Introductory note to 1908b

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Since antiquity mathematicians have spoken of classes, multitudes, and sets, but set theory as such is a product of the 19th century (Bernard Bolzano, Georg Cantor, Richard Dedekind, et al.). Not until the 20th century did set theory attain an axiomatic foundation (Ernst Zermelo, Thoralf Skolem, Johann von Neumann, Paul Bernays, Wilhelm Ackermann, et al.).

In this essay we discuss Zermelo's paper "Investigations in the foundations of set theory I" (*Investigations* for short). In order fully to appreciate Zermelo's research we must discuss the 19th century emergence of the concept of set as a *terminus technicus* of mathematics as well as the axiomatic method with all its many facets.

§1. The 19th century emergence of the concept of set

In the course of the 19th century mathematicians had to recognize that the continued development of mathematics required deeper foundations. In particular, the desire to accept proofs only on the basis of the logico-conceptual argumentation led increasingly to the inclusion of extensions of concepts (Begriffsumfänge) and then also of sets and classes in mathematical considerations.

In order to construct a theory of sets, it was first necessary clearly to formulate the fundamental concept of "set". This concept is one for which there is almost no intuitive foundation and whose intended content is rather difficult to grasp. In particular, the ontological status of sets had to be clarified. If sets are nothing more than "the mere being together" of various things, then it is questionable whether this coexistence can be treated as an autonomous independent object. In what manner do sets exist?

It was Bernard Bolzano (1781–1848), the Bohemian-Austrian mathematician, philosopher, and theologian, who first systematically studied the concept of set. He began to construct a theory of sets, publishing his results in his *Wissenschaftslehre* (*1837*). Further results appeared posthumously in his *Einleitung in die Größenlehre* (*1975*) and in his booklet *Paradoxien des Unendlichen* (*1851*).

While in previous centuries the word "set" (πλῆθος, multitudo, Vielheit, Menge, aggregate, ...) was used only to name an undetermined accumulation

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of objects, Bolzano added new meanings to this word. In fact, he added four attributes to the notion of a set. Without these new attributes it would not have been possible to transform the concept of a set from colloquial language into a mathematical *terminus technicus.*

The first attribute is that sets are uniquely determined by their extension, i.e., by the totality of their elements. In contrast to this, structured sets ("Inbegriffe" in Bolzano's terminology) are not determined by their extension alone. Notice also that in general the principle of extensionality is not valid for concepts. Bolzano wrote (*1975*, 152):

> Die Theile, aus denen eine Menge besteht, bestimmen sie, und zwar vollständig und alle auf einerley Art.[1]

About 40 years later Dedekind formulated this principle independently in the first paragraph of his celebrated essay *Was sind und was sollen die Zahlen?*, *1888*. Zermelo introduced the name "Axiom der Bestimmtheit". It is now customary to call this the axiom of extensionality, this being a heritage from Antoine Arnauld, who in 1662 distinguished in his *Logique de Port-Royal* between the extension and the intension of an idea. Cantor in *1878* and Dedekind in *1888* proposed referring to "elements" instead of "parts".

The second attribute is that sets can be constituted of objects from different species. Here the word "species" (εἶδος) should be understood in the Aristotelian sense. Since antiquity it was not usual in mathematics to collect things from different "genera" or "species" into a whole. For example, introducing the concept of number in his *Dictionaire Mathématique* (Amsterdam 1691), Jaques Ozanam (1640–1717) admitted only sets of objects of the same species. He wrote: "Le Nombre est l'assemblage de plusiers choses de même genre" (op. cit., 21). An exception can be found in the work of Gottfried Wilhelm Leibniz (1646–1716). In his *Dissertatio de arte combinatoria* (1667) he postulated the existence of finite multitudes of objects from different species as totalities. He wrote:

> ...liceat quotcunque res simul sumere, et tanquam unum totum supponere.[2]

Bolzano often stressed that sets can consist of objects of different species (cf. *1975*, 101–102).

The third attribute is that sets need not be definable. A set need not be the extension of a concept (Begriffsumfang, étendue de l'idée in the sense of Arnauld's *Logique de Port-Royal*). A set need not be thought of by some person and need not be given by an explicit enumeration of all its elements. In his booklet *Paradoxien des Unendlichen* (*1851*, §14, 17), Bolzano wrote:

[1] The parts out of which a set consists determine the set completely and all in a uniform manner.

[2] ...that it is permitted to comprehend anything whatsoever, and to assume that taken together they form a whole.

> Es gibt also Mengen..., auch ohne daß ein Wesen, welches sie denkt, da ist.[3]

The fourth attribute is that sets are "things" in the sense that they have an immutable form of existence. When previous authors spoke about multitudes or collections, they did not presume that these multitudes or collections were objects which have an independent existence. The word "multitude" (πλῆθος, multitudo, Vielheit, Menge, ...) always meant an undetermined number of several individual objects, but the multitude itself was not considered a substantial object.[4] The following quotation from a letter of Leibniz to Johann Bernoulli from February 21, 1699, may serve as an example for this widespread opinion (cf. *Leibniz 1856*, 575):

> Concedo multitudinem infinitam, sed haec multitudo non facit numerum seu unum totum.[5]

Bolzano defines sets as the mere being together of individual objects, and this coexistence of the objects is itself an object. For him such an object has an extra-mental reality. In his *Einleitung in die Größenlehre* he wrote (*1975*, 100–101):

> ... und ich bemerke, daß die erwähnten Dinge mit Wahrheit nicht zusammengedacht werden könnten, wenn sie nicht auch, ohne daß wir sie noch zusammendenken, schon zusammen wären, schon ... ein Ganzes bildeten, ein *Ding*, welchem gewisse Beschaffenheiten zukommen, die keinem der einzelnen Dinge, seinen Theilen, für sich allein zukommen (müssen). Ich behaupte also, daß Inbegriffe bestehen nicht dadurch, daß wir sie denken, sondern umgekehrt, daß wir nur dann sie mit Wahrheit denken können, wenn sie bestehen auch ohne daß wir sie denken.[6]

Bolzano's position here is comparable to that of the "realists" in the medieval debate on universals (Universalienstreit). However, the term "realism" means one thing with regard to sets (Mengenrealismus) and quite another

[3] Hence sets exist ... even if there is no one who thinks of them.

[4] Compare this with the detailed discussion of the word "multitude" (πλῆθος) in Plotinus' treatise *On Numbers* (Περὶ ἀριθμῶν), *Von den Zahlen*; cf. *Plotinus 1964*, section 12, 190–193.

[5] I concede that they (i.e. the terms of an infinite geometric series) do form an infinite multitude, but such a multitude has no number and is not a whole.

[6] ... and I note that the things mentioned above cannot truly be thought of together when they are not yet already together, even before our thinking of them as together, and when they are not yet a whole, a *thing*, whose properties are not (necessarily) properties of its elements. Hence, I claim that sets exist not through our thinking, but conversely, that we are able to think of them only if they also exist without our thinking of them.

with regard to universals (Universalienrealismus), because universals are intensionally determined objects while sets are extensionally determined.

Since then, other mathematicians have also consciously and explicitly asserted the objecthood of sets.[7] They have differed, however, in their conception of what is meant by "object". These differences did not affect the development of set theory in the 19th century, but they became quite important when the paradoxes and antinomies of set theory appeared. We shall come back to this point later.

In the very first paragraph of his essay *1888*, p. 1, Dedekind wrote:

> Ein solches System S (oder ein Inbegriff, eine Mannigfaltigkeit, eine Gesamtheit) ist als Gegenstand unseres Denkens ebenfalls ein Ding.[8]

Similarly, on the very first page of his book *Grundzüge der Mengenlehre* (*1914*), Felix Hausdorff wrote:

> Eine Menge ist eine Zusammenfassung von Dingen zu einem Ganzen, d. h. zu einem neuen Ding.[9]

The word "set" is now much more than a mere manner of speaking. Sets are now understood to have the attribute of objecthood. They may occur as elements in other sets. This was emphasized explicitly by Bolzano (*1975*, 102):

> Die Theile, aus denen ein Inbegriff bestehet, können selbst wieder Inbegriffe seyn.[10]

That new sets can be iteratively generated from existing sets would become a basic feature of set theory after Zermelo.

Bolzano gave a simple real definition ("Real-Definition") for the concept of set. For him sets were the "mere being together" of different objects. Some 60 years later, Cantor proposed a much more manageable definition. He wrote (*1895*, 481):

> Unter einer "Menge" verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die "Elemente" von M genannt werden) zu einem Ganzen.[11]

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[7] This is also emphasized by Michael Hallett in his book *1984*, 34–35. He writes (p. 299): "It was Cantor who first stressed the unity, the objecthood of sets, thereby marking them out from mere aggregations." But Cantor was not the first. Much earlier Bolzano and others pointed to the objecthood and unity of sets.

[8] As an object of our reasoning such a system (or set, multitude, totality) is also a thing.

[9] A set is a comprehension of things into a whole, i.e. into a new thing.

[10] The elements which constitute a set can themselves be sets.

[11] By a "set" we understand any comprehension M of definite, well-differentiated objects m of our intuition or our thought (which are called the "elements" of M) into a whole.

Cantor's definition of the concept of set is widely acknowledged and routinely cited. It is remarkable, however, that Cantor's is a "genetic" definition. It not only indicates what sets are, but also how they can be created. We shall return to this in more detail in §3.

§2. The appearance of the set-theoretic antinomies

Bolzano early on foresaw that working with objects that can theoretically contain anything and everything can cause inconsistencies. In his Mathematical Diary ("Mathematisches Tagebuch", Heft 22, p. 1968), he wrote:[12]

> *Das All der A*, wo *A* eine Vorstellung von beschränktem Umfange bezeichnet, z. B. Mensch — läßt sich recht wohl denken. Sobald man aber für *A* die weiteste aller Vorstellungen, nämlich die eines Etwas überhaupt setzt, so entsteht die Schwierigkeit, daß die Vorstellung *das All der Gegenstände* oder *das All von Allem* oder *das absolute All*, eigentlich auch sich selbst, weil ja dieses All auch wieder *Etwas* ist, umfassen sollte; welches doch ungereimt ist. Ich glaube deshalb, daß man diese Vorstellung in der Tat zu den widersprechenden (imaginären) zählen müsse, gerade wie die Vorstellung von der geschwindesten Bewegung. — Aber auch schon das All der Inbegriffe wäre eine solche sich selbst widersprechende Vorstellung.[13] [14]

It seems that Bolzano had some second thoughts. He added the following note:

> Responsio. Nicht doch! Diese Begriffe sind nicht widersprechend.[15]

Unfortunately, Bolzano did not pursue the antinomy of the universal set (i.e., the set of all sets), which he had thus adumbrated.

George Boole (1815–1864) also encountered the concept of the universal set ("class 1", as he called it) "which comprehends every conceivable class of objects," but he did not recognise its antinomic character (cf. *Boole 1847*, 15). The same is true of Dedekind when he referred to the totality of all things

[12] Compare also with *Bolzano 1827/44*, 29.
[13] Quoted in *Sebestik 2000*, 236.
[14] *The universe of all A*, where *A* is an idea of limited extension, e.g. a human being, can be thought of without any difficulties. But as soon as one takes for *A* the largest possible idea, namely the idea of an object in general, then a difficulty arises in that the idea of *the universe of everything*, or *the universe of all*, or *the absolute all*, will contain itself, since this universe is also *something*, which is absurd. Therefore, I think that in fact one should classify this idea along with other self-contradictory (imaginary) ideas, such as the idea of the fastest movement.—But already the universe of all collections would turn out to be such a self-contradictory idea.
[15] Answer. Hold on! These notions are not contradictory.

which might happen to be objects of his thinking ("die Gesamtheit S aller Dinge, welche Gegenstand (seines) Denkens sein können", *Dedekind 1888*, 14, art. 66).

Cantor also encountered the universal set ("Allmenge"). In contrast to Bolzano, Boole, and Dedekind, he acknowledged its antinomic character and tried to expel the concept of a universal set from his set theory. He distinguished in the years 1883 and 1887 three types of sets:

finite sets,
transfinite sets, and
absolutely infinite sets.

Infinite sets are transfinite if they can be embedded within sets which have a greater potency. Their size can be measured by the so-called alephs. In Cantor's opinion, both finite and transfinite sets can be used in mathematical reasoning. He was proud of having secured the domain of transfinite sets for mathematics. His comprehensive two-part treatise, *Cantor 1895* and *1897*, in which he presented his mature work in set theory, carried the highly significant title "Beiträge zur Begründung der transfiniten Mengenlehre" ("Contributions to the foundations of transfinite set theory").

As Cantor explained in a letter of October 5, 1883 to the philosopher Wilhelm Wundt, the absolutely infinite sets are those sets "die nicht mehr vergrößert werden können" ("which cannot be enlarged with regard to their potency"). The size of an absolutely infinite set cannot be measured by an aleph. The adjective "absolute" is meant to indicate that such infinities are so large that they transcend the scale of norms of comparable sizes.[16] Cantor spoke about the "absolute" and the "absolut infinite" for the first time in 1883 in his treatise "Grundlagen einer allgemeinen Mannigfaltigkeitslehre" (cf. *Cantor 1883b*, 556 and 587). He returned to this problem in his "Mitteilungen zur Lehre vom Transfiniten" (*Cantor 1887*, 91).

Cantor thought that the absolute infinite eludes mathematical determination (cf. *Cantor 1887*, 109). In his letter to Grace Chisholm Young of June 6, 1908, he wrote:

> Was über dem Finiten und Transfiniten liegt, ist ... das "Absolute", für den menschlichen Verstand Unfassbare, also der Mathematik gar nicht unterworfene, [das] Unmessbare.[17]

These are highly emotional sentences, but they are not quite accurate. It would have been more appropriate to say that, in the calculus of set theory,

[16] The Latin word "absolutus" means "unconditional, freed of all conditions, not depending on anything else, detached". The verb "solvere" means "to solve, to dissolve, to untie, to free".

[17] That which lies beyond the finite and the transfinite, ... is the absolute, for the human intellect ungraspable, and immensurable, hence not a subject of mathematics.

absolute infinities should not be treated in the same way as transfinite infinities. Cantor did not know how to introduce the concepts of finite and transfinite sets so that absolute infinite sets could be excluded from the domain of objects under consideration. Apparently, he would have liked to precondition his set theory with a ban on dealing with absolutely infinite sets, but such prohibitions are prohibited in the composition of a mathematical theory.

Cantor himself did not employ absolutely infinite sets in his mathematical reflections, but other mathematicians came across them and stumbled.

In 1897, Cesare Burali-Forti noticed a problem in Cantor's theory of order types. If Ω is the order type of the set S of all order types of well-ordered sets, then the set $S \cup \{S\}$ has order type $\Omega + 1$. But Ω is the order type of the set of all order types of well-ordered sets, hence nothing can be larger than Ω, a contradiction! Burali-Forti's paper was published in the spring of 1897.[18]

When Cantor became aware of Burali-Forti's paper, he was alarmed.[19] He had realized for quite some time that the domains of all ordinal numbers and of all cardinal numbers (Cantor always spoke of the "set" of all ordinal numbers and the "set" of all cardinal numbers) were absolutely infinite, but now he had to acknowledge that his genetic definition of the concept of set did not differentiate between the creation of a transfinite set and the creation of an absolutely infinite set. During the summer of 1897, he began to modify his definition of the concept of set. We will return to this in §4.

Cantor also noticed contradictions in Dedekind's work on the foundations of arithmetic. Dedekind's "Gedankenwelt" (the totality of the thinkable) is an absolute infinite totality whose size cannot be measured by any cardinal number (cf. *Dedekind 1888*, Theorem 66, and Cantor's letters to Hilbert of November 15, 1899, and January 27, 1900, reprinted in *Cantor 1991*). That this was also apparent to Zermelo can be seen in the account of Dedekind's essay that he wrote for Edmund Landau (see *Landau 1917b*).

It was quite easy for Zermelo to demonstrate that working with Bolzano's "Allmenge", Boole's "class 1", and Dedekind's "Gedankenwelt" leads to inconsistencies. All these classes M have the property that $\mathsf{P}(M) \subseteq M$ where

$$\mathsf{P}(M) = \mathfrak{U}(M) = \{X; X \subseteq M\}$$

[18] Burali-Forti (1861–1931) was an assistant to Guiseppe Peano in Turin in the years from 1894 until 1896 and hence familiar with the most recent developments in logic and set theory of his time. Burali-Forti presented his paper on transfinite ordinals in the meeting of the Circolo Matematico di Palermo on March 28, 1897. It appeared in print soon afterwards (cf. *Burali-Forti 1897*). The argument of Burali-Forti is strongly reminiscent of an argument which was given already by the Greek philosopher Plotinus (cf. *Plotinus 1964*, section 17–18, pp. 208–213) and also much later again by Bolzano (see above).

[19] Cantor knew Burali-Forti's paper, since in a letter to Dedekind of August 3, 1899, he talks about the Burali-Forti antinomy and even uses Burali-Forti's notation. However, he nowhere mentions Burali-Forti.

is the power set of M (in Zermelo's *Investigations* always denoted by $\mathfrak{U}(M)$ as the set of all "Untermengen" of M). According to Cantor this is impossible by reasons of cardinality (*Cantor 1890/91*). However, Zermelo realized that neither cardinality arguments nor the introduction of power sets (i.e. higher order objects) are necessary in this context. In fact, he showed that each set M has a subset A which cannot be an element of M. Such a subset is, for example, $A = \{x \in M; x \notin x\}$. If $A \in M$ were valid, it then would follow that $A \in A \Leftrightarrow A \notin A$, which is a contradiction. This argument appears in theorem 10 of the *Investigations*. Zermelo explained this elegant argument in April 16, 1902 to the philosopher Edmund Husserl, one of his Göttingen colleagues (cf. *Husserl 1979*, 399; see also *Rang and Thomas 1981*). Zermelo returned to this argument in *s1932d*, which suggests that Zermelo was quite pleased with it.[20]

In his letter of November 7, 1903 to Frege, Hilbert testified that Zermelo worked out his argument sometime around 1900 and made it public in Göttingen at that time. Hilbert's testimony is supported by a contemporary document, namely Zermelo's (unpublished) notes for his Göttingen lectures on set theory during the winter semester 1900/1901. In these notes there is an entry which shows that he gave a proof of Cantor's well-known theorem, stated here in terms of power sets rather than as Cantor did in terms of functions:

> Theorem (1890): *For every set M it holds that there is no surjection of M onto its power set* $\mathsf{P}(M)$.

But Zermelo gave a slightly different formulation of the theorem (see the last line on the first page of his §6):

> *For every set M there is no injection of* $\mathsf{P}(M)$ *into M.*

Proof. Suppose that an injective mapping ψ from $\mathsf{P}(M)$ into M existed. Consider $B = \{\psi(x);\, \psi(x) \notin x\}$.

If $\psi(B) \in B$ held, then it would follow that there is an $x \in \mathsf{P}(M)$ such that $\psi(B) = \psi(x)$ and $\psi(x) \notin x$. Since ψ is injective, we get $x = B$, and hence $\psi(B) \notin B$, a contradiction.

If $\psi(B) \notin B$ held, then $\psi(B)$ would satisfy the defining condition of B and hence $\psi(B) \in B$, again a contradiction.

Thus: $\psi(B) \in B \Leftrightarrow \psi(B) \notin B$. The assumption that an injective mapping from $\mathsf{P}(M)$ into M exists is therefore disproved. $\square$

Zermelo communicated his modification of Cantor's theorem to Gerhard Hessenberg as well, and he, following Zermelo's notes and notation, included it in his book *Grundbegriffe der Mengenlehre* (*Hessenberg 1906*, 41–42). This modification is also contained in Zermelo's *Investigations* (see his Theorem 32 on p. 276).

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[20] This argument is related to the ancient antinomy of the liar. A detailed account of its historical roots and its effects in later times is given in *Felgner 2009*.

Cantor's proof is faintly reminiscent of Euclid's proof that for any finite number of prime numbers there is always a further prime number (see book 9 of Euclid's *Elements*). Analogously, Cantor proved that for any mapping φ of M into $\mathsf{P}(M)$ there are always some additional subsets beyond the range of φ.

It seems that Zermelo's modification of Cantor's theorem resulted from an attempt to give direct proofs for the inconsistencies of Bolzano's "Allmenge", Boole's "class 1", and Dedekind's "Gedankenwelt" as "sets". In fact, when M is a class such that $\mathsf{P}(M) \subseteq M$, then the identity mapping is an injective mapping from $\mathsf{P}(M)$ into M and the proof of the modified version of Cantor's theorem immediately produces the set of all sets which do not contain themselves as elements, $A = \{x;\, x \notin x\}$. The equivalence $A \in A \Leftrightarrow A \notin A$ which follows immediately from the assumption that A is a set, was also observed by the English logician Bertrand Russell (1872–1970) at about the same time and hence will be called the Zermelo-Russellian antinomy.

It is remarkable that Bolzano had already foreseen that in the presence of some elementary assumptions on the concept of set the concept of universal set ("Allmenge") might lead to inconsistencies. In fact, it does lead to inconsistencies. As we have seen above, the equivalence $A \in A \Leftrightarrow A \notin A$ occurs only because we assumed that $A = \{x;\, x \notin x\}$ belongs to the domain of those objects through which the variable x ranges (i.e. to the domain of all sets). If we do not assume that $\exists x : x = A$ holds, then the argument given above cannot be carried through, since the variable x can no longer be assigned to A and the antinomy disappears. This gives us a hint of how to avoid the antinomy.

(a) In Zermelo's axiomatic set theory and also in the extended version proposed by Zermelo, Thoralf Skolem, and Adolf Abraham Fraenkel, called ZF (or more accurately ZSF), the above-mentioned antinomy can be avoided by introducing expressions of the form

$$\{x;\, \Phi(x)\} \quad (= \text{class of all } x\text{, that satisfy the property } \Phi)$$

as terms (and hence as "things") only when $\exists y \forall x (x \in y \leftrightarrow \Phi(x))$ is provable from the axioms. (Here $x, y, z, \ldots$ are variables running through the domain of all sets.) When

$$\exists y \forall x (x \in y \leftrightarrow \Phi(x))$$

is not provable from the axioms, then, according to a proposal made by Willard van Orman Quine, expressions of the form $\{x;\, \Phi(x)\}$ are still admitted, but they have to be treated in the strict nominalistic sense. This means that for arbitrary terms t the expressions $t \in \{x;\, \Phi(x)\}$ are admitted as formulas, but $\{x;\, \Phi(x)\}$ alone is not a term and is hence not treated as a *thing* (i.e. a set). In this case $\{x;\, \Phi(x)\}$ is called a "virtual class" (cf. *Quine 1963*, 15–21).

(b) In von Neumann-Bernays-Gödel axiomatic set theory, NBG, the antinomy is avoided by introducing two sorts of variables, variables for sets

$x, y, z, \ldots$ and variables for classes $X, Y, Z, \ldots$, and also by introducing two sorts of terms, set terms and class terms, where all comprehension terms $\{x; \Phi(x)\}$ are introduced as class terms. It follows from the axioms of the two-sorted predicate logic that for each class term $\{x; \Phi(x)\}$ we have $\exists X : X = \{x; \Phi(x)\}$. Accordingly, in NBG set theory class terms can be treated as "things". They are not "sets" but things of a different sort (or genus) and are called "classes".

Our discussion has shown that one should not treat all comprehension terms either as "things" or as "things of one and the same species". The fourth attribute, which was added to the notion of set by Bolzano (see above) makes possible the construction of a hierarchy of transfinite sets, but this attribute can also lead to inconsistencies if applied carelessly. Thus, if one does not want to exclude comprehensions, a careful use is possible, because they can be admitted either

(a) in a strictly nominalistic sense as linguistic objects,

or

(b) as "things" of a higher genus,

thus avoiding in both cases (at least directly) the appearance of the Zermelo-Russellian antinomy. But our discussion has also shown that one should not use ontological assumptions in too unrestricted a manner. A certain parsimony (parsimonia ontologiae) is always necessary. We conclude that the concept of "set" as defined by Bolzano, Cantor, and others is not as clear and unambiguous as they had maintained. One obtains clarity only by introducing the notion of set axiomatically.

§3. Remarks on the axiomatic method

Antiquity has bequeathed us two different methods of theory construction. One of them is the kind of axiomatics found in Euclid's *Elements* (Στοιχεῖα); the other—which is another kind of axiomatics—is the theory of science presented by Aristotle in the first book of his *Posterior analytics* (*Analytica posteriora*, Ἀναλυτικὰ ὕστερα).

In spite of many differences, both forms of axiomatics begin with explicit definitions of fundamental concepts and then proceed to postulates (Euclid's αἰτήματα) or hypotheses (Aristotle's ὑπόθεσεις). While in Euclid the status of the definitions is not very clear, it is quite clear and distinct in Aristotle.

Aristotle distinguishes among three different types of definitions (cf. *Analytica posteriora*, book 2, chap. 10, and *Topica*, books 6 and 7). The most important one is the *definitio essentialis* (ὅρος οὐσιώδης, Wesens-Definition, Essential-Definition), the definition of the essential features of an object. Such a definition names all the properties which an object has to satisfy in order to be defined as such. The properties named in such a definition are the essential properties of the objects under consideration. They define what the thing "is", what medieval philosophers called its *quiditas*. According to Aristotle the essence of a species of objects is defined when the next higher *genus* is specified (the so-called *genus proximum*) and also the property which isolates the species inside the genus (the *differentia specifica*) is stated.

