Math 311 Test I, Spring 2019

Dr Holmes

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This test will begin at 1:30 pm and end at 2:50 pm (I'll give a five minute warning at the official end of class).

You should not use anything but your copy of the test and a writing instrument. On any exam in this class, you may also use a ruler and compass, but I don't think this is particularly useful for this exam.

The questions come in pairs. In each pair (1-2, 3-4, 5-6, 7-8) , the one you (individually) do better on will count for 70 percent and the one you do worse on will count for 30 percent. Otherwise, the values of the questions are equal. Question 9 may be done in place of one of 7-8, or you can do all three of 7-8-9 and the best two will be used.

Lists of axioms, definitions, and logical rules are attached. You are welcome to tear these off for easy access.

1. Prove in our formal logic system that

$$((P \to Q) \land (Q \to R)) \to (P \to R).$$

2. Prove in our formal logic system that

$$((A \wedge B) \vee (C \wedge D)) \rightarrow (A \vee D).$$

Hint: expect to find uses for both the rule of proof by cases and the rule of addition.

3. Conditional statements: do two of the three items, or all three for possible additional credit.

   For the English sentence "If it rains, I will get wet", write down the converse and the contrapositive. Which of these is equivalent to the original statement?

   Rewrite the English sentence "All even numbers are composite" in a way which makes it clear that it is a conditional statement (write it as an if…then… statement about a general number x). Write the converse and the contrapositive of this statement.

   Present a truth table showing that the converse and the contrapositive of a conditional statement $A \to B$ are not logically equivalent. Highlight a row or column of the table which exhibits the failure of equivalence and comment briefly on why it does so.

4. Negations: write the negation of each of the following sentences in a natural way (it is not appopriate to use "It is not the case that" or to simply change a statement beginning with "all" to "not all": the idea is to demonstrate knowledge of the logical transformations we automatically apply (or should automatically apply) when we deny a complex statement).

   (a) If it rains, I will get wet.

   (b) All isosceles triangles are equilateral.

   (c) P lies on line L and P lies on line M.

5. Four interpretations of incidence geometry: we give four interpretations of the meanings of "point", "line" (the interpretation of "lies on" should be obvious in each case). One of them is a model of incidence geometry: identify it. For each of the others, say which axioms it satisfies and which axioms it does not satisfy, and give a briefly stated reason why it fails to satisfy each axiom that fails.

   (a) The points are the four corners of a rectangle and the lines are the sides of the rectangle.

   (b) The points are the four corners of a rectangle, and the lines are the sides and diagonals of the rectangle.

(c) The points are the numbers 1,2,3,4 and the lines are the sets $\{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}$ (all the subsets with three elements).

(d) The points are the numbers 1-5 and the lines are as shown in the diagram.

6. In the pictured model of incidence geometry, each of the three Parallel Postulates proposed for incidence geometry is false. Explain clearly why each of them fails.

7. Prove from either set of axioms (the proof is the same) that if L and M are distinct nonparallel lines, there is exactly one point P which lies on both L and M. Hint: be sure to prove both that there is at least one such point and that there is no more than one such point.

8. Prove from the section 3.2 axioms that for any points P, Q, $d(P, Q) = d(Q, P)$. Hint: remember that you need to do slightly different things in the case where P and Q are the same point and the case where they are different.

9. Prove from the axioms of incidence geometry that there are at least three distinct lines.

1 Logical Rules

rules for conjunction (and): the rule of conjunction: from A, B, deduce $A \wedge B$

the rule of simplification (two versions): from $A \wedge B$, deduce A; from $A \wedge B$, deduce B.

rules for implication (if…then…): the rule of modus ponens: from A and $A \rightarrow B$, deduce B

the rule of deduction: Assume A: argue in an indented block of statements until you reach B: close the indented block and conclude $A \rightarrow B$. You will justify this by something like "deduction, lines 3-6" (the numbers of the actual lines used, of course) and not refer to those lines again (because they depend on the assumption A you are no longer using).

rules for disjunction (or): The rule of addition (two flavors): from A, deduce $A \vee B$; from B deduce $A \vee B$.

The rule of proof by cases: In order to prove a conclusion C from a given statement $A \vee B$, first assume A and argue to C in an indented block (the first case) then assume B and argue to C in an indented block, then close both blocks and conclude C, justified by something like "proof by cases, 10, 10a-12a, 10b-13b", where 10 is the number of the assumption $A \vee B$ (different in your proof of course), and the two blocks refer to the indented blocks for the arguments in the two cases. Again, you won't refer back into those two blocks when you are done with the proof by cases, because they depend on assumptions you aren't making any more.

2 Axioms and definitions for geometry

Our primitive notions in incidence geometry are *point*, *line* and *lies on* (a point P may lie on a line L).

The axioms are

IA1 For any two distinct points P, Q, there is exactly one line L such that P lies on L and Q lies on L.

IA2 For any line L, there are at least two distinct points which lie on L.

IA3 There are three points A, B, C which are distinct and which have the property that for any line L, at least one of A, B, C does not lie on L.

We define "parallel" and give the three proposed parallel postulates.

definition of parallel: We say that lines L and M are parallel if and only if there is no point P such that P lies on L and P lies on M.

Euclidean parallel postulate: For each line L and each point P not on L, there is exactly one line M such that P lies on M and L is parallel to M.

hyperbolic parallel postulate: For each line L and each point P not on L, there are at least two lines M such that P lies on M and L is parallel to M.

Euclidean parallel postulate: For each line L and each point P not on L, there is no line M such that P lies on M and L is parallel to M (i.e., there are no parallel lines).

The primitive notions we use in section 3.2 are *point*. *line* (a line is a particular kind of set of points, and we say a point lies on a line iff it is an element of the line) and *distance* (for any points P, Q, $d(P, Q)$ is a real number, the distance from P to Q). Venema writes PQ instead of $d(P, Q)$: we do not like this.

The axioms are

Existence Postulate: There are at least two points.

Incidence Postulate: For any distinct points P, Q, there is exactly one line L such that $P \in L$ and $Q \in L$. We call this line $\overleftrightarrow{PQ}$.

Ruler Postulate: For any line L, there is a function f which is a bijection from L to the set of all real numbers with the property that for all $P, Q \in L$, $d(P, Q) = |f(P) - f(Q)|$. We call such a function f a *coordinate function* for L.