Induction Homework (Homework 11) Math 189 Fall 2023

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Because I vagued out about producing this, the due date is one week from today, Wednesday November 1. The extended time is provided not so that you can put it off but so that you have plenty of time to ask questions. Please use lots of paper: these are proofs in English with supporting calculations, they take up space.

1. Prove by mathematical induction that for every $n \geq 0$, $n^3 + 2n$ is divisible by 3.

2. Prove by mathematical induction that the sum of the first n cubes is $\left(\frac{n(n+1)}{2}\right)^2$. Use properties of summation notation that I have talked about.

3. For all large enough numbers, $n^5 < 2^n$. Find the largest n for which this is not true, and then prove by induction that this inequality holds for all larger numbers.

4. Prove by strong induction that every positive integer can be expressed as a sum of distinct Fibonacci numbers. The proof is quite similar to the similar result for powers of 2 which is an exercise in Levin with solution.

5. Do problem 23, section 2.5 in Levin.

6. Do problem 29, section 2.5 in Levin.

extra page

problem 1 $\quad 0^3+2\cdot 0$ div by 3 basis ✓

if k^3+2k is divisible by 3 (ind hyp)

$$(k+1)^3+2(k+1) = k^3+3k^2+3k+1+2k+2 =$$

$$\underbrace{k^3+2k}_{\text{ind hyp, divisible by 3}} + \underbrace{3k^2+3k+3}_{\text{divisible by 3}}$$

which is divisible by 3 because it is the sum of two things divisible by 3.

(3) $n=23$ is where it starts being true

we check that $2^{23} > 23^5$ with calculator

$8388608 > 6436343$

suppose $n \geq 23$ Basis already stated and established.

Assume for $k \geq 23$ that $2^k > k^5$.

Goal: $2^{k+1} > (k+1)^5$

$$(k+1)^5 = k^5+5n^4+10n^3+10n^2+5n+1$$

$$= k^5+n^4\underbrace{\left(5+\frac{10}{n}+\frac{10}{n^2}+\frac{5}{n^3}+\frac{1}{n^4}\right)}_{\text{this is } <6 \text{ for } n \geq 23}$$

$$< k^5+6n^4 < 2k^5 \underset{\text{(ind hyp)}}{<} 2\cdot 2^k = 2^{k+1}$$ ✓

extra page

#4 1 can be expressed as sum of one Fibonacci number, 1

Suppose for some k that $1 \le m \le k$ can be expressed as sums of distinct Fibonacci numbers. Let F_m be the largest Fibonacci number $\le n+1$. If $F_m = n+1$, $n+1 = F_m$ does it. ~~It~~ If $F_m < n+1$, we have $F_m + F_{m-1} > n+1$ so $(n+1) - F_m < F_{m-1}$,

$(n+1) - F_m < n+1$ so can be expressed as a sum of distinct Fibonacci numbers all less than F_m (because less than F_{m-1}!) so $n+1$ can be expressed

as (sum giving $(n+1) - F_m$) + F_m, a sum of distinct Fibonaccis.

typically people didn't show $(n+1) - F_m < F_m$, which is essential

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Basis ~~Suppose~~ $\sum_{k=0}^{0} \binom{0}{k} = 1 = 2^0$

Induction Suppose $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ (basis)

Goal $\sum_{k=0}^{n+1} \binom{n+1}{k} = 2^{n+1}$ use identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} = \binom{n+1}{0} + \sum_{k=1}^{n} \binom{n+1}{k} + \binom{n+1}{n+1} =$$

$$\binom{n+1}{0} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] + \binom{n+1}{n+1}$$

$$= 1 + \sum_{k=1}^{n} \binom{n}{k} + \sum_{k=1}^{n} \binom{n}{k-1} + 1$$

$$= 1 + (2^n - 1) + (2^n - 1) + 1 = 2 \cdot 2^n = 2^{n+1}$$

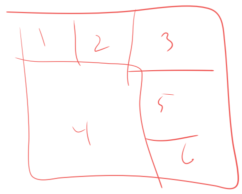
$\sum_{k=1}^{n} \binom{n}{k}$ is $\sum_{k=0}^{n} \binom{n}{k} - 1$ and $\sum_{k=1}^{n} \binom{n}{k-1} = \sum_{k=0}^{n-1} \binom{n}{k}$ is also $2^n - 1$

This is probably an advertisement for dots...
Lets see what you all do.

extra page

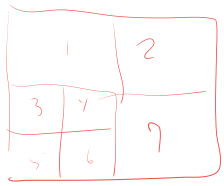
29 part of basis is shown

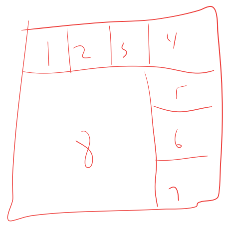
n = 6



n = 8

n = 7





Now to add 3 squares, take any one square in your division into (n−3) squares and divide into four parts

2. (algebra usually the issue. I felt steps were too large, not convincing.)

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 \checkmark$$

Induction assume $\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2$. Then $\sum_{i=1}^{k+1} i^3 = \left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 \overset{\text{ind hyp!}}{=}$

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{2} + \frac{4(k+1)^3}{4} = \frac{(k+1)^2\,(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} \checkmark$$