Math 414/514 Spring 2024 Sample Test I

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This document should have the look and feel of your actual exam. There are 8 questions, organized into pairs.

In each pair, you get 70 percent credit for the problem you do better on and 30 percent for the other.

Some reference information (axioms, theorems and definitions for reference) may be supplied on the actual test paper; for the practice test you can consult your book, but the actual test will be closed book, closed notes.

Problems on this exam should not be surprises: they should be or be very similar to things I have done in class or that you have been assigned in homework. Of course, they may be things that you chose not to do in homework...

The problems I choose for this practice test should give you a general idea of my thinking about the actual exam. Some of the questions on the actual exam may be exactly the same: don't assume that because I asked something on the practice exam I won't ask that exact question on the exam itself.

1 First Pair

abstract stuff about sets

1. Prove that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

   Recall that our strategy for showing that two sets are equal is to postulate an arbitrary element of the first, and show that it must belong to the second, then postulate an arbitrary element of the second and show that it belongs to the first.

Suppose $x \in A - (B \cup C)$.

It follows that $x \in A$ and $x \notin B \cup C$

It follows that $x \in A$ and $x \notin B$ and $x \notin C$

so ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$)

so $x \in A - B$ and $x \in A - C$

and so $x \in (A - B) \cap (A - C)$

Now suppose $x \in (A - B) \cap (A - C)$

This implies $x \in A - B$ and $x \in A - C$

and so ($x \in A$ and $x \notin B$) and ($x \in A \wedge x \notin C$)

and, so $x \in A$ and $x \notin B$ and $x \notin C$

so $x \in A$ and not ($x \in B$ or $x \in C$)

so $x \in A$ and not $x \in B \cup C$

so $x \in A - (B \cup C)$

We have shown that each of these sets is a subset of the other, so they are equal

2. Prove that for any function f and sets C and D, $f[C \cup D] = f[C] \cup f[D]$

Recall that $f[C]$ is defined as $\{f(c) : c \in C\}$. I prefer brackets to avoid even the possiblity of confusion between values of a function and elementwise images of a set under a function.

Suppose $x \in f[C \cup D]$.

This means for some $y \in C \cup D$, $x = f(y)$

so either $y \in C$ and $x = f(y)$ or $y \in D$ and $x = f(y)$

so either $y \in f[C]$ or $y \in f[D]$

so $y \in f[C] \cup f[D]$

Now suppose $x \in f[C] \cup f[D]$

then $x \in f[C]$ or $x \in f[D]$

so there is y s.t. $x = f(y)$ and $y \in C$ [case 1]

or y s.t. $x = f(y)$ and $y \in D$ [case 2]

in case 1, $y \in C \Rightarrow y \in C \cup D$ so $x \in f[C \cup D]$

in case 2, $y \in D \Rightarrow y \in C \cup D$ so $x \in f[C \cup D]$

So we have shown each of these sets is a subset of the other, so they are equal

2 Second Pair

mathematical induction

3. Prove by mathematical induction that $\sum_{i=0}^{n} ar^i = \frac{a-ar^{n+1}}{1-r}$

Basis $(n=0)$: $\sum_{i=0}^{0} ar^i = ar^0 = a \stackrel{?}{=} \frac{a-ar^1}{1-r} = \frac{a(1-r)}{1-r} = a$ ✓

Induction step:

Let $k \in \mathbb{Z}^+$ be chosen arbitrarily

and assume (ind hyp) that $\sum_{i=0}^{k} ar^i = \frac{a-ar^{k+1}}{1-r}$

Goal: show that $\sum_{i=0}^{k+1} ar^i = \frac{a-ar^{k+2}}{1-r}$

Proof: $\sum_{i=0}^{k+1} ar^i = \left(\sum_{i=0}^{k} ar^i\right) + ar^{k+1} =$ ind hyp

$\frac{a-ar^{k+1}}{1-r} + ar^{k+1} =$

$\frac{(a-ar^{k+1}) + (ar^{k+1}(1-r))}{1-r} =$

$\frac{a-ar^{k+1}+ar^{k+1}-ar^{k+2}}{1-r} = \frac{a-ar^{k+2}}{1-r}$ ✓

4. Prove by mathematical induction that a set of natural numbers which is nonempty must have a smallest element: hint, prove by strong induction that a set of natural numbers without a smallest element must be empty.

Suppose A is a set of natural numbers with no smallest element.

We prove by induction on n that for any n, for all $k \leq n$, $k \notin A$ [thus shows that A is empty]

Basis: for all $k \leq 1$, $k \notin A$, because $k \leq 1 \rightarrow k = 1$ and if $1 \in A$, 1 would be the smallest element of A $\times$

Induction Step: Let m be chosen arbitrarily. Suppose that for all $k \leq m$, $k \notin A$ [ind hyp]

Goal: show that for all $k \leq m+1$, $k \notin A$.

