function $g(\varphi_+^2,\varphi_-^2)$.

To handle the nonlinear Eqs. (2.12), (2.13), (2.18), and (2.19), we introduce the Banach spaces

$$\mathscr{H}_p=\{f\in\mathscr{H},\ \|f\|_{\mathscr{H}_p}\equiv\|f\|_{\mathscr{H}}+|||f|||_p<\infty\} \tag{3.11}$$

and

$$\mathscr{K}_p=\{g\in\mathscr{K},\ \|g\|_{\mathscr{K}_p}\equiv\|g\|_{\mathscr{K}}+|||g|||_p<\infty\}, \tag{3.12}$$

with $1\leqslant p\leqslant\infty$. It is not hard to check that T is a bounded linear operator from $\mathscr{H}_1$ to $\mathscr{H}_\infty$, $B_{\lambda,z}$ is a bounded linear operator on $\mathscr{H}_1$ and that $f\to f^n$ is a continuous map from $\mathscr{H}_\infty$ to $\mathscr{H}_1$ for any $n=2,3,\ldots$. Similarly, $\mathscr{T}$ is a bounded linear operator from $\mathscr{K}_1$ to $\mathscr{K}_\infty$, $\mathscr{B}_{\lambda,z}$ is a bounded linear operator on $\mathscr{K}_1$, and $g\to g^n$ is a continuous map from $\mathscr{K}_\infty$ to $\mathscr{K}_1$ for any $n=2,3,\ldots$.

LEMMA 3.1. (I) *$\zeta_{\lambda,z}\in\mathscr{H}_\infty$ for all $\lambda\in\mathbb{R}$ and $z=E+i\eta$ with $\eta>0$. The map $(\lambda,E,\eta)\to\zeta_{\lambda,E+i\eta}$ is continuous from $\mathbb{R}\times\mathbb{R}\times(0,\infty)$ to $\mathscr{H}_\infty$.*

(II) *If $|E|<\sqrt{K}$ we have $\zeta_{0,E}\in\mathscr{H}_\infty$ and*

$$\lim_{n\downarrow 0}\zeta_{0,E+i\eta}=\zeta_{0,E}\quad \text{in } \mathscr{H}_\infty. \tag{3.13}$$

(III) (2.13) *can be rewritten as a fixed point equation in $\mathscr{H}_\infty$:*

$$\zeta_{\lambda,z}=TB_{\lambda,z}\zeta_{\lambda,z}^K, \tag{3.14}$$

valid for all $\lambda \in \mathbb{R}$ *and* $z = E + i\eta$ *with* $\eta > 0$, *and also valid for* $\lambda = 0$ *and* $z = E$ *with* $|E| < \sqrt{K}$.

Proof. If $\eta > 0$, we clearly have $\{e^{iz\varphi'^2} h(\lambda\varphi'^2)[\zeta_{\lambda, z}(\varphi'^2)]^K\} \in \mathcal{H}_1$; hence it follows from (2.13) that (3.14) holds, so $\zeta_{\lambda, z} \in \mathcal{H}_\infty$ for all $\lambda \in \mathbb{R}$ and $z = E + i\eta$ with $\eta > 0$. To prove the continuity, we notice that for any fixed potential V $G_\lambda^{(0)}(E + i\eta)$ is a continuous function of $(\lambda, E, \eta) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ (by the same argument as in the paragraph preceding Theorem 1.4), with $|G_\lambda^{(0)}(E + i\eta)| \leqslant 1/\eta$. The continuity in (I) then follows from the dominated convergence theorem.

Part (II) is proven by explicit computations. The fact that (3.14) is valid for $\lambda = 0$ and $z = E$ with $|E| < \sqrt{K}$ is also checked by a computation. ∎

LEMMA 3.2. (I) $\xi_{\lambda, z} \in \mathcal{K}_\infty$ *for all* $\lambda \in \mathbb{R}$ *and* $z = E + i\eta$ *with* $\eta > 0$. *The map* $(\lambda, E, \eta) \to \xi_{\lambda, E + i\eta}$ *is continuous from* $\mathbb{R} \times \mathbb{R} \times (0, \infty)$ *to* $\mathcal{K}_\infty$.

