SECTION B

THEORY OF APPARENT VARIABLES

*9. EXTENSION OF THE THEORY OF DEDUCTION FROM LOWER TO HIGHER TYPES OF PROPOSITIONS

Summary of *9.

In the present number, we introduce two new primitive ideas, which may be expressed as "ϕx is always* true" and "ϕx is sometimes* true," or, more correctly, as "ϕx always" and "ϕx sometimes." When we assert "ϕx always," we are asserting all values of $\phi \hat{x}$, where "$\phi \hat{x}$" means the function itself, as opposed to an ambiguous value of the function (cf. pp. 15, 40); we are not asserting that ϕx is true for all values of x, because, in accordance with the theory of types, there are values of x for which "ϕx" is meaningless; for example, the function $\phi \hat{x}$ itself must be such a value. We shall denote "ϕx always" by the notation

$$(x) . \phi x,$$

where the "(x)" will be followed by a sufficiently large number of dots to cover the function of which "all values" are concerned. The form in which such propositions most frequently occur is the "formal implication," *i.e.* such a proposition as

$$(x) : \phi x . \supset . \psi x,$$

i.e. "ϕx always implies ψx." This is the form in which we express the universal affirmative "all objects having the property ϕ have the property ψ."

We shall denote "ϕx sometimes" by the notation

$$(\exists x) . \phi x.$$

Here "$\exists$" stands for "there exists," and the whole symbol may be read "there exists an x such that ϕx."

In a proposition of either of the two forms $(x) . \phi x$, $(\exists x) . \phi x$, the x is called an *apparent variable*. A proposition which contains no apparent variables is called "elementary," and a function, all whose values are elementary propositions, is called an elementary function. For reasons explained in Chapter II of the Introduction, it would seem that negation and disjunction and their derivatives must have a different meaning when applied to elementary propositions from that which they have when applied to such propositions as $(x) . \phi x$ or $(\exists x) . \phi x$. If $\phi \hat{x}$ is an elementary function, we will in this number call $(x) . \phi x$ and $(\exists x) . \phi x$ "first-order propositions." Then in virtue of the fact

* We use "always" as meaning "in all cases," not "at all times." A similar remark applies to "sometimes."

that disjunction and negation do not have the same meanings as applied to elementary or to first-order propositions, it follows that, in asserting the primitive propositions of $*1$, we must either confine them, in their application, to propositions of a single type, or we must regard them as the simultaneous assertion of a number of different primitive propositions, corresponding to the different meanings of "disjunction" and "negation." Likewise in regard to the primitive ideas of disjunction and negation, we must either, in the primitive propositions of $*1$, confine them to disjunctions and negations of elementary propositions, or we must regard them as really each multiple, so that in regard to each type of propositions we shall need a new primitive idea of negation and a new primitive idea of disjunction. In the present number, we shall show how, when the primitive ideas of negation and disjunction are restricted to elementary propositions, and the p, q, r of $*1$—$*5$ are therefore necessarily elementary propositions, it is possible to obtain definitions of the negation and disjunction of first-order propositions, and proofs of the analogues, for first-order propositions, of the primitive propositions $*1\cdot2$—$\cdot6$. ($*1\cdot1$ and $*1\cdot11$ have to be assumed afresh for first-order propositions, and the analogues of $*1\cdot7\cdot71\cdot72$ require a fresh treatment.) It follows that the analogues of the propositions of $*2$—$*5$ follow by merely repeating previous proofs. It follows also that the theory of deduction can be extended from first-order propositions to such as contain two apparent variables, by merely repeating the process which extends the theory of deduction from elementary to first-order propositions. Thus by merely repeating the process set forth in the present number, propositions of any order can be reached. Hence negation and disjunction may be treated in practice as if there were no difference in these ideas as applied to different types; that is to say, when "$\sim p$" or "$p \vee q$" occurs, it is unnecessary in practice to know what is the type of p or q, since the properties of negation and disjunction assumed in $*1$ (which are alone used in proving other properties) can be asserted, without formal change, of propositions of any order or, in the case of $p \vee q$, of any two orders. The limitation, in practice, to the treatment of negation or disjunction as single ideas, the same in all types, would only arise if we ever wished to assume that there is some one function of p whose value is always $\sim p$, whatever may be the order of p, or that there is some one function of p and q whose value is always $p \vee q$, whatever may be the orders of p and q. Such an assumption is not involved so long as p (and q) remain *real* variables, since, in that case, there is no need to give the same meaning to negation and disjunction for different values of p (and q), when these different values are of different types. But if p (or q) is going to be turned into an apparent variable, then since our two primitive ideas $(x) \,.\, \phi x$ and $(\exists x) \,.\, \phi x$ both demand some definite function ϕ, and restrict the apparent variable to possible arguments for ϕ, it follows that negation and disjunction must, wherever they occur in the expression in which p (or q) is an apparent variable, be restricted to the kind of negation or disjunction appropriate to a given type or pair of types. Thus, to take an instance, if we assert the law of excluded middle in the form

