Solutions

Math 189, section 3, Test II

Dr Holmes

October 31, 2021

The exam officially has the same length as the class period, with a bonus ten minutes: at 1:15 I will give a ten minute warning.

You may have one sheet of notebook paper with whatever you like written on it.

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1. Compare and contrast the membership and subset relations.

   In each part, fill in the dash with $\in$ (membership), $\subseteq$ (subset), both, or neither, as appropriate (write the word "both" if both membership and subset are correct, write "neither" if neither is correct).

   (a) $\emptyset$ $\subseteq$ $\mathbb{N}$

   (b) $\{1,3\}$ $\subseteq$ $\{1,2,3\}$

   (c) $\{a,b\}$ both $\{a,b,\{a,b\}\}$

   (d) 3 $\in$ $\mathbb{Z}^+$

   (e) 1 neither $\{\{1,2\}\}$

2. Set builder notation

   In each part, notation for a set is presented: you should write roster (list) notation for the same set, and answer any other question I ask.

   (a) Let A represent $\{1, 2, 3, 4\}$ (only in this part of this problem). Write the set
   $$B = \{S \in P(A) : |S| = 3\}$$
   in roster notation.

   How many elements are there in the set $P(A)$ from which B is being taken?

$\{\{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}\}$

$2^4 = 16$ elems in $P(\{1,2,3,4\})$

(b) Let C represent the set $\{1,2,3,4,5\}$ (only in this part of this problem). Write the set

$$D = \{(x,y) \in C \times C : y = x+1\}$$

in roster notation.

Is this set a function? If it is, what is its domain and what is its range?

$\{(1,2),(2,3),(3,4),(4,5)\}$

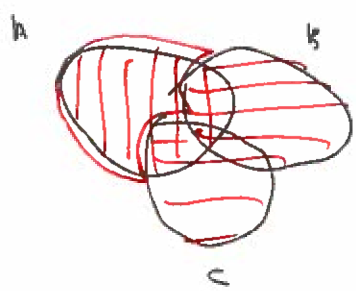
it is a function with domain $\{1,2,3,4\}$ and range $\{2,3,4,5\}$

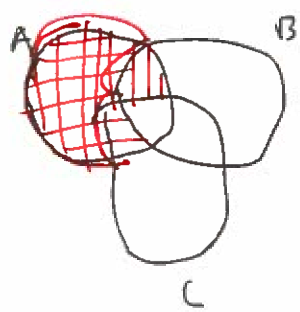
3. Two claimed set identities are given with supporting Venn diagrams. One is true, one is false.

If the identity is true, complete the Venn diagrams to illustrate this. Your diagrams should have informative shadings with a key and should have the result set clearly outlined in each diagram.

If the identity is false, complete the Venn diagrams to illustrate this. Your diagrams should have informative shadings with a key and should have the result set clearly outlined in each diagram. In addition, give a counterexample with small finite sets in roster notation which makes the claimed identity false, and show set calculations supporting this.

(a) $A - (B \cup C) = (A - B) \cap (A - C)$





||| A

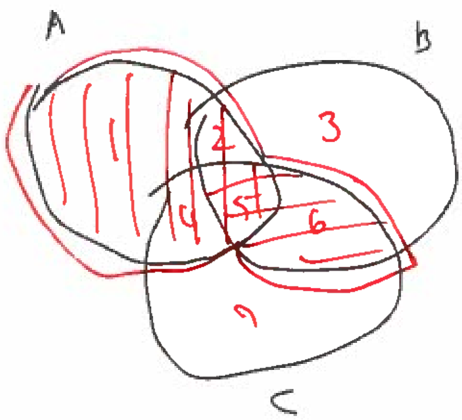
≡ B ∪ C

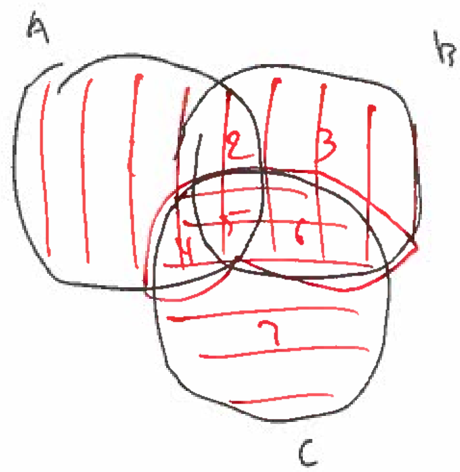
they are the same

|||| A − C

≡ A − B

(b) $A \cup (B \cap C) = (A \cup B) \cap C$





|||| A

≡ B∩C

A∪B ||||

C ≡

$A = \{1, 2, 4, 5\}$

$B = \{2, 3, 5, 6\}$

$C = \{4, 5, 6, 7\}$

$B \cap C = \{5, 6\}$

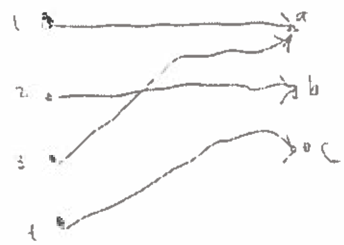
$A \cup (B \cup C) = \{1, 2, 4, 5, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$(A \cup B) \cap C = \{4, 5, 6\}$

4. A number of arrow diagrams are given. For each one answer the following questions: is it a picture of a function? If not, say why not; if it is, is the function one-to-one?; if it is not, say why not; if it is, give an arrow diagram of its inverse.

