Math 311 Test II, Spring 2019

Dr. Holmes

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The test will begin at 1:30 pm and end at 2:50 pm. You are allowed your test paper and writing instrument. There is a list of axioms and definitions attached to your paper (which you are welcome to separate from your paper so that you can use it while you work).

The problems in the test are presented in pairs (1 and 2, 3 and 4, 5 and 6, 7 and 8). In each pair, the problem you do better on (individually) will count for 70 percent of the value of the pair of problems and the problem you do worse on will count for 30 percent. It is possible that this percentage will be adjusted in your favor depending on class performance. It is also possible that the weight of the pair of problems you do worst on will be reduced; this depends on class performance as well.

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1. Prove that distinct lines L and M which are not parallel intersect at exactly one point. This proof would be the same in Venema's approach and in the alternative approach: this is a hint about what you need to prove it. (And of course this should not be unfamiliar!)

2. Prove that for any lines L, M, the intersection of the two lines will contain either 0, 1 or ∞ points. This is to be proved using Venema's axioms. You are allowed to use the theorem stated as problem 1 in your proof.

3. Prove that For any pair of distinct points P, Q there is a point M such that $P * M * Q$ and $d(P, M) = d(M, Q)$. Make sure you mention each axiom that is used in your proof. I do not require you to prove that there is only such such point; if you do prove this you may get some additional credit (don't attempt extra credit opportunities until you have done basic work on the entire exam!) This uses Venema's axioms, and makes essential use of the Ruler Postulate!

4. Prove that for any point P on a line L and real number d, there are exactly two points Q, R on the line L such that $d(P,Q) = d(P,R) = d$. This uses Venema's axioms and makes essential use of the Ruler Postulate.

5. Prove the Ray Theorem: if A is a point on line L and P is a point not on line L, and Q is a point on $\overrightarrow{AP}$ other than A, then P and Q are on the same side of A. This can be proved using either approach: it involves the definition of a ray and basic properties of betweenness, combined with the Plane Separation Axiom.

6. Prove Pasch's Axiom: If A, B, C are noncollinear points and none of them are on the line L , and L meets $\overline{AB}$, then L also meets either $\overline{AC}$ or $\overline{BC}$. This proof could be stated in terms of either approach and makes essential use of the Plane Separation Axiom (in fact, it uses nothing else, or hardly anything else). For extra credit, tell me why L cannot meet both $\overline{AC}$ and $\overline{BC}$.

7. Prove the Segment Subtraction theorem using Venema's approach: if $A * B * C$ and $A' * B' * C'$ and $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\overline{BC} \cong \overline{B'C'}$. Hint: all the concepts used are defined in terms of distances, and if you expand out the definitions the problem reduces to algebra.

8. Prove the Segment Subtraction theorem using the alternative approach: if $A * B * C$ and $A' * B' * C'$ and $\overline{AC} \cong \overline{A'C'}$ and $\overline{AB} \cong \overline{A'B'}$ then $\overline{BC} \cong \overline{B'C'}$. Hint: you need to use the point construction postulate and the segment addition axiom. The argument involves constructing a point which really ought to be C' in such a way that you can apply segment addition, then using the uniqueness part of point construction to show that the point really is C'.

1 Venema's Axioms, with some definitions and theorems

The official theory has as its primitive notions points, lines (which are sets of points) and the notion of distance: $d(P, Q)$ is a real number for points P, Q (Venema himself writes PQ instead of $d(P, Q)$), and half-planes, which are special sets of points described in the Plane Separation Axiom.

Existence Axiom: There are at least two points.

Incidence Axiom: For any pair of distinct points P, Q, there is exactly one line L such that $P \in L$ and $Q \in L$. This line is called $\overleftrightarrow{PQ}$.

Ruler Postulate: For each lne L, there is a function f from L to $\mathbb{R}$ which is one-to-one and onto (a bijection) and satisfies $|f(P) - f(Q)| = d(P, Q)$ for any $P, Q \in L$. Such a function f is called a coordinate function for the line L.