$$\text{“} \vdash . \, p \vee \sim p \text{”}$$

there is no need to place any restriction upon p: we may give to p a value of any order, and then give to the negation and disjunction involved those meanings which are appropriate to that order. But if we assert

$$\text{“} \vdash . \, (p) . \, p \vee \sim p \text{”}$$

it is necessary, if our symbol is to be significant, that "$p \vee \sim p$" should be the value, for the argument p, of a function ϕp; and this is only possible if the negation and disjunction involved have meanings fixed in advance, and if, therefore, p is limited to one type. Thus the assertion of the law of excluded middle in the form involving a real variable is more general than in the form involving an apparent variable. Similar remarks apply generally where the variable is the argument to a typically ambiguous function.

In what follows the single letters p and q will represent *elementary* propositions, and so will "ϕx," "ψx," etc. We shall show how, assuming the primitive ideas and propositions of $*1$ as applied to elementary propositions, we can define and prove analogous ideas and propositions as applied to propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$. By mere repetition of the analogous process, it will then follow that analogous ideas and propositions can be defined and proved for propositions of any order; whence, further, it follows that, in all that concerns disjunction and negation, so long as propositions do not appear as apparent variables, we may wholly ignore the distinction between different types of propositions and between different meanings of negation and disjunction. Since we never have occasion, in practice, to consider propositions as apparent variables, it follows that the hierarchy of propositions (as opposed to the hierarchy of functions) will never be relevant in practice after the present number.

The purpose and interest of the present number are purely philosophical, namely to show how, by means of certain primitive propositions, we can deduce the theory of deduction for propositions containing apparent variables from the theory of deduction for elementary propositions. From the purely technical point of view, the distinction between elementary and other propositions may be ignored, so long as propositions do not appear as apparent variables; we may then regard the primitive propositions of $*1$ as applying to propositions of any type, and proceed as in $*10$, where the purely technical development is resumed.

It should be observed that although, in the present number, we prove that the analogues of the primitive propositions of $*1$, if they hold for propositions containing n apparent variables, also hold for such as contain $n+1$, yet we must not suppose that mathematical induction may be used to infer that the analogues of the primitive propositions of $*1$ hold for propositions containing any number of apparent variables. Mathematical induction is a method of proof which is not yet applicable, and is (as will appear) incapable of being used freely until the theory of propositions containing apparent variables has been established. What we are enabled to do, by means of the propositions in the present number, is to prove our desired result for any assigned number of apparent variables—say ten—by ten applications of the same proof. Thus we can prove, concerning any assigned proposition, that it obeys the analogues of the primitive propositions of $*1$, but we can only do this by proceeding step by step, not by any such compendious method as mathematical induction would afford. The fact that higher types can only be reached step by step is essential, since to proceed otherwise we should need an apparent variable which would wander from type to type, which would contradict the principle upon which types are built up.