(II) *If* $|E| < \sqrt{K}$ *we have* $\xi_{0, E} \in \mathcal{K}_\infty$ *and*

$$\lim_{n \downarrow 0} \xi_{0, E + i\eta} = \xi_{0, E} \quad \text{in } \mathcal{K}_\infty. \tag{3.15}$$

(III) (2.19) *can be rewritten as a fixed point equation in* $\mathcal{K}_\infty$:

$$\xi_{\lambda, z} = \mathcal{T} \mathcal{B}_{\lambda, z} \xi_{\lambda, z}^K, \tag{3.16}$$

valid for all $\lambda \in \mathbb{R}$ *and* $z = E + i\eta$ *with* $\eta > 0$ *and, also, valid for* $\lambda = 0$ *and* $z = E$ *with* $|E| < \sqrt{K}$.

Proof. Same as for the previous lemma. ∎

LEMMA 3.3. *The map* $F\colon \mathbb{R} \times \mathbb{R} \times [0, \infty) \times \mathcal{H}_\infty \to \mathcal{H}_\infty$, *defined by*

$$F(\lambda, E, \eta, f) = TB_{\lambda, E + i\eta} f^K - f, \tag{3.17}$$

is continuous. F is continuously Frechet differentiable with respect to f, the partial derivative being

$$F_f(\lambda, E, \eta, f) = KTB_{\lambda, E + i\eta} M(f^{K-1}) - I. \tag{3.18}$$

Moreover, for any E such that $|E| < \sqrt{K}$ *we have* $F(0, E, 0, \zeta_{0, E}) = 0$ *and*

$$0 \notin \sigma(F_f(0, E, 0, \zeta_{0, E})). \tag{3.19}$$

Proof. The proof is straightforward except for (3.19). We have $F_f(0, E, 0, \zeta_{0, E}) = KA_{0, E} - I$, where $A_{0, E} = TB_{0, E} M(\zeta_{0, E}^{K-1})$. In the Hilbert space $\mathscr{H}$ we have $T^2 = I$ so $T^{-1} = T$ [8], and $B_{0,E} = M(e^{iE\varphi^2})$ is invertible with $B_{0,E}^{-1} = B_{0,-E}$. Let us consider the bounded operator on $\mathscr{H}$ given by

$$C_{0,E} \equiv M(\zeta_{0,E}^{K-1})\, TB_{0,E} = TB_{0,E} A_{0,E} B_{0,-E} T. \tag{3.20}$$

Acosta and Klein [3] proved (see their Propositions 3.2 and 3.3 and Theorem 3.5) that, for $|E| < \sqrt{K}$, the spectrum of $C_{0,E}$ in $\mathscr{H}$, $\sigma_{\mathscr{H}}(C_{0,E})$, consists of eigenvalues E_n; $n = 0, 1, 2, \ldots$, plus their limit point 0, where

$$E_n = \left(\frac{-E + i\sqrt{K - E^2}}{K} \right)^{2n}. \tag{3.21}$$

Notice $E_0 = 1$, and that for all $n = 1, 2, \ldots$ we have $|E_n| = 1/K^n$ and $\operatorname{Im} E_n \neq 0$, unless $E = 0$ when $E_1 = -1/K$. Since (3.20) shows that $A_{0,E}$ and $C_{0,E}$ are related by a similarity transformation, the same is true of $A_{0,E}$.

To prove (3.19), it now suffices to show that $A_{0,E}^2$ is a compact operator in $\mathscr{H}_\infty$, so it will follow that

$$\sigma(A_{0,E}) \equiv \sigma_{\mathscr{H}_\infty}(A_{0,E}) \subset \sigma_{\mathscr{H}}(A_{0,E}). \tag{3.22}$$

(We actually have equality since the all the eigenfunctions of $A_{0,E}$ in $\mathscr{H}$ are also in $\mathscr{H}_\infty$; see Proposition 3.2 in [3].) To prove the compactness, notice that $\zeta_{0,E}$, given in (2.15), and all its derivatives are exponentially decaying functions of φ^2. Thus, as in the proof of Lemma 3.8 in [3], we will show that

$$D_{0,E} \equiv B_{0,E} M(\zeta_{0,E}^{K-1})\, TB_{0,E} M(\zeta_{0,E}^{K-1}) \colon \mathscr{H}_\infty \to \mathscr{H}_1 \tag{3.23}$$

