Rubric for Homework 6

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1. Do project 5.3.

   $D = E$: D is the set of integers $3x$ where $x > 7$ is a positive integer; E is the set of positive integers. 1 for example belongs to E because it is a natural number but does not belong to D because it is not divisible by 3. So this is false.

   $C = G$: C is the set of $x + 7$ for x a positive integer. G is the set of integers greater than 7. Suppose y belongs to C: then $y = x + 7$ for some positive integer so $y - 7 = x \in \mathbf{N}$, so $7 < y$, that is $y > 7$ by our definition of order, so $y \in G$.

   Suppose $y \in G$. Then $y - 7 \in \mathbf{N}$. $(y - 7) + 7 = y$, so there is $x = y - 7 \in \mathbf{N}$ such that $x + 7 = y$, that is, $y \in C$. So these sets are equal.

   $D = B$ D is the set of $3x$ for x a positive integer greater than 7. B is the set of $3x + 21$ for x a positive integer.

   Suppose $z \in D$: then $z = 3x$ for $x > 7$, and $x = y + 7$ for some positive integer y. So $z = 3(y + 7) = 3y + 21$, so $z \in B$.

   Now suppose $u \in B$. Then there is $x \in \mathbf{N}$ such that $u = 3x + 21 = 3(x + 7)$. Now let $v = x + 7$: we have $v > 7$ and $u = 3v$ so $u \in D$.

   I gave credit for more informal descriptions of why the sets are the same, but this is really what I wanted.

2. Do Project 5.12

   i. If C is a subset of A and C is a subset of B, then C is a subset of $A \cup B$.

But the converse is not true: suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $C = \{1, 3\}$ is a subset of $A \cup B$ but not a subset of either A or B.

ii. The counterexample I gave to i. is also a counterexample to ii.

iii. This is valid. Suppose $C \subseteq A$ and $C \subseteq B$.

Then for any x if x is in C, it follow that it is in A and in B, so it is in $A \cap B$. So we have shown (since x was any element of C at all) that $C \subseteq A \cap B$.

Now suppose that $C \subseteq A \cap B$.

Let $x \in C$. Then $x \in A \cap B$, so $x \in A$. We have shown $C \subseteq A$.

Let $x \in C$. Then $x \in A \cap B$, so $x \in B$. We have shown $C \subseteq A$.

iv: If $C \subseteq A \cap B$ then $C \subseteq A$ and $C \subseteq B$, so $C \subseteq A$ or $C \subseteq B$.

But the converse is not true. Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$ and $C = \{2, 3\}$. $C \subseteq A$, so either $C \subseteq A$ or $C \subseteq B$, but it is not true that $C \subseteq A \cap B = \{3\}$.

3. Do project 5.16

$A - (B \cup C)$ is not in general equal to $(A - B) \cup (A - C)$: let A be $\{1, 2, 3\}$ B be $\{1, 2\}$, C be $\{2, 3\}$. $A - (B \cup C) = \{1, 2, 3\} - \{1, 2, 3\} = \emptyset$ whereas $A - B \cup A - C = \{3\} \cup \{1\} = \{1, 3\}$.

$A \cap (B - C) = (A \cap B) - (A \cap C)$ is true.

Suppose x is in $A \cap (B - C)$: then $x \in A$ and $x \in B$ and $x \notin C$.

So $x \in A$ and $x \in B$, so $x \in A \cap B$.

and $x \notin A \cap C$ because $x \notin C$ so $x \in (A \cap B) - (A \cap C)$.

Now suppose $x \in (A \cap B) - (A \cap C)$. $x \in A \cap B$, so $x \in A$ and $x \in B$. $x \notin A \cap C$, so we cannot have $x \in C$: if we did we would have $x \in A$ and $x \in C$ so $x \in A \cap C$, contradiction.

So we have $x \in A$ and $x \in B$ and $x \notin C$ so we have (by the last two statements) $x \in B - C$, so we have $x \in A \cap (B - C)$.

4. Venn diagram proof – Ill illustrate in class. Hard to produce here.

5. $\bigcup_{i=1}^{1} A_i = A_1$

$\bigcup_{i=1}^{k+1} A_i = (\bigcup_{i=1}^{k} A_i) \cup A_{k+1}$

$\bigcap_{i=1}^{1} A_i = A_1$

$\bigcap_{i=1}^{k+1} A_i = (\bigcap_{i=1}^{k} A_i) \cap A_{k+1}$

Most left out the basis parts (for 1) which are essential. Some of you left out the operation in the recursive part of the definition.

6. Do project 5.21

   i. is false: we can have something in $(A \cup C) \times (B \cup D)$ which is not in $(A \times B) \cup (C \times D)$.

   suppose $A = \{1\}, B = \{2\}, C = \{3\}, D = \{4\}$. $(1, 4)$ is in $(A \cup C) \times (B \cup D) = \{1, 3\} \times \{2, 4\}$ but not in $(A \times B) \cup (C \times D) = \{(1, 2), (3, 4)\}$

   ii. is true: if x is in $(A \times B) \cap (C \times D)$, then x is a pair (y, z) with $(y, z) \in A \times B$, so $y \in A$ and $z \in B$, and also $(y, z) \in C \times D$, so $y \in C$ and $z \in D$ so $y \in A \cap C$ and $z \in B \cap D$, so $x = (y, z) \in (A \cap C) \times (B \cap D)$.

   If $x \in (A \cap C) \times (B \cap D)$ then x is a pair (y, z) with $y \in A \cap C$ and $z \in B \cap D$, so $y \in A$ and $z \in B$ so $x = (y, z) \in A \times B$ and $y \in C$ and $z \in D$ so $x = (y, z) \in C \times D$, so $x \in (A \times B) \cap (C \times D)$.

7. If $A \times B = B \times A$ then $A = \emptyset$ or $B = \emptyset$ or $A = B$.

   Let A, B be arbitrarily chosen sets such that $A \times B = B \times A$:

   If $A = \emptyset$ we are done, so suppose $A \neq \emptyset$ so there is some $a \in A$.

   If $B = \emptyset$ we are done, so suppose $B \neq \emptyset$ so there is some $b \in B$.

   Now we need to prove $A = B$. Suppose $x \in A$. Then $(x, b) \in A \times B$, so $(x, b) \in B \times A$, so $x \in B$. Suppose $y \in B$. Then $(a, y) \in A \times B$, so $(a, y) \in B \times A$, so $y \in A$.