Riemann integrals of continuous functions exist

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In this snippet of notes, we will work toward proving that $\int_a^b f$ exists for any f continuous on $[a, b]$ along with prerequisite and related results

We begin by proving that if a function f is nondecreasing on $[a, b]$, $\int_b^a f$ exists.

Suppose that f is nondecreasing on $[a, b]$. (this means that for any $x, y \in [a, b]$ with $x < y$, $f(x) \leq f(y)$).

We will show that for any $\epsilon > 0$, there is a partition P such that $U(f, P) - L(f, P) < \epsilon$. It is a homework exercise in your current assignment that this is sufficient to establish that $\int_a^b f$ exists.

Let P be a partition $\{x_i\}_{0 \leq i \leq n}$ of $[a, b]$ such that there is a constant $\delta < \frac{\epsilon}{f(b)-f(a)}$ such that $x_i - x_{i-1} = \delta$ for each i for which this is defined: P determines a subdivision of $[a, b]$ into closed intervals all of the same length strictly less than ϵ.

Now

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} (x_i - x_{i-1})(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f)$$

$$= \sum_{i=1}^{n} \delta(f(x_i) - f(x_{i-1}))$$

[because the length of each interval in P is δ and $\sup_{[x_{i-1}]} f = f(x_i)$ and $\inf_{[x_{i-1}]} f = f(x_{i-1})$ because f is nondecreasing]

$$= \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta(f(b) - f(a)) < \epsilon$$

[the second equation holds because $\sum_{i=1}^{n}(f(x_i) - f(x_{i-1}))$ is a telescoping sum]

And this completes the proof that $\int_a^b f$ exists, mod the homework assignment mentioned.

I strongly recommend and may assign proving the same result for nonincreasing functions f.

The proof that $\int_a^b f$ exists if f is continuous on $[a, b]$ relies on the theorem that a function f continuous on a closed interval $[a, b]$ is uniformly continuous on $[a, b]$. We first explain what this statement means, then use it to prove that $\int_a^b f$ exists, then perhaps prove the prerequisite theorem.

That f is continuous on a set A means that for each $x \in A$, there is an $\epsilon > 0$ such that for any $y \in A$, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$. This follows from the usual definitions of limits and continuity which you should have known since undergraduate real anaysis if not since Calculus I.

That f is uniformly continuous on a set A means that for each $\epsilon > 0$, there is $\delta > 0$ such that for any $x, y \in A$, if $|x-y| < \delta$ then $|f(x)-f(y)| < \epsilon$.

The second assertion is stronger: it says that the tolerance of error δ you need such that if y is that close to x, $f(y)$ will be within ϵ of $f(x)$ does not depend on x: the same tolerance works everywhere in the set A.

The prerequisite theorem is "If f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$". For the moment we assume this and proceed to prove that $\int_a^b f$ exists.

Again, we will show that for any $\epsilon > 0$, there is a partition P such that $U(f, P) - L(f, P) < \epsilon$. It is a homework exercise in your current assignment that this is sufficient to establish that $\int_a^b f$ exists.

Choose $\epsilon > 0$ arbitrarily

Choose δ such that for any $x, y \in [a, b]$, if $|x-y| < \delta$, $|f(x)-f(y)| < \frac{\epsilon}{2(b-a)}$.

Let P be the partition of $[a, b]$ determined by $\{x_i\}_{0 \leq i \leq n}$ subdividing the interval into closed intervals all with equal length δ.

Now

$$U(f, P) - L(f, P) = \sum_{i=1}^{n}(x_i - x_{i-1})(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f)$$

$$= \sum_{i=1}^{n} \delta(\sup_{[x_{i-1}, x_i]} f - \inf_{[x_{i-1}, x_i]} f)$$

$$\leq \sum_{i=1}^{n} \delta \frac{\epsilon}{2(b-a)}$$

because $\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f \leq \frac{\epsilon}{2(b-a)}$ since the length of the interval is δ (any two points in the interval except x_i and x_{i-1} are at distance $< \delta$ and have values of f differing by less than $\frac{\epsilon}{2(b-a)}$; x_i and x_{i-1} are at distance exactly δ but continuity of f lets us see that the values of f at the endpoints might differ exactly by $\frac{\epsilon}{2(b-a)}$ but no more: so the difference between the largest and smallest value of the function on the interval is bounded above by $\frac{\epsilon}{2(b-a)}$ and $\sup_{[x_{i-1},x_i]} f - \inf_{[x_{i-1},x_i]} f$ is no greater than $\frac{\epsilon}{2(b-a)}$.

$$= \sum_{i=1}^{n} \frac{b-a}{n} \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{2} < \epsilon$$

Note that $\delta = \frac{b-a}{n}$.

I'll lecture the proof that a continuous function on a closed interval is uniformly continuous on Sept 6; notes on it will be added here eventually.

I lectured a series of results to get to the result on uniform continuity. The quality of these notes may suffer from the fact that I am ill as I write; please feel free to make any comments or ask any questions that you think are needed.

Monotone convergence theorem: Fir any sequence $\{x_i\}$ which is either nondecreasing ($i \leq j \rightarrow x_i \leq x_j$) and bounded above or nonincreasing ($i \leq j \rightarrow x_i \geq x_j$) and bounded below, the limit $\lim_{i\rightarrow\infty} x_i$ exists.

This theorem should be familiar to you since the second calculus course, and you should have seen a proof in your first real analysis course, but I support varying levels of preparation: I review it.

Proof: We cover only the case of $\{x_i\}$ nondecreasing: the proof in the other case is very similar.