Definition of Negation. We have first to define the negations of $(x) . \phi x$ and $(\exists x) . \phi x$. We define the negation of $(x) . \phi x$ as $(\exists x) . \sim \phi x$, *i.e.* "it is not the case that ϕx is always true" is to mean "it is the case that not-ϕx is sometimes true." Similarly the negation of $(\exists x) . \phi x$ is to be defined as $(x) . \sim \phi x$. Thus we put

$*9{\cdot}01.\quad \sim \{(x) . \phi x\} . = . (\exists x) . \sim \phi x \quad \text{Df}$

$*9{\cdot}02.\quad \sim \{(\exists x) . \phi x\} . = . (x) . \sim \phi x \quad \text{Df}$

To avoid brackets, we shall write $\sim (x) . \phi x$ in place of $\sim \{(x) . \phi x\}$, and $\sim (\exists x) . \phi x$ in place of $\sim \{(\exists x) . \phi x\}$. Thus:

$*9{\cdot}011.\quad \sim (x) . \phi x . = . \sim \{(x) . \phi x\} \quad \text{Df}$

$*9{\cdot}021.\quad \sim (\exists x) . \phi x . = . \sim \{(\exists x) . \phi x\} \quad \text{Df}$

Definition of Disjunction. To define disjunction when one or both of the propositions concerned is of the first order, we have to distinguish six cases, as follows:

$*9{\cdot}03.\quad (x) . \phi x . \lor . p : = . (x) . \phi x \lor p \quad \text{Df}$

$*9{\cdot}04.\quad p . \lor . (x) . \phi x : = . (x) . p \lor \phi x \quad \text{Df}$

$*9{\cdot}05.\quad (\exists x) . \phi x . \lor . p : = . (\exists x) . \phi x \lor p \quad \text{Df}$

$*9{\cdot}06.\quad p . \lor . (\exists x) . \phi x : = . (\exists x) . p \lor \phi x \quad \text{Df}$

$*9{\cdot}07.\quad (x) . \phi x . \lor . (\exists y) . \psi y : = : (x) : (\exists y) . \phi x \lor \psi y \quad \text{Df}$

$*9{\cdot}08.\quad (\exists y) . \psi y . \lor . (x) . \phi x : = : (x) : (\exists y) . \psi y \lor \phi x \quad \text{Df}$

(The definitions $*9{\cdot}07{\cdot}08$ are to apply also when ϕ and ψ are not both elementary functions.)

In virtue of these definitions, the true scope of an apparent variable is always the whole of the asserted proposition in which it occurs, even when, typographically, its scope appears to be only part of the asserted proposition. Thus when $(\exists x) . \phi x$ or $(x) . \phi x$ *appears* as *part* of an asserted proposition, it does not really occur, since the scope of the apparent variable really extends to the whole asserted proposition. It will be shown, however, that, so far as the theory of deduction is concerned, $(\exists x) . \phi x$ and $(x) . \phi x$ behave like propositions not containing apparent variables.

The definitions of implication, the logical product, and equivalence are to be transferred unchanged to $(x) . \phi x$ and $(\exists x) . \phi x$.

The above definitions can be repeated for successive types, and thus reach propositions of any type.

Primitive Propositions. The primitive propositions required are six in number, and may be divided into three sets of two. We have first two propositions, which effect the passage from elementary to first-order propositions, namely

$*9 \cdot 1.\quad \vdash : \phi x . \supset . (\exists z) . \phi z \quad$ Pp

$*9 \cdot 11.\quad \vdash : \phi x \vee \phi y . \supset . (\exists z) . \phi z \quad$ Pp

Of these, the first states that, if ϕx is true, then there is a value of $\phi \hat{z}$ which is true; *i.e.* if we can find an instance of a function which is true, then the function is "sometimes true." (When we speak of a function as "sometimes" true, we do not mean to assert that there is *more* than one argument for which it is true, but only that there is *at least* one.) Practically, the above primitive proposition gives the only method of proving "existence-theorems": in order to prove such theorems, it is necessary (and sufficient) to find some instance in which an object possesses the property in question. If we were to assume what may be called "existence-axioms," *i.e.* axioms stating $(\exists z) . \phi z$ for some particular ϕ, these axioms would give other methods of proving existence. Instances of such axioms are the multiplicative axiom ($*88$) and the axiom of infinity (defined in $*120 \cdot 03$). But we have not assumed any such axioms in the present work.