is a compact operator, so $A_{0,E}^2 = TD_{0,E}$ is a compact operator in $\mathscr{H}_\infty$. That $D_{0,E}$ is a compact operator in the Hilbert space $\mathscr{H}$ is just Proposition 9(i) in [20]. To show the compactness as an operator from $\mathscr{H}_\infty$ to $\mathscr{H}_1$, we first notice that it suffices to show that, given a continuously differentiable function β on $[0, \infty)$ with compact support, the operator $S = M(\beta(\varphi^2))\, TM(\beta(\varphi^2))\colon\ \mathscr{H}_\infty \to \mathscr{H}_1$ is compact, since $D_{0,E}$ can be approximated in norm by such operators. (In fact, the operator $B_{0,E} M(\zeta_{0,E}^{K-1})$ can be approximated in norm by operators of the form $M(\beta(\varphi^2))$, both as an operator from $\mathscr{H}_\infty$ to $\mathscr{H}_1$ and as an operator on $\mathscr{H}_1$, e.g., [3, 20].) Lemma 3.6 in [3] tells us that $M(\beta(\varphi^2))\, \mathscr{F} M(\beta(\varphi^2))$ is a compact operator in $C_b(\mathbb{R}^2)$, the Banach space of bounded continuous functions on $\mathbb{R}^2$ with the sup norm. Thus, given a sequence $\{f_n\}$ in $\mathscr{H}_\infty$ with $\sup_n \|f_n\|_{\mathscr{H}_\infty} < \infty$ and each f_n continuously differentiable on $[0, \infty)$, we can conclude, as in Lemma 3.8 in [3], that there exists a subsequence $\{f_{n_k}\}$ such that both $\{(Sf_{n_k})(\varphi^2)\}$ and $\{\partial(Sf_{n_k})(\varphi^2)\}$ are Cauchy sequences in $C_b(\mathbb{R}^2)$, all with support contained in the compact support of the function $\beta(\varphi^2)$. It follows that $\{(Sf_{n_k})(\varphi^2)\}$ is a Cauchy sequence in $\mathscr{H}_1$ so $S\colon \mathscr{H}_\infty \to \mathscr{H}_1$ is a compact operator. ∎

LEMMA 3.4. *The map $Q\colon \mathbb{R}\times\mathbb{R}\times[0,\infty)\times\mathscr{K}_\infty \to \mathscr{K}_\infty$, defined by*

$$Q(\lambda, E, \eta, g) = \mathscr{T}\mathscr{B}_{\lambda, E+i\eta} g^K - g, \tag{3.24}$$

is continuous. Q is continuously Frechet differentiable with respect to g, the partial derivative being

$$Q_g(\lambda, E, \eta, g) = K\mathscr{T}\mathscr{B}_{\lambda, E+i\eta} M(g^{K-1}) - I. \tag{3.25}$$

Moreover, for any E such that $|E| < \sqrt{K}$ we have $Q(0, E, 0, \xi_{0,E}) = 0$ and

$$0 \notin \sigma(Q_g(0, E, 0, \xi_{0,E})). \tag{3.26}$$

Proof. Again the proof is straightforward except for (3.26). We have $Q_g(0, E, 0, \xi_{0,E}) = K\mathscr{A}_{0,E} - I$, where $\mathscr{A}_{0,E} = \mathscr{T}\mathscr{B}_{0,E} M(\xi_{0,E}^{K-1})$. It follows from (2.22) that $\mathscr{A}_{0,E} = A_{0,E} \otimes \bar{A}_{0,E}$ as an operator in $\mathscr{K}$, where $\bar{A}_{0,E} = JA_{0,E}J$, with J being complex conjugation: $Jf = \bar{f}$ for any $f \in \mathscr{H}$. Since J is antiunitary on $\mathscr{H}$ we get

$$\sigma_{\mathscr{H}}(\bar{A}_{0,E}) = \overline{\sigma_{\mathscr{H}}(A_{0,E})}$$

and, hence,

$$\sigma_{\mathscr{K}}(\mathscr{A}_{0,E}) = \{\mathscr{E}_{i,j} = E_i \bar{E}_j;\, i, j = 0, 1, 2, ...\} \cup \{0\} \tag{3.27}$$

with E_i given by (3.21). The same argument as in the previous lemma shows that $\mathscr{A}_{0,E}^2$ is a compact operator on $\mathscr{K}_\infty$, so it follows that

$$\sigma(\mathscr{A}_{0,E}) \equiv \sigma_{\mathscr{K}_\infty}(\mathscr{A}_{0,E}) = \sigma_{\mathscr{K}}(\mathscr{A}_{0,E}). \tag{3.28}$$