Plane Separation Axiom: With each line L we can associate two sets H_1 and H_2 (the sides of the line) such that $H_1 \cap H_2 = H_1 \cap L = H_2 \cap L = \emptyset$ and $H_1 \cup H_2 \cup L$ is the entire plane (the three sets are a partition of the plane) **and** if P, Q are both in H_1 then $\overline{PQ} \subseteq H_1$ and if P, Q are both in H_2 then $\overline{PQ} \subseteq H_2$ [in these cases we say that P and Q are on the same side of the line] **and** if P is in H_1 and Q is in H_2, then $\overline{PQ}$ meets L [in this case we say that P and Q (or Q and P) are on opposite sides of the line]. The sets H_1 and H_2 are called *half-planes*.

We give some definitions.

lies on (both approaches): A point P lies on a line L iff $P \in L$.

parallel (both approaches): Lines L, M are parallel iff there is no point which lies both on L and on M.

collinear (both approaches): Points A, B, C are collinear iff they are distinct and there is a line L such that A, B, C all lie on L.

betweenness: $A * B * C$ (B is between A and C) iff A, B, C are collinear (and so distinct) and $d(A, B) + d(B, C) = d(A, C)$.

segment: $\overline{AB}$ is defined as $\{P : P = A \vee P = B \vee A * P * B\}$, where A, B are distinct points.

ray (both approaches): $\overrightarrow{AB}$ is defined as $\{P : P = A \vee P = B \vee A * P * B \vee A * B * P\}$, where A, B are distinct points.

congruence: $\overline{AB} \cong \overline{CD}$ is defined as holding iff $d(A, B) = d(C, D)$.

2 Alternative Approach Axioms, with some definitions and theorems

In the primitive notions we omit distance and add the primitive notions of segment (a segment is a set $\overline{AB}$ determined by a pair of distinct points A, B) and the notion of congruence (a relation $\overline{AB} \cong \overline{CD}$ between segments).

The existence, incidence and plane separation axioms are just as in Venema.

These are the additional axioms as listed initially.

basic segment axiom: $\{A, B\} \subseteq \overline{AB} = \overline{BA} \subseteq \overleftrightarrow{AB}$.

basic congruence axiom: Congruence is an equivalence relation (reflexive, symmetric, transitive).

definition of betweenness: $A * B * C$ holds iff A, B, C are distinct and $B \in \overline{AC}$. Notice that this implies that $A * B * C \leftrightarrow C * B * A$.

trichotomy: For any collinear A, B, C exactly one of $A*B*C, A*C*B, C*A*B$ holds.

segment partition: If $A*B*C$, then $\overline{AB} \cap \overline{BC} = \{B\}$, and $\overline{AB} \cup \overline{BC} = \overline{AC}$.

point construction: For any points A, B and segment $\overline{CD}$, there is a unique point E such that $E * A * B$ and $\overline{AE} \cong \overline{CD}$.

second point construction lemma (proved in the notes as a theorem from the othe For any points A, B and segment $\overline{CD}$, there is a unique point E in $\overrightarrow{AB}$ such that $\overline{AE} \cong \overline{CD}$. This is Venema's Point Construction Postulate (which he proves as a theorem). I use the "backwards" form as my axiom just because it is slightly easier to state; I could also use this as the axiom and prove the other one.

segment addition: If $A * B * C$ and $A' * B' * C'$ and $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

This may look like a lot of axioms, but remember that the Ruler Postulate allows Venema to import all the axioms for real numbers into his system!

Some lemmas.

first ray lemma: If $A * B * C$ or $A * C * B$ then $\overrightarrow{AC} = \overrightarrow{AB}$.

second ray lemma: If $A * B * C$, and D is on the same line, exactly one of A, C is on $\overrightarrow{BD}$.