The second of the above primitive propositions is only used once, in proving $(\exists z) . \phi z . \vee . (\exists z) . \phi z : \supset . (\exists z) . \phi z$, which is the analogue of $*1 \cdot 2$ (namely $p \vee p . \supset . p$) when p is replaced by $(\exists z) . \phi z$. The effect of this primitive proposition is to emphasize the ambiguity of the z required in order to secure $(\exists z) . \phi z$. We have, of course, in virtue of $*9 \cdot 1$,

$$\phi x . \supset . (\exists z) . \phi z \text{ and } \phi y . \supset . (\exists z) . \phi z .$$

But if we try to infer from these that $\phi x \vee \phi y . \supset . (\exists z) . \phi z$, we must use the proposition $q \supset p . r \supset p . \supset . q \vee r \supset p$, where p is $(\exists z) . \phi z$. Now it will be found, on referring to $*4 \cdot 77$ and the propositions used in its proof, that this proposition depends upon $*1 \cdot 2$, *i.e.* $p \vee p . \supset . p$. Hence it cannot be used by us to prove $(\exists x) . \phi x . \vee . (\exists x) . \phi x : \supset . (\exists x) . \phi x$, and thus we are compelled to assume the primitive proposition $*9 \cdot 11$.

We have next two propositions concerned with *inference* to or from propositions containing apparent variables, as opposed to implication. First, we have, for the new meaning of implication resulting from the above definitions of negation and disjunction, the analogue of $*1\cdot1$, namely

$*9\cdot12$. What is implied by a true premiss is true. Pp.

That is to say, given "$\vdash . p$" and "$\vdash . p \supset q$," we may proceed to "$\vdash . q$," even when the propositions p and q are not elementary. Also, as in $*1\cdot11$, we may proceed from "$\vdash . \phi x$" and "$\vdash . \phi x \supset \psi x$" to "$\vdash . \psi x$," where x is a real variable, and ϕ and ψ are not necessarily elementary functions. It is in this latter form that the axiom is usually needed. It is to be assumed for functions of several variables as well as for functions of one variable.

We have next the primitive proposition which permits the passage from a real to an apparent variable, namely "when ϕy may be asserted, where y may be any possible argument, then $(x) . \phi x$ may be asserted." In other words, when ϕy is true however y may be chosen among possible arguments, then $(x) . \phi x$ is true, *i.e.* all values of ϕ are true. That is to say, if we can assert a wholly ambiguous value ϕy, that must be because all values are true. We may express this primitive proposition by the words: "What is true in *any* case, however the case may be selected, is true in *all* cases." We cannot symbolise this proposition, because if we put

$$\text{“}\vdash : \phi y . \supset . (x) . \phi x\text{”}$$

that means: "However y may be chosen, ϕy implies $(x) . \phi x$," which is in general false. What we mean is: "If ϕy is true however y may be chosen, then $(x) . \phi x$ is true." But we have not supplied a symbol for the mere *hypothesis* of what is *asserted* in "$\vdash . \phi y$," where y is a real variable, and it is not worth while to supply such a symbol, because it would be very rarely required. If, for the moment, we use the symbol $[\phi y]$ to express this hypothesis, then our primitive proposition is

$$\vdash : [\phi y] . \supset . (x) . \phi x \quad \text{Pp.}$$

In practice, this primitive proposition is only used for *inference*, not for implication; that is to say, when we actually have an assertion containing a real variable, it enables us to turn this real variable into an apparent variable by placing it in brackets immediately after the assertion-sign, followed by enough dots to reach to the end of the assertion. This process will be called "turning a real variable into an apparent variable." Thus we may assert our primitive proposition, for technical use, in the form:

$*9\cdot13$. In any assertion containing a real variable, this real variable may be turned into an apparent variable of which all possible values are asserted to satisfy the function in question. Pp.