Since $\mathscr{E}_{i,j} \neq 1/K$ for any $i, j = 0, 1, 2, ...$, (3.26) follows. ∎

Lemmas 3.3 and 3.4 tell us that the hypotheses of the implicit function theorem (see Theorem B.1 in Appendix B) are satisfied by the functions $F(\lambda, E, \eta, f)$ and $Q(\lambda, E, \eta, g)$ at $(0, E, 0, \zeta_{0,E})$ and $(0, E, 0, \xi_{0,E})$, respectively, if $|E| < \sqrt{K}$. It follows that for each E such that $|E| < \sqrt{K}$ there exist $\lambda_E > 0$, $\varepsilon_E > 0$, $\eta_E > 0$, and $\delta_E > 0$, such that for each

$$(\lambda, E', \eta) \in (-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times [0, \eta_E)$$

there is a unique $\omega_{\lambda, E', \eta} \in \mathscr{K}_\infty$ with $\|\omega_{\lambda, E', \eta} - \xi_{0, E}\|_{\mathscr{K}_\infty} < \delta_E$, such that $Q(\lambda, E', \eta, \omega_{\lambda, E', \eta}) = 0$. Moreover, the map

$$(\lambda, E', \eta) \in (-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times [0, \eta_E) \to \omega_{\lambda, E', \eta} \in \mathscr{K}_\infty$$

is continuous. Similar statements hold for $F(\lambda, E, \eta, f)$.

THEOREM 3.5. *For any E such that $|E| < \sqrt{K}$ there exists $\lambda_E > 0$ and $\varepsilon_E > 0$, such that the maps*

$$(\lambda, E', \eta) \in (-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times (0, \infty) \to \xi_{\lambda, E' + i\eta} \in \mathscr{K}_\infty \tag{3.29}$$

and

$$(\lambda, E', \eta) \in (-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times (0, \infty) \to \zeta_{\lambda, E' + i\eta} \in \mathscr{H}_\infty \tag{3.30}$$

have continuous extensions to $(-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times [0, \infty)$ satisfying (3.16) *and* (3.14), *respectively.*

Proof. For the map given in (3.29) it suffices to prove that

$$\xi_{\lambda, E' + i\eta} = \omega_{\lambda, E', \eta} \quad \text{for all} \quad (\lambda, E', \eta) \in (-\lambda_E, \lambda_E) \times (E - \varepsilon_E, E + \varepsilon_E) \times (0, \eta_E). \tag{3.31}$$

But it follows from Lemma 3.2 that $\xi_{\lambda, E' + i\eta}$ is a continuous function of (λ, E', η) in the set $(\{0\} \times \{E'\} \times [0, \eta_1]) \cup (\mathbb{R} \times \mathbb{R} \times [\eta_1, \infty))$ for any $\eta_1 > 0$ which satisfies (3.16). Thus (3.31) follows from the uniqueness in Theorem B.1. The proof for the map in (3.30) is similar. ▮

Theorem 1.4 now follows from (2.18), (2.12), Theorem 3.5, the translation invariance of expectations, and a simple compactness argument.

4. A CRITERION FOR ABSOLUTELY CONTINUOUS SPECTRUM

We will now show that Theorem 1.1 follows from Theorem 1.3. We start with some general considerations. Let v be a finite measure on $\mathbb{R}$; its Stieltjes (or Borel) transform F is given by

$$F(z) = \int \frac{dv(t)}{t - z} \qquad \text{for} \quad z = E + i\eta \quad \text{with} \quad \eta > 0. \tag{4.32}$$

Given $a > 0$, we define v_a to be the restriction of v to the interval $(-a, a)$ and denote its Stieltjes transform by F_a. Our main result for Stieltjes transforms is the following criterion for absolute continuity.

THEOREM 4.1. *Suppose*

$$\liminf_{\eta \downarrow 0} \int_{-a}^{a} |F(E+i\eta)|^2 \, dE < \infty. \tag{4.33}$$

Then ν_a is absolutely continuous.

LEMMA 4.2. *Suppose* (4.33) *holds. Then*

$$\liminf_{\eta \downarrow 0} \int_{-\infty}^{\infty} |F_b(E+i\eta)|^2 \, dE < \infty \tag{4.34}$$

for any b such that $0 < b < a$.

