A snippet about the proof of the Archimedean property

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This is a snippet about the fact that I proved the Archimedean property in a way complementary to the way it is proved in the book.

We prove this: "Suppose that $\mathbb{F}$ is a complete ordered field. Then for every $t \in \mathbb{F}$, there is a positive integer n such that $t < n$".

The proof in the book: suppose otherwise. Then there is a t which is larger than every positive integer, and so is an upper bound for the set $\mathbb{Z}^+$.

Thus by the Completeness Axiom there is a least upper bound M for $\mathbb{Z}^+$.

Since M is the least upper bound for $\mathbb{Z}^+$, it follows that $M - 1$ is not an upper bound for $\mathbb{Z}^+$, so there is a positive integer $N > M - 1$.

But then $N + 1$ is a positive integer, and $N + 1 > M$, which contradicts the claim that M is the least upper bound of $\mathbb{Z}^+$, because it shows that M is not an upper bound for $\mathbb{Z}^+$ at all.

The alternative proof I came up with on my feet: consider the set $S = \{t \in \mathbb{F} : (\forall n \in \mathbb{Z}^+ : t > n\}$. This could be described as the set of strict upper bounds for $\mathbb{Z}^+$.

Suppose the statement "for every $t \in \mathbb{F}$, there is a positive integer n such that $t < n$" is false. Then S is nonempty. S is certainly bounded below (by any positive integer you like, such as 1) so it has a greatest lower bound (a corollary of the completeness property).

Let M be the greatest lower bound of S. Then $M + 1$ is not a lower bound of S, so there is $t \in S$ which is less than $M + 1$. This implies that $t - 1 < M \notin S$, from which it follows that there is an integer n such that $t - 1 < n$. But then $n + 1$ is an integer and $t < n + 1$, contradicting the assertion that $t \in S$.

If you look at it sideways, it is basically the same argument.

I hint that this is seriously relevant to problem 14b. Explain.