We have next two primitive propositions concerned with types. These require some preliminary explanations.

Primitive Idea: Individual. We say that x is "individual" if x is neither a proposition nor a function (cf. p. 51).

$*9{\cdot}131$. *Definition of "being of the same type."* The following is a step-by-step definition, the definition for higher types presupposing that for lower types. We say that u and v "are of the same type" if (1) both are individuals, (2) both are elementary functions taking arguments of the same type, (3) u is a function and v is its negation, (4) u is $\phi\hat{x}$ or $\psi\hat{x}$, and v is $\phi\hat{x} \vee \psi\hat{x}$, where $\phi\hat{x}$ and $\psi\hat{x}$ are elementary functions, (5) u is $(y) . \phi(\hat{x}, y)$ and v is $(z) . \psi(\hat{x}, z)$, where $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ are of the same type, (6) both are elementary propositions, (7) u is a proposition and v is $\sim u$, or (8) u is $(x) . \phi x$ and v is $(y) . \psi y$, where $\phi\hat{x}$ and $\psi\hat{x}$ are of the same type.

Our primitive propositions are:

$*9{\cdot}14$. If "ϕx" is significant, then if x is of the same type as a, "ϕa" is significant, and vice versa. Pp. (Cf. note on $*10{\cdot}121$, p. 140.)

$*9{\cdot}15$. If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa. Pp.

It will be seen that, in virtue of the definitions,

$$(x) . \phi x . \supset . p \text{ means } \sim(x) . \phi x . \vee . p, \ \textit{i.e.} \ (\exists x) . \sim\phi x . \vee . p,$$
$$\textit{i.e.} \ (\exists x) . \sim\phi x \vee p, \ \textit{i.e.} \ (\exists x) . \phi x \supset p$$
$$(\exists x) . \phi x . \supset . p \text{ means } \sim(\exists x) . \phi x . \vee . p, \ \textit{i.e.} \ (x) . \sim\phi x . \vee . p,$$
$$\textit{i.e.} \ (x) . \sim\phi x \vee p, \ \textit{i.e.} \ (x) . \phi x \supset p$$

In order to prove that $(x) . \phi x$ and $(\exists x) . \phi x$ obey the same rules of deduction as ϕx, we have to prove that propositions of the forms $(x) . \phi x$ and $(\exists x) . \phi x$ may replace one or more of the propositions p, q, r in $*1{\cdot}2$—$\cdot 6$. When this has been proved, the previous proofs of subsequent propositions in $*2$—$*5$ become applicable. These proofs are given below. Certain other propositions, required in the proofs, are also proved.

$*9{\cdot}2$. $\vdash : (x) . \phi x . \supset . \phi y$

The above proposition states the principle of deduction from the general to the particular, *i.e.* "what holds in all cases, holds in any one case."

Dem.

$$\vdash . *2{\cdot}1 . \supset \vdash . \sim\phi y \vee \phi y \quad (1)$$
$$\vdash . *9{\cdot}1 . \supset \vdash : \sim\phi y \vee \phi y . \supset . (\exists x) . \sim\phi x \vee \phi y \quad (2)$$
$$\vdash . (1) . (2) . *1{\cdot}11 . \supset \vdash . (\exists x) . \sim\phi x \vee \phi y \quad (3)$$
$$[(3).(*9{\cdot}05)] \quad \vdash : (\exists x) . \sim\phi x . \vee . \phi y \quad (4)$$
$$[(4).(*9{\cdot}01.*1{\cdot}01)] \quad \vdash : (x) . \phi x . \supset . \phi y$$

In the second line of the above proof, "$\sim\phi y \vee \phi y$" is taken as the value, for the argument y, of the function "$\sim\phi x \vee \phi y$," where x is the argument. A similar method of using $*9{\cdot}1$ is employed in most of the following proofs.