Proof. Let $0 < b < a$, then

$$\begin{aligned} \int_{-a}^{a} [\operatorname{Im} F_b(E+i\eta)]^2 \, dE &\leqslant \int_{-a}^{a} [\operatorname{Im} F(E+i\eta)]^2 \, dE \\ &\leqslant \int_{-a}^{a} |F(E+i\eta)|^2 \, dE, \end{aligned} \tag{4.35}$$

since by (4.32),

$$\begin{aligned} \operatorname{Im} F_b(E+i\eta) &= \int_{-b}^{b} \frac{\eta}{(t-E)^2+\eta^2} \, d\nu(t) \\ &\leqslant \int_{-\infty}^{\infty} \frac{\eta}{(t-E)^2+\eta^2} \, d\nu(t) = \operatorname{Im} F(E+i\eta). \end{aligned} \tag{4.36}$$

On the other hand, for $|E| \geqslant a$,

$$\operatorname{Im} F_b(E+i\eta) \leqslant \int_{-b}^{b} \frac{\eta}{(b-E)^2+\eta^2} \, d\nu(t) \leqslant \frac{\eta}{(|E|-b)^2} \, \nu((-b, b)), \tag{4.37}$$

so

$$\int_{|E| \geqslant a} [\operatorname{Im} F_b(E+i\eta)]^2 \, dE \leqslant \frac{2\nu((-b, b))^2}{3(a-b)^3} \, \eta^2. \tag{4.38}$$

From (4.34), (4.35), and (4.38) we get

$$\liminf_{\eta \downarrow 0} \int_{-\infty}^{\infty} [\operatorname{Im} F_b(E+i\eta)]^2 \, dE < \infty. \tag{4.39}$$

But

$$2[\operatorname{Im} F_b(E+i\eta)]^2 = |F_b(E+i\eta)|^2 - \operatorname{Re}[F_b(E+i\eta)^2] \tag{4.40}$$

and

$$\int_{-\infty}^{\infty} F_b(E+i\eta)^2 \, dE = 0, \tag{4.41}$$

since the function $g(\omega) \equiv F_b(\omega + i\eta)$ is analytic in $\{\omega \in \mathbb{C}; \operatorname{Im} \omega > -\eta\}$ and

$$|g(\omega)| \leqslant \frac{\nu((-b, b))}{|\omega| - b} \qquad \text{if} \quad |\omega| > b.$$

The lemma is proved. ∎

The following lemma is closely related to Kato-smoothing [16] (see also Section XIII.7 in [27]).

LEMMA 4.3. *Let ν be a finite measure on $\mathbb{R}$, $F(E+i\eta)$ its Stieltjes transform, and let $\hat{\nu}(k) = \int e^{-ikt} \, d\nu(t)$ be its characteristic function. Then*

$$\int_{-\infty}^{\infty} |\hat{\nu}(k)|^2 \, dk = \frac{1}{\pi} \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} |F(E+i\eta)|^2 \, dE. \tag{4.42}$$

Proof. Let $g_\eta(t) = -(t+i\eta)^{-1}$, so $F(E+i\eta) = (g_\eta * \nu)(E)$. Since

$$g_\eta(t) = -i \int_0^{\infty} e^{-\eta k + itk} \, dk, \tag{4.43}$$

it follows that $\hat{g}_\eta(k) = 2\pi i \chi_{(0,\infty)}(k) \, e^{-\eta k}$, where $\hat{f}(k) = \int e^{-ikt} f(t) \, d(t)$ for any given function f. Thus $\hat{F}_\eta(k) = -i \sqrt{2\pi} \, \chi_{(0,\infty)}(k) \, e^{-\eta k} \hat{\nu}(k)$ with $F_\eta(E) = F(E+i\eta)$. Using the Plancherel theorem plus the fact that $\hat{\nu}(-k) = \overline{\hat{\nu}(k)}$, we get

$$\begin{aligned} \int_{-\infty}^{\infty} |F(E+i\eta)|^2 \, dE &= 2\pi \int_0^{\infty} e^{-2\eta k} \, |\hat{\nu}(k)|^2 \, dk \\ &= \pi \int_{-\infty}^{\infty} e^{-2\eta |k|} \, |\hat{\nu}(k)|^2 \, dk. \end{aligned} \tag{4.44}$$

The monotone convergence theorem and (4.44) give

$$\int_{-\infty}^{\infty} |\hat{v}(k)|^2 \, dk = \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} e^{-2\eta |k|} \, |\hat{v}(k)|^2 \, dk$$

$$= \frac{1}{\pi} \lim_{\eta \downarrow 0} \int_{-\infty}^{\infty} |F(E+i\eta)|^2 \, dE. \quad \blacksquare \tag{4.45}$$