Concepts for which there are no higher (i.e. more general) concepts cannot have a "Wesens-Definition", a *definitio essentialis.* The usual examples for such undefinable concepts are the concepts of "equality" and "being". Mathematicians differ over whether or not some of the fundamental concepts of mathematics are also undefinable, e.g., basic concepts of geometry such as "point", "space", etc. and of set theory such as "set" and "elementhood".

Blaise Pascal (1623–1662) was perhaps the first thinker who was convinced of the undefinability of fundamental mathematical notions and who carefully argued in favor of his conviction. In an essay written in the years 1655–1658 (published posthumously), conventionally entitled *De l'esprit géométrique*, he outlined his proposals for a consistent development of arithmetic and geometry.

Pascal thought that mathematicians should first of all seek those concepts that are immediately understood by everyone without any attempts to precisely define them. These notions he called "mots primitifs". According to Pascal, examples of such "primitive notions" are the concepts of "space", "time", "equality", "number", "existence", etc. He took them to be trans-subjective self-evident truths immediately comprehensible by everyone capable of speaking a language. This kind of intuitive awareness lies beyond the rational mind, and is in Pascal's words a "*sentiment du coeur*", i.e. a kind of intuitive knowledge of principles of the heart. The "heart" seems here to play the role of the Platonic "Nous" (νοῦς) for the Jansenist Pascal.

On the basis of the "mots primitifs", all other mathematical concepts can be introduced by means of nominal definitions. Starting from the concept of number, which according to Pascal remains undefined, one can define, e.g., the concept of even positive integer, rightly calling it a nominal definition.

The difficult problem of introducing the fundamental concepts of arithmetic and geometry by means of *definitiones essentiales* or any other device was finessed by Pascal with the somehow audacious remark that these concepts cannot be defined and are in any case well-known to everyone. In particular, Pascal was convinced that one need not define the concept of number and that an adequate definition is, in any event, impossible.

Pascal for his time clearly saw the problems connected with the introduction of primitive concepts. However, he erred when he thought that the whole content of primitive concepts is given to us by intuition (*la lumière naturelle*). What is this "intuition", and how can we control it? Pascal does not answer these questions.

The essay *De l'esprit géométrique* made an enormous impression on many mathematicians. As late as the 19th and the 20th centuries there were still adherents of Pascal's method. For example, in his *Leçons sur les fonctions de* *variables réelles* (*1905c*) Émile Borel had this to say about the fundamental concept of set:

> L'idée d'ensemble est une notion *primitive* dont nous ne donnerons pas de définition. Citons seulement quelques exemples d'ensembles: l'ensemble des points d'une droite, etc.[21]

In his works on set theory, Hausdorff also adopted Pascal's and Borel's point of view. In the first pages of the second edition of his book on set theory (*Mengenlehre*, *1927*), he wrote:

> Eine Menge entsteht durch Zusammenfassung von Einzeldingen zu einem Ganzen. Eine Menge ist eine Vielheit, als Einheit gedacht. Wenn diese oder ähnliche Sätze Definitionen sein wollten, so würde man mit Recht einwenden, daß sie idem per idem oder gar obscurum per obscurius definieren. Wir können sie aber als Demonstrationen gelten lassen, als Verweisungen auf einen *primitiven*, allen Menschen vertrauten Denkakt, der einer Auflösung in noch ursprünglichere Akte vielleicht weder fähig noch bedürftig ist.[22]

It is clearly unsatisfactory, when called upon to state definitions or axioms to respond that they are unnecessary. The problem of how to demarcate a "primitive act of thinking which is familiar to everyone" remains unsolved, yet such demarcation is necessary if one hopes to avoid the antinomies of set theory.

In contrast to Pascal, other mathematicians and philosophers were convinced that all mathematical disciplines should start from explicit definitions of their fundamental concepts. It may perhaps be advisable to proceed not from *definitiones essentiales* but rather from *causal definitions* ("Kausal-Definitionen" in Aristotle's sense), i.e. from definitions that give the cause for the existence of the objects in question. The problem of whether in mathematics one begins—or should begin—from causal definitions was heatedly discussed by Scholastic and Renaissance mathematicians and philosophers. The utility of such definitions was vigorously challenged by the English philosopher Thomas Hobbes (1588–1679) in his work *De corpore* (London 1655). The German philosopher and mathematician Ehrenfried Walter von Tschirnhaus (1651–1708) took up the problem and published a theory of science based on causal definitions. In his book *Medicina mentis* (Amsterdam 1687, Leipzig

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[21] The concept of a set is a primitive notion which we do not define here. It suffices to give a few examples of sets: the set of all points on a line, etc.

[22] A set is created when by an act of comprehension individual entities are aggregated into a whole. A set is a multitude, considered as a unit. If such a sentence or a similar sentence were offered as a definition, one could rightly object that it defines idem per idem or even obscurum per obscurius. However, we may accept them as demonstrations, as references to a *primitive* act of thinking which is familiar to everyone, but which is neither capable of nor in need of being resolved into more primitive acts.

1695), he called those definitions *genetic definitions* in which the objects are not only defined by their essential properties, but also by the processes of their generation. In order truly to comprehend an object, one must be able to reconstruct the object mentally. Therefore, the definitions of the fundamental notions of a science must include a statement of the methods of their mental construction or reconstruction.

To construct a theory axiomatically means—according to Tschirnhaus—to give its fundamental concepts in the form of genetic definitions. From the contents of these genetic definitions one extracts the axioms of the theory, and from the axioms one deduces the theorems as usual by applying purely logical principles. Hence, the truth of the axioms is *ex terminis* known for certain, for what is expressed in the axioms has to be clear from that which is deposited in the definitions. In particular, the existence of the objects with which the theory deals, is secured. All this differs greatly from Aristotle's concept of a theory built upon axioms. For Aristotle, the truth of an axiom (he spoke of a hypothesis) must be clear, either immediately or by induction from empirical experience.

In the 18th century the axiomatic method as conceived by Tschirnhaus was brought into a more distinct form by the German philosopher Christian Wolff (1679–1754). In all his mathematical textbooks, he employed this form, which resulted in its widespread diffusion.

A prominent example of the Hobbes-Tschirnhaus-Wolffian concept of axiomatics is Cantorian set theory. At the beginning of his final treatise "Beiträge zur Begründung der transfiniten Mengenlehre" (*1895*, *1897*), Cantor stated his famous definition of the concept of set, which we have already quoted in §1. This definition is obviously an attempt at a genetic definition since on the one hand it indicates what a set "is", namely a multitude of various things, and on the other hand it indicates how a set is generated, namely by an act of comprehension. Cantor, however, does not set up axioms or postulates. After defining the concept of set, he immediately begins to elaborate the content of his definition in a long and admirable sequence of theorems.

In §2 we have already indicated that Cantor's set theory was unable to prevent the appearance of antinomies. The process of generating sets is in general not a finite process. Accordingly, it requires a clarifying metatheory of sets. Thus, genetic definitions of the fundamental concepts of set theory are not very persuasive.

In the course of the 19th century, it became more and more apparent that in geometry as well it is impossible to provide genetic definitions for some of the fundamental concepts such as "point" and "space", although it is certainly possible to give genetic definitions for many other concepts (cf. Heron's book on definitions [23]). This brings out that geometry—like set theory—cannot be based on genetic definitions of fundamental concepts.

[23] Cf. *Heronis Alexandrini Opera quae supersunt omnia*, vol. 4: *Definitiones*, edited by Johan Ludvig Heiberg, Teubner Verlag Leipzig 1912.

Indeed, in the 19th century it became increasingly evident that the fundamental concepts of geometry (point, space, ...) and of set theory (set, elementhood, ...) are not explicitly definable because they are not contained in higher concepts. If one extends the Aristotelian table of categories to include mathematical concepts, then these concepts must be listed at the outset.

However, during the 19th century it also became increasingly clear that it is still possible to introduce the fundamental concepts of a theory by some kind of definition even if they are not explicitly definable. They can be introduced via *implicit definitions*.

If one wishes to introduce the fundamental concepts of a theory via implicit definitions, this can be done by first setting up a language in which each fundamental concept is represented by a sign, e.g. by a letter. The axioms of the theory are formulated in this language. The signs representing the fundamental concepts are thus "folded" into the axioms. But the signs are not treated as symbols, i.e., as signs with a prescribed meaning.[24] Only that which can be derived from the axioms determines the meaning of that for which the signs stand. The totality of all axioms limits the possibilities of interpreting the signs and hence of assigning meanings to them. In this way the fundamental notions are implicitly "defined" and delimited.

This was the approach of Johann von Neumann (1903–1957), whose axiomatization of set theory relied upon implicitly defined fundamental concepts. In his paper *1925*, 220, he wrote:

> Man konstruiert eine Reihe von Postulaten, in denen das Wort "Menge" zwar vorkommt, aber ohne jede Bedeutung. Unter "Menge" wird hier ... nur ein Ding verstanden, von dem man nicht mehr weiß und nicht mehr wissen will, als aus den Postulaten über es folgt.[25]

In von Neumann's theory of sets, the whole content of the Bolzano-Cantorian concept of set is neither presupposed nor used. His axiom system is a *formal system* which gives rules for using set theory's fundamental concepts. Von Neumann produced his theory of sets from a *formal* rather than from a *contentual standpoint*.

The formal standpoint was originally developed in algebra in the circle of George Peacock (1791–1858), Duncan Farquharson Gregory (1813–1844), George Boole (1815–1864), and their associates (see *Pycior 1981*), becoming the dominant position only through the axiomatizations of the arithmetic of natural numbers (Richard Dedekind, 1888), of projective geometry (Gino Fano, 1892), of the algebraic concept of a field (Heinrich Weber, 1893), and

[24] The Greek word "symbol" (symbolon, σύμβολον) is composed from "syn" (σύν = together) and "bállein" (βάλλειν = to throw). A "symbol" is, hence, a sign to which a certain meaning is attached.

[25] A series of postulates is contructed in which the word "set" appears, but without any meaning. By a "set" nothing else than an object is understood, about which we do not know and do not want to know anything more than what follows from the postulates.

of classical geometry (David Hilbert, 1899). In none of these examples was there any attempt to describe the "nature" of natural numbers, of geometric and algebraic objects, but only to define the respective content of the objects dealt with in these theories.

In the next section, we will have a look at Zermelo's axiom system in order to discern the paradigm which he followed. At first, we will briefly discuss the work of those who prepared the path for axiomatizing set theory.

§4. The axiomatization of set theory

When Cantor in the summer of 1897 began to revise the foundations of his set theory, he thought that he had found a way out of the crisis by distinguishing between "*consistent*" and "*inconsistent*" sets. This pair of concepts had been introduced shortly before by Ernst Schröder in his *Vorlesungen über Algebra der Logik* (*1890*, 211–213). However, Cantor could not find a way out of the crisis by this means because the consistency proofs which are required here were far out of reach.

Some time later, Cantor tried another approach, namely by distinguishing between *finished* (*fertigen*) and *unfinished* (*unfertigen*) sets. In a letter to Hilbert of October 10, 1898, Cantor proposed placing the following definition at the beginning of set theory (cf. *Cantor 1991*, 396; see also Cantor's letter of August 3, 1899, to Dedekind in *Cantor 1932*, 443–444):

> *Definition.* Unter einer *fertigen Menge* verstehe man jede Vielheit, bei welcher alle Elemente *ohne Widerspruch* als zusammenseiend und daher als *ein Ding für sich* gedacht werden können.[26]

In practice, as I have already indicated, this definition by itself is not workable. Nevertheless, Cantor thought it possible to derive the following principles from the content of the above definition. He wrote (cf. *Cantor 1991*, 396):

> Aus d⟦ieser⟧ Definition ... ergeben sich mancherlei Sätze, unter Anderm diese:[27]

This shows once again that Cantor was still following the Tschirnhaus-Wolffian paradigm. The propositions he obtained were the following ones:

(A1) Ist M eine fert. Menge, so ist auch jede Theilmenge von M eine fert. Menge.
(If M is a finished set, then also each subset of M is a finished set.)

[26] *Definition.* By a *finished set* ⟦or *ready set*⟧ one should understand any multitude all of whose elements can be thought of *without contradiction* as being together and thereby forming *a thing in itself.*

[27] From this definition we obtain many propositions, amongst them the following ones:

(A2) Substituirt man in einer fert. M. an Stelle der Elemente fertige Mengen, so ist die hieraus resultierende Vielheit eine fertige M.
(If in a finished set all its elements are substituted by other finished sets then the resulting multitude is a finished set.)

(A3) Ist von zwei aequivalenten Vielheiten die eine eine fert. M., so ist es auch die andere.
(If one of two equivalent multitudes is a finished set, then the other one is one as well.)

(A4) Die Vielheit aller Theilmengen einer fertigen Menge M ist eine fertige Menge.
(The multitude of all subsets of a finished set is a finished set.)

Adding a principle of infinity, he wrote:

> Daß die "*abzählbaren*" Vielheiten fertige Mengen sind, scheint mir ein axiomatisch sicherer Satz zu sein.
> (That all countable multitudes are finished sets seems to me to be an axiomatically irrefutable sentence.)

These are early versions of the axioms of separation, union, replacement, power set and infinity as they later appear in the axiomatic system of Zermelo-Skolem-Fraenkel, ZSF. Only the axiom of union (A2) is not quite correctly formulated. In his letter of August 3, 1899, to Dedekind, Cantor gave the following correct formulation (cf. *Cantor 1991*, 407):

> Jede Menge von Mengen ist, wenn man die letzteren in ihre Elemente auflöst, auch eine Menge.[28]

Émile Borel in his *Leçons sur la théorie des fonctions* (*1898*, 104) reported that in August 1897 he had met Cantor at the International Congress of Mathematicians in Zurich, Switzerland, and that they had discussed set-theoretic topics at great length. Borel's introduction to set theory in his *Leçons* (*1898*, 1–20) is perhaps based on recollections of these discussions. In his *Leçons*, Borel discusses some of the above mentioned axioms. In particular, he discusses the replacement axiom (A3), saying that a multitude which results from a *"given set"* (*"d'une ensemble donné"*) by replacing each element by another element also has to be considered a set. Borel, however, explicitly refused to define the concept of set.

Cantor's definition of a "finished set" is unsuitable as a definition and falls short of its goal. Its inadequacy is apparent. The multitude of all sets can in fact "be thought of without contradiction as being together and hence forming a thing in itself". In the von Neumann-Bernays-Gödel set theory NBG this multitude exists as an object. But this object must not be treated as if it were a true set. However, it is remarkable that it was Cantor who in 1898 first presented a list of set-theoretic axioms.

[28] Each set of sets, when resolved into its elements, is again a set.

Gregory Moore (*1976*) referred to a 1905 paper by the English mathematician A. E. Harward which appeared some years before Zermelo published his axiom system. This paper included some of the Zermeloan axioms, and so Moore asked the "tantalizing question" whether some of Zermelo's ideas had been in the air. Presumably the question has a positive answer, as can be seen by Cantor's letters to Hilbert of October 10, 1898, and to Dedekind of August 3, 1899, and also by Borel's *Leçons*. Zermelo's notebooks also show that he had worked on a formulation of axioms for set theory between 1904 and 1906. Harward in 1905 merely formulated some of Cantor's axioms and also a certain multiplicative axiom. It is not plausible that Harward knew Cantor's letters. Zermelo did not know them either, for otherwise he would not have omitted the axiom of replacement in his system of axioms.

The first persuasive axiomatization of set theory was produced by Zermelo. He submitted his paper in July 1907; it appeared in print in 1908. He had begun preparatory work between 1904 and 1906, and in his lectures on "Set Theory and the Concept of Number" ("Mengenlehre und Zahlbegriff") in the summer semester 1906, he had presented a preliminary version.

Zermelo proposed seven axioms. These are the Cantorian axioms, without the axiom of replacement, but with an axiom of extensionality, an axiom of elementary sets, and an axiom of choice, none of which had been formulated by Cantor. In the following, we shall discuss these axioms in some detail.

An **axiom of extensionality (Axiom I)** was originally formulated by Bolzano in his *Größenlehre*, but this book was published posthumously in 1975 and could not have influenced Zermelo. Independently of Bolzano, Dedekind formulated this axiom in his essay *1888*, §1, art. 2. Presumably, Zermelo took this axiom from Dedekind, calling it "Axiom der Bestimmtheit" ("axiom of decisiveness").

The **existence of an empty set** (also called "null set" ("Nullmenge") by Zermelo and others) is postulated in **Axiom II**. It may be a bit surprising that Zermelo spoke of an *improper set* (*uneigentliche Menge*). But this cautious mode of expression has its cause in Cantor's definition of the concept of set. In fact, it is not clear from Cantor's definition whether the empty class really is a set. Sets, according to Cantor, are generated by an act of aggregating various entities into a whole. An empty class, however, is not generated by such an act. How can an empty class be an object, when it is not created by an act? Bertrand Russell in his *Principles of Mathematics* (*1903*, 74) posed the question: can a non-empty set remain a set when all its elements are removed?[29] He concluded (op. cit., 75) that "*there is no actual null-class*", but, since it is useful to have an empty set at one's disposal, it is nonetheless

[29] Compare this with the cat in Lewis Carroll's *Alice's adventures in wonderland*, London 1865/1866, which "vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone. 'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in all my life!'."

introduced. From this point of view, it may be understandable that Zermelo spoke of an "improper" set. To emphasize that this was of some importance for Zermelo, as he explained in a 1921 letter to Fraenkel (1891–1965):[30]

> Nebenbei wird mir selbst die Berechtigung dieser "Nullmenge" immer zweifelhafter. Könnte sie nicht entbehrt werden bei geeigneter Einschränkung des "Aussonderungsaxioms"? Tatsächlich dient sie doch nur zur formalen Vereinfachung.[31]

Similarly, Fraenkel in his book *Einleitung in die Mengenlehre* (*1923a*, 15), introduced the concept of an "empty set" for purely formal reasons. The same is true of Hessenberg (*1906*), Hausdorff (*1914*), and many others. In time, however, the empty set assumed the same authority and legitimacy as all the other sets.

The **axiom of separation (Axiom III)** will be discussed in the next section in connection with the difficult problem of "definite properties". Let us mention here only that Zermelo named it "Axiom der Aussonderung".

The **power set axiom (Axiom IV)** postulates that for each set M there is another set whose elements are the (definable and the undefinable) subsets of M. For that reason the size of the power set of a set is almost ungraspable and its power cannot be unambiguously determined. Nonetheless, the power set axiom renders a typical feature of the Bolzano-Cantorian concept of a set (see §1, the third attribute) and also (in conjunction with the axiom of separation in its full strength) makes possible the existence of different transfinite cardinalities.

The **axiom of union (Axiom V)**, Zermelo's "Axiom der Vereinigung", is unproblematic.

The formulation of the **axiom of choice (Axiom VI)** is Zermelo's major creative accomplishment. With the introduction of this axiom, he made a significant step beyond Cantor and Dedekind. The formulation of this axiom took place in conversations with Erhard Schmidt. In the second part of his *Investigations* Zermelo demonstrated that quite a number of set-theoretic problems can be solved with the aid of this axiom. Accordingly, the axiom of choice is indispensable.

In the formulation of the **axiom of infinity (Axiom VII)** Zermelo profited from Dedekind's having provided a definition of infinite set.[32] Dedekind in his essay *1888*, §5, called a set M infinite if there is a one-to-one mapping from M onto a proper subset of M. In his lecture on "Set Theory and the Concept of Number" ("Mengenlehre und Zahlbegriff", summer-semester

[30] Quoted in *Ebbinghaus 2007b*, 285.

[31] By the way, the justification of the "empty set" is becoming more and more doubtful for me. Would it be possible to dispense with it by a suitable limitation of the axiom of separation? Actually, the empty set serves only for some formal simplification.

[32] An account of the history of this finiteness definition is given in the marginal note 20 in *Hausdorff 2002*, 588.

1906), Zermelo formulated the axiom of infinity as follows: there is a set which is equivalent to (i.e. equipollent with) a proper subset of itself. In his *Investigations*, however, Zermelo gave the following more concrete version of the axiom: there is a set Z ("Z" for "Zahlen") with the following properties, $\emptyset \in Z$ and $\{a\} \in Z$ for each $a \in Z$. Here we have $\emptyset \neq \{\emptyset\}$ since $\emptyset$ is empty and $\{\emptyset\}$ is non-empty. We also have $\{\emptyset\} \neq \{\{\emptyset\}\}$, etc.

In Zermelo's system of axioms, an axiom of replacement and an axiom of foundation were missing. Both axioms were introduced shortly after 1920 and added to Zermelo's system.

The **axiom of replacement (Ersetzungsaxiom)**. In a letter to Zermelo of May 6, 1921, Fraenkel had posed the following question (quoted in *Ebbinghaus 2007b*, 136 and 286):

> Es sei Z_0 eine unendliche Menge ... und $\mathfrak{U}(Z_0) = Z_1, \mathfrak{U}(Z_1) = Z_2$, usw. Wie folgt dann aus Ihrer Theorie (Grundl. d. M. I), daß $\{Z_0, Z_1, Z_2, \ldots\}$ eine Menge ist, daß also die Vereinigungsmenge existiert? Würde Ihre Theorie zu einem solchen Beweis nicht genügen, so wäre offenbar z. B. die Existenz von Mengen von der Kardinalzahl $\aleph_\omega$ nicht beweisbar.[33]

In his answer of May 9, 1921, Zermelo wrote (cf. *Ebbinghaus 2007b*, 286) that he had missed this point when writing his *Investigations*, and he continues:

> In der Tat wird da wohl noch ein Axiom nötig sein, aber welches? Man könnte es so versuchen: Sind die Dinge $A, B, C, \ldots$ durch eineindeutige Beziehung den Dingen $a, b, c, \ldots$ zugeordnet, welche die Elemente einer Menge m bilden, so sind auch die Dinge $A, B, C, \ldots$ Elemente einer Menge M.[34]

Zermelo called this axiom "*Zuordnungs-Axiom*" (axiom of assignment, or axiom of coordination), which did not please him very much because he felt that the concept of assignment was not definite enough ("zu wenig definit"). In 1921 (at a meeting, published as *1921*), Fraenkel renamed this axiom the "axiom of replacement" ("Ersetzungsaxiom"), cf. *Fraenkel 1922b*. He adopted Zermelo's formulation almost verbatim, but also allowing replacements which are not one-to-one. However, the nature of these replacements remained unclear. It is remarkable that Skolem in 1922 (at a meeting, published in 1923) too, independently of Zermelo and Fraenkel, also proposed an axiom of replacement (cf. *Skolem 1923*, 145–146). In Skolem's formulation, however, the

[33] Let Z_0 be an infinite set ... and $\mathfrak{U}(Z_0) = Z_1, \mathfrak{U}(Z_1) = Z_2$, etc. How can you show in your theory ... that $\{Z_0, Z_1, Z_2, \ldots\}$ is a set, and, hence, that, e.g., the union of this set exists? If your theory is insufficient to allow such a proof, then obviously the existence of, say, sets of cardinality $\aleph_\omega$ cannot be proved.

[34] Indeed, a new axiom is necessary here, but which axiom? One could try to formulate it in the following manner: If the objects $A, B, C, \ldots$ are assigned to the objects $a, b, c, \ldots$ by a one-to-one relation, and the latter objects form a set, then $A, B, C, \ldots$ are also elements of a set M.

replacements are given by formulas of the basic set-theoretic language. Skolem also emphasized that such an axiom is necessary if one wished to prove the existence of sets of power $\aleph_\omega$.

An axiom of replacement is also needed, e.g. in the proof of the determinacy of all Borel sets of natural numbers, as was shown by Tony Martin in 1975 (cf. *Martin 1975*). Harvey Friedman proved in 1971 that Borel determinacy is not provable in Zermelo's original system of axioms (cf. *Friedman 1971*). An axiom of replacement is also needed, for instance, in the proof of the existence of the set of all hereditarily-finite sets, as was shown by Adrian Mathias in *2001*. It is the axiom of replacement which in general permits the creation of sets via transfinite recursion, as was shown by von Neumann in *1928d* in his axiomatic version of set theory.

Although the axiom of replacement was formulated already in 1898 by Cantor, it was von Neumann who recognized its fundamental importance. This has been pointed out conclusively by Michael Hallett in his book *1984*, 95 and 280–286.

The **axiom of foundation (Fundierungsaxiom)**. In order to exclude sets A which have the property $A \in A$, or $A \in B \in A$ for a suitable B, etc., in 1923 Skolem proposed an axiom of foundation according to which *all descending* $\in$*-chains are finite.* According to this formulation, an axiom of foundation can be introduced only after the introduction of the concepts of finiteness and of function. Von Neumann proposed the following elegant formulation: *each non-empty set* M *contains an element which has no element in common with* M (cf. *von Neumann 1925*). In the course of the 20th century, other variants of the axiom of foundation were proposed. A discussion of these variants can be found in *Felgner 2002b.*

§5. The problem of "definite" properties

Zermelo's axiomatization made it possible, after a period of uncertainty caused by the appearance of the antinomies, for set theory to be developed further on a solid basis. For that, Zermelo's axiomatization has always been praised. However, some of the Zermeloan axioms met with skepticism. The axiom of choice in particular led to a number of well-known controversies. That need not detain us now. The axiom of separation, however, was also hotly debated because of the notion of "definite property", which occurs in the formulation of that axiom. A few words are required to explain the debate.

