Math 311 Test II, Spring 2013

Dr. Holmes

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The test lasts officially from 10:30 am to 11:45 am. It is likely that there will be a short period after 11:45 am when you can still work. You are allowed your test paper, your writing instrument and drawing tools if you wish.

Cell phones must be turned off and inaccessible to you.

Problems 5 and 6 have two alternative parts. There is no expectation that you do both parts. You may benefit from doing both parts, but it would not be good to do this until you are done with the entire exam.

You are also permitted to use one of the parts of 5 or 6 to replace another problem on the test (not the other one of 5 and 6). If you are doing this, you need to say you are doing it and your test will be marked that way (I will not be surprised to see you trading one of the parts of 5 or 6 for problem 7, but I'm also interested to see if there is success on problem 7).

Your grade will be posted using the ID number on the first inside page on the class announcements page.

There are reference materials at the end. There are certainly things that I have not included there. If you think I left something out by mistake, you are welcome to ask me, but I may have left it out on purpose.

Be aware that I know perfectly well that this material is difficult. My intentions are friendly; the grading formula may be adjusted at my discretion if the exam is too long or if student performance on particular problems is poor across the entire class (that is why I have not said anything about what it is). It is to your advantage to do as much as you can, and also not to spend too long on one problem that you are having trouble with.

1. Prove using the axioms only that there are at least two points on any line. Show every use of an axiom.

2. Prove using the axioms only that there are three noncollinear points. Show every use of an axiom.

3. In the Euclidean plane with lines defined in the usual way and using the taxicab metric ($d((a,b),(c,d)) = |a-c| + |b-d|$), let A be the point (0,0) and B be the point (2,2). Identify two different points which satisfy the equation $d(A,C) + d(C,B) = d(A,B)$. Show calculations supporting your claim.

   Are both of these points "between" A and B in terms of the definition? Explain. Make sure you do give at least one point "between" A and B which satisfies the equation.

   Suppose I made a stupid typo in the definition above and wrote $d((a,b),(c,d)) = |a-b| + |c-d|$. Show that this does not define a metric.

4. Coordinate functions. You need to do both parts of this problem, but the one you do better on will count more than the other.

   (a) Prove that if f is a coordinate function for a line L, so is g, where $g(P)$ is defined as $1 - f(P)$ for any point P.

   You do need to prove that g is a bijection (it is part of the definition) but there is a quite short argument for this.

   (b) Prove using facts about coordinate functions that it cannot be true both that $A * B * C$ and that $A * C * B$.

5. Do one of the two proofs. If you do both your best work will count. If you do both very well, you might earn extra credit. Never attempt extra credit opportunities unless you are finished with the rest of the exam.

   (a) Prove the Ray Theorem. If A is a point on line L and B is a point external to L then every point $C \neq A$ on $\overrightarrow{AB}$ on the same side of L as B.

(b) Prove Pasch's "Axiom": if ΔABC is a triangle and L is a line on which none of A, B, C are incident, if L intersects $\overline{AB}$ then either L intersects $\overline{BC}$ or L intersects $\overline{AC}$.

6. Do one of the two proofs. If you do both your best work will count. If you do both very well, you might earn extra credit. Never attempt extra credit opportunities unless you are finished with the rest of the exam.

   (a) Prove the Z-theorem. If A and B are distinct points on a line L, and C and D are points external to L and on opposite sides of L, then $\overrightarrow{AC}$ does not intersect $\overrightarrow{BD}$ at any point.

(b) Prove the Segment Addition Theorem. If $A * B * C$ and $D * E * F$, and moreover $\overline{AB} \cong \overline{DE}$ and $\overline{BC} \cong \overline{EF}$, then $\overline{AC} \cong \overline{DF}$.

7. Suppose that D is a point in the interior of $\angle BAC$ (for both parts)

   Show that A, B, C are noncollinear. (this comes from the definition of an angle and its interior)

   Suppose E is a point which is neither on the angle $\angle BAC$ nor in its interior. Show that $\overline{DE}$ intersects $\angle BAC$. (Hint: this is nothing more than the definitions of an angle and its interior, the Plane Separation Postulate, and some logic of cases; I will be checking that you have covered all possibilities)

1 Selected References

I am under no obligation to supply every theorem and definition; I may have left some out on purpose. But you are welcome to ask if you think I left something out; I might have left it out by accident.

Existence Postulate: There are at least two points.

Incidence Postulate: Every line is a set of points. If A and B are distinct points, there is one and only one line L such that A and B both belong to L (lie on L).

Ruler Postulate: With every pair of points A, B we associate a real number $d(A, B)$ called the distance from A to B.

Definition of a coordinate function: A function f is a coordinate function on a line L iff it is a bijection from L to the set of all real numbers and for any points P, Q on L we have $d(P, Q) = |f(P) - f(Q)|$.

For any line L, there is a coordinate function on L.

Remark: I know it is odd that the definition of a coordinate function is in the middle of the statement of the Ruler Postulate. But you need the first sentence in the Postulate to state the definition, and it is very nice to have the definition to state the second part.

Definition of betweenness: A point C is between points A and B, written $A * C * B$ iff A, B, C are distinct and collinear and $d(A, C) + d(C, B) = d(A, C)$. Notice that $A * C * B$ is exactly equivalent to $B * C * A$.

Betweenness Theorem for Points: If f is a coordinate function on L and A, B, C are on L, then $A * C * B$ iff either $f(A) < f(C) < f(B)$ or $f(B) < f(C) < f(A)$.

Trichotomy for betweenness (3.2.19): If A, B, C are distinct collinear points, exactly one of $A * B * C, A * C * B, B * A * C$ is true.

Ruler Placement Postulate: If A, B are distinct points on L there is a coordinate function f for L such that $0 = f(A) < f(B)$.

Plane Separation Postulate: Let L be a line. There are two sets H_1 and H_2 which we call the half-planes cut by L such that L, H_1 and H_2 are pairwise disjoint and their union is the entire plane, and for any points A, B both in H, one of the half-planes, $\overline{AB} \subseteq H$, and for any A, B such that $A \in H_1$ and $B \in H_2$, $\overline{AB}$ intersects L. Points in the same half-plane are said to be on the same side of L and points in different half-planes are said to be on opposite sides of L.

definition of angle: Suppose that A, B, C are distinct and it is not the case that $B * A * C$. Then we define $\angle BAC$ as the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

definition of the interior of an angle: A point D is in the interior of $\angle BAC$ if and only if it is in the half-plane cut by line AB which contains C and it is in the half-plane cut by line AC which includes B.