Proof of Theorem 4.1. It suffices to apply Lemma 4.3 to the finite measure v_b for any b such that $0<b<a$; it then follows from (4.42) and (4.34) that $\hat{v}_b \in L^2(\mathbb{R}, dk)$, so v_b is absolutely continuous with a square integrable density. ∎

We can now prove Theorem 1.1. Let $0<E<\sqrt{K}$ and $\lambda(E)>0$ as in Theorem 1.3, so (1.5) holds. For $|\lambda|<\lambda(E)$ and any $x \in \mathbb{B}$ we use Fubini's theorem and Fatou's lemma to obtain

$$\mathbb{E}\left(\liminf_{\eta \downarrow 0} \int_{-E}^{E} |G_\lambda(x, x; E' + i\eta)|^2 \, dE' \right)$$

$$\leqslant \liminf_{\eta \downarrow 0} \int_{-E}^{E} \mathbb{E}(|G_\lambda(x, x; E' + i\eta)|^2) \, dE' < \infty. \tag{4.46}$$

Thus, we must have

$$\liminf_{\eta \downarrow 0} \int_{-E}^{E} |G_\lambda(x, x; E' + i\eta)|^2 \, dE' < \infty \tag{4.47}$$

with probability one. Since $G_\lambda(x, x; E+i\eta)$ is the Stieltjes transform of the measure $dv_{\lambda, x}(E) = \langle x \, |dP_\lambda(E)| \, x \rangle$, where $dP_\lambda(E)$ is the spectral measure of the operator H_λ, it now follows from Theorem 4.1 that for each $x \in \mathbb{B}$ the restriction of $v_{\lambda, x}$ to the interval $(-E, E)$ is purely absolutely continuous with probability one, so Theorem 1.1 is proven.

APPENDIX A: THE SUPERSYMMETRIC REPLICA TRICK

In this appendix we give a more intuitive derivation of Eqs. (2.9), (2.10), (2.13), and (2.19). The volume $\mathbb{B}_\ell$ will consist of all sites in $\mathbb{B}$ whose distance from the origin is less than or equal to ℓ; similarly $\mathbb{B}_\ell^{(x \mid y)}$, $\mathbb{B}_\ell^{(x)}$ will denote all sites in $\mathbb{B}^{(x\,|\,y)}$, $\mathbb{B}^{(x)}$, respectively, whose distance from x is less than or equal to ℓ. (For convenience we also allow $\ell = \infty$, in which case it may be omitted from the notation.) We will write $H_{\lambda,\ell}$, $H_{\lambda,}^{(x\,|\,y)}$ and $H_{\lambda,\ell}^{(x)}$ for $H_{\lambda,\mathbb{B}_\ell}$, $H_{\lambda,\mathbb{B}_\ell^{(x\,|\,y)}}$, and $H_{\lambda,\mathbb{B}_\ell^{(x)}}$, respectively. Similarly, we will use $G_{\lambda,\ell}(x, y; z)$ for $G_{\lambda,\mathbb{B}_\ell}(x, y; z)$ and $G_{\lambda,\ell}(z)$, $G_{\lambda,\ell}^{(x\,|\,y)}(z)$, $G_{\lambda,\ell}^{(x)}(z)$ for $G_{\lambda,\ell}(0, 0; z)$, $G_{\lambda,\mathbb{B}_\ell^{(x\,|\,y)}}(x, x; z)$, $G_{\lambda,\mathbb{B}_\ell^{(x)}}(x, x; z)$, respectively. We have (see Proposition 1.2 in [3])

$$\lim_{\ell\to\infty} G_{\lambda,\ell}(x, y; z) = G_\lambda(x, y; z) \qquad \text{for any} \quad x, y \in \mathbb{B},\ E \in \mathbb{R}, \text{ and } \eta > 0, \tag{A.1}$$

with similar limits for $G_{\lambda,\ell}(z)$, $G_{\lambda,\ell}^{(x\,|\,y)}(z)$, and $G_{\lambda,\ell}^{(x)}(z)$.