In 1882, Cantor spoke of "*well-defined multitudes*" ("wohldefinierte Mannichfaltigkeiten"), meaning sets for which (*1882*, 114)

> auf Grund ihrer Definition und infolge des logischen Prinzips vom ausgeschlossenen Dritten es als intern bestimmt angesehen werden muß, sowohl ob irgendein derselben Begriffssphäre angehöriges Objekt zu der gedachten Mannigfaltigkeit als Element gehört oder nicht,

> als auch, ob zwei zur Menge gehörige Objekte, trotz formaler Unterschiede in der Art des Gegebenseins, einander gleich sind oder nicht.[35]

At first, Zermelo adopted Cantor's terminology. In his notes for his lecture on set theory in the winter semester 1900/1901, which also contain additions from the years 1904–1906, we find in §2 the formulation of some axioms: e.g., an elementary axiom of summation,[36] a weak axiom of foundation,[37] an axiom of power set, and an axiom of separation. In these notes, Zermelo spoke of *well-defined sets* (wohldefinierte Mengen) and *well-defined properties* (wohldefinierte Eigenschaften). In his paper offering a "New proof for the possibility of a well-ordering" (*1908a*) which he finished on July 14, 1907, he formulated the axiom of separation as follows:

> Alle diejenigen Elemente einer Menge M, denen eine für jedes einzelne Element wohldefinierte Eigenschaft $\mathfrak{E}$ zukommt, bilden die Elemente einer zweiten Menge $M_{\mathfrak{E}}$, einer "Untermenge" von M.[38]

But even in his 1906 Göttingen lecture "Set theory and the concept of number" (*Mengenlehre und Zahlbegriff*), we find an axiom in which the word "definite" appears and which was meant to express the fact that all things of the basic domain (Grundbereich) are pairwise distinguishable (cf. *Ebbinghaus 2007b*, 83, and *Peckhaus 1990a*, 96). Zermelo's formulation reads as follows:

> Sind a, b Dinge, so ist entschieden, "definit", ob $a = b$ oder $a \neq b$.[39]

Finally, in his *Investigations* Zermelo formulated the axiom of separation with the less commonly used word "definite" instead of the less precise phrase "well-defined".

Zermelo borrowed the word "definite" from Edmund Husserl, who introduced it shortly after 1900 and discussed it in some detail in a series of two lectures at the Göttingen Mathematical Society in the winter semester 1901/1902. Husserl also discussed it twelve years later in great detail in his *Ideen zu einer reinen phänomenologischen Philosophie* (cf. *Husserl 1950*, 167), noting in a footnote (loc. cit., 168) that the word had entered mathematical discourse without revealing its source. Husserl probably also had Zermelo in mind, for Zermelo had used the word in his axiom of separation without indicating its source.

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[35] Because of their definition and because of the logical principle of the excluded third one has to consider as internally determined whether or not any object of the same genus belongs to the set, and also whether or not two objects which belong to the set are equal despite formal differences in the way they are given.

[36] It postulates that for each object m the set $\{m\}$ exists whose only element is m, and that in addition for any well-defined set M the set M augmented by the new element m exists and is well-defined.

[37] It states that a well-defined set never contains itself as an element.

[38] All elements of a set M that have a property $\mathfrak{E}$ well-defined for every single element are the elements of another set, $M_{\mathfrak{E}}$, a "subset" of M.

[39] If a and b are objects, then it is determined, "definite", whether $a = b$ or $a \neq b$.

Using the expression "definite", Husserl intended to indicate that "definite properties" are mathematically exhaustive defined properties (cf. *Husserl 1950*, 167, and also §31 in Husserl's *Formale und Transzendentale Logik*, *Husserl 1974*, 98–102). Therefore, a concept is "definite" in the sense of Husserl when it permits a detailed analysis (not of its content, but) of its extension. Zermelo outlined his understanding of the word "definite" roughly as follows. A property $\mathfrak{E}$ is "definite", provided that for each element of a given set M whether $\mathfrak{E}$ holds or not can be decided without any arbitrariness on the basis of the $\in$-relation which holds among the objects of the domain.

Hermann Weyl was among the first who expressed doubts about the clarity and distinctness of the concept of "*Definitheit*". In *1910*, he proposed removing the word "definite" from the formulation of the axiom of separation and admitting it only for those properties which can be formulated

> aufgrund der beiden Beziehungen = and $\in$
> (on the basis of the relations = and $\in$)

as the only non-logical signs. In his pamphlet *Das Kontinuum* (*1918*, 36) he stated more clearly that he was thinking of first-order logic. A few years later Skolem in *1923* made the same proposal and in a rather similar manner von Neumann eliminated the notion of "definiteness". In his letter to Zermelo of August 15, 1923 he wrote that in his axiom system he does not introduce the notion of "definiteness" explicitly but instead states all the admissible schemes for the formation of functions and sets. Von Neumann's letter is reprinted in *Meschkowski 1967*, 271–273.

By requiring "definiteness" Zermelo wanted to prevent the use of properties in the axiom of separation which had led to well-known semantic antinomies such as the antinomy of Jules Richard, 1905, the antinomy of G. G. Berry, 1906, etc. In later years Zermelo came back to the problem of "definiteness". He discussed it in his 1929 essay "Über den Begriff der Definitheit in der Axiomatik" (*1929a*) without, however, reaching a final solution (cf. *Ebbinghaus 2007b*, 179–183).

§6. The background of Zermelo's system of axioms

Zermelo's requirement that the properties which are allowed to occur in the axiom of separation should be "definite" shows that he did not take a formalist position. His position was always the *contentual standpoint* (der inhaltliche Standpunkt). Whether a property is "definite" or not, cannot be decided formally by an application of a general rule, but only by an examination of the content and the meaning of the property.

The fact that Zermelo took the contentual standpoint is apparent everywhere in his work. One sees this, for example, by the way in which he treated the $\in$-sign. Axioms are formulated in everyday language with the minimal use of signs, namely a sign for the elementhood relation and a few comprehension terms. In Zermelo's axiom system, the ∈-sign is not an undefined sign whose meaning is defined implicitly by the totality of all axioms, but rather a symbol, i.e., a sign to which the presupposed meaning of true elementhood is attached (see footnote 24).

That Zermelo employed the sign "∈" not as an undefined sign but as a symbol can be seen from a remark made by Zermelo in the sequel to his Theorem 10. There he argues that, since the domain $\mathfrak{B}$ of sets itself is not a set, the Zermelo-Russellian antinomy can simply be set aside (Zermelo's *Investigations*, 265). His argument is valid, however, only when the relation "∈" is the actual true elementhood relation.

The problem of the "domains" (Bereiche). At the beginning of his essay, Zermelo indicates that set theory is concerned with a "domain" of objects (*einem "Bereich"* $\mathfrak{B}$ *von Objekten*). In the formulation of some of the axioms, reference is made to these "domains". This provoked considerable confusion and a number of controversies. Is the concept of a "domain" another primitive notion of set theory? In what respect are "domains" different from "sets"? Would it be permissible simply to omit all references to these "domains" or is it necessary to provide axioms for "domains"?

It is striking that only in §1 of his *Investigations* does Zermelo refer to "domains". The word appears there no fewer than eleven times. In the much larger §2, the word "domain" does not appear at all. One gets the impression that the term would be superfluous. This seems in fact to be the case. In his paper "Über Grenzzahlen und Mengenbereiche" (*1930a*), Zermelo formulates all the axioms without any reference to "domains". It becomes clear that speaking of "domains" is only a way of referring informally to subdomains of the world of finite and transfinite sets (probably including urelements) in which all the Zermeloan (or Zermelo-Skolem-Fraenkelian) axioms are valid. Validity in this context means that the ∈-sign is to be interpreted by the true elementhood relation and the concept of power set by the true power set.

In his axiomatization, Zermelo departs from the content of the concept of set as elaborated by Bolzano and Cantor. However, Zermelo derives from Bolzano and Cantor only those principles (or axioms), which lead to unproblematic sets. He does not start his set theory with an explicit definition of the concept of set, but the Bolzano-Cantor definition of the concept of set is the unmentioned basis for his axiom system. In his paper on the "New proof for the existence of a well-ordering" (*1908a*), Zermelo indicates that in order to avoid the antinomies he will rely upon "specialized versions of Cantor's definition". Until the "correct" definition of the concept of set can be found, the axioms should mention particular characteristics of such a definition. This indicates clearly that Zermelo in his axiomatization is following the classical paradigm (Aristotle, Euclid, Hobbes, Tschirnhaus, Wolff, et al.), but he himself does not yet feel able to present the "correct" definition of what a set is.

As late as 1929–1932 Zermelo was searching for an adequate definition (cf. *Ebbinghaus 2007b*, 213–217).

Although Zermelo worked closely in Göttingen together with Hilbert, and although he was obviously influenced by Hilbert's early axiomatization program, he never adopted Hilbert's formalist standpoint. (In the literature, it is quite frequently claimed wrongly that he did.) Zermelo's standpoint was the contentual standpoint. In fact, he repeatedly criticized the formalist standpoint.

The modification and extension of Zermelo's system proposed by Skolem and Fraenkel is now widely accepted. The system of axioms consists of all of Zermelo's axioms, except for the axiom of choice, supplemented with the axiom of replacement and the axiom of foundation. This system is usually denoted by ZF (or more accurately by ZSF) and is called the Zermelo-Fraenkel (or Zermelo-Skolem-Fraenkel) axiom system. These axioms are formulated in a first-order language whose only non-logical sign is the $\in$-sign. A sign whose intended interpretation is "is a set" is not necessary because all objects referred to in the axioms should be considered "sets".

The question in which sense the sets of the theory ZSF are "things" (as discussed above in §1) has a very liberal answer. In ZSF sets are treated formally as things, which should be taken to mean that they are all included in the domain of values of the individual variables.

The modification of the Zermeloan axiom system proposed by Skolem yielded a change in the underlying philosophical standpoint: the contentual standpoint was replaced by the formalist standpoint. Zermelo did not agree with that change and he expressed his disagreement in harsh polemics.

§7. Cardinal arithmetic in Zermelo's paper

In the second part of his *Investigations* Zermelo treats the notions of finite and transfinite number. In the years around 1900 many aspects of the number concept had become confused and some of them had been threatened by paradoxes and antinomies. It was Zermelo's aim to construct a sound arithmetic of finite and infinite cardinal numbers. To begin with, let us first recall the issues and problems that had emerged with the number concept.

(1) Firstly, we have to mention the concept of natural number. The exposition of *Dedekind 1888* lost credibility because Dedekind posited the existence of an infinite class of all possible thoughts. (This was discussed above in §2, where it was noted that this is an absolute infinite proper class.) Dedekind acknowledged the difficulties connected with his procedure and for quite some time was unwilling to allow a reprint of his essay *1888*.

Frege's approach to the arithmetic of natural numbers was also unsuccessful. He thought, mistakenly, that he could introduce the natural numbers in a canonical way as logical objects, i.e., as classes of concepts related to the concept of equinumerability. In 1902, when Russell found an inconsistency in Frege's theory, Frege was greatly distressed. He confessed that the basis on which he intended to build arithmetic was shaken.

On the basis of his axiom system, Zermelo was able to postulate the existence of a set Z_0 representing the natural numbers. Z_0 is the intersection of all inductive subsets of a set whose existence is guaranteed by the axiom of infinity. A set M is called *inductive* whenever it contains the empty set as an element, and with each element a also contains the singleton $\{a\}$. The elements of Z_0 are hence $0 = \emptyset, 1 = \{0\}, 2 = \{1\} = \{\{0\}\}, \ldots, n+1 = \{n\}, \ldots$.

(2) Secondly, we have to mention the concept of the cardinal number $\mathsf{Card}(M)$ of an arbitrary set M. In his construction of cardinal numbers Cantor referred to an act of a twofold abstraction, but he was unable to say what objects are created by this procedure (cf. *Cantor 1895*, 481).

In 1903, Russell defined the cardinality of a set M as the class of all sets which are equipotent with M. He wrote (*1903*, 115):

> This method [...], to define as the number of a class the class of all classes similar to the given class [...], is an irreproachable definition of the number of a class in purely logical terms.

It is incomprehensible why Russell, who displays in full detail the antinomy of the set of all sets which do not contain themselves as elements (*1903*, Chapter X, 101–107), did not realize that his definition of cardinality ran into the same antinomy. (The number 1 e.g. is the set of all singletons and we have hence $\{\{x\}; x \subseteq 1\} \subseteq 1$, which leads to a contradiction in almost the same way as in the Zermeloan argument, see §2.)

When writing his *Investigations*, Zermelo did not yet know how to introduce set-theoretic representatives for the sizes of infinite sets. Therefore, he only treated the notion of equipollence (also called equivalence). Additionally, a suitable set-theoretic representation of the general notion of a function was not yet known to Zermelo (such a definition was first given by Hausdorff in 1914, cf. *Felgner 2002c*), he had to work with equipollent disjoint sets. His theorems 10, 19, and 28 are designed to make this possible.

A successful definition of the notion of ordinal number and of cardinal number was discovered by Zermelo in the years 1913–1915. This is apparent from a footnote in *Bernays 1941*, 6. (See also the careful documentations in *Hallett 1984*, 271–280, and in *Ebbinghaus 2007b*, 133–134.) Independently of Zermelo, von Neumann also introduced the same objects as ordinal numbers. These definitions can be found in all monographs on set theory. A detailed discussion of all the difficulties which are connected with the definition of cardinal numbers can be found in *Felgner 2002a*.

(3) Thirdly, there is the problem of whether or not all cardinal numbers are alephs, i.e., cardinal numbers of well-orderable sets. Cantor repeatedly claimed that this was the case. He tried to prove this with the argument that

the "set" of all alephs is an absolute infinite totality (cf. Cantor's letter of September 26, 1897, to Hilbert, see *Cantor 1991*, 388–390). Philip Jourdain also published a "proof", which failed. It was Zermelo, who in 1904 published the first correct proof of the well-orderability of each set by assuming a new axiom, the axiom of choice. Zermelo's proof was strongly attacked by many mathematicians. In order to make apparent the correctness of his proof, Zermelo sought to make clear the grounds upon which his proof rests.

(4) Fourthly, there is the problem of trichotomy. Cantor made the claim that for any cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$ one has either $\mathfrak{a} < \mathfrak{b}$, or $\mathfrak{a} = \mathfrak{b}$, or $\mathfrak{a} > \mathfrak{b}$. Cantor announced a proof, but he did not publish it nor reveal to anyone the details of this alleged proof (cf. *Cantor 1895*, 484, *Cantor 1932*, 351, remark 2). The truth of the law of trichotomy was doubted occasionally by skeptics. Russell, for example, wrote in his *Principles of mathematics* (*Russell 1903*, 323):

> [...] and it may be that $2^{\aleph_0}$ is neither greater nor less than $\aleph_1$ and $\aleph_2$ and their successors.

Felix Bernstein (*1901*) and Godfrey Harold Hardy (*1904*) were able to prove $\aleph_1 \leq 2^{\aleph_0}$, but their proofs relied implicitly upon the axiom of choice, which was not yet isolated as a set-theoretic principle. Zermelo formulated the axiom of choice in 1904 and thereby deduced the well-ordering principle from it. Thus, the law of trichotomy follows from the axiom of choice and hence is provable in Zermelo's axiom system.

It should be noted that, conversely, the law of trichotomy implies the well-ordering theorem as was shown by Friedrich Hartogs (*1915*, 436–443). Thus, all three statements—the axiom of choice, the law of trichotomy, and the well-ordering principle—are equivalent on the basis of the remaining axioms.

(5) Fifthly, there is the annoying "result" announced by Julius König in 1904 at the International Congress of Mathematicians in Heidelberg. He offered a "proof" of the non-wellorderability of the continuum. In his "proof" he used the Bernstein aleph-formula $\aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta} \cdot \aleph_\alpha$. His "result" was sensational and seemed entirely to topple Cantor's set theory, but, a few days after König's lecture, Hausdorff and Zermelo independently found the mistake in König's "proof". In fact, Bernstein's aleph-formula is valid only for finite ordinals α (cf. *Ebbinghaus 2007a* and *Purkert 2002*, 9–12). In consequence, König withdrew his "proof". The analysis of König's faulty argument led Hausdorff to his recursion formula (Rekurrenz-Formel) and Zermelo to the Zermelo-König inequality, discussed below.

In §2 of the *Investigations* Zermelo aimed at corrections of these misleading and erroneous "results". He sought also to set to rights the arithmetic of finite and transfinite cardinal numbers. He rounded off his paper with the presentation of a few further classical results, for which he was able to present elegant new proofs.

The Dedekind-Bernstein equivalence theorem

Theorems 25–27 present a proof of the important equivalence theorem, which is usually ascribed to Cantor, Bernstein, and Schröder. Cantor, however, in 1882/1883, formulated the theorem only for well-orderable sets. Bernstein found a proof in the winter of 1896/1897 which he included in his PhD dissertation (Göttingen 1901). He published this proof in 1905 (*1905c*) after Borel had incorporated it in his *Leçons sur la théorie des fonctions* (*1898*, 104–105). The logician Ernst Schröder also attempted, in 1898, to prove the theorem, but Alwin Reinhold Korselt, in 1902, found a mistake in Schröder's proof. A totally different proof was given by Dedekind in 1887, but this proof remained unpublished. The manuscript was found by Jean Cavaillès in Dedekind's posthumous papers. (See *Dedekind 1932*, 447–449.) In addition, Dedekind communicated his proof in a letter of August 29, 1899 to Cantor. The letter was published in Cantor's Collected Works (*Cantor 1932*, 449 and 451).

In the first decade of the twentieth century, Zermelo knew only Bernstein's proof. When he lectured on set theory in Göttingen during the winter semester 1900/1901, he included Bernstein's proof of the equivalence theorem. However, he added a little note saying that Bernstein's proof rests heavily on the construction of a chain of the following form $Q, \varphi(Q), \varphi(\varphi(Q)), \varphi(\varphi(\varphi(Q))), \ldots$. We follow Zermelo's notation in the proof of his theorem 25 and consider sets $M' \subseteq M_1 \subseteq M$, a one-to-one mapping φ from M onto M' and put $Q = M_1 - M'$. It is clear that $A_0 = Q \cup \bigcup_{1 \leq n \in \omega} \varphi^n(Q)$ and $\bigcup_{1 \leq n \in \omega} \varphi^n(Q)$ are equipollent, and this simple fact is the basis of Bernstein's proof.

According to Dedekind, such a set A_0 can also be described "from above" as the intersection of all φ-closed sets which contain the initial object Q (as a subset),

$$A_0 = \bigcap \{X \subseteq M;\, Q \subseteq X \;\&\; \varphi(X) \subseteq X\}\,.$$

This opens the possibility of another proof of the equivalence theorem. Zermelo had probably realized this by the spring of 1905. He outlined a new proof on a postcard to Hilbert written at the end of June 1905 (cf. *Ebbinghaus 2007b*, 89 and 279–280). As Zermelo indicated, this new proof rests on Dedekind's theory of chains (cf. *Dedekind 1888, art. 37, 44, 63*). This new and elegant proof is used to establish theorem 25. Much later, Zermelo discovered that Dedekind himself had almost the same proof in 1887/1899.[40] We note in passing that in Zermelo's first proof of the well-ordering theorem (*Zermelo 1904*) the well-ordering is obtained "from below" as a union, and that in his second proof (*Zermelo 1908a*) the well-ordering is obtained "from above" as an intersection. This second proof rests on Dedekind's theory of chains.

[40] See the introductory note to *Zermelo 1901* for more on the history of the equivalence theorem.

The multiplication theorem

It is rather surprising that Zermelo does not prove the laws of idempotency for the addition and the multiplication of cardinal numbers, $\aleph_\alpha + \aleph_\alpha = \aleph_\alpha$ and $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$ for all alephs. This is surprising because all other laws about addition and multiplication easily follow from these laws, and it was Zermelo who first announced these laws without proof at the end of his 1904 paper on the well-ordering theorem. He communicated his proof to Hessenberg and probably also to some other colleagues. In 1906, Hessenberg found his own proof and published it in his book *Grundbegriffe der Mengenlehre* (*1906*). On page 109 he included the following remark:

> In jüngster Zeit hat mir Herr Zermelo einen Beweis mitgeteilt, der von dem hier gegebenen wesentlich verschieden ist und demnächst an anderer Stelle erscheinen wird.[41]

But Zermelo had not yet published his proof. Moreover, in his papers no sketch of a proof has been found so far. (See *Deiser 2005* for more on the history of the multiplication theorem.)

The Zermelo-König inequality

The highlight of the second part of Zermelo's *Investigations* is the proof of the so-called Zermelo-König inequality (theorem 33). From that inequality, the original lemma of König and also the Cantor inequality $\mathfrak{m} < 2^{\mathfrak{m}}$ follow. A diagonal argument which is used in Cantor's proof of his inequality is also used in the proof of the Zermelo-König inequality. From a historical point of view it is remarkable that the first diagonal argument was given by Paul du Bois-Reymond in 1873 in connection with his "Infinitär-Kalkül" (cf. *du Bois-Reymond 1873*).

In his book on transfinite numbers, Ivan Ivanovich Shegalkin independently discovered a proof of the above-mentioned Zermelo-König inequality (cf. *Shegalkin 1907*). However, Zermelo had already proved the inequality in 1904. In his *Investigations* he proudly added (p. 279):

> Das vorstehende (Ende 1904 der Göttinger Mathematischen Gesellschaft von mir mitgeteilte) Theorem ist der allgemeinste bisher bekannte Satz über das Größer und Kleiner der Mächtigkeiten, aus dem alle übrigen sich ableiten lassen.[42]

That the Zermelo-König inequality also plays a fundamental role in the proofs of the laws of the exponentiation of alephs was shown, much later, by Alfred Tarski in *1925*.

[41] Recently, Zermelo informed me of his proof, which is essentially different from the one given here and which will appear somewhere else in the near future.

[42] This theorem (communicated by me to the Göttingen Mathematical Society at the end of 1904) is the most general theorem now known concerning the comparison of cardinalities, one from which all the others can be derived.

§8. Concluding remarks

Zermelo wrote his *Investigations* in Chesières, a village situated between Montreux and Martinach (Martigny), near Aelen (Aigle), in the Alpine range at the edge of the Rhône valley. He spent the summer of 1907 there in order to recuperate from his pulmonary illness. Here he also wrote his paper on the "New proof of the possibility of a well-ordering".

Both papers are connected. In order to secure his much-criticized proofs of the well-ordering theorem, he carefully analyzed the grounds upon which his proofs rest. (This has been discussed in detail by Moore in *1978*). But this was surely not Zermelo's main concern. What he wanted to achieve was to secure cardinal arithmetic and in particular to publish his own contributions in this field.

To the title of Zermelo's paper was added the words "Part I", but no Part II has ever been published. In the preface to Part I Zermelo wrote that

Untersuchungen über die Grundlagen der Mengenlehre I

1908b

Die Mengenlehre ist derjenige Zweig der Mathematik, dem die Aufgabe zufällt, die Grundbegriffe der Zahl, der Anordnung und der Funktion in ihrer ursprünglichen Einfachheit mathematisch zu untersuchen und damit die logischen Grundlagen der gesamten Arithmetik und Analysis zu entwickeln; sie bildet somit einen unentbehrlichen Bestandteil der mathematischen Wissenschaft. Nun scheint aber gegenwärtig gerade diese Disziplin in ihrer ganzen Existenz bedroht durch gewisse Widersprüche oder „Antinomieen", die sich aus ihren scheinbar denknotwendig gegebenen Prinzipien herleiten lassen und bisher noch keine allseitig befriedigende Lösung gefunden haben. Angesichts namentlich der „*Russell*schen Antinomie" von der „Menge aller Mengen, welche sich selbst nicht als Element enthalten"* scheint es heute nicht mehr zulässig, einem beliebigen logisch definierbaren Begriffe eine „Menge" oder „Klasse" als seinen „Umfang" zuzuweisen. Die ursprüngliche Cantorsche Definition einer „Menge" als einer „Zusammenfassung von bestimmten wohlunterschiedenen Objekten unserer Anschauung oder

[1] B. Russell, „The Principles of Mathematics", vol. I, p. 366–368, 101–107.

> ein zweiter Artikel, der die Lehre von der Wohlordnung und ihre[r] Anwendung auf die endlichen Mengen und die Prinzipien der Arithmetik im Zusammenhang entwickeln soll, in Vorbereitung [sei].[43]

The second paper announced here was not written. However Zermelo gave a talk on the intended topic at the International Congress of Mathematicians 1908 in Rome, and published its main contents in the proceedings of the congress, cf. *Zermelo 1909b*. A slightly extended version of that paper appeared in 1909 in the Swedish journal *Acta mathematica* (cf. *Zermelo 1909a*). These two papers may be considered as a substitute for the abandoned second paper.

[43] A second paper, which will develop the theory of well-ordering together with its application to finite sets and the principles of arithmetic, is in preparation.

