Solutions

Math 275 sections 1,5,6 Fall 2020: Test I

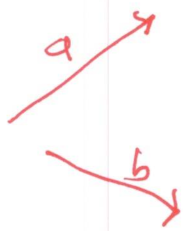
Dr Holmes

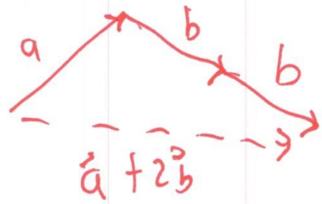
Sept 18-20, 2020

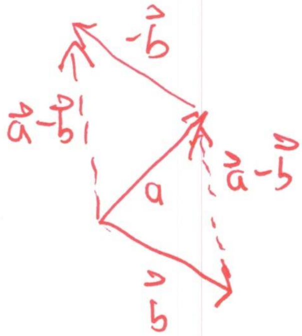
Please work this exam out on paper, in private, without seeking advice from anyone but the instructor (from whom you may seek clarification of the meaning of a question or to whom you may report a suspected error in a question).

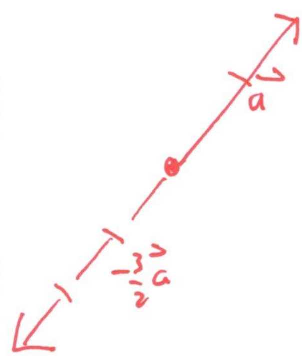
Please return your paper to the instructor by email before 11:55 pm on Sunday the 20th of September. You should scan it and send me the electronic file (pictures taken with your phone are acceptable if that is all you can do). The file or files that you send me should have informative names: I should be able to tell from the file name what your name is, that you are a student in Math 275 section x, and that this is a Test I paper.

1. Draw on your paper vectors **a** and **b** which are not parallel. Then construct copies of $\mathbf{a} + 2\mathbf{b}$, $\mathbf{a} - \mathbf{b}$, and $-\frac{3}{2}\mathbf{a}$ in convincing ways. You may copy any vector (preserving magnitude and direction as best you can) but be sure to label copies of a vector so that I can tell what you intend.









2. Compute the scalar and vector projections of $\langle 1, 3, 5\rangle$ onto $\langle 1, -1, 2\rangle$.

   Express $\langle 1, 3, 5\rangle$ as the sum of a vector parallel to $\langle 1, -1, 2\rangle$ and a vector orthogonal to $\langle 1, -1, 2\rangle$. Your work should include a check that the second variable is in fact orthogonal to $\langle 1, -1, 2\rangle$.

$$\mathrm{comp}_{\vec{b}}(\vec{a}) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \frac{1\cdot 1 - 3\cdot 1 + 5\cdot 2}{\sqrt{6}} = \frac{8}{\sqrt{6}}$$

$$\mathrm{proj}_{\vec{b}}(\vec{a}) = \frac{8}{\sqrt{6}}\left\langle \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle = \left\langle \frac{8}{6}, -\frac{8}{6}, \frac{16}{6} \right\rangle$$

$$= \left\langle \frac{4}{3}, -\frac{4}{3}, \frac{8}{3} \right\rangle$$

obtained by subtraction

$$\langle 1, 3, 5\rangle = \left\langle \frac{4}{3}, -\frac{4}{3}, \frac{8}{3} \right\rangle + \left\langle -\frac{1}{3}, \frac{13}{3}, \frac{7}{3} \right\rangle$$

parallel to $\langle 1, -1, 2\rangle$ $\qquad$ $\perp$ to $\langle 1, -1, 2\rangle$

check :

$$\left(-\frac{1}{3}\cdot 1\right) + \left(\frac{13}{3}\cdot -1\right) + 2\cdot\frac{7}{3}$$

$$= -\frac{1}{3} - \frac{13}{3} + \frac{14}{3} = 0 \checkmark$$

3. Coordinates of three points are given: $A = (2, 1, 2); B = (1, 3, 5); C = (4, 4, 4)$.

   Determine the angle $\angle BAC$ in degrees with two decimal places of accuracy.

   Compute the area of $\triangle ABC$.

$$\vec{AB} = \langle -1, 2, 3 \rangle$$

$$\vec{AC} = \langle 2, 3, 2 \rangle$$

$$\cos(\theta) = \frac{-2+6+6}{\sqrt{14}\,\sqrt{17}} = \frac{10}{\sqrt{14}\,\sqrt{17}}$$

$$\cos^{-1}\left(\frac{10}{\sqrt{14}\,\sqrt{17}}\right) \approx 49.59^\circ$$

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 3 & -1 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{array}$$

$$-5\vec{i} + 8\vec{j} - 7\vec{k} = \langle -1, 2, 3 \rangle \times \langle 2, 3, 2 \rangle$$

$$\frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \frac{1}{2}\sqrt{5^2 + 8^2 + 7^2}$$

$$= \frac{1}{2}\sqrt{138} \approx 5.87$$

4. Let A, B, C be the same points as in the previous problem.

Give scalar parametric equations and symmetric equations for line AC.

Determine an equation for the plane in which points A, B, C lie, in the form $ax + by + cz = d$.

The vector parametric equation

is $\underbrace{\langle 2, 1, 2\rangle}_{\text{point } A} + t\underbrace{\langle 2, 3, 2\rangle}_{\text{vector } \overrightarrow{AC}}$

$\langle 4, 4, 4\rangle + t\langle 2,3,2\rangle$ also works

This is not what is asked for!

