Riemann integrals of continuous functions exist

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In this snippet of notes, we will work toward proving that $\int_a^b f$ exists for any f continuous on $[a, b]$ along with prerequisite and related results

We begin by proving that if a function f is nondecreasing on $[a, b]$, $\int_b^a f$ exists.

Suppose that f is nondecreasing on $[a, b]$. (this means that for any $x, y \in [a, b]$ with $x < y$, $f(x) \leq f(y)$).

We will show that for any $\epsilon > 0$, there is a partition P such that $U(f, P) - L(f, P) < \epsilon$. It is a homework exercise in your current assignment that this is sufficient to establish that $\int_a^b f$ exists.

Let P be a partition $\{x_i\}_{0 \leq i \leq n}$ of $[a, b]$ such that there is a constant $\delta < \frac{\epsilon}{f(b)-f(a)}$ such that $x_i - x_{i-1} = \delta$ for each i for which this is defined: P determines a subdivision of $[a, b]$ into closed intervals all of the same length strictly less than ϵ.

Now

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f)$$

$$= \sum_{i=1}^{n} \delta(f(x_i) - f(x_{i-1}))$$

[because the length of each interval in P is δ and $\sup_{[x_{i-1}]} f = f(x_i)$ and $\inf_{[x_{i-1}]} f = f(x_{i-1})$ because f is nondecreasing]

$$= \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \epsilon$$

[the second equation holds because $\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))$ is a telescoping sum]

And this completes the proof that $\int_a^b f$ exists, mod the homework assignment mentioned.

I strongly recommend and may assign proving the same result for nonincreasing functions f.

The proof that $\int_a^b f$ exists if f is continuous on $[a,b]$ relies on the theorem that a function f continuous on a closed interval $[a,b]$ is uniformly continuous on $[a,b]$. We first explain what this statement means, then use it to prove that $\int_a^b f$ exists, then perhaps prove the prerequisite theorem.

That f is continuous on a set A means that for each $x \in A$, there is an $\epsilon > 0$ such that for any $y \in A$, if $|y-x| < \delta$ then $|f(y)-f(x)| < \epsilon$. This follows from the usual definitions of limits and continuity which you should have known since undergraduate real anaysis if not since Calculus I.

That f is uniformly continuous on a set A means that for each $\epsilon > 0$, there is $\delta > 0$ such that for any $x, y \in A$, if $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$.

The second assertion is stronger: it says that the tolerance of error δ you need such that if y is that close to x, $f(y)$ will be within ϵ of $f(x)$ does not depend on x: the same tolerance works everywhere in the set A.

The prerequisite theorem is "If f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$". For the moment we assume this and proceed to prove that $\int_a^b f$ exists.

Again, we will show that for any $\epsilon > 0$, there is a partition P such that $U(f,P) - L(f,P) < \epsilon$. It is a homework exercise in your current assignment that this is sufficient to establish that $\int_a^b f$ exists.

Choose $\epsilon > 0$ arbitrarily

Choose δ such that for any $x, y \in [a,b]$, if $|x-y| < \delta$, $|f(x)-f(y)| < \frac{\epsilon}{2(b-a)}$.

Let P be the partition of $[a,b]$ determined by $\{x_i\}_{0 \leq i \leq n}$ subdividing the interval into closed intervals all with equal length δ.

Now

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(x_i - x_{i-1})(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f)$$

$$= \sum_{i=1}^{n} \delta(\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f)$$

$$\leq \sum_{i=1}^{n} \delta \frac{\epsilon}{2(b-a)}$$

because $\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \leq \frac{\epsilon}{2(b-a)}$ since the length of the interval is δ (any two points in the interval except x_i and x_{i-1} are at distance $< \delta$ and have values of f differing by less than $\frac{\epsilon}{2(b-a)}$; x_i and x_{i-1} are at distance exactly δ but continuity of f lets us see that the values of f at the endpoints might differ exactly by $\frac{\epsilon}{2(b-a)}$ but no more: so the difference between the largest and smallest value of the function on the interval is bounded above by $\frac{\epsilon}{2(b-a)}$ and $\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f$ is no greater than $\frac{\epsilon}{2(b-a)}$.

$$= \sum_{i=1}^{n} \frac{b-a}{n} \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{2} < \epsilon$$

Note that $\delta = \frac{b-a}{n}$.

I'll lecture the proof that a continuous function on a closed interval is uniformly continuous on Sept 6; notes on it will be added here eventually.