Investigations in the foundations of set theory I

1908b

Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions "number", "order", and "function", taking them in their pristine, simple form, and to develop thereby the logical foundations of all of arithmetic and analysis; thus it constitutes an indispensable component of the science of mathematics. At present, however, the very existence of this discipline seems to be threatened by certain contradictions, or "antinomies", that can be derived from its principles—principles necessarily governing our thinking, it seems—and to which no entirely satisfactory solution has yet been found. In particular, in view of the "*Russell* antinomy" of the set of all sets that do not contain themselves as elements,[1] it no longer seems admissible today to assign to an arbitrary logically definable notion a "set", or "class", as its "extension". Cantor's original definition of a "set" as "a collection, gathered into a whole, of certain well-distinguished

[1] *Russell 1903*, pp. 366–368 and 101–107.

unseres Denkens zu einem Ganzen"[2] bedarf also jedenfalls einer Einschränkung, ohne daß es doch schon gelungen wäre, sie durch eine andere, ebenso einfache zu ersetzen, welche zu keinen solchen Bedenken mehr Anlaß gäbe. Unter diesen Umständen bleibt gegenwärtig nichts anderes übrig, als den umgekehrten Weg einzuschlagen und, ausgehend von der historisch bestehenden „Mengenlehre", die Prinzipien aufzusuchen, welche zur Begründung dieser mathematischen Disziplin erforderlich sind. Diese Aufgabe muß in der Weise gelöst werden, daß man die Prinzipien einmal eng genug einschränkt, um alle Widersprüche auszuschließen, gleichzeitig aber auch weit genug ausdehnt, um alles Wertvolle dieser Lehre beizubehalten.

In der hier vorliegenden Arbeit gedenke ich nun zu zeigen, wie sich die ge-
262 samte von *G. Cantor* und *R. Dedekind* geschaffene Theorie auf | einige wenige Definitionen und auf sieben anscheinend voneinander unabhängige „Prinzipien" oder „Axiome" zurückführen läßt. Die weitere, mehr philosophische Frage nach dem Ursprung und dem Gültigkeitsbereiche dieser Prinzipien soll hier noch unerörtert bleiben. Selbst die gewiß sehr wesentliche „Widerspruchslosigkeit" meiner Axiome habe ich noch nicht streng beweisen können, sondern mich auf den gelegentlichen Hinweis beschränken müssen, daß die bisher bekannten „Antinomieen" sämtlich verschwinden, wenn man die hier vorgeschlagenen Prinzipien zugrunde legt. Für spätere Untersuchungen, welche sich mit solchen tiefer liegenden Problemen beschäftigen, möchte ich hiermit wenigstens eine nützliche Vorarbeit liefern.

Der nachstehende Artikel enthält die Axiome und ihre nächsten Folgerungen, sowie eine auf diese Prinzipien gegründete Theorie der Äquivalenz, welche die formelle Anwendung der Kardinalzahlen vermeidet. Ein zweiter Artikel, der die Lehre von der Wohlordnung und ihre Anwendung auf die endlichen Mengen und die Prinzipien der Arithmetik im Zusammenhange entwickeln soll, ist in Vorbereitung.

§ 1.
Grundlegende Definitionen und Axiome

1. Die Mengenlehre hat zu tun mit einem „*Bereich*" $\mathfrak{B}$ von Objekten, die wir einfach als „*Dinge*" bezeichnen wollen, unter denen die „Mengen" einen Teil bilden. Sollen zwei Symbole a und b dasselbe Ding bezeichnen, so schreiben wir $a = b$, im entgegengesetzten Falle $a \neq b$. Von einem Dinge a sagen wir, es „existiere", wenn es dem Bereiche $\mathfrak{B}$ angehört; ebenso sagen wir von einer Klasse $\mathfrak{K}$ von Dingen, „es gebe Dinge der Klasse $\mathfrak{K}$", wenn $\mathfrak{B}$ mindestens ein Individuum dieser Klasse enthält.

2. Zwischen den Dingen des Bereiches $\mathfrak{B}$ bestehen gewisse „*Grundbeziehungen*" der Form $a \varepsilon b$. Gilt für zwei Dinge a, b die Beziehung $a \varepsilon b$, so sagen wir, „a sei *Element* der Menge b" oder „b enthalte a als Element" oder „besitze

[2] G. Cantor, Math. Annalen Bd. 46, p. 481.

objects of our perception or our thought"[2] therefore certainly requires some restriction; it has not, however, been successfully replaced by one that is just as simple and does not give rise to such reservations. Under these circumstances there is at this point nothing left for us to do but to proceed in the opposite direction and, starting from "set theory" as it is historically given, to seek out the principles required for establishing the foundations of this mathematical discipline. In solving the problem we must, on the one hand, restrict these principles sufficiently to exclude all contradictions and, on the other, take them sufficiently wide to retain all that is valuable in this theory.

Now in the present paper I intend to show how the entire theory created by *G. Cantor* and *R. Dedekind* can be reduced to a few definitions and seven "principles", or "axioms", which appear to be mutually independent. The further, more philosophical, question about the origin of these principles and the extent to which they are valid will not be discussed here. I have not yet even been able to prove rigorously that my axioms are "consistent", though this is certainly very essential; instead I have had to confine myself to pointing out now and then that the "antinomies" discovered so far vanish one and all if the principles here proposed are adopted as a basis. But I hope to have done at least some useful spadework hereby for subsequent investigations in such deeper problems.

The present paper contains the axioms and their most immediate consequences, as well as a theory of equivalence based upon these principles that avoids the formal use of cardinal numbers. A second paper, which will develop the theory of well-ordering together with its application to finite sets and the principles of arithmetic, is in preparation.

§ 1.
Fundamental definitions and axioms

1. Set theory is concerned with a "*domain*" $\mathfrak{B}$ of individuals, which we shall call simply "*objects*" and among which are the "sets". If two symbols, a and b, denote the same object, we write $a = b$, otherwise $a \neq b$. We say of an object a that it "exists" if it belongs to the domain $\mathfrak{B}$; likewise we say of a class $\mathfrak{K}$ of objects that "there exist objects of the class $\mathfrak{K}$" if $\mathfrak{B}$ contains at least one individual of this class.

2. Certain "*fundamental relations*" of the form $a \varepsilon b$ obtain between the objects of the domain $\mathfrak{B}$. If for two objects a and b the relation $a \varepsilon b$ holds, we say "a is an *element* of the set b", "b contains a as an element", or "b possesses

[2] *Cantor 1895*, p. 481.

das Element a“. Ein Ding b, welches ein anderes a als Element enthält, kann immer als eine *Menge* bezeichnet werden, aber auch nur dann — mit einer einzigen Ausnahme (Axiom II).

3. Ist jedes Element x einer Menge M gleichzeitig auch Element der Menge N, so daß aus $x \,\varepsilon\, M$ stets $x \,\varepsilon\, N$ gefolgert werden kann, so sagen wir, „M sei *Untermenge* von N“, und schreiben $M \mathrel{\underline{\in}} N$.[1] Es ist stets $M \mathrel{\underline{\in}} M$, und aus
263 $M \mathrel{\underline{\in}} N$ und $N \mathrel{\underline{\in}} R$ folgt immer $M \mathrel{\underline{\in}} R$. „*Elementenfremd*“ | heißen zwei
Mengen M, N, wenn sie keine „gemeinsamen“ Elemente besitzen, oder wenn kein Element von M gleichzeitig Element von N ist.

4. Eine Frage oder Aussage $\mathfrak{E}$, über deren Gültigkeit oder Ungültigkeit die Grundbeziehungen des Bereiches vermöge der Axiome und der allgemeingültigen logischen Gesetze ohne Willkür entscheiden, heißt „*definit*“. Ebenso wird auch eine „Klassenaussage“ $\mathfrak{E}(x)$, in welcher der variable Term x alle Individuen einer Klasse $\mathfrak{K}$ durchlaufen kann, als „definit“ bezeichnet, wenn sie für *jedes einzelne* Individuum x der Klasse $\mathfrak{K}$ definit ist. So ist die Frage, ob $a \,\varepsilon\, b$ oder nicht ist, immer definit, ebenso die Frage, ob $M \mathrel{\underline{\in}} N$ oder nicht.

Über die Grundbeziehungen unseres Bereiches $\mathfrak{B}$ gelten nun die folgenden „*Axiome*“ oder „*Postulate*“.

Axiom I. Ist jedes Element einer Menge M gleichzeitig Element von N und umgekehrt, ist also gleichzeitig $M \mathrel{\underline{\in}} N$ und $N \mathrel{\underline{\in}} M$, so ist immer $M = N$. Oder kürzer: jede Menge ist durch ihre Elemente bestimmt.

(Axiom der Bestimmtheit.)

Die Menge, welche nur die Elemente $a, b, c, \ldots, r$ enthält, wird zur Abkürzung vielfach mit $\{a, b, c, \ldots, r\}$ bezeichnet werden.

Axiom II. Es gibt eine (uneigentliche) Menge, die „*Nullmenge*“ 0, welche gar keine Elemente enthält. Ist a irgend ein Ding des Bereiches, so existiert eine Menge $\{a\}$, welche a und nur a als Element enthält; sind a, b irgend zwei Dinge des Bereiches, so existiert immer eine Menge $\{a, b\}$, welche sowohl a als b, aber kein von beiden verschiedenes Ding x als Element enthält.

(Axiom der Elementarmengen.)

5. Nach I sind die „Elementarmengen“ $\{a\}$, $\{a, b\}$ immer eindeutig bestimmt, und es gibt nur eine einzige „Nullmenge“. Die Frage, ob $a = b$ oder nicht, ist immer definit (Nr. 4), da sie mit der Frage, ob $a \,\varepsilon\, \{b\}$ ist, gleichbedeutend ist.

6. Die Nullmenge ist Untermenge jeder Menge M, $0 \mathrel{\underline{\in}} M$; eine gleichzeitig von 0 und M verschiedene Untermenge von M wird als „*Teil*“ von M bezeichnet. Die Mengen 0 und $\{a\}$ besitzen keine Teile.

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[1] Dieses „Subsumptions“-Zeichen wurde von *E. Schröder* („Vorlesungen über Algebra der Logik“ Bd. I) eingeführt, Herr *G. Peano* und ihm folgend *B. Russell*, *Whitehead* u.a. brauchen dafür das Zeichen $\supset$.

the element a". An object b may be called a *set* if and—with a single exception (Axiom II)—only if it contains another object, a, as an element.

3. If every element x of a set M is also an element of the set N, so that from $x \varepsilon M$ it always follows that $x \varepsilon N$, we say that "M is a *subset* of N" and we write $M \subseteq N$.[3] We always have $M \subseteq M$ and from $M \subseteq N$ and $N \subseteq R$ it always follows that $M \subseteq R$. Two sets M and N are said to be "*disjoint*" if they possess no common element, or if no element of M is an element of N.

4. A question or assertion $\mathfrak{E}$ is said to be "*definite*" if the fundamental relations of the domain, by means of the axioms and the universally valid laws of logic, determine without arbitrariness whether it holds or not. Likewise a "propositional function" $\mathfrak{E}(x)$, in which the variable term x ranges over all individuals of a class $\mathfrak{K}$, is said to be "definite" if it is definite for *each single* individual x of the class $\mathfrak{K}$. Thus the question whether $a \varepsilon b$ or not is always definite, as is the question whether $M \subseteq N$ or not.

The fundamental relations of our domain $\mathfrak{B}$, now, are subject to the following "*axioms*", or "*postulates*".

Axiom I. If every element of a set M is also an element of N and vice versa, if, therefore, both $M \subseteq N$ and $N \subseteq M$, then always $M = N$; or, more briefly: Every set is determined by its elements.

(Axiom of extensionality.)

The set that contains only the elements $a, b, c, \dots, r$ will often be denoted briefly by $\{a, b, c, \dots, r\}$.

Axiom II. There exists a (fictitious) set, the "*null set*", 0, that contains no element at all. If a is any object of the domain, there exists a set $\{a\}$ containing a and only a as element; if a and b are any two objects of the domain, there always exists a set $\{a, b\}$ containing as elements a and b but no object x distinct from both.

(Axiom of elementary sets.)

5. According to Axiom I, the "elementary sets" $\{a\}$ and $\{a, b\}$ are always uniquely determined and there is only a single "null set". The question whether $a = b$ or not is always definite (No. 4), since it is equivalent to the question whether or not $a \varepsilon \{b\}$.

6. The null set is a subset of every set M: $0 \subseteq M$; a subset of M that differs from both 0 and M is called a "part" of M. The sets 0 and $\{a\}$ do not have parts.

[3] ⟦Zermelo uses the sign "$\overline{\in}$" for inclusion. He comments here:⟧ This sign ⟦$\overline{\in}$⟧ of inclusion was introduced by *Schröder* (*1890*). *Peano* and, following him, *Russell*, *Whitehead*, and others use the sign Ↄ instead.

Axiom III. Ist die Klassenaussage $\mathfrak{E}(x)$ definit für alle Elemente einer Menge M, so besitzt M immer eine Untermenge $M_{\mathfrak{E}}$, welche alle diejenigen Elemente x von M, für welche $\mathfrak{E}(x)$ wahr ist, und nur solche als Elemente enthält.

(Axiom der Aussonderung.)

Indem das vorstehende Axiom III in weitem Umfange die Definition neuer Mengen gestattet, bildet es einen gewissen Ersatz für die in der Einleitung angeführte und als unhaltbar aufgegebene allgemeine Mengendefinition, von der es sich durch die folgenden Einschränkungen unterscheidet: Erstens dürfen mit Hilfe dieses Axio-
264 mes | niemals Mengen *independent definiert*, sondern immer nur als Untermengen aus bereits gegebenen *ausgesondert* werden, wodurch widerspruchsvolle Gebilde wie „die Menge aller Mengen“ oder „die Menge aller Ordinalzahlen“ und damit nach dem Ausdrucke des Herrn *G. Hessenberg* („Grundbegriffe der Mengenlehre“ XXIV) die „ultrafiniten Paradoxieen“ ausgeschlossen sind. Zugleich muß zweitens das bestimmende Kriterium $\mathfrak{E}(x)$ im Sinne unserer Erklärung Nr. 4 immer „definit“ d. h. für jedes einzelne Element x von M durch die „Grundbeziehungen des Bereiches“ entschieden sein, und hiermit kommen alle solchen Kriterien wie „durch eine endliche Anzahl von Worten definierbar“ und damit die „Antinomie Richard“ oder die „Paradoxie der endlichen Bezeichnung“ (*Hessenberg* a. a. O. XXIII, vergl. dagegen *J. König*, Math. Ann. Bd. 61, p. 156) für unseren Standpunkt in Wegfall. Hieraus folgt aber auch, daß, streng genommen, vor jeder Anwendung unseres Axioms III immer erst das betreffende Kriterium $\mathfrak{E}(x)$ als „definit“ nachgewiesen werden muß, was denn auch in den folgenden Entwicklungen bei jeder Gelegenheit, wo es nicht ganz selbstverständlich ist, immer geschehen soll.

7. Ist $M_1 \mathrel{\underline{\in}} M$, so besitzt M immer eine weitere Untermenge $M - M_1$, die „*Komplementärmenge* von M_1“, welche alle diejenigen Elemente von M umfaßt, die *nicht* Elemente von M_1 sind. Die Komplementärmenge von $M - M_1$ ist wieder M_1. Die Komplementärmenge von $M_1 = M$ ist die Nullmenge 0, die Komplementärmenge jedes „Teiles“ M_1 von M (Nr. 6) ist wieder ein „Teil“ von M.

8. Sind M, N irgend zwei Mengen, so bilden nach III diejenigen Elemente von M, welche gleichzeitig Elemente von N sind, die Elemente einer Untermenge D von M, welche auch Untermenge von N ist und alle M und N *gemeinsamen* Elemente umfaßt. Diese Menge D wird der „gemeinsame Bestandteil“ oder der „*Durchschnitt*“ der Mengen M und N genannt und mit $[M, N]$ bezeichnet. Ist $M \mathrel{\underline{\in}} N$, so ist $[M, N] = M$; ist $N = 0$ oder sind M und N „elementenfremd“ (Nr. 3), so ist $[M, N] = 0$.

9. Ebenso existiert auch für mehrere Mengen $M, N, R, \ldots$ immer ein „Durchschnitt“ $D = [M, N, R, \ldots]$. Ist nämlich T irgend eine Menge, deren Elemente selbst Mengen sind, so entspricht nach III jedem Dinge a eine gewisse Untermenge $T_a \mathrel{\underline{\in}} T$, welche alle diejenigen Elemente von T umfaßt, die a als Element enthalten. Es ist somit für jedes a definit, ob $T_a = T$ ist, d. h. ob a gemeinsames Element aller Elemente von T ist, und ist A ein beliebiges Element von T, so bilden alle Elemente a von A, für welche $T_a = T$

Axiom III. Whenever the propositional function $\mathfrak{E}(x)$ is definite for all elements of a set M, M possesses a subset $M_{\mathfrak{E}}$ containing as elements precisely those elements x of M for which $\mathfrak{E}(x)$ is true.

(Axiom of separation.)

By giving us a large measure of freedom in defining new sets, Axiom III in a sense furnishes a substitute for the general definition of set that was cited in the introduction and rejected as untenable. It differs from that definition in that it contains the following restrictions. In the first place, sets may never be *independently defined* by means of this axiom but must always be *separated* as subsets from sets already given; thus contradictory notions such as "the set of all sets" or "the set of all ordinal numbers", and with them the "ultrafinite paradoxes", to use Mr. *G. Hessenberg*'s expression (*1906*, chap. 24), are excluded. In the second place, moreover, the defining criterion must always be "definite" in the sense of our definition in No. 4 (that is, for each single element x of M the "fundamental relations of the domain" must determine whether it holds or not), with the result that, from our point of view, all criteria such as "definable by means of a finite number of words", hence the "Richard antinomy" and the "paradox of finite denotation" (*Hessenberg 1906*, chap. 23; on the other hand, see *J. König 1905c*) vanish. But it also follows that we must, prior to each application of our Axiom III, prove the criterion $\mathfrak{E}(x)$ in question to be "definite", if we wish to be rigorous; in the considerations developed below this will indeed be proved whenever it is not altogether evident.

7. If $M_1 \subseteq M$, then M always possesses another subset, $M - M_1$, the "*complement* of M_1", which contains all those elements of M that are *not* elements of M_1. The complement of $M - M_1$ is M_1 again. If $M_1 = M$, its complement is the null set, 0; the complement of any "part" (No. 6) M_1 of M is again a "part" of M.

8. If M and N are any two sets, then according to Axiom III all those elements of M that are also elements of N are the elements of a subset D of M; D is also a subset of N and contains all elements *common* to M and N. This set D is called the "common component", or "*intersection*", of the sets M and N and is denoted by $[M, N]$. If $M = N$, then $[M, N] = M$; if $N = 0$ or if M and N are "disjoint" (No. 3), then $[M, N] = 0$.

9. Likewise, for several sets $M, N, R, \ldots$ there always exists an "intersection" $D = [M, N, R, \ldots]$. For, if T is any set whose elements are themselves sets, then according to Axiom III there corresponds to every object a a certain subset T_a of T that contains all those elements of T that contain a as an element. Thus it is definite for every a whether $T_a = T$, that is, whether a is a common element of all elements of T; if A is an arbitrary element of T, all elements a of A for which $T_a = T$ are the elements of a subset D of A that

ist, die Elemente einer Untermenge D von A, welche alle diese gemeinsamen Elemente umfaßt. Diese Menge D wird „der zu T gehörende Durchschnitt" genannt und mit $\mathfrak{D}T$ bezeichnet. Besitzen die Elemente von T keine gemeinsamen Elemente, so ist $\mathfrak{D}T = 0$, und dies ist z. B. immer der Fall, wenn ein Element von T keine Menge oder die Nullmenge ist.

10. *Theorem.* Jede Menge M besitzt mindestens eine Untermenge M_0, welche nicht Element von M ist.

265 | *Beweis.* Für jedes Element x von M ist es definit, ob $x \varepsilon x$ ist oder nicht; diese Möglichkeit $x \varepsilon x$ ist an und für sich durch unsere Axiome nicht ausgeschlossen. Ist nun M_0 diejenige Untermenge von M, welche gemäß III alle solchen Elemente von M umfaßt, für die *nicht* $x \varepsilon x$ ist, so kann M_0 nicht Element von M sein. Denn entweder ist $M_0 \varepsilon M_0$ oder nicht. Im ersteren Falle enthielte M_0 ein Element $x = M_0$, für welches $x \varepsilon x$ wäre, und dieses widerspräche der Definition von M_0. Es ist also sicher *nicht* $M_0 \varepsilon M_0$, und es müßte somit M_0, wenn es Element von M wäre, auch Element von M_0 sein, was soeben ausgeschlossen wurde.

Aus dem Theorem folgt, daß nicht alle Dinge x des Bereiches $\mathfrak{B}$ Elemente einer und derselben Menge sein können; d. h. *der Bereich* $\mathfrak{B}$ *ist selbst keine Menge*, — womit die „*Russell*sche Antinomie" für unseren Standpunkt beseitigt ist.

Axiom IV. Jeder Menge T entspricht eine zweite Menge $\mathfrak{U}T$ (die „*Potenzmenge*" von T), welche alle Untermengen von T und nur solche als Elemente enthält.

(Axiom der Potenzmenge.)

Axiom V. Jeder Menge T entspricht eine Menge $\mathfrak{S}T$ (die „*Vereinigungsmenge*" von T), welche alle Elemente der Elemente von T und nur solche als Elemente enthält.

(Axiom der Vereinigung.)

11. Ist kein Element von T eine von 0 verschiedene Menge, so ist natürlich $\mathfrak{S}T = 0$. Ist $T = \{M, N, R, \ldots\}$, wo die $M, N, R, \ldots$ sämtlich Mengen sind, so schreibt man auch $\mathfrak{S}T = M + N + R + \cdots$ und nennt $\mathfrak{S}T$ die „*Summe* der Mengen $M, N, R, \ldots$", ob einige dieser Mengen $M, N, R, \ldots$ nun gemeinsame Elemente besitzen oder nicht. Es ist immer $M = M + 0 = M + M = M + M + \cdots$.

12. Für die soeben definierte „Addition" der Mengen gilt das „kommutative" und das „assoziative" Gesetz:

$$M + N = N + M\,, \quad M + (N + R) = (M + N) + R\,.$$

contains all these common elements. This set D is called "the intersection associated with T" and is denoted by $\mathfrak{D}T$. If the elements of T do not possess a common element, $\mathfrak{D}T = 0$, and this is always the case if, for example, an element of T is not a set or if it is the null set.

10. *Theorem.* Every set M possesses at least one subset M_0 that is not an element of M.

Proof. It is definite for every element x of M whether $x \varepsilon x$ or not; the possibility that $x \varepsilon x$ is not in itself excluded by our axioms. If now M_0 is the subset of M that, in accordance with Axiom III, contains all those elements of M for which it is *not* the case that $x \varepsilon x$, then M_0 cannot be an element of M. For either $M_0 \varepsilon M_0$ or not. In the first case, M_0 would contain an element $x = M_0$ for which $x \varepsilon x$, and this would contradict the definition of M_0. Thus M_0 is surely *not* an element of M_0, and in consequence M_0, if it were an element of M, would also have to be an element of M_0, which was just excluded.

It follows from the theorem that not all objects x of the domain $\mathfrak{B}$ can be elements of one and the same set; that is, *the domain* $\mathfrak{B}$ *is not itself a set*, and this disposes of the "*Russell* antinomy" so far as we are concerned.

Axiom IV. To every set T there corresponds another set $\mathfrak{U}T$, the "*power set*" of T, that contains as elements precisely all subsets of T.

(Axiom of the power set.)

Axiom V. To every set T there corresponds a set $\mathfrak{S}T$, the "*union*" of T, that contains as elements precisely all elements of the elements of T.

(Axiom of the union.)

11. If no element of T is a set different from 0, then, of course, $\mathfrak{S}T = 0$. If $T = \{M, N, R, \ldots\}$, where $M, N, R, \ldots$ all are sets, we also write $\mathfrak{S}T = M + N + R + \cdots$ and call $\mathfrak{S}T$ the "*sum*" of the sets $M, N, R, \ldots$, whether some of these sets $M, N, R, \ldots$ contain common elements or not. Always $M = M + 0 = M + M = M + M + \cdots$.

12. For the "addition" of sets that we have just defined, the "commutative" and "associative" laws hold:

$$M + N = N + M\,, \quad M + (N + R) = (M + N) + R\,.$$

Endlich gilt für „Summen“ und „Durchschnitte“ (Nr. 8) auch das „distributive“ Gesetz in doppelter Form:

$$[M+N, R] = [M, R] + [N, R]\,,$$
$$[M, N] + R = [M+R, N+R]\,.$$

Den Beweis führt man mit Hilfe von I, indem man zeigt, daß jedes Element der linksstehenden Menge zugleich Element der rechtsstehenden Menge ist und umgekehrt.[1]

266 | 13. *Einführung des Produktes.* Ist M eine von 0 verschiedene Menge und a irgend eines ihrer Elemente, so ist nach Nr. 5 definit, ob $M = \{a\}$ ist oder nicht. *Es ist also immer definit, ob eine vorgelegte Menge aus einem einzigen Element besteht oder nicht.*

Es sei nun T eine Menge, deren Elemente $M, N, R, \ldots$ lauter (untereinander elementenfremde) Mengen sein mögen, und S_1 irgend eine Untermenge ihrer „Vereinigungsmenge“ $\mathfrak{S}T$. Dann ist für jedes Element M von T definit, ob der Durchschnitt $[M, S_1]$ aus einem einzigen Element besteht oder nicht. Somit bilden alle diejenigen Elemente von T, welche mit S_1 genau ein Element gemein haben, die Elemente einer gewissen Untermenge T_1 von T, und es ist wieder definit, ob $T_1 = T$ ist oder nicht. Alle Untermengen $S_1 \Subset \mathfrak{S}T$, welche mit jedem Elemente von T genau ein Element gemein haben, bilden also nach III die Elemente einer Menge $P = \mathfrak{P}T$, welche nach III und IV Untermenge von $\mathfrak{U}\mathfrak{S}T$ ist und als die „zu T gehörende *Verbindungsmenge*“ oder als „das *Produkt* der Mengen $M, N, R, \ldots$“ bezeichnet werden soll. Ist $T = \{M, N\}$, oder $T = \{M, N, R\}$, so schreibt man abgekürzt $\mathfrak{P}T = MN$ oder $= MNR$.