The supersymmetric replica trick (see [17]) says that if $x_1, x_2 \in \mathbb{B}_\ell$, with ℓ a *finite* positive integer, then for all $z \in \mathbb{C}$ with $\eta = \operatorname{Im} z > 0$, we have

$$G_{\lambda,\ell}(x_1, x_2; z)$$
$$= i \int \psi(x_1)\,\bar{\psi}(x_2) \exp\left\{ -i \sum_{x\in\mathbb{B}_\ell} \Phi(x)\cdot[(H_{\lambda,\ell} - z)\,\Phi](x)\right\} D_{\mathbb{B}_\ell}\Phi, \tag{A.2}$$

where $\Phi(x) = (\varphi(x), \psi(x), \bar{\psi}(x))$ with $\varphi(x) \in \mathbb{R}^2$ and $\psi(x)$, $\bar{\psi}(x)$ anticommuting "variables" (i.e., elements of a Grassman algebra),

$$\Phi(x)\cdot\Phi(y) = \varphi(x)\cdot\varphi(y) + \tfrac{1}{2}(\bar{\psi}(x)\,\psi(y) + \bar{\psi}(y)\,\psi(x))$$

and

$$D_\Lambda \Phi = \prod_{x\in\Lambda} d\Phi(x) \qquad \text{with} \quad d\Phi(x) = \frac{1}{\pi}\, d\bar{\psi}(x)\, d\psi(x)\, d^2\varphi(x).$$

To compute functions of ψ, $\bar{\psi}$ we expand in a power series that terminates after a finite number of terms due to the anticommutativity. The linear functional denoted by integration against $d\bar{\psi}(x)\,d\psi(x)$ is defined by

$$\int (a_0 + a_1\psi(x) + a_2\bar{\psi}(x) + a_3\bar{\psi}(x)\,\psi(x))\, d\bar{\psi}(x)\, d\psi(x) = -a_3. \tag{A.3}$$

We will use the (bad, but convenient) notation $\Phi(x)^2 = \Phi(x)\cdot\Phi(x)$ and $\varphi(x)^2 = \varphi(x)\cdot\varphi(x)$. We will also denote a generic $\Phi(x) = (\varphi(x), \psi(x), \bar{\psi}(x))$ by $\Phi = (\varphi, \psi, \bar{\psi})$. Notice that if $f\colon [0, \infty) \to \mathbb{C}$ is continuously differentiable, then $f(\Phi^2) = f(\varphi^2) + f'(\varphi^2)\,\bar{\psi}\psi$.

In particular, for finite ℓ and $\operatorname{Im} z > 0$, we get

$$G_{\lambda,\ell}(z) = i \int \psi(0)\, \bar{\psi}(0)\, e^{i(z - V(0))\, \Phi(0)^2}$$

$$\times e^{\{-i \sum_{x\,:\,d(x,0)=1} (\Phi(0)\cdot\Phi(x) + \sum_{y \in \mathbb{B}_{\ell-1}^{(x\mid 0)}} \Phi(y)\cdot[(H_{\lambda,\ell-1}^{(x\mid 0)} - z)\,\Phi](y))\}}\, D_{\mathbb{B}_\ell}\Phi$$

$$= i \int \psi(0)\, \bar{\psi}(0)\, e^{i(z - \lambda V(0))\, \Phi(0)^2}$$

$$\times \left[\prod_{x:\, d(x,0)=1} \int e^{\{-i\Phi(0)\cdot\Phi(x) - i \sum_{y \in \mathbb{B}_{\ell-1}^{(x\mid 0)}} \Phi(y)\cdot[(H_{\lambda,\ell-1}^{(x\mid 0)} - z)\,\Phi](y))\}} \right.$$

$$\left. \times D_{\mathbb{B}_{\ell-1}^{(x\mid 0)}}\Phi \right] d\Phi(0), \tag{A.4}$$

where we used the fact that Δ (and, hence, H_λ) has only zero matrix elements between sites in different branches of the tree.

By an explicit computation,

$$\int e^{\{-i\Phi(0)\cdot\Phi(x) - i \sum_{y \in \mathbb{B}_{\ell-1}^{(x\mid 0)}} \Phi(y)\cdot[(H_{\lambda,\ell-1}^{(x\mid 0)} - z)\,\Phi](y))\}}\, D_{\mathbb{B}_{\ell-1}^{(x\mid 0)}}\Phi$$

$$= e^{(i/4)\, G_{\lambda,\ell-1}^{(x\mid 0)}(z)\, \Phi(0)^2}. \tag{A.5}$$

Thus,

$$G_{\lambda,\ell}(z) = i \int \psi(0)\, \bar{\psi}(0)\, e^{i(z - \lambda V(0))\, \Phi(0)^2}$$

$$\times \exp\left\{ \frac{i}{4} \sum_{x;\, d(x,0)=1} G_{\lambda,\ell-1}^{(x\mid 0)}(z)\, \Phi(0)^2 \right\} d\Phi(0). \tag{A.6}$$