(a) :



~~not a fnctn~~

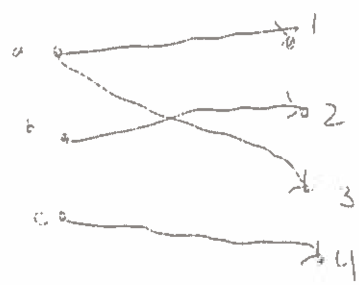
~~both 1 and 3 map~~

a fnctn,

not one to one

1, 3 both map to a

(b) :

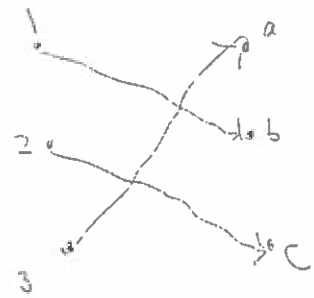


not a fnctn,

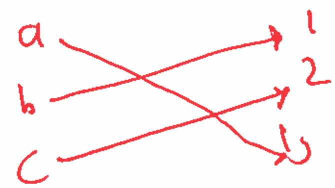
a maps to

both 1 and 3

(c) :

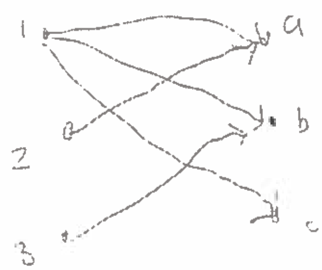


a function one to one



inverse

(d) :



not a function,

1 maps to a, b, c

5. In each part, explain whether what is presented is a bijection from $\mathbb{R}$ to $\mathbb{R}$ or not. If it isn't, briefly explain why not (a number of things can go wrong: it might not be a function; it might not have the right domain; it might not be one-to-one; it might not be onto $\mathbb{R}$); support what you say with numerical counterexamples where appropriate). If it is, determine its inverse.

(a) $f(x) = \frac{1}{x^3-8}$

not a fctn from $\mathbb{R}$ to $\mathbb{R}$:
not defined at 2 or -2

(b) $g(x) = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ (Hint: this is a very familiar graph).

not a fctn at all:
$(0,1)$ and $(0,-1)$ are both in it
(circle)

(c) $h(x) = 3x - 2$

one-to-one and onto: $h(y) = x$

$3y - 2 = x$

$3y = x + 2$

$y = \frac{x+2}{3}$

$h^{-1}(x) = \frac{x+2}{3}$

(d) $k(x) = x^2 + 1$

not one-to-one:
$k(2) = k(-2) = 3$

6. Some relations are given, some by diagrams and some mathematical relations. For each one, say which of the following properties it has: reflexive, irreflexive, or neither; symmetric, antisymmetric, or neither, transitive or not. Give brief explanations or counterexamples.

(a) The relation $\leq$ (less than or equal to) on natural numbers.

$x \leq x$ reflexive

not irreflexive

if $x \leq y$ and $y \leq x$, then $x = y$ antisymmetric

$2 \leq 3$ but $3 \nleq 2$ not symmetric

if $x \leq y$ and $y \leq z$ then $x \leq z$ - transitive

(b) The relation $x\,N\,y$ on real numbers defined as holding when

$$|x - y| \leq 2:$$

x and y are at distance at most 2 from one another.

$|x-x| \leq 2$ reflexive

not irreflexive

if $|x-y| \leq 2$ then $|y-x| \leq 2$, symmetric

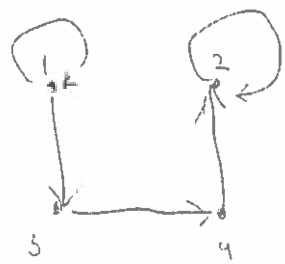
~~if~~ $|1-2| = 1 \leq 2$ and $|2-1| = 1 \leq 2$

but $1 \neq 2$, not antisymmetric

$|1-3| \leq 2$ and $|3-5| \leq 2$ but $|1-5| \nleq 2$

not transitive

(c) :