Um nun den Satz zu gewinnen, daß *ein Produkt mehrerer Mengen nur dann verschwinden* (d. h. der Nullmenge gleich sein) *kann, wenn ein Faktor verschwindet*, brauchen wir ein weiteres Axiom.

Axiom VI. Ist T eine Menge, deren sämtliche Elemente von 0 verschiedene Mengen und untereinander elementenfremd sind, so enthält ihre Vereinigung $\mathfrak{S}T$ mindestens eine Untermenge S_1, welche mit jedem Elemente von T ein und nur ein Element gemein hat.

(Axiom der Auswahl.)

Man kann das Axiom auch so ausdrücken, daß man sagt, es sei immer möglich, aus jedem Elemente $M, N, R, \ldots$ von T ein einzelnes Element $m, n, r, \ldots$ *auszuwählen* und alle diese Elemente zu einer Menge S_1 zu vereinigen.[1]

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[1] Diese vollständige Theorie dieser „logischen Addition und Multiplikation“ findet sich in *E. Schröders* „Algebra der Logik“, Bd. I.

[1] Über die Berechtigung dieses Axiomes vgl. meine Abhandlung Math. Ann. Bd. 65, p. 107–128, wo im § 2 p. 111 ff. die bezügliche Literatur erörtert wird.

Finally, for "sums" and "intersections" (No. 8) the "distributive law" also holds, in the two forms:

$$[M+N,R]=[M,R]+[N,R]$$

and

$$[M,N]+R=[M+R,N+R]\,.$$

The proof is carried out by means of Axiom I and consists in a demonstration that every element of the set on the left is also an element of the set on the right, and conversely.[4]

13. *Introduction of the product.* If M is a set different from 0 and a is any one of its elements, then according to No. 5 it is definite whether $M=\{a\}$ or not. *It is therefore always definite whether a given set consists of a single element or not.*

Now let T be a set whose elements, $M, N, R, \ldots$, are various (mutually disjoint) sets, and let S_1 be any subset of its "union" $\mathfrak{S}T$. Then it is definite for every element M of T whether the intersection $[M, S_1]$ consists of a single element or not. Thus all those elements of T that have exactly one element in common with S_1 are the elements of a certain subset T_1 of T, and it is again definite whether $T_1 = T$ or not. All subsets S_1 of $\mathfrak{S}T$ that have exactly one element in common with each element of T then are, according to Axiom III, the elements of a set $P=\mathfrak{P}T$, which, according to Axioms III and IV, is a subset of $\mathfrak{U}\mathfrak{S}T$ and will be called the "*connection set* associated with T" or "the *product* of the sets $M, N, R, \ldots$". If $T=\{M,N\}$, or $T=\{M,N,R\}$, we write $\mathfrak{P}T=MN$, or $\mathfrak{P}T=MNR$, respectively, for short.

In order, now, to obtain the theorem that *the product of several sets can vanish* (that is, be equal to the null set) *only if a factor vanishes* we need a further axiom.

Axiom VI. If T is a set whose elements all are sets that are different from 0 and mutually disjoint, its union $\mathfrak{S}T$ includes at least one subset S_1 having one and only one element in common with each element of T.

(Axiom of choice.)

We can also express this axiom by saying that it is always possible to *choose* a single element from each element $M, N, R, \ldots$ of T and to combine all the chosen elements, $m, n, r, \ldots$, into a set S_1.[5]

[4] The complete theory of this "logical addition and multiplication" can be found in *Schröder 1890*.

[5] For the justification of this axiom see my *1908a*, where in § 2, pp. 111 ff, the relevant literature is discussed.

Die vorstehenden Axiome genügen, wie wir sehen werden, um alle wesentlichen Theoreme der allgemeinen Mengenlehre abzuleiten. Um aber die Existenz „unendlicher" Mengen zu sichern, bedürfen wir noch des folgenden, seinem wesentlichen Inhalte von Herrn *R. Dedekind*[2] herrührenden Axiomes.

Axiom VII. Der Bereich enthält mindestens eine Menge Z, welche die Nullmenge als Element enthält und so beschaffen ist, daß jedem ihrer
267 | Elemente a ein weiteres Element der Form $\{a\}$ entspricht, oder welche mit jedem ihrer Elemente a auch die entsprechende Menge $\{a\}$ als Element enthält.

(Axiom des Unendlichen.)

14_{VII}.[1] Ist Z eine beliebige Menge von der in VII geforderten Beschaffenheit, so ist für jede ihrer Untermengen Z_1 definit, ob sie die gleiche Eigenschaft besitzt. Denn ist a irgend ein Element von Z_1, so ist definit, ob auch $\{a\} \varepsilon Z_1$ ist, und alle so beschaffenen Elemente a von Z_1 bilden die Elemente einer Untermenge $Z_1{}'$, für welche definit ist, ob $Z_1{}' = Z_1$ ist oder nicht. Somit bilden alle Untermengen Z_1 von der betrachteten Eigenschaft die Elemente einer Untermenge $T \Subset \mathfrak{U} Z$, und der ihnen entsprechende Durchschnitt (Nr. 9) $Z_0 = \mathfrak{D} T$ ist eine Menge von der gleichen Beschaffenheit. Denn einmal ist 0 gemeinsames Element aller Elemente Z_1 von T, und andererseits, wenn a gemeinsames Element aller dieser Z_1 ist, so ist auch $\{a\}$ allen gemeinsam und somit gleichfalls Element von Z_0.

Ist nun Z' irgend eine andere Menge von der im Axiom geforderten Beschaffenheit, so entspricht ihr in genau derselben Weise wie Z_0 dem Z eine kleinste Untermenge $Z_0{}'$ von der betrachteten Eigenschaft. Nun muß aber auch der Durchschnitt $[Z_0, Z_0{}']$, welcher eine gemeinsame Untermenge von Z und Z' ist, die gleiche Beschaffenheit wie Z und Z' haben und als Untermenge von Z den Bestandteil Z_0, sowie als Untermenge von Z' den Bestandteil $Z_0{}'$ enthalten. Nach I folgt also, daß $[Z_0, Z_0{}'] = Z_0 = Z_0{}'$ sein muß, und daß somit Z_0 *der gemeinsame Bestandteil aller möglichen wie Z beschaffenen Mengen* ist, obwohl diese nicht die Elemente einer Menge zu bilden brauchen. Die Menge Z_0 enthält die Elemente $0, \{0\}, \{\{0\}\}$ usw. und möge als „*Zahlenreihe*" bezeichnet werden, weil ihre Elemente die Stelle der Zahlzeichen vertreten können. Sie bildet das einfachste Beispiel einer „abzählbar unendlichen" Menge (Nr. 36).

[2] „Was sind und was sollen die Zahlen?" § 5 Nr. 66. Der von Herrn Dedekind hier versuchte „Beweis" dieses Prinzips kann nicht befriedigen, da er von der „Menge alles Denkbaren" ausgeht, während für unseren Standpunkt nach Nr. 10 der Bereich $\mathfrak{B}$ selbst *keine* Menge bildet.

[1] Die Indizes VI oder VII an der Nummer eines Theorems sollen ausdrücken, daß hier das Axiom VI oder VII explizit oder implizit zur Anwendung kommt.

The preceding axioms suffice, as we shall see, for the derivation of all essential theorems of general set theory. But in order to secure the existence of infinite sets we still require the following axiom, which is essentially due to Mr. *R. Dedekind*.[6]

Axiom VII. There exists in the domain at least one set Z that contains the null set as an element and is so constituted that to each of its elements a there corresponds a further element of the form $\{a\}$, in other words, that with each of its elements a it also contains the corresponding set $\{a\}$ as an element.

(Axiom of infinity.)

14_{VII}.[7] If Z is an arbitrary set constituted as required by Axiom VII, it is definite for each of its subsets Z_1 whether it possesses the same property. For, if a is any element of Z_1, it is definite whether $\{a\}$, too, is an element of Z_1, and all elements a of Z_1 that satisfy this condition are the elements of a subset $Z_1{}'$ for which it is definite whether $Z_1{}' = Z_1$ or not. Thus all subsets Z_1 having the property in question are the elements of a subset T of $\mathfrak{U}Z$, and the intersection (No. 9) $Z_0 = \mathfrak{D}T$ that corresponds to them is a set constituted in the same way. For, on the one hand, 0 is a common element of all elements Z_1 of T, and, on the other, if a is a common element of all of these Z_1, then $\{a\}$ is also common to all of them and is thus likewise an element of Z_0.

Now if Z' is any other set constituted as required by the axiom, there corresponds to it a smallest subset Z_0' having the same property, exactly as Z_0 corresponds to Z. But now the intersection $[Z_0, Z_0{}']$, which is a common subset of Z and Z', must be constituted in the same way as Z and Z'; and just as, being a subset of Z, it must contain the component Z_0, so, as a subset of Z', it must contain the component $Z_0{}'$. According to Axiom I it then necessarily follows that $[Z_0, Z_0{}'] = Z_0 = Z_0{}'$ and that Z_0 thus is the *common component of all possible sets constituted like* Z, even though these need not be elements of a set. The set Z_0 contains the elements 0, $\{0\}$, $\{\{0\}\}$, and so forth, and it may be called the "*number sequence*", because its elements can take the place of the numerals. It is the simplest example of a "denumerably infinite" set (below, No. 36).

[6] *Dedekind 1888*, §5, art. 66. The "proof" that Mr. Dedekind there attempts to give of this principle cannot be satisfactory, since it takes its departure from "the set of everything thinkable", whereas from our point of view the domain $\mathfrak{B}$ itself, according to No. 10, does *not* form a set.

[7] The subscript VI, or VII, on the number of a section indicates that explicit or implicit use has been made of Axiom VI, or VII, respectively, in establishing the theorem of that section.

§ 2.
Theorie der Äquivalenz

Die „Äquivalenz" zweier Mengen[2] läßt sich für unseren Standpunkt zunächst nur für den Fall definieren, wo die Mengen „elementenfremd" (Nr. 3) sind, und kann erst nachträglich auf den allgemeinen Fall ausgedehnt werden.

15. *Definition A.* Zwei elementenfremde Mengen M und N heißen „*unmittelbar äquivalent*", $M \sim N$, wenn ihr Produkt MN (Nr. 13) mindestens eine solche Untermenge Φ besitzt, daß jedes Element von $M + N$ in einem
268 | und nur einem Elemente $\{m,\, n\}$ von Φ als Element erscheint. Eine Menge $\Phi \Subset MN$ von der betrachteten Beschaffenheit heißt eine „*Abbildung von M auf N*"; zwei Elemente m, n, welche in einem Elemente von Φ vereinigt erscheinen, heißen „aufeinander abgebildet", sie „entsprechen einander", das eine ist „das Bild" des anderen.

16. Ist Φ irgend eine Untermenge von MN, also Element von $\mathfrak{U}(MN)$, und x irgend ein Element von $M + N$, so ist es immer definit (Nr. 4), ob die x enthaltenden Elemente von Φ eine Menge bilden, die aus einem einzigen Element besteht (Nr. 13). Somit ist auch definit, ob *alle* Elemente x von $M + N$ diese Eigenschaft besitzen, d. h. ob Φ eine „Abbildung" von M auf N darstellt oder nicht. Die sämtlichen Abbildungen Φ bilden also nach III die Elemente einer gewissen Untermenge Ω von $\mathfrak{U}(MN)$, und es ist definit, ob Ω von 0 verschieden ist oder nicht. *Für zwei elementenfremde Mengen M, N ist es also immer definit, ob sie äquivalent sind oder nicht.*

17. Sind zwei äquivalente elementenfremde Mengen M, N durch Φ aufeinander abgebildet, so entspricht auch jeder Untermenge $M_1 \Subset M$ eine äquivalente Untermenge $N_1 \Subset N$ vermöge einer Abbildung Φ_1, welche eine Untermenge von Φ ist.

Denn für jedes Element $\{m, n\}$ von Φ ist es definit, ob $m \,\varepsilon\, M_1$ ist oder nicht, und alle in dieser Weise zu M_1 gehörenden Elemente von Φ bilden somit die Elemente einer Untermenge $\Phi_1 \Subset \Phi$. Bezeichnet man nun mit N_1 den Durchschnitt (Nr. 8) von $\mathfrak{S}\Phi_1$ mit N, so erscheint jedes Element von $M_1 + N_1$ nur in einem einzigen Elemente von Φ_1 als Element, weil es sonst auch in Φ mehrfach vorkommen würde, und es ist nach Nr. 15 in der Tat $M_1 \sim N_1$.

18. Sind zwei elementenfremde Mengen M und N einer und derselben dritten Menge R gleichzeitig elementenfremd und äquivalent, oder ist $M \sim R, R \sim R', R' \sim N$, wobei je zwei aufeinander folgende Mengen elementenfremd sein sollen, so ist auch immer $M \sim N$.

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[2] G. Cantor, Math. Annalen Bd. 46, p. 483.

§ 2.
Theory of equivalence

From our point of view, the "equivalence" of two sets[8] cannot be defined at first except for the case in which the sets are "disjoint" (No. 3); it is only afterward that the definition can be extended to the general case.

15. *Definition A.* Two disjoint sets M and N are said to be "*immediately equivalent*", $M \sim N$, if their product MN (No. 13) possesses at least one subset Φ such that each element of $M + N$ occurs as an element in one and only one element $\{m, n\}$ of Φ. A subset Φ of MN thus constituted is called a "*mapping of M onto N*"; two elements m and n that occur together in one element of Φ are said to be "mapped onto each other"; they "correspond to each other", or one is the "image" of the other.

16. If Φ is any subset of MN and therefore an element of $\mathfrak{U}(MN)$ and if x is any element of $M+N$, it is always definite (No. 4) whether the elements of Φ that contain x form a set consisting of a single element (No. 13). Thus it is also definite whether *all* elements x of $M + N$ possess this property, that is, whether Φ represents a "mapping" of M onto N or not. According to Axiom III, all of the mappings Φ therefore are the elements of a certain subset Ω of $\mathfrak{U}(MN)$, and it is definite whether Ω differs from 0 or not. *It is therefore always definite for two disjoint sets M and N whether they are equivalent or not.*

17. If two equivalent disjoint sets M and N are mapped onto each other by Φ, there also corresponds to each subset M_1 of M an equivalent subset N_1 of N under a mapping Φ_1, that is a subset of Φ.

For it is definite for every element $\{m, n\}$ of Φ whether $m \,\varepsilon\, M_1$ or not, and therefore all elements of Φ thus associated with M_1 are the elements of a subset Φ_1 of Φ. If we now denote by N_1 the intersection (No. 8) of $\mathfrak{S}\Phi_1$ with N, each element of $M_1 + N_1$ occurs as an element in only a single element of Φ_1, since otherwise it would occur more than once in Φ as well; and, according to No. 15, we in fact have $M_1 \sim N_1$.

18. If two disjoint sets M and N are disjoint from and equivalent to one and the same third set, R, or if $M \sim R$, $R \sim R'$, and $R' \sim N$, where each of these pairs of equivalent sets is assumed to be disjoint, then always also $M \sim N$.

[8] *Cantor 1895*, p. 483.

Es seien $\Phi ⋹ MR$, $\mathsf{X} ⋹ RR'$, $\Psi ⋹ R'N$ drei „Abbildungen" (Nr. 15), welche bezw. M auf R, R auf R' und R' auf N abbilden. Ist dann $\{m, n\}$ irgend ein Element von MN, so ist definit, ob es ein Element $r \mathrel{\varepsilon} R$ und ein Element $r' \mathrel{\varepsilon} R'$ gibt, so daß gleichzeitig $\{m, r\} \mathrel{\varepsilon} \Phi$, $\{r, r'\} \mathrel{\varepsilon} \mathsf{X}$ und $\{r', n\} \mathrel{\varepsilon} \Psi$ ist. Alle Elemente $\{m, n\}$ von dieser Beschaffenheit bilden somit die Elemente einer Menge $\Omega ⋹ MN$, welche eine Abbildung von M auf N darstellt. Ist nämlich etwa m irgend ein Element von M, so entspricht ihm immer ein einziges Element $r \mathrel{\varepsilon} R$, ein einziges $r' \mathrel{\varepsilon} R'$ und somit auch ein einziges $n \mathrel{\varepsilon} N$ von der verlangten Beschaffenheit; das Analoge gilt für jedes Element n von N. Jedem Elemente von $M + N$ entspricht also in der Tat ein einziges Element $\{m, n\}$ von Ω, und es ist wirklich $M \sim N$.

269 | 19. *Theorem.* Sind M und N irgend zwei Mengen, so gibt es immer eine Menge M', welche der einen M äquivalent und der anderen N elementenfremd ist.

Beweis. Es sei $S = \mathfrak{S}(M + N)$ gemäß V die Menge, welche die Elemente der Elemente von $M + N$ umfaßt, und r gemäß Nr. 10 ein Ding, welches nicht Element von $M + S$ ist. Dann sind die Mengen M und $R = \{r\}$ elementenfremd, und das Produkt $M' = MR$ besitzt die im Theorem verlangte Eigenschaft. In der Tat ist dann jedes Element von M' nach Nr. 13 eine Menge der Form $m' = \{m, r\}$, wo $m \mathrel{\varepsilon} M$ ist, und niemals Element von $M + N$, weil sonst r Element eines Elementes von $M + N$ und damit gemäß V Element von S wäre gegen die Annahme. Also ist M' beiden Mengen M und N elementenfremd.

Ferner entspricht jedem Element m von M ein und nur ein Element $m' = \{m, r\}$, und umgekehrt enthält jedes m' nur ein einziges Element m von M als Element, da r kein Element von M sein sollte. Jedem Element von $M + M'$ entspricht also ein einziges Element $\{m, m'\}$ von MM', für welches $m' = \{m, r\}$ ist, und wenn man alle so beschaffenen Paare $\{m, m'\}$ zu einer Untermenge $\Phi ⋹ MM'$ rechnet, so ist nach Nr. 15 Φ eine Abbildung von M auf M' und $M \sim M'$.

Aus unserem Satze folgt, daß *die sämtlichen Mengen, welche einer nicht verschwindenden Menge M äquivalent sind, nicht die Elemente einer Menge T bilden können*; denn ist T eine beliebige Menge, so gibt es immer eine Menge $M' \sim M$, welche der Vereinigung $\mathfrak{S}T$ elementenfremd und daher *nicht* Element von T ist.

20. Sind M und N irgend zwei Mengen, so ist es immer definit, ob es eine Menge R gibt, welche beiden Mengen M und N gleichzeitig elementenfremd und äquivalent ist.

Es sei nämlich M' gemäß Nr. 19 eine Menge, welche M äquivalent und $M + N$ elementenfremd ist. Dann ist nach Nr. 16 definit, ob $M' \sim N$ ist oder nicht. Im ersteren Falle ist $R = M'$ eine Menge von der verlangten Beschaffenheit, im entgegengesetzten Falle kann es eine solche Menge R überhaupt nicht geben, da nach Nr. 18 aus $M' \sim M$, $M \sim R$ und $R \sim N$ immer $M' \sim N$ folgen müßte gegen die Annahme.

Let the subset Φ of MR, the subset X of RR', and the subset Ψ of $R'N$ be three "mappings" (No. 15) that map M onto R, R onto R', and R' onto N, respectively. If then $\{m, n\}$ is any element of MN, it is definite whether there exist an element r of R and an element r' of R' such that $\{m, r\} \, \varepsilon \, \Phi$, $\{r, r'\} \, \varepsilon \, \mathsf{X}$, and $\{r', n\} \, \varepsilon \, \Psi$. All elements $\{m, n\}$ for which this is the case therefore are the elements of a subset Ω of MN, which represents a mapping of M onto N. For, if m is any element of M, there always correspond to it a single element r of R, a single element r' of R', and therefore also a single element n of N that satisfy the required condition; an analogous statement holds for each element n of N. Therefore to each element of $M + N$ there actually corresponds a single element $\{m, n\}$ of Ω, and we in fact have $M \sim N$.

19. *Theorem.* If M and N are any two sets, there always exists a set M' that is equivalent to one, M, and disjoint from the other, N.

Proof. Let $S = \mathfrak{S}\{M + N\}$, in accordance with Axiom V, be the set that contains the elements of $M + N$, and let r, in accordance with No. 10, be an object that is not an element of $M + S$. Then the sets M and $R = \{r\}$ are disjoint, and the product $M' = MR$ possesses the property required by the theorem. Indeed, every element of M' is then, according to No. 13, a set m' of the form $\{m, r\}$ (where $m \, \varepsilon \, M$) but never an element of $M + N$, since otherwise r would be an element of an element of $M + N$, hence, by Axiom V, an element of S, contrary to the assumption. Thus M' is disjoint from both sets, M and N.

Further, there corresponds to each element m of M one and only one element $m' = \{m, r\}$, and conversely each m' contains as an element only a single element m of M, since r was assumed not to be an element of M. To each element of $M + M'$, therefore, there corresponds a single element $\{m, m'\}$ of MM' for which $m' = \{m, r\}$, and if all pairs $\{m, m'\}$ that are so constituted are assumed to form a subset Φ of MM', then, according to No. 15, Φ is a mapping of M onto M', and $M \sim M'$.

It follows from our theorem that *it is not possible for all sets equivalent to a nonempty set M to be the elements of a set T*; for if T is an arbitrary set, there always exists a set M', equivalent to M, that is disjoint from the union $\mathfrak{S}T$ and therefore *not* an element of T.

20. If M and N are any two sets, it is always definite whether there is a set R that is simultaneously disjoint from and equivalent to both sets, M and N.

For let M', in accordance with No. 19, be a set that is equivalent to M and disjoint from $M + N$. Then, according to No. 16, it is definite whether $M' \sim N$ or not. If $M' \sim N$, then $R = M'$ is a set constituted as required; otherwise, such a set R cannot exist at all, since, according to No. 18, it would always necessarily follow from $M' \sim M$, $M \sim R$, and $R \sim N$ that $M' \sim N$, contrary to the assumption.

Das vorstehende Theorem in Verbindung mit Nr. 18 berechtigt uns jetzt zu der folgenden Erweiterung unserer Definition A:

21. *Definition B.* Zwei beliebige (nicht elementenfremde) Mengen M und N heißen „*mittelbar äquivalent*“, $M \sim N$, wenn es eine dritte Menge R gibt, welche ihnen beiden elementenfremd und im Sinne der Definition A beiden „unmittelbar äquivalent“ ist.

Eine solche durch R „vermittelte“ Äquivalenz zweier Mengen M und N
270 wird gegeben durch *zwei simultane* „Abbildungen“ $\Phi \mathrel{\underline{\in}} MR$ und $|$ $\Psi \mathrel{\underline{\in}} NR$, und zwei Elemente $m \,\varepsilon\, M$ und $n \,\varepsilon\, N$ heißen „entsprechend“ oder „aufeinander abgebildet“, wenn sie einem und demselben dritten Elemente $r \,\varepsilon\, R$ entsprechen, so daß gleichzeitig $\{m, r\} \,\varepsilon\, \Phi$ und $\{n, r\} \,\varepsilon\, \Psi$ ist. Auch bei einer solchen vermittelten Abbildung entspricht wie in Nr. 17 jeder Untermenge M_1 von M eine äquivalente Untermenge R_1 von R und somit wieder eine äquivalente Untermenge $N_1 \mathrel{\underline{\in}} N$.

Wegen Nr. 18 kann diese Definition B auch auf elementenfremde Mengen M, N angewendet werden, und nach Nr. 20 ist es immer *definit, ob zwei beliebige Mengen* im Sinne dieser Definition *äquivalent sind oder nicht.*

22. Jede Menge ist sich selbst äquivalent. Sind zwei Mengen M, N einer dritten R äquivalent, so sind sie einander selbst äquivalent.

Ist nämlich gemäß Nr. 19 M' eine Menge, welche M elementenfremd und äquivalent ist, so ist gleichzeitig $M \sim M'$ und $M' \sim M$, also nach Nr. 21 wirklich $M \sim M$.

Ist ferner die Äquivalenz der Mengen M und R vermittelt durch M', sowie die Äquivalenz von R und N vermittelt durch N', wobei M' zu M und R, sowie N' zu N und R elementenfremd sein soll, so wählen wir gemäß Nr. 19 eine sechste Menge R', welche $\sim R$ und der Summe $M + N + R$ elementenfremd ist, und haben dann wegen Nr. 18

$$M \sim M' \sim R \sim R', \quad \text{also} \quad M \sim R'$$

und

$$N \sim N' \sim R \sim R', \quad \text{also} \quad N \sim R',$$

so daß nach Nr. 21 die Äquivalenz von M und N durch R' vermittelt ist.

23. Die Nullmenge ist nur sich selbst äquivalent. Jede Menge der Form $\{a\}$ ist jeder anderen Menge $\{b\}$ derselben Form und keiner sonstigen Menge äquivalent.