$*1{\cdot}11$ is used, as in the third line of the above proof, in almost all steps except such as are mere applications of definitions. Hence it will not be further referred to, unless in cases where its employment is obscure or specially important.

***9·21.** $\vdash :. (x) . \phi x \supset \psi x . \supset : (x) . \phi x . \supset . (x) . \psi x$

I.e. if ϕx always implies ψx, then "ϕx always" implies "ψx always." The use of this proposition is constant throughout the remainder of this work.

Dem.

$\vdash . *2{\cdot}08 . \supset \vdash : \phi z \supset \psi z . \supset . \phi z \supset \psi z$ (1)

$\vdash . (1) . *9{\cdot}1 . \supset \vdash : (\exists y) : \phi z \supset \psi z . \supset . \phi y \supset \psi z$ (2)

$\vdash . (2) . *9{\cdot}1 . \supset \vdash :. (\exists x) :. (\exists y) : \phi x \supset \psi x . \supset . \phi y \supset \psi z$ (3)

$\vdash . (3) . *9{\cdot}13 . \supset \vdash :: (z) :: (\exists x) :. (\exists y) : \phi x \supset \psi x . \supset . \phi y \supset \psi z$ (4)

$[(4).(*9{\cdot}06)] \quad \vdash :: (z) :: (\exists x) :. \phi x \supset \psi x . \supset : (\exists y) . \phi y \supset \psi z$ (5)

$[(5).(*1{\cdot}01.*9{\cdot}08)] \quad \vdash :. (\exists x) . \sim(\phi x \supset \psi x) : \lor : (z) : (\exists y) . \sim \phi y \lor \psi z$ (6)

$[(6).(*9{\cdot}08)] \quad \vdash :. (\exists x) . \sim(\phi x \supset \psi x) : \lor : (\exists y) . \sim \phi y . \lor . (z) . \psi z$ (7)

$[(7).(*1{\cdot}01)] \quad \vdash :. (x) . \phi x \supset \psi x . \supset : (y) . \phi y . \supset . (z) . \psi z$

This is the proposition to be proved, since "$(y) . \phi y$" is the same proposition as "$(x) . \phi x$," and "$(z) . \psi z$" is the same proposition as "$(x) . \psi x$."

***9·22.** $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

I.e. if ϕx always implies ψx, then if ϕx is sometimes true, so is ψx. This proposition, like *9·21, is constantly used in the sequel.

Dem.

$\vdash . *2{\cdot}08 . \supset \vdash : \phi y \supset \psi y . \supset . \phi y \supset \psi y$ (1)

$\vdash . (1) . *9{\cdot}1 . \supset \vdash : (\exists z) : \phi y \supset \psi y . \supset . \phi y \supset \psi z$ (2)

$\vdash . (2) . *9{\cdot}1 . \supset \vdash :. (\exists x) :. (\exists z) : \phi x \supset \psi x . \supset . \phi y \supset \psi z$ (3)

$\vdash . (3) . *9{\cdot}13 . \supset \vdash :: (y) :: (\exists x) :. (\exists z) : \phi x \supset \psi x . \supset . \phi y \supset \psi z$ (4)

$[(4).(*9{\cdot}06)] \quad \vdash :: (y) :: (\exists x) :. \phi x \supset \psi x . \supset : (\exists z) . \phi y \supset \psi z$ (5)

$[(5).(*1{\cdot}01.*9{\cdot}08)] \quad \vdash :: (\exists x) . \sim(\phi x \supset \psi x) : \lor : (y) : (\exists z) . \phi y \supset \psi z$ (6)

$[(6).(*1{\cdot}01.*9{\cdot}07)] \quad \vdash :: (\exists x) . \sim(\phi x \supset \psi x) : \lor : (y) . \sim \phi y . \lor . (\exists z) . \psi z$ (7)

$[(7).(*1{\cdot}01.*9{\cdot}01{\cdot}02)] \vdash :. (x) . \phi x \supset \psi x . \supset : (\exists y) . \phi y . \supset . (\exists z) . \psi z$

This is the proposition to be proved, because $(\exists y) . \phi y$ is the same proposition as $(\exists x) . \phi x$, and $(\exists z) . \psi z$ is the same proposition as $(\exists x) . \psi x$.