not reflexive
not irreflexive
some self loops there
some missing

not symmetric $2 \to 4$
but not $4 \to 2$

it is antisymmetric
not reciprocal arrows
between different nodes

it is not transitive
$1 \to 3$, $3 \to 4$ but not
$1 \to 4$

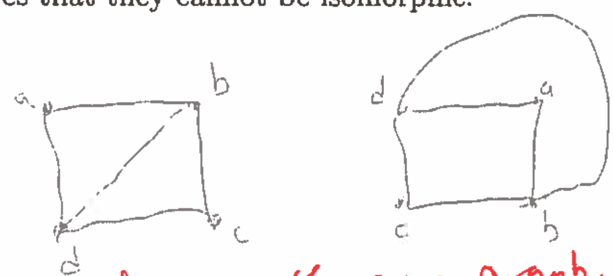
(d) :

reflexive
not irreflexive
symmetric (every relation holds!)
not antisymmetric ($a \to b$ and $b \to a$)

transitive (every relation holds!)

7. In each part two graphs are given: for each pair of graphs, either point out (correctly) that they are the same graph, or present an isomorphism between them or state a property one has which the other does not which ensures that they cannot be isomorphic.

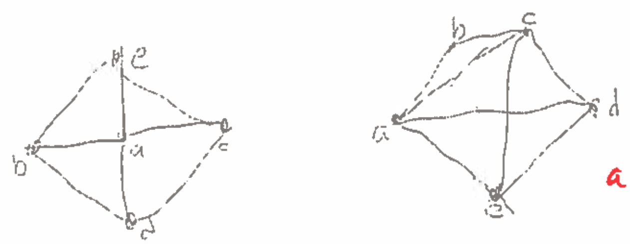
(a) :



these are the same graph.

(b) :

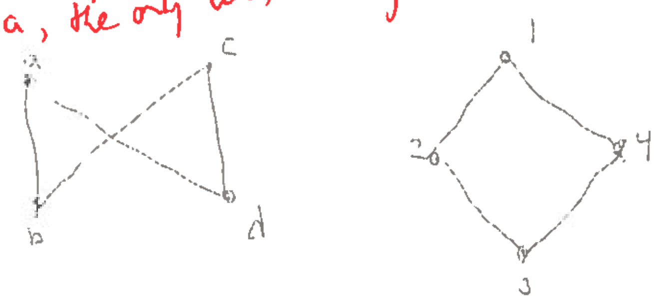
Since the graph has one vertex of degree 4 and the other has two, they can't be isomorphic



a, c are both vertices of degree 4

~~these~~ ↑ a, the only vertex of degree 4

(c) :



a → 1

b → 2 is an isomorphism

c → 3 there are a lot of possibilities

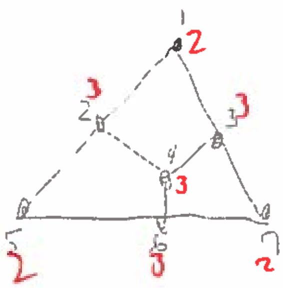
d → 4

8. Two graphs are pictured. One has an Eulerian walk and one does not. For the one which does, exhibit the walk as a sequence of vertices.

   For the one which does not, give a reason why it cannot have an Eulerian walk. There is a theorem discussed in class to be applied here.

   Definitions: A walk is a finite sequence of vertices in which each sucessive pair of vertices form an edge. An Eulerian walk is one in which each edge in the graph appears exactly once, and in which each vertex appears (a vertex might be visited more than once).

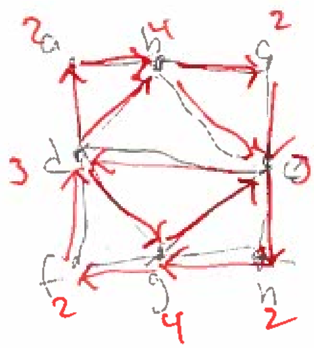
   (a) :



the graph has 4 vertices of odd degree; to have an Eulerian walk a graph must have 0 or 2 (and be connected)

don't just draw arrows - make a list of vertices you visit in order.

   (b) :



d b e d a b c e h g f d g e

Eulerian walk many possible

9. There are two parts. The one you do better on will count 7 points, and the one you do worse on 3.

   (a) Let f be a function from A to B which is one-to-one. Let g be a function from B to C which is one-to-one. What is the domain of $g \circ f$? A What is a reasonable target for $g \circ f$? C
   Prove that $g \circ f$ is one-to-one.

Suppose $x, y \in A$ arbitrary
and $(g \circ f)(x) = (g \circ f) y$
our goal is to show $x = y$
~~$g\circ$~~ $g(f(x)) = g(f(y))$ def composition
$f(x) = f(y)$ because g is one-to-one
$x = y$ because f is one to one

and it is shown.

(b) Exhibit a regular undirected graph of degree 3 with seven vertices, or explain why there cannot be one.

a regular graph with 7 vertices and edge 3 would have 21 total degree, and total degree is twice the number of edges and so cannot be an odd number, contradiction.