Denn da das Produkt $0 \cdot M$ immer $= 0$ ist, so kann keine Menge $M \neq 0$ im Sinne der Nr. 15 der Nullmenge (unmittelbar) und somit auch keine Menge M' im Sinne von Nr. 21 ihr „mittelbar“ äquivalent sein.

The preceding theorem, in combination with No. 18, now justifies the following extension of our Definition A:

21. *Definition B.* Two arbitrary (not disjoint) sets M and N are said to be "*mediately equivalent*", $M \sim N$, if there exists a third set, R, that is disjoint from both and "immediately equivalent" to both in the sense of Definition A.

Such an equivalence, "mediated" by R, of two sets M and N is given by means of *two simultaneous* mappings, a subset Φ of MR and a subset Ψ of NR, and two elements, m of M and n of N, are said to "correspond" or "be mapped onto each other" if they correspond to one and the same element r of R, so that both $\{m, r\} \,\varepsilon\, \Phi$ and $\{n, r\} \,\varepsilon\, \Psi$. In the case of such a mediated mapping, too, there corresponds to each subset M_1 of M, as in No. 17, an equivalent subset R_1 of R, and consequently again an equivalent subset N_1 of N.

On account of No. 18, Definition B may also be applied to disjoint sets M and N, and according to No. 20 *it is always definite whether two arbitrary sets are equivalent or not* in the sense of this definition.

22. Every set is equivalent to itself. If two sets, M and N, are equivalent to a third, R, they are equivalent to each other.

For if, in accordance with No. 19, M' is a set that is disjoint from and equivalent to M, both $M \sim M'$ and $M' \sim M$; therefore, according to No. 21, we in fact have $M \sim M$.

If, furthermore, the equivalence of the sets M and R is mediated by M', and that of R and N by N', where M' is assumed to be disjoint from M and R, and N' to be disjoint from N and R, then we choose, in accordance with No. 19, a sixth set, R', equivalent to R and disjoint from the sum $M + N + R$, and we now have, on account of No. 18,

$$M \sim M' \sim R \sim R', \quad \text{therefore} \quad M \sim R',$$

and

$$N \sim N' \sim R \sim R', \quad \text{therefore} \quad N \sim R',$$

so that according to No. 21 the equivalence of M and N is mediated by R'.

23. The null set is equivalent only to itself. Every set of the form $\{a\}$ is equivalent to all other sets $\{b\}$ of the same form, and to no other set.

For, since the product $0 \cdot M$ is always equal to 0,[9] no set $M \neq 0$ can be (immediately) equivalent to the null set in the sense of No. 15, and therefore no set M' can be "mediately" equivalent to it in the sense of No. 21.

[9] ⟦Up to this point Zermelo has used only juxtaposition for the product; from here on he occasionally uses a dot.⟧

Ist ferner $\{a\}$ elementenfremd zu M, d. h. a nicht $\varepsilon\, M$, so sind alle Elemente des Produktes $\{a\}\, M$ von der Form $\{a, m\}$, und wenn M außer m noch ein weiteres Element p enthielte, so wären $\{a, m\}$ und $\{a, p\}$ nicht elementenfremd, wie in Nr. 15 für jede „Abbildung“ $\Phi \mathrel{\underline{\in}} \{a\}M$ gefordert. Dagegen ist $\{a\} \cdot \{b\} = \{a, b\}$ stets eine Abbildung von $\{a\}$ auf $\{b\}$.

24. *Theorem.* Ist $M \sim M'$ und $N \sim N'$, während M und N einerseits, M' und N' andererseits einander elementenfremd sind, so ist immer

$$M + N \sim M' + N' .$$

Beweis. Wir betrachten zunächst den Fall, wo $M + N$ und $M' + N'$ elementenfremd sind. Dann ist auf beide Äquivalenzen $M \sim M'$ und $N \sim N'$
271 die Definition A Nr. 15 anwendbar, und es gibt zwei | Abbildungen $\Phi \mathrel{\underline{\in}} MM'$ und $\Psi \mathrel{\underline{\in}} NN'$, deren Summe $\Phi + \Psi$ die verlangte Abbildung von $M + N$ auf $M' + N'$ darstellt. Ist nämlich $p\, \varepsilon\, (M + N)$, so ist entweder $p\, \varepsilon\, M$ oder $p\, \varepsilon\, N$, aber wegen $[M, N] = 0$ nicht beides gleichzeitig, und im einen Falle enthält Φ, im anderen Ψ ein einziges Element der Form $\{p, q\}$. Ebenso entspricht auch jedem Elemente q von $M' + N'$ ein und nur ein Element $\{p, q\}$ in $\Phi + \Psi$.

Sind $M + N$ und $M' + N'$ nicht selbst elementenfremd, so gibt es gemäß Nr. 19 eine Menge $S'' \sim M' + N'$, welche der Summe $M + N + M' + N'$ elementenfremd ist, und bei einer Abbildung X von $M' + N'$ auf S'' mögen wegen Nr. 17 den beiden Teilen M' und N' die äquivalenten und elementenfremden Teile M'' und N'' von S'' entsprechen. Dann ist $M \sim M' \sim M''$ sowie $N \sim N' \sim N''$ und, da jetzt $M + N$ und $M'' + N''$ elementenfremd sind, nach dem soeben Bewiesenen

$$M + N \sim M'' + N'' = S'' \sim M' + N' ,$$

also wieder

$$M + N \sim M' + N' .$$

25. *Theorem.* Ist eine Menge M einem ihrer Teile M' äquivalent, so ist sie auch jedem anderen Teile M_1 äquivalent, welcher M' als Bestandteil enthält.

Beweis. Es sei

$$M \sim M' \mathrel{\underline{\in}} M_1 \mathrel{\underline{\in}} M \quad \text{und} \quad Q = M_1 - M' .$$

Wegen der vorausgesetzten Äquivalenz $M \sim M'$ gibt es gemäß Nr. 21 eine Abbildung $\{\Phi, \Psi\}$ von M auf M', vermittelt etwa durch M''. Ist nun A eine beliebige Untermenge von M, so entspricht ihr bei der betrachteten Abbildung eine bestimmte Untermenge A' von M', und es ist definit, ob $A' \mathrel{\underline{\in}} A$ ist oder nicht. Somit bilden alle solchen Elemente A von $\mathfrak{U}\, M$, für welche gleichzeitig $Q \mathrel{\underline{\in}} A$ und $A' \mathrel{\underline{\in}} A$ ist, nach III die Elemente einer gewissen Menge $T \mathrel{\underline{\in}} \mathfrak{U}\, M$, und es ist namentlich M selbst Element von T. Der gemeinsame Bestandteil $A_0 = \mathfrak{D}\, T$ aller Elemente von T (Nr. 9) besitzt nun die folgenden

If, furthermore, $\{a\}$ is disjoint from M, that is, if a is not an element of M, then all elements of the product $\{a\}M$ are of the form $\{a, m\}$ and, if M were to contain, besides m, another element, p, then $\{a, m\}$ and $\{a, p\}$ would not be disjoint, and this, according to No. 15, would prevent any subset Φ of $\{a\}M$ from being a "mapping". On the other hand, $\{a\} \cdot \{b\} = \{a, b\}$ is always a mapping of $\{a\}$ onto $\{b\}$.

24. *Theorem*. If $M \sim M'$ and $N \sim N'$, while M and N on the one hand, and M' and N' on the other, are mutually disjoint, then always

$$M + N \sim M' + N' .$$

Proof. We first consider the case in which $M+N$ and $M'+N'$ are disjoint. Then Definition A (No. 15) is applicable to both of the equivalences $M \sim M'$ and $N \sim N'$, and there are two mappings, the subset Φ of MM' and the subset Ψ of NN', whose sum $\Phi + \Psi$ represents the required mapping of $M + N$ onto $M' + N'$. For, if $p \, \varepsilon (M + N)$, either $p \, \varepsilon \, M$ or $p \, \varepsilon \, N$, but, since $[M, N] = 0$, not both; and Φ in one case, and Ψ in the other, contains a single element of the form $\{p, q\}$. Likewise there corresponds to each element q of $M' + N'$ one and only one element $\{p, q\}$ of $\Phi + \Psi$.

If $M + N$ and $M' + N'$ are not themselves disjoint, there exists, according to No. 19, a set S'' equivalent to $M' + N'$ and disjoint from the sum $M + N + M' + N'$; and, given a mapping X of $M' + N'$ onto S'', equivalent and disjoint parts M'' and N'' of S'' will, according to No. 17, correspond to the two parts M' and N'. Then $M \sim M' \sim M''$, as well as $N \sim N' \sim N''$, and, since now $M + N$ and $M'' + N''$ are disjoint,

$$M + N \sim M'' + N'' = S'' \sim M' + N'$$

according to what has just been proved; therefore again

$$M + N \sim M' + N' .$$

25. *Theorem*. If a set M is equivalent to one of its parts, M', it is also equivalent to any other part M_1, that includes M' as component.

Proof. Let

$$M \sim M' \subseteq M_1 \subseteq M \quad \text{and} \quad Q = M_1 - M' .$$

Because of the equivalence $M \sim M'$ that has been assumed, there exists, according to No. 21, a mapping $\{\Phi, \Psi\}$ of M onto M', mediated by, say, M''. If now A is an arbitrary subset of M, a certain subset A' of M' will correspond to it under the mapping in question, and it is definite whether $A' \subseteq A$ or not. Thus all elements A of $\mathfrak{U} M$ for which we have both $Q \subseteq A$ and $A' \subseteq A$ are, according to Axiom III, the elements of a certain subset T of $\mathfrak{U} M$, and, in particular, M is itself an element of T. The common component $A_0 = \mathfrak{D} T$

Eigenschaften: 1) $Q \mathrel{\underline{\in}} A_0$, weil Q eine gemeinsame Untermenge aller $A \,\varepsilon\, T$ ist, 2) $A_0{}' \mathrel{\underline{\in}} A_0$, weil jedes Element x von A_0 gemeinsames Element aller $A \,\varepsilon\, T$ und somit auch sein Bild $x' \,\varepsilon\, A' \mathrel{\underline{\in}} A$ gemeinsames Element aller A ist. Wegen 1) und 2) ist also auch $A_0 \,\varepsilon\, T$. Endlich ist 3) $A_0 = Q + A_0{}'$. Da nämlich $A_0{}' \mathrel{\underline{\in}} A_0$ und gleichzeitig $\mathrel{\underline{\in}} M' \mathrel{\underline{\in}} M - Q$ ist, so ist einmal $A_0{}' \mathrel{\underline{\in}} A_0 - Q$. Andererseits ist aber auch jedes Element r von $A_0 - Q$ ein Element von $A_0{}'$ und daher $A_0 - Q \mathrel{\underline{\in}} A_0{}'$. In der Tat, wäre r nicht $\varepsilon\, A_0{}'$, so würde auch $A_1 = A_0 - \{r\}$ noch $A_0{}'$ und a fortiori $A_1{}'$ als Bestandteil enthalten und, da es immer noch Q enthält, selbst Element von T sein, während es doch nur ein *Teil* von $A_0 = \mathfrak{D}\, T$ ist. Es ist also

$$M_1 = Q + M' = (Q + A_0{}') + (M' - A_0{}') = A_0 + (M' - A_0{}')\,,$$

272 | wo die beiden Summanden rechts keine Elemente gemein haben, weil Q und M' elementenfremd sind. Da nun aber $A_0 \sim A_0{}'$ und $M' - A_0{}'$ sich selbst äquivalent ist, so folgt nach Nr. 24

$$M_1 \sim A_0{}' + (M' - A_0{}') = M' \sim M\,,$$

d. h. wie behauptet, $M_1 \sim M$.

26. *Folgerung.* Ist eine Menge M einem ihrer Teile M' äquivalent, so ist sie auch jeder Menge M_1 äquivalent, welche aus M durch Fortlassung oder Hinzufügung eines einzelnen Elementes entsteht.

Es sei

$$M \sim M' = M - R \quad \text{und} \quad M_1 = M - \{r\}\,,$$

wo $r \,\varepsilon\, R$ sein möge. Dann ist

$$M' = M - \{r\} - (R - \{r\}) \mathrel{\underline{\in}} M - \{r\} = M_1$$

und nach dem vorigen Satze $M \sim M_1$.

Ist ferner

$$M_2 = M - \{a\}\,, \quad \text{wo} \quad a \,\varepsilon\, M' = M - R$$

ist, so sei

$$M_0 = M - \{a, r\}\,,$$

und wir haben nach Nr. 23 und 24

$$M_2 = M_0 + \{r\} \sim M_0 + \{a\} = M_1 \sim M\,,$$

also auch

$$M_2 \sim M\,.$$

Ist endlich

$$M_3 = M + \{c\}\,,$$

wo c nicht $\varepsilon\, M$ ist, so folgt aus $M \sim M'$ wieder nach Nr. 24

$$M_3 = M + \{c\} \sim M' + \{c\} = M - R + \{c\} = M_3 - R\,,$$

(No. 9) of all elements of T now possesses the following properties: 1) $Q \subseteq A_0$, since Q is a common subset of all elements A of T; 2) ${A_0}' \subseteq A_0$, because every element x of A_0 is a common element of all elements A of T and its map $x' \,\varepsilon\, A' \subseteq A$ is thus also a common element of all A. On account of 1) and 2), therefore, also $A_0 \,\varepsilon\, T$. Finally we have 3) $A_0 = Q + {A_0}'$. For, since ${A_0}' \subseteq A_0$ and also ${A_0}' \subseteq M' \subseteq M - Q$, on the one hand ${A_0}' \subseteq A_0 - Q$. On the other hand, however, every element r of $A_0 - Q$ is also an element of ${A_0}'$, and therefore $A_0 - Q \subseteq {A_0}'$. Indeed, if r were not an element of A_0', then $A_1 = A_0 - \{r\}$ would still have ${A_0}'$, and a fortiori ${A_1}'$, as a component, and, since it still includes Q, it would itself be an element of T, whereas it is in fact only a *part* of $A_0 = \mathfrak{D}T$. Therefore

$$M_1 = Q + M' = (Q + {A_0}') + (M' - {A_0}') = A_0 + (M' - {A_0}')\,,$$

where the two summands on the right have no element in common, since Q and M' are disjoint. But now, since A_0 is equivalent to ${A_0}'$ and $M' - {A_0}'$ is equivalent to itself, it follows according to No. 24 that

$$M_1 \sim {A_0}' + (M' - {A_0}') = M' \sim M\,;$$

that is, $M_1 \sim M$ as asserted.

26. *Corollary.* If a set M is equivalent to one of its parts, M', it is also equivalent to any set M_1 that is obtained from M when a single element is removed or added.

Let

$$M \sim M' = M - R \quad \text{and} \quad M_1 = M - \{r\}\,,$$

where r is some element of R. Then

$$M' = M - \{r\} - (R - \{r\}) \subseteq M - \{r\} = M_1\,,$$

and, according to the previous theorem, $M \sim M_1$.

If, furthermore,

$$M_2 = M - \{a\}\,, \quad \text{where} \quad a \,\varepsilon\, M' = M - R\,.$$

let

$$M_0 = M - \{a, r\}\,,$$

and we have, according to Nos. 23 and 24,

$$M_2 = M_0 + \{r\} \sim M_0 + \{a\} = M_1 \sim M\,;$$

therefore also

$$M_2 \sim M\,.$$

If, finally,

$$M_3 = M + \{c\}\,,$$

where c is not an element of M, it follows from $M \sim M'$, again according to No. 24, that

$$M_3 = M + \{c\} \sim M' + \{c\} = M - R + \{c\} = M_3 - R\,,$$

und nach dem vorher Bewiesenen weiter

$$M = M_3 - \{c\} \sim M_3\,,$$

womit der Satz in allen seinen Teilen bewiesen ist.

27. *Äquivalenzsatz.* Ist jede von zwei Mengen M, N einer Untermenge der anderen äquivalent, so sind M und N selbst äquivalent.

Es sei $M \sim M' \mathrel{\underline{\in}} N$ und $N \sim N' \mathrel{\underline{\in}} M$. Dann entspricht wegen Nr. 21 der Untermenge M' von N eine äquivalente Untermenge $M'' \mathrel{\underline{\in}} N' \mathrel{\underline{\in}} M$, und es ist $M \sim M' \sim M''$, also nach dem Theorem Nr. 25 auch $M \sim N' \sim N$, q. e. d.[1]

273 | 28. *Theorem.* Ist T eine beliebige Menge, deren Elemente M, N, R, ..., sämtlich Mengen sind, so kann man sie alle gleichzeitig abbilden auf äquivalente Mengen M', N', R', ..., welche die Elemente einer neuen Menge T' bilden und unter sich sowohl wie einer gegebenen Menge Z elementenfremd sind.

Beweis. Es sei $S = \mathfrak{S}\,T = M + N + R + \cdots$ nach V die Summe aller Elemente von T, und gemäß Nr. 19 sei T'' eine Menge, welche T äquivalent und der Summe $T + S + \mathfrak{S}\,(S + Z)$ elementenfremd ist, so daß vermöge einer Abbildung Ω jedem Elemente $M, N, R, \ldots$ von T ein bestimmtes Element $M'', N'', R'', \ldots$ von T'' entspricht. Ein beliebiges Element des Produktes ST'' (Nr. 13) ist dann von der Form $\{s, M''\}$, wo $s \,\varepsilon\, S$ und $M'' \,\varepsilon\, T''$ ist, und für jedes solche Element ist es definit (Nr. 4), ob $s \,\varepsilon\, M$ ist, wo M das dem M'' vermöge Ω entsprechende Element von T, d. h. gemäß Nr. 15 $\{M, M''\} \,\varepsilon\, \Omega$ sein soll. Alle so beschaffenen Elemente des Produktes bilden somit wegen III die Elemente einer Untermenge S' von ST'', und diese Menge S' ist $S + Z$ elementenfremd, weil sonst ein $M'' \,\varepsilon\, T''$ als Element von $\{s, M''\}$ Element eines Elementes von $S + Z$, also wegen V Element von $\mathfrak{S}\,(S + Z)$ wäre gegen die über T'' gemachte Annahme. Ist ferner M ein beliebiges Element von T, und M'' das entsprechende von T'', so bilden diejenigen Elemente $\{s, M''\}$ von S', welche M'' als Element enthalten, nach III eine gewisse Untermenge $M' \mathrel{\underline{\in}} S'$, und es ist $M' \sim M$ vermöge einer Abbildung $\mathsf{M} \mathrel{\underline{\in}} MM' \mathrel{\underline{\in}} SS'$, in

[1] Der hier in den Nrn. 25 und 27 gegebene Beweis des „Äquivalenzsatzes" (auf Grund meiner brieflichen Mitteilung vom Jan. 1906 zuerst publiziert von Herrn *H. Poincaré* in der Revue de Métaphysique et de Morale t. 14, p. 314) beruht lediglich auf der *Dedekind*schen Kettentheorie (Was sind und was sollen die Zahlen? § 4) und vermeidet im Gegensatz zu den älteren Beweisen von *E. Schröder* und *F. Bernstein*, sowie zu dem letzten Beweise von *J. König* (Comptes Rendus t. 143, 9 VII 1906) jede Bezugnahme auf geordnete Reihen vom Typus ω oder das Prinzip der vollständigen Induktion. Einen ganz ähnlichen Beweis veröffentlichte ungefähr gleichzeitig Herr *G. Peano* („Super Teorema de Cantor-Bernstein", Rendiconti del Circolo Matematico XXI sowie Revista de Mathematica VIII, p. 136), wo in der letztgenannten Note zugleich auch der von Herrn *H. Poincaré* gegen meinen Beweis gerichtete Einwand erörtert wird. Vgl. meine Note Math. Ann. Bd. 65, p. 107–128, § 2 b.

and furthermore, according to what was proved previously,

$$M = M_3 - \{c\} \sim M_3\,;$$

thus the theorem is proved in its entirety.

27. *Equivalence theorem*. If each of the two sets M and N is equivalent to a subset of the other, M and N are themselves equivalent.

Let $M \sim M' \subseteq N$ and $N \sim N' \subseteq M$. Then on account of No. 21 there corresponds to the subset M' of N an equivalent set M'' such that $M'' \subseteq N' \subseteq M$, and we have $M \sim M' \sim M''$; therefore, according to the theorem of No. 25, also $M \sim N' \sim N$, q. e. d.[10]

28. *Theorem*. If all the sets $M, N, R, \ldots$ are elements of an arbitrary set T, they can all be simultaneously mapped onto ⟦respectively⟧ equivalent sets $M', N', R', \ldots$ that are the elements of a new set, T', and are disjoint from one another as well as from a given set Z.

Proof. Let $S = \mathfrak{S}T = M + N + R + \cdots$, in accordance with Axiom V, be the sum of all elements of T, and, in accordance with No. 19, let T'' be a set equivalent to T and disjoint from the sum $T + S + \mathfrak{S}(S + Z)$, so that under a mapping Ω to each element of T there corresponds a certain element of T'': $M'', N'', R'', \ldots$ to $M, N, R, \ldots$, respectively. An arbitrary element of the product ST'' (No. 13) will then have the form $\{s, M''\}$, where $s \,\varepsilon\, S$ and $M'' \,\varepsilon\, T''$, and for every such element it is definite (No. 4) whether $s \,\varepsilon\, M$, where M is assumed to be the element of T corresponding to M'' under Ω, that is, according to No. 15, to be such that $\{M, M''\} \,\varepsilon\, \Omega$. All elements of the product that are so constituted then are, on account of Axiom III, the elements of a subset S' of ST'', and this set S' is disjoint from $S + Z$, since otherwise an element M'' of T'', being an element of $\{s, M''\}$, would be an element of an element of $S + Z$ and thus, on account of Axiom V, also an element of $\mathfrak{S}(S + Z)$, contrary to the assumption made about T''. If, furthermore, M is an arbitrary element of T and M'' the corresponding one of T'', those elements $\{s, M''\}$ of S' that contain M'' as an element form, according to Axiom III, a certain subset M' of S', and $M' \sim M$ by virtue of a mapping M ($\mathsf{M} \subseteq MM' \subseteq SS'$) under which to each element m of M there

[10] The proof of the "equivalence theorem" given here in Nos. 25 and 27 (first published by Mr. *H. Poincaré* (*1906b*, p. 314) on the basis of a letter that I wrote in January 1906) rests solely upon *Dedekind*'s chain theory (*1888*, §4) and, unlike the older proofs by *E. Schröder* and *F. Bernstein* as well as the latest proof by *J. König* (*1907*), avoids any reference to ordered sequences of order ω or to the principle of mathematical induction. At approximately the same time Mr. *G. Peano* (*1906a*) published a proof that was quite similar; the paper containing that proof also contains a discussion of the objection directed by Mr. *H. Poincaré* against my proof. See § 2b in my *1908a*.

welcher jedem Elemente m von M ein Element $m' = \{m, M''\}$ von M' entspricht und umgekehrt. Ebenso gehört auch zu jedem anderen Element $N \varepsilon T$ eine äquivalente Untermenge $N' ⋲ S'$ und eine Abbildung $\mathsf{N} ⋲ NN' ⋲ SS'$, durch welche jedem Elemente n von N ein Element $\{n, N''\}$ von N' entspricht. Die beiden Untermengen M' und N', welche zu zwei verschiedenen Elementen M und N von T gehören, sind aber immer elementenfremd, denn wäre etwa

$$\{m, M''\} = \{n, N''\}$$

ein gemeinsames Element von M' und N', so müßte M'' als Element von $\{n, N''\}$ entweder $= N''$ oder $= n$ sein, und im ersten Falle wäre auch $M = N$, im zweiten aber wären T'' und S nicht elementenfremd, gegen die Annahme. Die Untermengen $M', N', R', \ldots$ von S', welche vermöge der Abbildungen
274 $\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots$ den Elementen $M, N, R, \ldots$, von | T äquivalent sind, sind also in der Tat sowohl unter sich als auch, weil S' es ist, der Menge Z elementenfremd. Endlich ist von jeder Untermenge ${S_1}' ⋲ S'$, welche ein Element $\{s, M''\}$ enthält, immer definit, ob sie mit der entsprechenden Menge M' identisch ist oder nicht, und alle diese $M', N', R', \ldots$ bilden gemäß III und IV die Elemente einer gewissen Menge $T' ⋲ \mathfrak{U}S'$; der Satz ist also in allen seinen Teilen bewiesen.

29_{VI}. *Allgemeines Auswahlprinzip.* Ist T eine Menge, deren Elemente $M, N,\ R, \ldots$ sämtlich von Null verschiedene Mengen sind, so gibt es immer Mengen P, welche nach einer bestimmten Vorschrift jedem Element M von T eines seiner Elemente $m \varepsilon M$ eindeutig zuordnen.

Beweis. Man wende auf T' das in der vorhergehenden Nr. 28 angegebene Verfahren an, wobei $Z = 0$ gesetzt werden kann, und hat dann alle Mengen $M, N, R, \ldots$ gleichzeitig abgebildet auf die äquivalenten Mengen $M', N', R', \ldots$, welche unter sich elementenfremd sind und die Elemente einer Menge T' bilden. Ist nun P gemäß VI eine solche Untermenge von $\mathfrak{S}\, T'$, welche mit jedem Element von T' genau ein Element gemein hat, so leistet P die verlangte Zuordnung. Ist nämlich M irgend ein Element von T und ist M' das entsprechende Element von T', so enthält P nur ein einziges Element m' von M', und diesem entspricht wieder ein ganz bestimmtes Element m von M.

