Solution for study for Fall 2024 Test II

Math 189, Fall 2022, Test I

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The test will begin at 130 pm and officially end at 245 pm.

You are allowed your test paper, your writing instrument, and a non-graphing calculator. You will need the calculator; make sure you have a non-graphing scientific calculator for the exam.

There is no use of your book or notes allowed, nor is there a sheet of formulas. If I think a specific fomula should be supplied, I will supply it with the problem. For the most part, you are expected to know the formulas you are working with.

1. set questions

   (a) Fill in each sentence with $\in$ or $\subseteq$ in such a way as to make it true. If both work, say both, if neither work, say neither.

      i. $\{1,2\}$___$\{1,\{1,2\},2\}$

      ii. $\emptyset$___$\{1,2,3\}$

      iii. 3___$\mathbb{N}$

      iv. 1___$\{\{1,2\}\}$

   not on this test

   (b) Let A be the set $\{5,6,7,8,9,10\}$

      Give a definition of A in the form $\{x \in \mathbb{N} : \ldots\}$

      Give the set $\{x \in A : 3x \geq 20\}$ in list notation.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$.

   (a) How many functions from A to B are there? 3^4

   (b) Give one of these functions as a set of ordered pairs, using list notation. would ask for this

   (c) How many injective functions from A to B are there? none

   (d) How many injective functions from B to A are there? $4 \cdot 3 \cdot 2$

   (e) How many nondecreasing functions from A to B are there (hint: this is a stars and bars question) ← this is why I might ask this

   $n = 3 \quad k = 4$

   $$\binom{3+4-1}{4} = \binom{3+4-1}{2} = 15$$

3. (a) Lauren has three blouses, seven skirts and ten one-piece dresses she can wear. How many outfits can she make?

hot on [illegible]
[illegible]

(b) Suppose in addition that she has three patterns of striped socks she can wear with a short skirt. Three of her seven skirts are short, and five of her one-piece dresses have short skirts. How many outfits does she have now (with a long skirt she will always wear plain white socks).

4. In a sophomore class of 23 students at a small school, every student takes at least one of English, Math, French. 10 take English, 18 take Math and 13 take French. 7 take English and Math, 10 take Math and French, and 5 take English and French. How many brave students are taking all three subjects?

not on this test

5. combinations and permutations

(a) A group of 12 children is to divide into three teams of four, the Lions, the Tigers and the Bears. In addition, each team needs a captain. How many ways are there to do this?

$$\binom{12}{4}\binom{8}{4}\binom{4}{4} = 34650$$

(b) How would the answer differ if I just asked how many ways the 12 children can divide themselves into three teams of four, each with a captain? Explain.

$$\frac{34650}{3!} = 5775$$

6. Each of these four questions about k choices from n alternatives is answered in a different way, because of different combinations of conditions: in some we are allowed to repeat choices, and in some we are not; in some the order in which we make choice matters and in others it does not. Briefly answer each question, and **include calculations and brief explanation of the conditions which apply.**

   (a) A committee with 8 members wants to choose a three member executive committee. In how many ways can this be done?

   order doesn't matter
   repetition not allowed
   $\binom{8}{3}$

   (b) 8 scrabble tiles with different letters on them are on the table in front of you. You idly make a 5 letter "word" using these tiles (no requirement that it be in the dictionary or even possible to pronounce). How many ways can you do this?

   order matters
   repetitions are not allowed
   $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$

(c) How many four letter "words" (they don't need to be in the dictionary or even pronouncable) are possible for you to make, assuming that that you have at least 4 of each letter in your bag of letter tiles?

26^4

order matters
repetitions are allowed

(d) You go to the florist and order a bunch of a dozen roses. There are pink, white, red and exotic genetically engineered blue roses. How many bunches of a dozen are possible?

order does not matter
repetitions are allowed

$$\binom{12+4-1}{3} = \binom{12+4-1}{12}$$

7. The sequence of Lucas numbers is defined by $L_0 = 1, L_1, = 3, L_{n+2} = L_n + L_{n+1}$

Determine the Lucas numbers up to L_8.

A sequence S_n is defined by $S_n = \sum_{i=0}^{n} L_i$. Compute the S_i's up to S_8.

	0	1	2	3	4	5	6	7	8
L_n	1	3	4	7	11	18	29	47	76
$\sum_{i=0}^{n} L_i$	1	4	8	15	26	44	73	120	196

assuming I can add!

8. Two sequences are given, one arithmetic and one geometric. The indexing in both cases starts with 0.

   Write a formula for the nth term of each sequence.

   Compute the sum of the first 15 terms of each sequence, using a formula rather than computing the first 15 terms and adding them up.

   2, 6, 10, 14... is the first sequence.

arithmetic

$2+4n$

notice, 15th term has n = 14!

$$\left(\frac{2+(2+4\cdot 14)}{2}\right)15$$

$$= \left(\frac{2+58}{2}\right)15 = (30)(15) = 450$$

   2, 6, 18, 54... is the second sequence.

$a_n = 2\cdot 3^n$

really 15 here, n+1

$$\frac{2\cdot 3^{15}-2}{3-1}$$

9. polynomial fitting or exponential recurrence relations (choose). Indexing in the sequences in both parts starts at 0.

   Do one of the following. If you do both you can earn extra credit.

   (a) 0,1,5,14,30,55...

   Use the method of finite differences to determine its degree.

   Find a formula for the polynomial.

   Give a description $a_n = \sum_{i=0}^{n}(\text{something})$ for the nth term of this sequence when $n > 0$.

$a_n = \sum_{i=0}^{n} (i+1)^2$ no as n can't be 0!

2 2

3 5 7

1 4 9 16 25

0 1 5 14 30 55

third degree

$a_n = \binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}$

$= \frac{6n}{6} + \frac{9n^2 - 9n}{6} + \frac{2(n)(n-1)(n-2)}{6}$ ($2n^3 - 6n^2 + 4n$)

$a_n = \frac{2n^3 + 3n^2 + n}{6}$

(b) Determine a closed form for the sequence defined by $a_0 = 2, a_1 = 6, a_{n+2} = 6a_{n+1} - 8a_n$. Show work developing this from a characteristic polynomial.

$r^2 = 6r - 8$

$r^2 - 6r + 8 = 0$

$(r-4)(r-2)$

$a_n = 4^n + 2^n$

$a_n = A4^n + B2^n$

$a_0 = A4^0 + B2^0 = A + B = 2$

$a_1 = A4^1 + B2^1 = 2A + 4B = 6$

$2A + 2B = 4$

$2B = 2$

$B = 1$

$A = 1$

10. Do both parts. Proofs by mathematical induction are expected. The part on which you do better will count 70 percent and the part you do worse on 30 percent.

    In both parts, be sure to clearly identify the basis step, the induction hypothesis, the induction goal, and show where the induction hypothesis is used in the proof of the induction goal.

    (a) Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. State the theorem using summation notation, then prove it by mathematical induction.

These I will do in class only.

(b) Prove using mathematical induction that $n^3 + 5n$ is divisible by 3 for each natural number n.

These I will do in class only.