***9·23.** $\vdash : (x) . \phi x . \supset . (x) . \phi x$ [Id . *9·13·21]

***9·24.** $\vdash : (\exists x) . \phi x . \supset . (\exists x) . \phi x$ [Id . *9·13·22]

***9·25.** $\vdash :. (x) . p \lor \phi x . \supset : p . \lor . (x) . \phi x$ [*9·23 . (*9·04)]

We are now in a position to prove the analogues of *1·2—6, replacing one of the letters p, q, r in those propositions by $(x) . \phi x$ or $(\exists x) . \phi x$. The proofs are given below.

$*9\cdot3.\ \vdash :. (x) . \phi x . \vee . (x) . \phi x : \supset . (x) . \phi x$

Dem.

$\vdash . *1\cdot2 . \quad \supset \vdash . \phi x \vee \phi x . \supset . \phi x$ (1)

$\vdash . (1) . *9\cdot1 . \quad \supset \vdash : (\exists y) : \phi x \vee \phi y . \supset . \phi x$ (2)

$\vdash . (2) . *9\cdot13 . \quad \supset \vdash :. (x) :. (\exists y) : \phi x \vee \phi y . \supset . \phi x$ (3)

$[(3) . (*9\cdot05\cdot01\cdot04)] \quad \vdash :. (x) :. \phi x . \vee . (y) . \phi y : \supset . \phi x$ (4)

$\vdash . (4) . *9\cdot21 . \quad \supset \vdash :. (x) : \phi x . \vee . (y) . \phi y : \supset . (x) . \phi x$ (5)

$[(5) . (*9\cdot03)] \quad \vdash :. (x) . \phi x . \vee . (y) . \phi y : \supset . (x) . \phi x :. \supset \vdash . \mathrm{Prop}$

$*9\cdot31.\ \vdash :. (\exists x) . \phi x . \vee . (\exists x) . \phi x : \supset . (\exists x) . \phi x$

This is the only proposition which employs $*9\cdot11$.

Dem.

$\vdash . *9\cdot11\cdot13 . \quad \supset \vdash : (y) : \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (1)

$[(1) . (*9\cdot03\cdot02)] \quad \vdash : (\exists y) . \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (2)

$\vdash . (2) . *9\cdot13 . \supset \vdash : (x) : (\exists y) . \phi x \vee \phi y . \supset . (\exists z) . \phi z$ (3)

$[(3) . (*9\cdot03\cdot02)] \quad \vdash :. (\exists x) : (\exists y) . \phi x \vee \phi y : \supset . (\exists z) . \phi z$ (4)

$[(4) . (*9\cdot05\cdot06)] \quad \vdash :. (\exists x) . \phi x . \vee . (\exists y) . \phi y : \supset . (\exists z) . \phi z$

$*9\cdot32.\ \vdash :. q . \supset : (x) . \phi x . \vee . q$

Dem.

$\vdash . *1\cdot3 . \quad \supset \vdash :. q . \supset : \phi x . \vee . q$ (1)

$\vdash . (1) . *9\cdot13 . \supset \vdash :. (x) :. q . \supset : \phi x . \vee . q$

$[*9\cdot25] \quad \supset \vdash :. q . \supset : (x) : \phi x . \vee . q$ (2)

$[(2) . (*9\cdot03)] \quad \vdash :. q . \supset : (x) . \phi x . \dot{\vee} . q$

$*9\cdot33.\ \vdash :. q . \supset : (\exists x) . \phi x . \vee . q$ [Proof as above]

$*9\cdot34.\ \vdash :. (x) . \phi x . \supset : p . \vee . (x) . \phi x$

Dem.