30_{VI}. *Theorem.* Sind zwei äquivalente Mengen T und T', deren Elemente $M,\ N,\ R, \ldots$ bzw. $M',\ N',\ R', \ldots$ unter sich elementenfremde Mengen sind, so aufeinander abgebildet, daß jedem Element M der einen Menge eine äquivalente Menge M' als Element der anderen entspricht, so sind auch die zugehörigen Summen $\mathfrak{S}\, T$ und $\mathfrak{S}\, T'$, sowie die entsprechenden Produkte $\mathfrak{P}\, T$ und $\mathfrak{P}\, T'$ einander äquivalent.

Beweis. Wir beweisen den Satz zunächst unter der Annahme, daß $S = \mathfrak{S}\, T$ und $S' = \mathfrak{S}\, T'$ einander elementenfremd sind, in welchem Falle auch jedes Element von T jedem Element von T' elementenfremd sein muß. Aus $M \sim M'$ folgt dann gemäß Nr. 15, daß $\mathfrak{U}(MM')$ eine von 0 verschiedene Untermenge A_M besitzt, welche die sämtlichen möglichen Abbildungen $\mathsf{M}, \mathsf{M}', \mathsf{M}'', \ldots$

corresponds an element $m' = \{m, M''\}$ of M' and conversely. Likewise, with any other element N of T there are associated an equivalent subset N' of S' and a mapping N ($\mathsf{N} \subseteq NN' \subseteq SS'$) under which to each element n of N there corresponds an element $\{n, N''\}$ of N'. The two subsets M' and N', which correspond to two different elements M and N of T, are, however, always disjoint, for, if, say,

$$\{m, M''\} = \{n, N''\}$$

were a common element of M' and N', then M'', being an element of $\{n, N''\}$, would have to be equal either to N'' or to n, and in the first case M would also be equal to N, and in the second T'' and S would not be disjoint, contrary to the assumption. The subsets $M', N', R', \ldots$ of S', which by virtue of the mappings $\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots$ are equivalent to the elements $M, N, R, \ldots$ of T, are thus indeed disjoint from one another, and they are also disjoint from the set Z since S' is. Finally, it is always definite for every subset ${S_1}'$ of S' containing an element $\{s, M''\}$ whether that subset is identical with the corresponding set M', and all of those among $M', N', R' \ldots$ for which this is the case are, according to Axioms III and IV, the elements of a certain subset T' of $\mathfrak{U}S'$. The theorem is therefore proved in its entirety.

29_{VI}. *General principle of choice*. If T is a set whose elements $M, N, R, \ldots$ all are sets different from the null set, there always exist sets P that, according to a certain rule, uniquely correlate with each element M of T one element m of that M.

Proof. Apply to T[11] the procedure specified in No. 28 above, letting $Z = 0$. This yields simultaneous mappings of all sets $M, N, R, \ldots$ onto the equivalent sets $M', N', R', \ldots$, which are mutually disjoint and form the elements of a set T'. If now P, in accordance with Axiom VI, is a subset of $\mathfrak{S}T'$ that has exactly one element in common with each element of T', then P provides the desired correlation. For, if M is any element of T and M' the corresponding element of T', P contains only a single element m' of M', and to this there again corresponds a well-determined element m of M.

30_{VI} *Theorem*. Let T and T' be two equivalent sets containing as elements the mutually disjoint sets $M, N, R, \ldots$ and the mutually disjoint sets $M', N', R', \ldots$, respectively. If T and T' are mapped onto each other in such a way that to each element M of one set there corresponds an equivalent set M' as an element of the other, the associated sums $\mathfrak{S}T$ and $\mathfrak{S}T'$, as well as the corresponding products $\mathfrak{P}T$ and $\mathfrak{P}T'$, are also equivalent.

Proof. We first prove the theorem on the assumption that $S = \mathfrak{S}T$ and $S' = \mathfrak{S}T'$ are mutually disjoint, in which case each element of T would have to be disjoint from each element of T', too. It then follows from $M \sim M'$, according to No. 15, that $\mathfrak{U}(MM')$ possesses a subset A_M, different from 0, that contains as elements all possible mappings $\mathsf{M}, \mathsf{M}', \mathsf{M}'', \ldots$ of M onto M'.

[11] ⟦Zermelo erroneously writes "T′" instead of "T".⟧

von M auf M' als Elemente enthält. Ebenso entspricht jedem anderen Element N von T eine Menge $\mathsf{A}_N \mathrel{\underline{\in}} \mathfrak{U}(NN')$, welche die sämtlichen Abbildungen von N auf N' umfaßt, und auch A_N ist $\neq 0$. Alle diese Abbildungsmengen $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$ sind Untermengen von $\mathfrak{U}(SS')$ und bilden daher wegen III und IV die Elemente einer gewissen Untermenge $\mathsf{T} \mathrel{\underline{\in}} \mathfrak{U}\mathfrak{U}(SS')$. Da nun die Elemente von T sämtlich von 0 verschiedene Mengen und unter sich elementenfremd sind (weil aus der Elementenfremdheit von MM' und NN' auch die ihrer Untermengen folgt), so ist nach Axiom VI auch das Produkt
275 $\mathfrak{P}\mathsf{T} \neq 0$, und ein beliebiges Ele- | ment Θ von $\mathfrak{P}$ ist eine Menge der Form $\Theta = \{\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots\}$, welche von jeder der Mengen $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$ genau ein Element enthält. Die Existenz einer solchen „kombinierten Abbildung“ hätten wir kürzer auch aus dem Theorem Nr. 29 schließen können. Bilden wir nun gemäß V die Vereinigung

$$\Omega = \mathfrak{S}\Theta = \mathsf{M} + \mathsf{N} + \mathsf{P} + \cdots \mathrel{\underline{\in}} SS',$$

so liefert Ω die verlangte Abbildung von S auf S'. Denn jedes Element s von S muß einem und nur einem Elemente von T, etwa M, als Element angehören und daher in einem einzigen Element der entsprechenden Abbildung M als Element erscheinen, während in allen übrigen Summanden $\mathsf{N}, \mathsf{P}, \ldots$ kein Element von M mehr vorkommt. Das analoge gilt auch für jedes Element s' von S', und nach der Definition Nr. 15 ist somit in der Tat $S \sim S'$.

Durch dasselbe Ω und seine Untermengen wird wegen Nr. 17 auch jede Untermenge p von S auf eine äquivalente Untermenge p' von S' abgebildet, und ist insbesondere $p = \{m, n, \ldots\}$ gemäß Nr. 13 ein Element von $P = \mathfrak{P}T$, so ist die ihm entsprechende Untermenge $p' = \{m', n', \ldots\}$ von S' auch ein Element von $\mathfrak{P}T'$. Ist nämlich M' ein beliebiges Element von T', und M das entsprechende Element von T, so enthält p ein und nur ein Element $m \,\varepsilon\, M$ und p' das entsprechende Element $m' \,\varepsilon\, M'$, aber auch kein weiteres Element von M', da ein solches auch einem zweiten Elemente von M in p entsprechen müßte. Ebenso entspricht jedem Elemente $p' \,\varepsilon\, \mathfrak{P}T'$ ein und nur ein Element $p \,\varepsilon\, \mathfrak{P}T$, und wir erhalten in der Tat eine bestimmte Untermenge $\mathsf{\Pi} \mathrel{\underline{\in}} \mathfrak{P}T \cdot \mathfrak{P}T'$ als Abbildung von $\mathfrak{P}T$ auf $\mathfrak{P}T'$, so daß auch diese beiden Produkte einander äquivalent sind.

Sind nun aber S und S' nicht mehr elementenfremd, so können wir gemäß Nr. 19 eine dritte Menge S'' einführen, welche S' äquivalent und $S + S'$ elementenfremd ist. Dann entspricht wegen Nr. 17 einer Untermenge $M' \mathrel{\underline{\in}} S'$ eine äquivalente Untermenge $M'' \mathrel{\underline{\in}} S''$, und da die $M', N', R', \ldots$ untereinander elementenfremd sind, so gilt das gleiche auch von den entsprechenden $M'', N'', R'', \ldots$. Da ferner jedes Element s'' von S'' einem Elemente s' von S' entspricht, welches einer der Mengen $M', N', R', \ldots$ angehört, so ist S'' die *Summe* aller dieser $M'', N'', R'', \ldots$, welche die Elemente einer gewissen Untermenge $T'' \mathrel{\underline{\in}} \mathfrak{U}S''$ bilden. Nun haben wir aber $M \sim M' \sim M''$, $N \sim N' \sim N''$, $\ldots$; es ist also jedes Element M von T dem entsprechenden Element M'' von T'' äquivalent, und da jetzt $S'' = \mathfrak{S}T''$ beiden Summen

Likewise, to any other element N of T there corresponds a subset A_N of $\mathfrak{U}(NN')$ that contains all mappings of N onto N', and A_N is also different from 0. All these mapping-sets $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$ are subsets of $\mathfrak{U}(SS')$ and therefore are, according to Axioms III and IV, the elements of a certain subset T of $\mathfrak{U}\mathfrak{U}(SS')$. Since, now, all elements of T are sets that are different from 0 and mutually disjoint (because from the fact that MM' and NN' are disjoint it follows that their subsets are disjoint), according to Axiom VI the product $\mathfrak{P}\mathsf{T}$ is also different from 0, and an arbitrary element Θ of $\mathfrak{P}\mathsf{T}$ is a set of the form $\{\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots\}$ that contains exactly one element of each of the sets $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$. We could also have inferred the existence of such a "combined" mapping from the theorem of No. 29, and more quickly at that. If we now form, in accordance with Axiom V, the union

$$\Omega = \mathfrak{S}\Theta = \mathsf{M} + \mathsf{N} + \mathsf{P} + \cdots \subseteq SS',$$

then Ω provides the required mapping of S onto S'. For each element s of S must belong as an element to one and only one element of T, say M, and must therefore occur as an element in a single element of the corresponding mapping M, while no element of M occurs in any of the remaining summands $\mathsf{N}, \mathsf{P}, \ldots$. An analogous statement holds for every element of s' of S', and according to Definition A (No. 15) we thus have $S \sim S'$.

By means of the same Ω and its subsets, each subset p of S is also mapped, on account of No. 17, onto an equivalent subset p' of S', and if in particular $p = \{m, n, \ldots\}$ is, in accordance with No. 13, an element of $P = \mathfrak{P}T$, the subset $p' = \{m', n', \ldots\}$ of S' corresponding to it is an element of $\mathfrak{P}T'$. For, if M' is an arbitrary element of T' and M the corresponding element of T, p contains one and only one element m of M and p' contains the corresponding element m' of M' but no other element of M', since such an element would also have to correspond to a second element of M in p. Likewise, there corresponds to each element p' of $\mathfrak{P}T'$ one and only one element p of $\mathfrak{P}T$, and we indeed obtain a certain subset Π of $\mathfrak{P}T \cdot \mathfrak{P}T'$ as a mapping of $\mathfrak{P}T$ onto $\mathfrak{P}T'$, so that these two products are mutually equivalent.

But if now S and S' are no longer assumed to be disjoint, we can, according to No. 19, introduce a third set, S'', that is equivalent to S' and disjoint from $S + S'$. Then, on account of No. 17, there corresponds to a subset M' of S' an equivalent subset M'' of S'', and, since $M', N', R', \ldots$ are mutually disjoint, the same holds of the corresponding $M'', N'', R'', \ldots$. Since, furthermore, every element s'' of S'' corresponds to an element s' of S' belonging to one of the sets $M', N', R', \ldots$, S'' is the *sum* of all these $M'', N'', R'', \ldots$, which are the elements of a certain subset T'' of $\mathfrak{U}S''$. But now we have $M \sim M' \sim M''$, $N \sim N' \sim N'', \ldots$; thus every element M of T is equivalent to the corresponding element M'' of T'', and, since now $S'' = \mathfrak{S}T''$ is disjoint from both of the sums $S' = \mathfrak{S}T'$ and $S = \mathfrak{S}T$, it follows according to what

$S' = \mathfrak{S}T'$ und $S = \mathfrak{S}T$ elementenfremd ist, so folgt nach dem oben Bewiesenen:

$$\mathfrak{S}T \sim \mathfrak{S}T'' \sim \mathfrak{S}T' \quad \text{und} \quad \mathfrak{P}T \sim \mathfrak{P}T'' \sim \mathfrak{P}T',$$

womit der Satz in voller Allgemeinheit bewiesen ist.

276 31. *Definition.* Ist eine Menge M einer Untermenge der Menge N | äquivalent, aber nicht umgekehrt N einer Untermenge von M, so sagen wir, M sei „von *kleinerer Mächtigkeit* als N“, und schreiben abgekürzt $M < N$.

Folgerungen. a) Da es nach Nr. 21 für irgend zwei Mengen definit ist, ob sie einander äquivalent sind oder nicht, so ist es auch definit, ob M mindestens einem Element von $\mathfrak{U}N$, sowie ob N irgend einem Element von $\mathfrak{U}M$ äquivalent ist. *Es ist also immer definit, ob $M < N$ ist oder nicht.*

b) Die drei Beziehungen $M < N$, $M \sim N$, $N < M$ schließen einander aus.

c) Ist $M < N$ und $N < R$ oder $N \sim R$, so ist immer auch $M < R$.

d) Ist M einer Untermenge von N äquivalent, so ist entweder $M \sim N$ oder $M < N$. Dies ist eine Folge des „Äquivalenzsatzes“ Nr. 27.

e) Die Nullmenge ist von kleinerer Mächtigkeit als jede andere Menge, ebenso jede aus einem einzigen Elemente bestehende Menge $\{a\}$ von kleinerer Mächtigkeit als jede Menge M, welche echte Teile besitzt. Vergl. Nr. 23.

32. *Satz von Cantor.* Ist M eine beliebige Menge, so ist immer $M < \mathfrak{U}M$. *Jede Menge ist von kleinerer Mächtigkeit als die Menge ihrer Untermengen.*

Beweis. Jedem Element m von M entspricht eine Untermenge $\{m\} \mathrel{\underline{\in}} M$. Da es nun für jede Untermenge $M_1 \mathrel{\underline{\in}} M$ definit ist, ob sie nur ein einziges Element enthält (Nr. 13), so bilden alle Untermengen der Form $\{m\}$ die Elemente einer Menge $U_0 \mathrel{\underline{\in}} \mathfrak{U}M$, und es ist $M \sim U_0$.

Wäre umgekehrt $U = \mathfrak{U}M$ äquivalent einer Untermenge $M_0 \mathrel{\underline{\in}} M$, so entspräche vermöge einer Abbildung Φ von U auf M_0 jeder Untermenge $M_1 \mathrel{\underline{\in}} M$ ein bestimmtes Element m_1 von M_0, so daß $\{M_1, m_1\} \,\varepsilon\, \Phi$ wäre, und es wäre immer definit, ob $m_1 \,\varepsilon\, M_1$ ist oder nicht. Alle solchen Elemente m_1 von M_0, für welche *nicht* $m_1 \,\varepsilon\, M_1$ ist, bildeten also die Elemente einer Untermenge $M' \mathrel{\underline{\in}} M_0 \mathrel{\underline{\in}} M$, welche gleichfalls Element von U wäre. Dieser Menge $M' \mathrel{\underline{\in}} M$ kann aber kein Element m' von M_0 entsprechen. Wäre nämlich $m' \varepsilon M'$, so widerspräche dies der Definition von M'. Wäre aber m' *nicht* $\varepsilon\, M'$, so müßte nach derselben Definition M' auch dieses Element m' enthalten, widersprechend der Annahme. Es ergibt sich also, daß U keiner Untermenge von M äquivalent sein kann, und in Verbindung mit dem zuerst Bewiesenen, $M < \mathfrak{U}M$.

Der Satz gilt für *alle* Mengen M, z. B. auch für $M = 0$, und es ist in der Tat

$$0 < \{0\} = \mathfrak{U}(0)\,.$$

Ebenso ist auch für jedes a

$$\{a\} < \{0, \{a\}\} = \mathfrak{U}\{a\}\,.$$

has been proved above that

$$\mathfrak{S}T \sim \mathfrak{S}T'' \sim \mathfrak{S}T' \quad \text{and} \quad \mathfrak{P}T \sim \mathfrak{P}T'' \sim \mathfrak{P}T',$$

thus the theorem is proved in full generality.

31. *Definition.* If a set M is equivalent to a subset of the set N, but N is not equivalent to a subset of M, we say that M is "*of lower cardinality* than N", and we write $M < N$ for short.

Corollaries. a) Since according to No. 21 it is definite for any two sets whether they are equivalent or not, it is also definite whether M is equivalent to at least one element of $\mathfrak{U}N$, as well as whether N is equivalent to some element of $\mathfrak{U}M$. *It is therefore always definite whether $M < N$ or not.*

b) The three relations $M < N$, $M \sim N$, and $M > N$ are mutually exclusive.

c) Whenever we have $M < N$ and either $N < R$ or $N \sim R$, we have $M < R$.

d) If M is equivalent to a subset of N, then either $M \sim N$ or $M < N$. This is a consequence of the "equivalence theorem" (No. 27).

e) The null set is of lower cardinality than any other set; likewise, every set $\{a\}$ consisting of a single element is of lower cardinality than any set M that has strict parts (see No. 23).

32. *Cantor's theorem.* If M is an arbitrary set, then always $M < \mathfrak{U}M$. *Every set is of lower cardinality than the set of its subsets.*

Proof. To each element m of M there corresponds a subset $\{m\}$ of M. Now since it is definite for each subset M_1 of M whether it contains only a single element (No. 13), all subsets of the form $\{m\}$ are the elements of a subset U_0 of $\mathfrak{U}M$, and $M \sim U_0$.

If on the other hand $U = \mathfrak{U}M$ were equivalent to a subset M_0 of M, then under a mapping Φ of U onto M_0 there would correspond to each subset M_1 of M a certain element m_1 of M_0, so that $\{M_1, m_1\}$ would be an element of Φ, and it would always be definite whether $m_1 \,\varepsilon\, M_1$ or not. All those elements m_1 of M_0 for which m_1 is *not* an element of M_1 would then be the elements of a set M' ($M' \subseteq M_0 \subseteq M$), which likewise would be an element of U. But no element m' of M_0 could correspond to this subset M' of M. For if m' were an element of M', this would contradict the definition of M'. But if m' were *not* an element of M', then, according to the same definition, M' would also have to contain this element m', contrary to the assumption. Thus it follows that U cannot be equivalent to any subset of M, and, in combination with what was proved first, $M < \mathfrak{U}M$.

The theorem holds for *all* sets M, even, for instance, for $M = 0$, and indeed

$$0 < \{0\} = \mathfrak{U}0 .$$

Likewise, for every a,

$$\{a\} < \{0, \{a\}\} = \mathfrak{U}\{a\} .$$

Aus dem Satze folgt endlich, daß es zu jeder beliebigen Menge T von
277 | Mengen $M, N, R, \ldots$ immer noch Mengen von größerer Mächtigkeit gibt; z. B. die Menge

$$P = \mathfrak{U}\,\mathfrak{S}T > \mathfrak{S}T \gtrsim M, N, R, \ldots$$

besitzt diese Eigenschaft.

33_{VI}. *Theorem.* Sind zwei äquivalente Mengen T und T', deren Elemente M, N, $R, \ldots$ bzw. M', N', $R', \ldots$ unter sich elementenfremde Mengen sind, so aufeinander abgebildet, daß jedes Element M von T von kleinerer Mächtigkeit ist als das entsprechende Element M' von T', so ist auch die Summe $S = \mathfrak{S}T$ aller Elemente von T von kleinerer Mächtigkeit als das Produkt $P' = \mathfrak{P}T'$ aller Elemente von T'.

Beweis. Es genügt, den Satz für den Fall zu beweisen, wo die beiden Summen $S = \mathfrak{S}\,T$ und $S' = \mathfrak{S}\,T'$ elementenfremd sind. Die Ausdehnung auf den allgemeinen Fall vollzieht sich dann analog wie bei dem Theorem Nr. 30 und mit Hilfe desselben durch Einschaltung einer dritten Menge $S'' \sim S$, welche S' elementenfremd ist.

Zunächst ist zu zeigen, daß S einer Untermenge von P' äquivalent ist. Wegen $M < M'$ existiert eine von 0 verschiedene Untermenge $\mathsf{A}_M \Subset \mathfrak{U}(MM')$, deren sämtliche Elemente $\mathsf{M}, \mathsf{M}', \mathsf{M}'', \ldots$ Abbildungen sind, welche M auf Untermengen ${M_1}', {M_2}', \ldots$ von M' abbilden. Solche Abbildungsmengen $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$ existieren für je zwei entsprechende Elemente $\{M, M'\}$, $\{N, N'\}$, $\{R, R'\}, \ldots$ von T und T', und jedes Element $\Theta = \{\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots\}$ ihres Produktes $\mathfrak{P}\mathsf{T} = \mathsf{A}_M \cdot \mathsf{A}_N \cdot \mathsf{A}_R \cdots$ liefert, analog wie in Nr. 30, eine simultane Abbildung sämtlicher Elemente $M, N, R, \ldots$ von T auf äquivalente Untermengen ${M_1}'$, ${N_1}'$, ${R_1}'$, $\ldots$ der entsprechenden Elemente von T'. Durch $\Omega = \mathfrak{S}\Theta \Subset SS'$ wird also jedes Element s von S auf ein Element s' von S' abgebildet, wenn auch nicht umgekehrt jedes von S' auf eines von S.

Nun sind aber die Komplementärmengen $M' - {M_1}', N' - {N_1}', R' - {R_1}', \ldots$, welche die Elemente einer Menge ${T_1}' \Subset \mathfrak{U}S'$ bilden, sämtlich von 0 verschieden, weil wegen $M < M'$ der Fall $M \sim {M_1}' = M'$ immer ausgeschlossen ist. Somit ist auch das Produkt $\mathfrak{P}{T_1}' \neq 0$, und es existiert mindestens eine Menge $q\,\varepsilon\,\mathfrak{P}{T_1}'$ von der Form $q = \{{m_0}', {n_0}', {r_0}', \ldots\} \Subset S'$, welche mit jeder der Mengen $M' - {M_1}', N' - {N_1}', \ldots$ genau ein Element gemein hat und daher auch Element von P' ist.

Ist nun s irgend ein Element von S, und s' das vermöge Ω ihm entsprechende Element von S', so entspricht ihnen beiden noch ein Element s_0 von $q \Subset S'$ in der Weise, daß s' und ${s_0}'$ immer einem und demselben Elemente von T' angehören und somit für $s\,\varepsilon\,M$ immer ${s_0}' = {m_0}'$ ist usw. Da aber im Falle $s'\,\varepsilon\,{M_1}'$ stets ${s_0}'\,\varepsilon\,(M' - {M_1}')$ ist, so sind s' und ${s_0}'$ immer voneinander verschieden. Bilden wir nun die Menge

$$q_s = q - \{{s_0}'\} + \{s'\}\,,$$

Finally, it follows from the theorem that for every arbitrary set T of sets $M, N, R, \ldots$ there always exist further sets of higher cardinality; for example, the set

$$P = \mathfrak{U}\mathfrak{S}T > \mathfrak{S}T \gtrsim M, N, R, \ldots$$

possesses this property.

33_{VI}. *Theorem.* Let T and T' be two equivalent sets containing as elements the mutually disjoint sets $M, N, R, \ldots$ and the mutually disjoint sets $M', N', R', \ldots$, respectively. If T and T' are mapped onto each other in such a way that each element M of T is of lower cardinality than the corresponding element M' of T', the sum $S = \mathfrak{S}T$ of all elements of T is also of lower cardinalty than the product $P' = \mathfrak{P}T'$ of all elements of T'.

Proof. It suffices to prove the theorem for the case in which the two sums $S = \mathfrak{S}T$ and $S' = \mathfrak{S}T'$ are disjoint. The extension to the general case is then accomplished by a method analogous to that used for the theorem in No. 30, through interposition of a third set, S'', equivalent to S and disjoint from S', and by means of that theorem.

First it must be shown that S is equivalent to a subset of P'. Because $M < M'$, there exists a subset A_M of $\mathfrak{U}(MM')$, different from 0, all of whose elements $\mathsf{M}, \mathsf{M}', \mathsf{M}'', \ldots$ are mappings that map M onto subsets ${M_1}', {M_2}', \ldots$ of M'. Such mapping-sets $\mathsf{A}_M, \mathsf{A}_N, \mathsf{A}_R, \ldots$ exist for any two corresponding elements $\{M, M'\}, \{N, N'\}, \{R, R'\}, \ldots$ of T and T', and each element $\Theta = \{\mathsf{M}, \mathsf{N}, \mathsf{P}, \ldots\}$ of their product $\mathfrak{P}\mathsf{T} = \mathsf{A}_M \cdot \mathsf{A}_N \cdot \mathsf{A}_R \ldots$ furnishes, just as in No. 30, a simultaneous mapping of all elements $M, N, R, \ldots$ of T onto equivalent subsets ${M_1}', {N_1}', {R_1}', \ldots$ of the corresponding elements of T'. By means of $\Omega = \mathfrak{S}\Theta \subseteq SS'$, therefore, every element s of S is mapped onto an element s' of S', even though not every element of S' is mapped onto one of S.

But now the complements $M' - {M_1}'$, $N' - {N_1}'$, $R' - {R_1}', \ldots$, which are the elements of a subset T_1' of $\mathfrak{U}S'$, all are different from 0, since, because $M < M'$, the case $M \sim {M_1}' = M'$ is always excluded. Thus also the product $\mathfrak{P}{T_1}' \neq 0$, and there exists at least one set $q \,\varepsilon\, \mathfrak{P}{T_1}'$ of the form $\{{m_0}', {n_0}', {r_0}', \ldots\} \subseteq S'$ that has exactly one element in common with each of the sets $M' - {M_1}'$, $N' - {N_1}', \ldots$, and is therefore also an element of P'.