Proof: If $k \leq m+1$, then either $k \leq m$ (in which case $k \notin A$) or $k = m+1$ (which is impossible because $m+1 \in A$ and $(\forall k \leq m : k \notin A)$ together imply that m is the smallest element of A, contrary to hypothesis.

Now we have shown that for all $n \in \mathbb{N}$, for all $k \leq n$, $k \notin A$ from which it follows that $n \leq n \notin A$.

So a set with no smallest element is empty

So a nonempty set has a smallest element.

3 Third Pair

5. Prove from the axioms given in the book that if $0 < x < y$ then $x^2 < y^2$. You may be a little informal about equational algebra but your order reasoning must be directly from the axioms he actually gives for order (definitions 1.1.1 and 1.1.7).

If $0 < x < y$ then (since $x > 0$)

$$0x < xx < xy$$

so $x^2 < xy$

and also (since $y > 0$)

$$0y < xy < yy$$

so $xy < y^2$

so $x^2 < xy$ and $xy < y^2 \rightarrow x^2 < y^2$ by transitivity.

6. Let $A \subseteq B$ be nonempty sets of real numbers. Suppose that B is bounded above and below.

   Argue that A must be bounded above and below.

   Show that $\texttt{inf}(B) \leq \texttt{inf}(A) \leq \texttt{sup}(A) \leq \texttt{sup}(B)$

   This has to be proved from basic properties of order and sets and the definitions of inf and sup.

$\inf(B)$ is a lower bound of B.
for any $a \in A$, $a \in B$ because $A \subseteq B$
thus $a \leq \inf(B)$ so $\inf(B)$ is a lower bound for A
so $\inf(B) \leq \inf(A)$ because $\inf(A)$ is the greatest lower bound of A.

A is nonempty so there is $a \in A$.
$\inf(A) \leq a$ because $\inf(A)$ is a lower bound for A
$a \leq \sup(A)$ because $\sup(A)$ is an upper bound for A
thus $\inf(A) \leq \sup(A)$ by transitivity.

$\sup(B)$ is an upper bound for B. if $a \in A$ then $a \in B$
and so $a \leq \sup(B)$ so $\sup(B)$ is an upper bound
for A. Thus $\sup(A) \leq \sup(B)$ because $\sup(B)$
is the least upper bound of A.

All three inequalities have been verified.

4 Fourth Pair

7. Show that $|x - y| < \epsilon$ if and only if $x - \epsilon < y < x + \epsilon$

   This should not need anything but the definition of absolute value and the most basic properties of order. You need to prove an implication in one direction then the other. You might have a use for proof by cases.

Suppose $|x-y| < \varepsilon$.

It follows that ① $x - y < \varepsilon$ (either $x-y = |x-y| < \varepsilon$ or $x-y \le 0 < \varepsilon$)

and ② $y-x < \varepsilon$ (either $y-x = |x-y| < \varepsilon$ ~~for~~ $y-x \le 0 < \varepsilon$)

So by ① $x - \varepsilon < y$

and by ② $y < \varepsilon + x$ so $x-\varepsilon < y < x+\varepsilon$

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if $x-\varepsilon < y < x+\varepsilon$

then $x-\varepsilon < y$ so $x-y < \varepsilon$

and $y < x+\varepsilon$ so $y-x < \varepsilon$

so $|x-y| < \varepsilon$ because it is equal to one of $x-y$ and $y-x$.

8. Prove that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions that $\texttt{inf}_{x\in D}[f(x) + g(x)] \geq \texttt{inf}_{x\in D} f(x) + \texttt{inf}_{x\in D} g(x)$

Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded functions.

Suppose m is a lower bound for $f[D]$: $m \leq f(x)$ for all $x \in D$

and M is a lower bound for $g[D]$: $M \leq g(x)$ for all $x \in D$

These exist because f and g are bounded.

Then for any x, $f(x) + g(x) \geq m + M$ (obvious)

so $m + M$ is a lower bound for $\{ f(x) + g(x) : x \in D \}$

and so $m + M \leq \{ f(x) + g(x) : x \in D \} = \inf_{x \in D} [f(x) + g(x)]$

because the latter is the greatest lower bound.

This works for any lower bounds m and M of

$f[D]$ and $g[D]$ respectively, so in particular

for $m = \inf_{x \in D} [f(D)]$ and $M = \inf_{x \in D} [g(D)]$

so $\inf_{x \in D} [f(D)] + \inf_{x \in D} [g(D)] \leq \inf_{x \in D} [f(x) + g(x)]$.