The scalar parametric equations are

$x = 2+2t$ or $4+2t$

$y = 1+3t$ or $4+3t$

$z = 2+2t$ or $4+2t$

The symmetric equations are

$$\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{2} \quad \text{or} \quad \frac{x-4}{2} = \frac{y-4}{3} = \frac{z-4}{2}$$

The equation for the plane - normal vector $\langle -5, 8, -7\rangle$ from the previous problem

$$-5x + 8y - 7z = (-5)(2) + \underbrace{8(1)}_{\text{plug in a point}} - 7(2) = -16$$

or $5x - 8y + 7z = 16$ also works

$-5(x-2) + 8(y-2) - 7(z-1) = 0$ is ~~an~~ useful equation and can be converted in to the desired form

5. The line with symmetric equations

$$\frac{x-5}{2}=\frac{y-4}{1}=\frac{z-7}{3}$$

intersects the line with symmetric equations

$$\frac{x-6}{1}=\frac{y}{-1}=\frac{z-7}{1}.$$

Write vector and scalar parametric equations for each line.

Find the point of intersection of the two lines.

$x = 5+2t$ $\quad$ $x = 6+s$

$y = 4+t$ $\quad$ $y = -s$

$z = 7+3t$ $\quad$ $z = 7+s$

$5+2t = 6+s$

$4+t = -s$

$9+3t = 6$

$3t = -3$

$t = -1$

$y = 4-1 = 3 = -s$

so $s = -3$

Now $5+2\cdot(-1) = 5+2t = 3 = 6+s = 6+(-3)$

$4+(-1) = 4+t = 3 = -s = -(-3)$

$7+3(-1) = 7+3t = 4 = 7+s = 7+(-3)$

$(3,3,4)$ is the point of intersection

Most successful work look like this.

I solved directly for x an y.

6. The planes $x+y+z=6$ and $x-2y-2z=-9$ share the point (1,2,3).

Find vector parametric equations for the line of intersection of these two planes.

What is the angle between the two planes?

normal vectors $\langle 1,1,1\rangle$, $\langle 1,-2,-2\rangle$

take their cross product to get a vector parallel to the line of intersection

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & -2 & 1 & -2 \end{array}$$

$$0\vec{i} + 3\vec{j} - 3\vec{k}$$

$\langle 1,2,3\rangle + t\langle 0,3,-3\rangle$ is the vector parametric equation

Symmetric equations are not symmetric:

$$x = 1;\ \frac{y-2}{3} = \frac{z-3}{-3}$$

not asked for, but if you divided by 0 I nicked you slightly.

The angle between the two planes (half the question)

is $\cos^{-1}\left(\dfrac{-3}{\sqrt{3}\,\sqrt{9}}\right) = 125.26^\circ$ ← angle between $\langle 1,1,1\rangle$ and $\langle 1,-2,-2\rangle$

negative, this is an obtuse angle

7. The equation $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t\rangle$ parameterizes a helix.

Determine the tangent vector to this helix at the point $(0, 3, 2\pi)$.

Determine vector parametric equations for the line tangent to this curve at $(0, 3, 2\pi)$.

Determine the arc length of the portion of this helix between $(3, 0, 0)$ and $(0, 3, 2\pi)$.

$$\vec{r}\,'(t) = \langle -3\sin(t), 3\cos(t), 4\rangle$$

$\vec{r}(t) = \langle 0, 3, 2\pi\rangle$ when $4t = 2\pi$, $\boxed{t = \frac{\pi}{2}}$

$\vec{r}\,'(\frac{\pi}{2}) = \langle -3\sin(\frac{\pi}{2}), 3\cos(\frac{\pi}{2}), 4\rangle = \boxed{\langle -3, 0, 4\rangle}$ tangent vector

$\boxed{\langle 0, 3, 2\pi\rangle + t\langle -3, 0, 4\rangle}$ vector equation of tangent line

arc length (half the question!)

not 2π

$$\int_0^{\frac{\pi}{2}} \sqrt{9\sin^2(t) + 9\cos^2(t) + 16}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{9 + 16}\, dt$$

I marked off if I saw no evidence of actually computing the integral symbolically.

$$= \left[5t\right]_0^{\frac{\pi}{2}} = 5\frac{\pi}{2}$$

$$= \boxed{\frac{5\pi}{2}}$$

8. The acceleration of a particle at time t is given by $\mathbf{a}(t) = \langle 1, t, -t \rangle$.
The velocity of the particle when $t = 0$ is $\langle -1, 0, 1 \rangle$.
The position of the particle when $t = 0$ is $\langle 1, 1, 1 \rangle$.
Determine the position of the particle at any time t. Determine its position when $t = 2$.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \left\langle t + C_1, \frac{t^2}{2} + C_2, -t + C_3 \right\rangle$$

$$\vec{v}(0) = \langle C_1, C_2, C_3 \rangle = \langle -1, 0, 1 \rangle$$

so $\vec{v}(t) = \left\langle t - 1, \frac{t^2}{2}, -t + 1 \right\rangle$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \left\langle \frac{t^2}{2} - t + C_1, \frac{t^3}{6} + C_2, -\frac{t^3}{6} + t + C_3 \right\rangle$$

$$\vec{r}(0) = \langle C_1, C_2, C_3 \rangle = \langle 1, 1, 1 \rangle$$

so $\vec{r}(t) = \left\langle \frac{t^2}{2} - t + 1, \frac{t^3}{6} + 1, -\frac{t^3}{6} + t + 1 \right\rangle$

$$\vec{r}(2) = \left\langle 1, \frac{7}{3}, \frac{5}{3} \right\rangle$$