Since $\operatorname{Im} z > 0$, we can let $\ell \to \infty$, obtaining

$$G_\lambda(z) = i \int \psi \bar{\psi} e^{i(z - \lambda V(0))\, \Phi^2} \exp\left\{ \frac{i}{4} \sum_{x:\, d(x,0)=1} G_\lambda^{(x\mid 0)}(z)\, \Phi^2 \right\} d\Phi. \tag{A.7}$$

Integrating over the anticommuting variables we get (2.9).

If in (A.5) we repeat the argument used in (A.4) and let $\ell \to \infty$, we get

$$e^{(i/4)\, G_\lambda^{(x\mid y)}(z)\, \Phi^2} = \int e^{-i\Phi\cdot\Phi'} e^{i(z - \lambda V(x))\, \Phi'^2}$$

$$\times \exp\left\{ \frac{i}{4} \sum_{x':\, d(x',x)=1,\, x' \neq y} G_\lambda^{(x'\mid x)}(z)\, \Phi'^2 \right\} d\Phi', \tag{A.8}$$

from which we get (2.10). If we take expectations in (A.8), we get

$$\zeta_{\lambda, z}(\Phi^2) = \int e^{-i\Phi \cdot \Phi'} e^{iz\Phi'^2} h(\lambda \Phi'^2)[\zeta_{\lambda, z}(\Phi'^2)]^K \, d\Phi', \tag{A.9}$$

in the supersymmetric formulation; (2.13) follows.

Similarly to (A.9), we have

$$\xi_{\lambda, z}(\Phi_+^2, \Phi_-^2) = \int e^{-i(\Phi_+ \cdot \Phi'_+ - i\Phi_- \cdot \Phi'_-)} e^{iE(\Phi'^2_+ - \Phi'^2_-)} \times h(\lambda(\Phi'^2_+ - \Phi'^2_-))[\xi_{\lambda, z}(\Phi'^2_+, \Phi'^2_-)]^K \, d\Phi'_+ \, d\Phi'_-, \tag{A.10}$$

which gives (2.19).

Notice that in the supersymmetric formulation we have [8]

$$(Tf)(\Phi^2) = \int e^{-i\Phi \cdot \Phi'} f(\Phi'^2) \, d\Phi' \tag{A.11}$$

and $(M(g) f)(\Phi^2) = g(\Phi^2) f(\Phi^2)$.

APPENDIX B: THE IMPLICIT FUNCTION THEOREM

We used the following version of the implicit function theorem (see 2.7.2 in [25]):

THEOREM B.1 (Implicit function theorem). *Let $\mathscr{M}$ be a complete metric space, let $\mathscr{X}$ be a Banach space, and let f be a continuous function from an open set $U \subset \mathscr{M} \times \mathscr{X} \to \mathscr{X}$ which has a Frechet derivative with respect to $x \in \mathscr{X}$, $f_x(m, x)$, which is continuous in U. Suppose $f(m_0, x_0) = 0$ for some $(m_0, x_0) \in U$ and that $0 \notin \sigma_{\mathscr{X}}(f_x(m_0, x_0))$ (i.e., $f_x(m_0, x_0)$ is a Banach space isomorphism of $\mathscr{X}$). Then*

(i) *There exist $r, \delta > 0$ such that for each $m \in \{m' \in \mathscr{M}; d(m', m_0) < r\}$ there exists a unique $u(m) \in \{x \in \mathscr{X}; \|x - x_0\| < \delta\}$, such that $f(m, u(m)) = 0$.*

(ii) *The map $m \in \{m' \in \mathscr{M}; d(m', m_0) < r\} \to u(m) \in \mathscr{X}$ is continuous.*

This statement of the implicit function theorem is slightly stronger than the one given in Theorem 2.7.2 in [25], but it is the result actually proved in [25].

ACKNOWLEDGMENTS

It is a pleasure to thank Svetlana Jitomirskaya for many discussion and suggestions. I am grateful to Barry Simon for Lemma 4.3, which simplified my original proof of Theorem 4.1. I thank Michael Aizenman for several stimulating discussions about the Bethe lattice.

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Printed in Belgium