Math 414 1/17/2024 Whiteboard

a set is an object with elements.

$A \in B$ means "A is an element of B"

$\in$ is a primitive notion.

I. if $x \in A$ then A is a set. Only sets have elements.

II. Identity criteria for sets:

if A and B are sets

$$A = B \text{ iff for any } x,\ x \in A \leftrightarrow x \in B.$$

sets are equal iff they have the same elements.

III. Corollary: there is only one set $\emptyset$ with no elements.
Any non-sets may be distinct from one another and $\emptyset$ and all have no elements.

IV List notation for finite sets

example: $\{0,1,2\}$ is the set whose elements are exactly $0,1,2$

How many elements does $\{a,b\}$ have?
if $a=b$ it has one element; if $a \neq b$ it has two elements.

V. $x \notin y$ means $\neg\, x \in y$

VI. Sets can have any sort of thing as elements
Usually, the elements of sets we are interested in will be of a common sort.

VII. $x \subseteq y$ is defined as x is a set and y is a set and $(\forall z: z \in x \rightarrow z \in y)$

Liebler used $\subset$ for subset, for most this "is" proper subset.

	me	Liebler
subset	$\subseteq$	$\subset$
proper subset	$\subsetneqq$	$\subsetneq$

$A \subsetneqq B$ means $A \subseteq B$ and $A \neq B$.

VIII. Elements are not parts.

Part to whole is transitive

if A is part of B, and B is part of C
then is A is part of C.

$1 \in \{1,2\}$ $\quad$ $2 \in \{1,2\}$

$\{1,2\} \in \{\{1,2\}\}$

$1 \notin \{\{1,2\}\}$ $\quad$ [if by some horrible chance $1 = \{1,2\} \in \{\{1,2\}\}$ then $2 \notin \{1,2\}$, $2 \notin \{\{1,2\}\}$]

$\in$ is not transitive and so it is not the relation of part to whole.

Subset is transitive.

If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

If sets can be said to have parts - the subsets are the parts...

There is a general tendency to confuse elementhood/membership with subset -

to avoid this, I don't use A in B for either relation

$A \in B$ $\quad$ A is contained in B

$A \subseteq B$ $\quad$ A is included in B

I don't guarantee to be perfect

IX Subsets and the empty set.

Theorem: for any set A, $\emptyset \subseteq A$.

Proof: $(\forall x: \underbrace{x \in \emptyset}_{F} \to x \in A)$

$\emptyset \in A$ may be true or false.

$x \in y$

$x \subseteq y$ means x is a set, y is a set and

$$(\forall z: z \in x \to z \in y)$$

$\{1\} \notin \{1,2\}$ but $\{1\} \subseteq \{1,2\}$

at least we don't think so

$$E = \{n \in \mathbb{N} : (\exists m \in \mathbb{N} : 2m = n)\}$$

$E \subseteq \mathbb{N}$ but $E \notin \mathbb{N}$.

X. Set builder notation (Axiom of Separation)

If A is a set and $P(x)$ is a <u>sentence</u> about a variable x,

$\{x \in A : P(x)\}$ is a set – and

$$\forall a: \quad a \in \{x \in A : P(x)\} \Leftrightarrow a \in A \land P(a).$$

$$E = \{x \in \mathbb{N} : (\exists m \in \mathbb{N} : 2m = n)\}$$

$2 \in E$ since $2 \in \mathbb{N}$ and $(\exists m \in \mathbb{N} : 2m = 2)$

namely $m := 1$.

$3 \in E$ since $(\underbrace{3 \in \mathbb{N}}_{T} \land \underbrace{(\exists m \in \mathbb{N} : 2m = 3)}_{F})$

$\{0,1,2\} = \{x \in \mathbb{Z} : x = 0 \lor x = 1 \lor x = 2\}$

$\boxed{\text{XI}}$ Specific sets $\quad \mathbb{N} = \{1, 2, 3 \ldots\}$ [$\mathbb{N} = \{0, 1, 2 \ldots\}$ is a frequent alternate]

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$$

$$\mathbb{Q} = \{\tfrac{m}{n} : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$$

that is

$$\mathbb{Q} = \{x \in \mathbb{R} : (\exists m, n \in \mathbb{N} : \tfrac{m}{n} = x)\}$$ [not this implies $n \neq 0$]

$$\{\underline{f(x)} : x \in A\}$$

$$\{y \in B : (\exists x : x \in A \land y = f(x))\}$$

In axiomatic set there is a separate much stronger axiom asserting existence of $\{f(x) : x \in A\}$

$B^c = A \setminus B$ if A is the entire universe ?!

or the collection of all things we are currently interested

$R : \{x \in A : x \notin x\}$ does not belong to A.

$R \in R \leftrightarrow R \in A \land R \notin R$

so $R \notin A$. so no set can be the universe.

If we were for example talking only about sets of real numbers we might use the notation B^c for $\mathbb{R} - B$.

I will generally avoid B^c in favor of explicitly writing $A - B$ for the appropriate A.

$$\left.\begin{aligned} A \cap B &= \{x \in A : x \in B\} \\ A - B &= \{x \in A : x \notin B\} \end{aligned}\right\} \text{ these follow from Separation.}$$

$A \cup B$ doesn't follow from just Separation

$$A \cup B = \{x \in \underset{\dots}{???} : x \in A \vee x \in B\}$$

the existence of unions is a separate axiom

$A \cap B = \emptyset$ ~~means~~ is read A, B are disjoint.

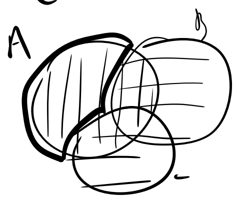
If $A_1, \dots, A_n$ are collection of sets we say $\{A_1, \dots, A_n\}$ is pairwise disjoint iff for each i, j $A_i \cap A_j = \emptyset$.

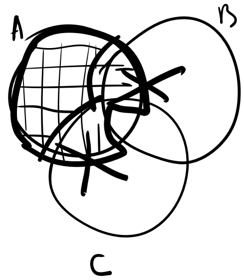
if $\mathcal{Q}$ is ~~a~~ possibly infinite collection of sets $\mathcal{Q}$ is pairwise disjoint iff

$$(\forall A, B : A \in \mathcal{Q} \wedge B \in \mathcal{Q} \rightarrow A \cap B = \emptyset)$$

de Morgan's law (for union and intersection)

$$A-(B\cup C) = (A-B)\,\underline{\cap}\,(A-C) \qquad (B\cup C)^c = B^c\cap C^c$$





||| $A-B$

$\equiv$ $A-C$

$A-B \underline{\underline{\cap}} A-C$

not $\cup$

$$A-(B\cap C) = (A-B)\cup(A-C)$$

$$\bigcup_{n=1}^{\infty} A_n = \{x : (\exists n : x\in A_n)\}$$

$$\bigcap_{n=1}^{\infty} A_n = \{x\in A_1 : (\forall n : x\in A_n)\}$$

if $\mathcal{A}$ is a collection of sets

$$\bigcup\mathcal{A} = \{x : (\exists A\in\mathcal{A} : x\in A)\}$$

we actually need an axiom to assert existence of these generalized union.

$$\bigcap\mathcal{A} = \{x\in A : (\forall B\in\mathcal{A} : x\in B)\} \quad (\text{if } A\in\mathcal{A})$$

~~$\bigcap\emptyset$ is the universe - there is no~~ such set.