Uncountable Sets

Recall that a set is said to be <u>countable</u>

iff and only if it is either finite (the same size as $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$, or empty) or the same size as $\mathbb{N}$

Recall that sets A, B are said to be the same size iff there is a bijection from A to B.

Theorem For any set A, $|A| \nleq |\mathcal{P}(A)|$.

Certainly $|A| \leq |\mathcal{P}(A)|$. The map sending each $a \in A$ to $\{a\}$ is injective and is a map from A to the power set of A.

Suppose that A and $\mathcal{P}(A)$ are the same size.

suppose $f : A \to \mathcal{P}(A)$ is a bijection.

Consider the set $R = \{a \in A : a \notin f(a)\}$.

$R \in \mathcal{P}(A)$

Let $r = f^{-1}(R)$ [an element of A]

$$r \in R \leftrightarrow \underbrace{r \in A}_{\text{it is}} \land r \notin f(r)$$

that is $r \in R \leftrightarrow \text{true} \land r \notin f(f^{-1}(R)) = R$

$\times$

What I've show is that $|A| \le |\mathcal{P}(A)|$

and $|\mathcal{P}(A)| \neq |A|$.

This does mean there is no injection from $\mathcal{P}(A)$ into A.

$\underline{|A| < |\mathcal{P}(A)|}$.

$\mathbb{N}$

$\mathcal{P}(\mathbb{N}) \leftarrow$ larger

$\mathcal{P}^2(\mathbb{N}) \leftarrow$ larger

$\mathcal{P}^3(\mathbb{N}) \leftarrow$ larger

Interesting fact $\mathcal{P}(\mathbb{N})$ is the same as as $\mathbb{R}$.

$\mathcal{P}(\mathbb{N})$ has a bijection into $[0,1]$.

send $A \subseteq \mathbb{N}$ to $\sum_{i \in A} \frac{2}{3^i}$ .. base $\underline{\underline{3}}$ $.20220220\ldots$

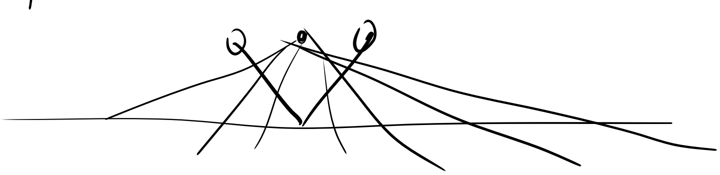
$\cancel{.01111\ldots}$ base 2

$.1\cancel{0000}\ldots$

$\mathcal{P}(\mathbb{N})$ is the same se as a subset of $[0,1]$

$[0,1]$ is the same se as the $\mathbb{R}$

$(0,1)$



This does show that $\mathbb{R}$ is uncountable because it is at least as large as $\mathcal{P}(\mathbb{N})$ which is larger than $\mathbb{N}$.

Theorem (Cantor's original argument, I believe)

$\mathbb{R}$ is uncountable.

We actually show that $[0,1]$ is uncountable (which is enough).

Suppose for the sake of a contradiction that $[0,1]$ is countable. Then we have a sequence $a_1, a_2, a_3, a_4 \ldots$ whose range is $[0,1]$.

$a_1, a_2, a_3, a_4 \ldots$ exhaust all elements of $[0,1]$.

We show that it doesn't.

We do this by a recursive construction.

We have a sequence $\{a_i\}$ assumed to have range $[0,1]$.

We will construct sequences $\{b_i\}, \{c_i\}$ which we use to describe closed intervals on $[0,1]$.

$b_1 = 0 \quad c_1 = 1 \qquad [b_1, c_1] = [0,1]$.

When we have constructed $[b_i, c_i]$ which (ind hyp) is an interval of length $\frac{1}{3^{i-1}}$ (check that this is true at the basis)

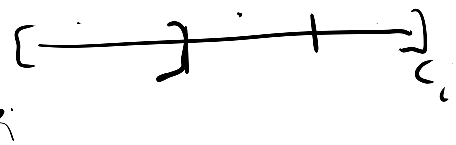
we choose $[b_{i+1}, c_{i+1}]$ so as to exclude a_i

$[b_i, c_i]$ is divided into three intervals

$[b_i, \frac{1}{3}(c_i - b_i) + b_i]$

$[b_i + \frac{1}{3}(c_i - b_i), b_i + \frac{2}{3}(c_i - b_i)]$

$[b_i + \frac{2}{3}(c_i - b_i), c_i]$



each of these is of length $\frac{1}{3^i}$

Define $[b_{i+1}, c_{i+1}]$ as the first of these 3 intervals which does not contain a_i

Nested interval theorem: a sequence of nested closed intervals whose length goes to 0 has an intersection - a single point.

The sequence of b_i's is nondecreasing (is increasing) and bounded above by c_1 so it has a limit.

The sequence of c_i's is decreasing, bounded below by b_1, so it has a limit.

The sequence of $c_i - b_i$'s goes to 0.

so $\lim c_i = \lim b_i$ call this d_∞

d_∞ belongs to each $[b_i, c_i]$ so it is not equal to any a_i,ben $a_i \notin [b_{i+1}, c_{i+1}]$

$[0,1)$ is uncountable

Suppose otherwise

$$
\begin{aligned}
a_1 &= .d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15} \dots \\
a_2 &= .d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25} \\
a_3 &= .d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35} \\
a_4 &= .d_{41}\, d_{42}\, d_{43}\, d_{44}\, d_{45} \dots
\end{aligned}
$$

$e_1 \neq d_{11}$

$e_2 \neq d_{22}$

$e_3 \neq d_{33}$

$\vdots$

$e_i = d_{ii} + 2 \mod 10$

chose a number ~~with~~ $e = .e_1\, e_2\, e_3\, e_4\, e_5 \dots$

$.5\,0\,0\,0\,0\,0$

$a_i = .4\,9\,9\,9\,9\,9 \dots$

We will need to pay attention on occasion to whether set of reals we talk about are countable or uncountable.