If now s is any element of S, and s' is the element of S' corresponding to it under Ω, there corresponds to both of them yet another element s_0 of the subset q of S', such that s' and ${s_0}'$ always belong to one and the same element of T', and consequently for $s \,\varepsilon\, M$ always ${s_0}' = {m_0}'$, and so forth. But, since ${s_0}' \,\varepsilon (M' - {M_1}')$ whenever $s' \,\varepsilon\, {M_1}'$, s' and ${s_0}'$ are always distinct. If we now form the set

$$q_s = q - \{{s_0}'\} + \{s'\} ,$$

278 | welche aus q entsteht, indem wir das eine Element $s_0{}'$ durch das andere s' ersetzen, so erhalten wir wieder ein Element von P', nämlich eine Untermenge von S', welche mit jeder der Mengen $M', N', R', \ldots$ genau ein Element gemein hat. Diese Elemente q_s von P', welche die Elemente einer Untermenge $P_0{}' \Subset P'$ bilden, sind aber sämtlich voneinander verschieden. Denn sind etwa m_1 und m_2 zwei verschiedene Elemente *derselben* Menge $M \,\varepsilon\, T$, so sind auch die entsprechenden Elemente $m_1{}'$ und $m_2{}'$ von $M_1{}'$, welche an die Stelle von s' treten, voneinander verschieden, und somit auch

$$q_{m_1} = q - \{m_0{}'\} + \{m_1{}'\} \neq q - \{m_0{}'\} + \{m_2{}'\} = q_{m_2},$$

da q außer $m_0{}'$ kein weiteres Element mit M' gemein hat. Sind aber m und n zwei Elemente von S, welche *verschiedenen* Mengen M und N angehören, so hat $q_m = q - \{m_0{}'\} + \{m'\}$ mit M' ein Element m' von $M_1{}'$, dagegen $q_n = q - \{n_0{}'\} + \{n'\}$ mit M' nur das Element $m_0{}' \,\varepsilon\, (M' - M_1{}')$ gemeinsam, und beide Mengen sind gleichfalls voneinander verschieden. Somit bilden die Paare $\{s, q_s\}$ die Elemente einer Menge $\Phi \Subset SP_0{}'$, welche gemäß Nr. 15 den Charakter einer Abbildung besitzt, und es ist in der Tat $S \sim P_0{}' \Subset P'$.

Andererseits kann aber P' keiner Untermenge S_0 von S äquivalent sein. Wäre dies nämlich der Fall, so müßte vermöge einer Abbildung $\Psi \Subset S_0 P' \Subset SP'$ jedem Elemente $s \,\varepsilon\, S_0$ ein Element $p_s \,\varepsilon\, P'$ entsprechen. Betrachten wir insbesondere diejenigen Elemente p_m, welche Elementen m des Durchschnittes $M_0 = [M, S_0]$ entsprechen. Jedes dieser p_m enthält dabei ein Element $m'' \,\varepsilon\, M'$, nämlich dasjenige, welches p_m als Element von P' mit M' gemeinsam hat; die zu verschiedenen m gehörenden m'' brauchen aber nicht immer verschieden zu sein. Jedenfalls bilden alle m'', die zu den Elementen m von M_0 gehören, die Elemente einer Untermenge $M_2{}'$ von M', welche von M' selbst verschieden ist, da sonst M' einer Untermenge von $M_0 \Subset M$ äquivalent wäre gegen die Voraussetzung $M < M'$.[1] In derselben Weise gehören zu *allen* Elementen $M, N, R, \ldots$ von T gewisse echte Teilmengen $M_2{}', N_2{}', R_2{}', \ldots$ der entsprechenden Elemente $M', N', R', \ldots$ von T'. Die zugehörigen Komplementärmengen $M' - M_2{}', N' - N_2{}', R' - R_2{}', \ldots$ sind also sämtlich von 0 verschieden und bilden die Elemente einer Menge $T_2{}' \Subset \mathfrak{U}S'$. Ist nun $p_0{}'$ irgend ein Element von $\mathfrak{P}\,T_2{}' \neq 0$, so ist es gleichzeitig auch Element von P', kann aber bei der vorausgesetzten Abbildung Ψ keinem Elemente s von S_0 entsprechen. Wäre nämlich etwa $p_0{}' = p_m$, entspräche also $p_0{}'$ einem Elemente von M_0, so müßte es nach der gemachten Annahme mit M' ein Element $m'' \,\varepsilon\, M_2{}'$ gemein haben, während in Wirklichkeit $p_0{}'$ mit M' kein anderes
279 | Element als eines von $M' - M_2{}'$ gemein haben kann. Ebensowenig kann $p_0{}'$ irgend einem Elemente von $N_0, R_0, \ldots$ entsprechen, entspricht also überhaupt keinem Elemente von $S_0 \Subset S$, und die Annahme $P' \sim S_0$ führt auf einen Widerspruch, womit der Beweis der Behauptung $S < P'$ vollendet ist.

Das vorstehende (Ende 1904 der Göttinger Mathematischen Gesellschaft von mir mitgeteilte) Theorem ist der allgemeinste bisher bekannte Satz über das Größer

[1] Auch hier kommt das Auswahlaxiom VI zur Anwendung.

which we obtain from q when we replace one of these elements, ${s_0}'$, by the other, s', we again obtain an element of P', namely, a subset of S' that has exactly one element in common with each of the sets $M', N', R', \ldots$. But the elements of P' such as q_s which are the elements of a subset ${P_0}'$ of P', all are distinct. For if, say, m_1 and m_2 are two distinct elements of the *same* element M of T, the corresponding elements ${m_1}'$ and ${m_2}'$ of ${M_1}'$, which take the place of s', are also distinct, and thus

$$q_{m_1} = q - \{{m_0}'\} + \{{m_1}'\} \neq q - \{{m_0}'\} + \{{m_2}'\} = q_{m_2}\,,$$

since, except for ${m_0}'$, q has no element in common with M'. But if m and n are two elements of S that belong to *distinct* sets M and N, then $q_m = q - \{{m_0}'\} + \{m'\}$ has one element m' of ${M_1}'$ in common with M', while $q_n = q - \{{n_0}'\} + \{n'\}$ has only the element ${m_0}'$ of $(M' - {M_1}')$ in common with M', and the two sets are likewise distinct. Thus the pairs $\{s, q_s\}$ are the elements of a subset Φ of SP_0', which, according to No. 15, possesses the character of a mapping, and we in fact have $S \sim {P_0}' \subseteq P'$.

On the other hand, P' cannot be equivalent to any subset S_0 of S. For if this were the case, there would have to correspond to every element s of S_0 an element p_s of P' under a mapping Ψ ($\Psi \subseteq S_0P' \subseteq SP'$). Let us consider in particular those elements p_m that correspond to elements m of the intersection $M_0 = [M, S_0]$. Each of these p_m contains, then, an element m'' of M', namely, the one that p_m, as an element of P', has in common with M'; but the m'' belonging to distinct m are not necessarily always distinct. In any case, all m'' belonging to the elements m of M_0 are the elements of a subset ${M_2}'$ of M' that is distinct from M' itself, since otherwise M' would be equivalent to a subset of M_0, which in turn is a subset of M, contrary to the assumption that $M < M'$.[12] Similarly, there answer to *all* elements $M, N, R, \ldots$ of T certain strict partial sets ${M_2}', {N_2}', {R_2}', \ldots$ of the corresponding elements $M', N', R', \ldots$ of T'. The respective complements $M' - {M_2}'$, $N' - {N_2}'$, $R' - {R_2}', \ldots$, therefore, all are different from 0 and are the elements of a subset ${T_2}'$ of $\mathfrak{U}S'$. If, now, ${p_0}'$ is any element of $\mathfrak{P}{T_2}' \neq 0$, it is also an element of P', but under the mapping Ψ whose existence was assumed it cannot correspond to any element s of S_0. For if, say, ${p_0}'$ were equal to p_m, if, therefore, ${p_0}'$ were to correspond to an element of M_0, then according to the assumption made it would necessarily have an element m'' of ${M_2}'$ in common with M', while actually ${p_0}'$ cannot have any element in common with M' other than one of $M' - {M_2}'$. Nor can ${p_0}'$ correspond to any element of $N_0, R_0, \ldots$; thus it corresponds to no element of the subset S_0 of S at all, and the assumption that $P' \sim S_0$ leads to a contradiction, which completes the proof of the assertion that $S < P'$.

This theorem (communicated by me to the Göttingen Mathematical Society at the end of 1904) is the most general theorem now known concerning the comparison

[12] Here, too, use is made of Axiom VI (axiom of choice).

und Kleiner der Mächtigkeiten, aus dem alle übrigen sich ableiten lassen. Der Beweis beruht auf einer Verallgemeinerung des von Herrn *J. König* für einen speziellen Fall (siehe unten) angewandten Verfahrens.

34_{VI}. *Folgerung* (*Satz von J. König*). Ist eine Menge T, deren Elemente sämtlich Mengen und untereinander elementenfremd sind, in der Weise auf eine Untermenge T' von T abgebildet, daß jedem Elemente M von T ein Element M' von T' von größerer Mächtigkeit ($M < M'$) entspricht, so ist immer $\mathfrak{S}\,T < \mathfrak{P}\,T$, sofern $\mathfrak{P}\,T \neq 0$ ist.[1]

Nach dem Theorem Nr. 33 ist in dem betrachteten Falle immer $\mathfrak{S}T < \mathfrak{P}T'$; es bleibt also nur noch zu zeigen, daß hier $\mathfrak{P}T'$ einer Untermenge von $\mathfrak{P}T$ äquivalent ist. Für $T' = T$ ist dies trivial; im anderen Falle ist aber $\mathfrak{P}(T - T') \neq 0$, weil sonst wegen VI die Nullmenge ein Element von $T - T'$ und gegen die Annahme $\mathfrak{P}T = 0$ wäre. Ist aber q irgend ein Element von $\mathfrak{P}(T - T')$ und $p' \,\varepsilon\, \mathfrak{P}T'$, so ist $p' + q$ Element von $\mathfrak{P}T$, nämlich eine Untermenge von $\mathfrak{S}T' + \mathfrak{S}(T - T') = \mathfrak{S}T$, welche mit jedem Elemente von T' sowohl als von $T - T'$ genau ein Element gemein hat. Somit entspricht bei festgehaltenem q jedem Element p' von $\mathfrak{P}T'$ ein bestimmtes Element $p' + q$ von $\mathfrak{P}T$, und alle diese $p' + q$ bilden die Elemente einer gewissen Untermenge P_q von $\mathfrak{P}T$, welche $\sim \mathfrak{P}T'$ ist.

35. Auch der Cantorsche Satz Nr. 32 läßt sich als besonderer Fall aus dem allgemeinen Theorem Nr. 33 gewinnen.

Es sei M eine beliebige Menge, M' gemäß Nr. 19 eine M äquivalente und elementenfremde Menge und $\Phi \Subset MM'$ eine beliebige „Abbildung“ von M auf M'. Jedem Element m von M entspricht dann ein bestimmtes Element $\{m, m'\}$ von Φ und es ist immer gemäß Nr. 31e

$$\{m\} < \{m, m'\}\,.$$

Diese Mengen $\{m\}$ bilden offenbar die Elemente einer weiteren Menge $T \sim M$, und es ist nach dem Theorem Nr. 33

$$M = \mathfrak{S}T < \mathfrak{P}\Phi\,.$$

Es bleibt also nur noch zu zeigen, daß $\mathfrak{P}\Phi \sim \mathfrak{U}M$ ist. Nun ist jedes Element von $\mathfrak{P}\Phi$ eine Menge der Form $M_1 + (M' - M_1{}')$, wo M_1 eine Untermenge
280 von M, und $M_1{}'$ die entsprechende von M' bedeutet. Somit | entspricht in der Tat jedem Elemente M_1 von $\mathfrak{U}M$ ein und nur ein Element von $\mathfrak{P}\Phi$ und umgekehrt, und es ist, wie behauptet,

$$M < \mathfrak{P}\Phi \sim \mathfrak{U}M\,.$$

[1] *J. König*, Math. Ann. Bd. 60, p. 177 für den besonderen Fall, wo die Elemente von T nach ihrer Mächtigkeit geordnet eine Reihe vom Typus ω bilden.

of cardinalities, one from which all the others can be derived. The proof rests upon a generalization of a procedure applied by Mr. *J. König* in a special case (see below).

34_{VI}. *Corollary* (*J. König's theorem*). If a set T whose elements all are sets that are mutually disjoint is mapped onto a subset T' of T in such a way that to each element M of T there corresponds an element M' of T' of higher cardinality ($M < M'$), then $\mathfrak{S}T < \mathfrak{P}T$ whenever $\mathfrak{P}T \neq 0$.[13]

According to the theorem of No. 33, we always have $\mathfrak{S}T < \mathfrak{P}T'$ in the case under consideration; it therefore remains to be shown only that $\mathfrak{P}T'$ is here equivalent to a subset of $\mathfrak{P}T$. When $T' = T$ this is trivial; but in the other case we have $\mathfrak{P}(T - T') \neq 0$, else, on account of Axiom VI, the null set would be an element of $T - T'$, and, contrary to the assumption, $\mathfrak{P}T$ would equal 0. But if q is any element of $\mathfrak{P}(T - T')$ and p' any element of $\mathfrak{P}T'$, then $p' + q$ is an element of $\mathfrak{P}T$, namely, a subset of $\mathfrak{S}T' + \mathfrak{S}(T - T') = \mathfrak{S}T$ that has exactly one element in common with each element of T' as well as of $T - T'$. Thus for a fixed q there corresponds to each element p' of $\mathfrak{P}T'$ a certain element $p'+q$ of $\mathfrak{P}T$, and all these $p'+q$ are the elements of a certain subset P_q of $\mathfrak{P}T$ that is equivalent to $\mathfrak{P}T'$.

35. Cantor's theorem (No. 32) can also be obtained as a special case of the general theorem of No. 33.

Let M be an arbitrary set; let M' be a set—and according to No. 19 such a set does exist—equivalent to and disjoint from M, and let the subset Φ of MM' be an arbitrary mapping of M onto M'. Then to each element m of M there corresponds a definite element $\{m, m'\}$ of Φ and, according to No. 31e, always

$$\{m\} < \{m, m'\}\,.$$

These sets $\{m\}$ obviously are the elements of a new set, T, that is equivalent to M, and, according to the theorem of No. 33,

$$M = \mathfrak{S}T < \mathfrak{P}\Phi\,.$$

It thus remains to be shown only that $\mathfrak{P}\Phi \sim \mathfrak{U}M$. Now every element of $\mathfrak{P}\Phi$ is a set of the form $M_1 + (M' - M_1{}')$, where M_1 is a subset of M and $M_1{}'$ the corresponding subset of M'. Then there indeed corresponds to each element M_1 of $\mathfrak{U}M$ one and only one element of $\mathfrak{P}\Phi$ and conversely; and, as asserted,

$$M < \mathfrak{P}\Phi \sim \mathfrak{U}M\,.$$

[13] *J. König 1905b* for the special case in which the elements of T, when ordered according to cardinality, form a sequence of type ω.

36_{VII}. *Theorem.* Die „Zahlenreihe“ Z_0 (Nr. 14) ist eine „unendliche“ Menge d. h. eine solche, welche einem ihrer Teile äquivalent ist. Umgekehrt enthält auch jede „unendliche“ Menge M einen Bestandteil M_0, welcher „abzählbar unendlich“, d. h. der Zahlenreihe äquivalent ist.

Beweis. Es sei Z eine beliebige Menge, welche gemäß VII das Element 0 und mit jedem ihrer Elemente a auch das entsprechende Element $\{a\}$ enthält, und diese Menge Z sei durch eine Abbildung $\Omega \Subset ZZ'$ gemäß Nr. 19 abgebildet auf eine ihr äquivalente und elementenfremde Menge Z'. Ist nun $\{z, x'\}$ ein beliebiges Element von ZZ', und $\{x, x'\}$ Element von Ω für dasselbe x', so ist immer definit, ob $z = \{x\}$ ist oder nicht. Alle solchen Elemente $\{\{x\}, x'\}$ von ZZ' bilden also nach III die Elemente einer gewissen Untermenge $\mathsf{\Phi} \Subset ZZ'$, und $\mathsf{\Phi}$ ist eine „Abbildung“ von Z' auf $Z_1 \Subset Z$, wo Z_1 alle Elemente der Form $z = \{x\}$ umfaßt. In der Tat entspricht jedem $x' \,\varepsilon\, Z'$ ein bestimmtes $\{x\} \,\varepsilon\, Z_1$ und umgekehrt, d. h. jedes Element von $Z_1 + Z'$ erscheint in einem und nur einem Elemente von $\mathsf{\Phi}$. Es ist also nach Nr. 21 $Z \sim Z' \sim Z_1$, wo Z_1, weil es das Element 0 nicht enthält, nur ein *Teil* von Z ist; und jede wie Z beschaffene Menge, also auch Z_0 ist „unendlich“.

Um nun auch die zweite Hälfte des Theorems zu beweisen, betrachten wir eine beliebige „unendliche“ Menge M, die wir aber mit Rücksicht auf Nr. 19 unbeschadet der Allgemeinheit als elementenfremd zu Z_0 annehmen können. Es sei also $M \sim M' = M - R$, r ein beliebiges Element von $R \neq 0$ und $\{\mathsf{\Phi}, \mathsf{\Psi}\}$ gemäß Nr. 21 eine Abbildung, bei welcher jedem Elemente $m \,\varepsilon\, M$ ein Element $m' \,\varepsilon\, M'$ entspricht und umgekehrt. Ferner sei A eine Untermenge des Produktes MZ_0, welche die folgenden Eigenschaften besitzt: 1) sie enthält das Element $\{r, 0\}$; und 2), ist $\{m, z\}$ irgend ein Element von A, so enthält A auch das weitere Element $\{m', z'\}$, wo m' das dem m entsprechende Element von M', und $z' = \{z\}$ wegen Nr. 14 gleichfalls Element von Z_0 ist. Ist nun $\mathsf{A}_0 = \mathfrak{D}\mathsf{T}$ der gemeinsame Bestandteil aller wie A beschaffenen Untermengen von MZ_0, welche wegen III, IV die Elemente einer gewissen Menge $\mathsf{T} \Subset \mathfrak{U}(MZ_0)$ bilden, so besitzt auch A_0, wie man ohne weiteres erkennt, gleichfalls die Eigenschaften 1) und 2), ist also ebenfalls Element von T. Ferner ist, mit alleiniger Ausnahme von $\{r, 0\}$, jedes Element von A_0 auch von der Form $\{m', z'\}$; denn im entgegengesetzten Falle könnten wir es fortlassen, und der Rest von A_0 besäße immer noch die Eigenschaften 1) und 2), ohne doch, wie alle Elemente von T, den Bestandteil A_0 zu
281 enthalten. | Hieraus folgt zunächst, daß das Element $\{r, 0\}$ allen übrigen Elementen von A_0 elementenfremd ist, da weder $r = m' \,\varepsilon\, M'$ noch $0 = \{z\} = z'$ sein und somit kein weiteres Element $\{m', z'\}$ eines der Elemente r oder 0 enthalten kann. Ist ferner ein Element $\{m, z\}$ von A_0 allen übrigen elementenfremd, so gilt das gleiche auch von dem entsprechenden Elemente $\{m', z'\}$, da zu jedem Elemente der Form $\{m', z_1'\}$ oder $\{m_1', z'\}$ ein weiteres Element $\{m, z_1\}$ oder $\{m_1, z\}$ gehören müßte. Alle Elemente von A_0, welche allen übrigen elementenfremd sind, bilden also die Elemente einer Untermenge A_0' von A_0, welche die Eigenschaften 1) und 2) besitzt und daher als Element von T umgekehrt A_0 als Untermenge enthält, d. h. mit A_0 identisch ist. Jedes

36_{VII}. *Theorem.* The "number sequence" Z_0 (No. 14) is an "infinite" set, that is, one that is equivalent to one of its parts. Conversely, also, every "infinite" set M contains a "denumerably infinite" component M_0, that is, one that is equivalent to the number sequence.

Proof. Let Z be an arbitrary set that, in accordance with Axiom VII, contains the element 0 and, for each of its elements a, also the corresponding element $\{a\}$ and let this set Z be mapped by means of the subset Ω of ZZ' onto a set Z' equivalent to and disjoint from it, which, according to No. 19, is possible. Now, whenever $\{z, x'\}$ is an arbitrary element of ZZ' and $\{x, x'\}$ an element of Ω for the same x', it is definite whether $z = \{x\}$ or not. All elements of ZZ' that in fact have the form $\{\{x\}, x'\}$ thus are, according to Axiom III, the elements of a certain subset $\mathsf{\Phi}$ of ZZ', and $\mathsf{\Phi}$ is a "mapping" of Z' onto the subset Z_1 of Z that contains all elements z of the form $\{x\}$. In fact, to every element x' of Z' there corresponds a certain element $\{x\}$ of Z_1 and conversely; that is, each element of $Z_1 + Z'$ occurs in one and only one element of $\mathsf{\Phi}$. Thus, according to No. 21, $Z \sim Z' \sim Z_1$, where Z_1, since it does not contain the element 0, is only a *part* of Z; and every set constituted like Z, hence also Z_0, is "infinite".

To prove now the second half of the theorem as well, we consider an arbitrary "infinite" set M, which, however, we may in view of No. 19 assume without loss of generality to be disjoint from Z_0. Thus let $M \sim M' = M - R$, let r be an arbitrary element of $R \neq 0$, and let $\{\mathsf{\Phi}, \mathsf{\Psi}\}$ be a mapping, whose existence is possible according to No. 21, under which there corresponds to each element m of M an element m' of M' and conversely. Furthermore let A be a subset of MZ_0 that possesses the following properties: 1) it contains the element $\{r, 0\}$; and 2) if $\{m, z\}$ is any element of A, then A also contains the further element $\{m', z'\}$, where m' is the element of M' corresponding to m, and $z' = \{z\}$ is likewise an element of Z_0 on account of No. 14. If now $\mathsf{A}_0 = \mathfrak{D}\mathsf{T}$ is the common component of all subsets of MZ_0 constituted like A, which, on account of Axioms III and IV, are the elements of a certain subset T of $\mathfrak{U}(MZ_0)$, then A_0 also possesses properties 1) and 2), as we see immediately, and is thus likewise an element of T. Furthermore, with the sole exception of $\{r, 0\}$, every element of A_0, too, has the form $\{m', z'\}$; for in the contrary case we could remove it, and the remainder of A_0 would still possess properties 1) and 2), without, however, including the component A_0, which all elements of T do include. From this it follows first that the element $\{r, 0\}$ is disjoint from all other elements of A_0, since neither $r = m' \,\varepsilon\, M'$ nor $0 = \{z\} = z'$ is possible, and therefore no further element $\{m', z'\}$ can contain one of the elements r or 0. Furthermore, if an element $\{m, z\}$ of A_0 is disjoint from all the remaining ones, the same also holds for the corresponding element $\{m', z'\}$, since to each element of the form $\{m', z_1'\}$ or $\{m_1', z'\}$ there would have to correspond a further element, $\{m, z_1\}$ or $\{m_1, z\}$. All those elements of A_0 that are disjoint from all the others, therefore, are the elements of a subset A_0' of A_0 possessing properties 1) and 2); hence, being an element of T, A_0' now includes A_0 as a subset, that is, is identical with A_0. Every

Element von

$$\mathfrak{S}\mathsf{A}_0 = M_0 + Z_{00} \mathrel{\underline{\in}} M + Z_0\,,$$

wo wir mit M_0 und Z_{00} die gemeinsamen Bestandteile von $\mathfrak{S}\mathsf{A}_0$ mit M, bezw. Z_0 bezeichnen, kann also nur in einem einzigen Elemente von A_0 als Element figurieren, und es ist (wegen Nr. 15) $M_0 \sim Z_{00}$. Nun ist aber Z_{00} eine Untermenge von Z_0, welche das Element 0 und mit jedem ihrer Elemente z auch das zugehörige $z' = \{z\}$ enthält; Z_{00} muß also wegen Nr. 14 die ganze Zahlenreihe Z_0 als Bestandteil enthalten, d. h. es ist $Z_{00} = Z_0$ und, wie behauptet, $Z_0 \sim M_0 \mathrel{\underline{\in}} M$.

Chesières, den 30. Juli 1907.

element of

$$\mathfrak{S}\mathsf{A}_0 = M_0 + Z_{00} \subseteq M + Z_0\,,$$

where M_0 is the common component of $\mathfrak{S}\mathsf{A}_0$ and M, and Z_{00} that of $\mathfrak{S}\mathsf{A}_0$ and Z_0, can therefore figure as an element in only a single element of A_0, and, on account of No. 15, $M_0 \sim Z_{00}$. But now Z_{00} is a subset of Z_0 containing the element 0 and, for every one of its elements z, also the associated $z' = \{z\}$; Z_{00} must therefore on account of No. 14 contain the entire number sequence Z_0 as a component; that is, we have $Z_{00} = Z_0$ and, as asserted, $Z_0 \sim M_0 \subseteq M$.

Chesières, on the 30th of July 1907.