Suppose that $\{x_i\}$ is a nondecreasing sequence and bounded above. This means there is b such that for every i, $x_i \leq b$. This implies that the set $\{x_i : i \in \mathbb{N}\}$ is nonempty (it contains x_1) and bounded above by b. This means that it has a least upper bound L.

We claim that $\lim_{x\rightarrow\infty} x_i = L$, that is, $(\forall \epsilon > 0 : (\exists N \in \mathbb{N} : (\forall i \in \mathbb{N} : i \geq N \rightarrow |x_i - L| < \epsilon)))$.

Choose $\epsilon > 0$. $L - \epsilon$ is not an upper bound of $\{x_i : i \in \mathbb{N}\}$, so there is N such that $x_N > L - \epsilon$. Now for any $i > N$, we have $x_N \leq x_i$ (nonincreasing) so $L - \epsilon < x_N \leq x_i \leq L < L + \epsilon$, so $|x_i - L| < \epsilon$, which is what we need.

Bolzano-Weierstrass Theorem: For any $a < b$ real numbers, and any sequence $\{x_i\}$ of elements of $[a, b]$, there is a convergent subsequence of $\{x_i\}$, that is, there is a strictly increasing sequence $\{s_i\}$ of natural numbers such that the sequence $y_i = x_{s_i}$ converges.

Proof: We define sequences $\{A_i\}$ and $\{B_i\}$ recursively.

$A_0 = a$ and $B_0 = b$.

Suppose A_i and B_i have been defined, and there are infinitely many j such that $A_i \leq x_j \leq B_i$ [notice that this is true for $i = 0$].

If there are infinitely many j such that $A_i \leq \frac{A_i+B_i}{2}$, we define A_{i+1} as A_i and B_{i+1} as $\frac{A_i+B_i}{2}$.

Otherwise, there will be infinitely many j such that $\frac{A_i+B_i}{2} \leq B_i$, and we define A_{i+1} as $\frac{A_i+B_i}{2}$ and B_{i+1} as B_i.

Notice that we enforce the hypothesis of the recursion on both cases, so we will be able to define A_i and B_i for each $i \in \mathbb{N}$.

More facts can be seen by induction on i: $A_i \leq A_{i+1}$ and $B_i \geq B_{i+1}$ will always hold, and $B_i - A_i = \frac{b-a}{2^i}$.

By the monotone convergence theorem, $\{A_i\}$ converges to a limit L (nondecreasing and bounded above by b) and $\{B_i\}$ converges to a limit M (nonincreasing and bounded below by a). By the subtraction property of limits of sequences, $M - L = \lim_{i\to\infty} B_i - A_i = \lim_{i\to\infty} \frac{b-a}{2^i} = 0$, so $L = M$.

We define s_0 as 0 and define s_{i+1} as the smallest $j > s_i$ such that $x_j \in [A_j, B_j]$: infinitely many values of j make the last statement true, so we can find one bigger than s_i.

The sequence $\{x_{s_i}\}_{i\in\mathbb{N}}$ is a subsequence of $\{x_i\}$ and it converges to L. because for any i, $A_{s_i} \leq x_{s_i} \leq B_{s_i}$, and as $i \to \infty$, $A_{s_i} \to L$ and $B_{s_i} \to L$, so $x_{s_i} \to L$ by the very familiar Squeeze Theorem.

Definition: A function f is continuous on a set A iff $(\forall x \in A : \forall \epsilon > 0 : \exists \delta > 0 : \forall y \in A : |x - y| < \delta \to |f(x) - f(y)| < \epsilon)$.

A function f is uniformly continuous on a set A iff $(\forall\epsilon > 0 : \forall\delta > 0 : \forall x, y \in A : |x - y| < \delta \to |f(x) - f(y)| < \epsilon)$.

Notice that uniform continuity is a stronger condition: it allows you to select your δ given ϵ independently of where you are in the set A.

Uniform Continuity Theorem: If $a < b$ are real numbers, and f is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.

Proof: Suppose that $a < b$ and f is continuous on $a, b]$.

Suppose for the sake of a contradiction that f is not uniformly continuous on $[a, b]$.

Then there is an $\epsilon > 0$ such that for each δ we can choose $x, y \in [a, b]$ such that $|x - y| \leq \delta$ but $|f(x) - f(y)| \geq \epsilon$.

In particular, for each $k \in \mathbb{N}$, we can choose $x_k, y_k \in [a, b]$ such that $|x_k - y_k| < \frac{1}{k}$ and $|f(x_k) - f(y_k)| \geq \epsilon$.

By the Bolzano Weierstrass Theorem, there is a sequence $U_k = x_{s_k}$ (s strictly increasing) which has a limit L.

Define V_k as y_{s_k}. By the Bolzano Weirstrass theorem there is a sequence $Y_k = V_{t_k}$ (t strictly increasing) such that Y_k has a limit M. Define X_k as U_{t_k}: being a subsequence of U, it has the same limit L that U has.

Now $L - M = \lim_{k\to\infty}(X_k - Y_k) = 0$, because $|X_k - Y_k| = |x_{s_{t_k}} - y_{s_{t_k}}| < \frac{1}{s_{t_k}}$, which approaches 0 as k goes to infinity. So $L = M$.

Because f is continuous, $\lim_{i\to\infty} f(X_i) = \lim_{i\to\infty} f(Y_i) = f(L)$.

This implies that $\lim_{i\to\infty}(f(X_i) - f(Y_i)) = f(L) - f(L) = 0$.

But this is impossible, because $|f(X_i) - f(Y_i)| = |f(x_{s_{t_i}}) - f(x_{s_{t_i}})| \geq \epsilon$ for every i.

So our assumption that f was not uniformly continuous must be false.