$\vdash . *1\cdot3 . \quad \supset \vdash : \phi x . \supset . p \vee \phi x$ (1)

$\vdash . (1) . *9\cdot13 . \quad \supset \vdash : (x) : \phi x . \supset . p \vee \phi x$ (2)

$\vdash . (2) . *9\cdot21 . \quad \supset \vdash : (x) . \phi x . \supset . (x) . p \vee \phi x$ (3)

$\vdash . (3) . (*9\cdot04) . \supset \vdash . \mathrm{Prop}$

$*9\cdot35.\ \vdash :. (\exists x) . \phi x . \supset : p . \vee . (\exists x) . \phi x$ [Proof as above]

$*9\cdot36.\ \vdash :. p . \vee . (x) . \phi x : \supset : (x) . \phi x . \vee . p$

Dem.

$\vdash . *1\cdot4 . \quad \supset \vdash : p \vee \phi x . \supset . \phi x \vee p$ (1)

$\vdash . (1) . *9\cdot13\cdot21 . \quad \supset \vdash : (x) . p \vee \phi x . \supset . (x) . \phi x \vee p$ (2)

$\vdash . (2) . (*9\cdot03\cdot04) . \supset \vdash . \mathrm{Prop}$

$*9\cdot361.\ \vdash :. (x) . \phi x . \vee . p : \supset : p . \vee . (x) . \phi x$ [Similar proof]

$*9\cdot37.\ \vdash :. p . \vee . (\exists x) . \phi x : \supset : (\exists x) . \phi x . \vee . p$ [Similar proof]

$*9\cdot371.\ \vdash :. (\exists x) . \phi x . \vee . p : \supset : p . \vee . (\exists x) . \phi x$ [Similar proof]

*9·4. $\vdash :: p : \lor : q . \lor . (x) . \phi x :. \supset :. q : \lor : p . \lor . (x) . \phi x$

Dem.

$\vdash . *1{\cdot}5 . *9{\cdot}21 . \supset \vdash :. (x) : p . \lor . q \lor \phi x : \supset : (x) : q . \lor . p \lor \phi x$ (1)

$\vdash . (1) . (*9{\cdot}04) . \supset \vdash . \text{Prop}$

*9·401. $\vdash :: p : \lor : q . \lor . (\exists x) . \phi x :. \supset :. q : \lor : p . \lor . (\exists x) . \phi x$ [As above]

*9·41. $\vdash :: p : \lor : (x) . \phi x . \lor . r :. \supset :. (x) . \phi x : \lor : p \lor r$ [As above]

*9·411. $\vdash :: p : \lor : (\exists x) . \phi x . \lor . r :. \supset :. (\exists x) . \phi x : \lor : p \lor r$ [As above]

*9·42. $\vdash :: (x) . \phi x : \lor : q \lor r :. \supset :. q : \lor : (x) . \phi x . \lor . r$ [As above]

*9·421. $\vdash :: (\exists x) . \phi x : \lor : q \lor r :. \supset :. q : \lor : (\exists x) . \phi x . \lor . r$ [As above]

*9·5. $\vdash :: p \supset q . \supset :. p . \lor . (x) . \phi x : \supset : q . \lor . (x) . \phi x$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. p \supset q . \supset : p \lor \phi y . \supset . q \lor \phi y$ (1)

$\vdash . (1) . *9{\cdot}1 . (*9{\cdot}06) . \quad \supset \vdash :. p \supset q . \supset : (\exists x) : p \lor \phi x . \supset . q \lor \phi y$ (2)

$\vdash . (2) . *9{\cdot}13 . (*9{\cdot}04) . \supset \vdash :: p \supset q . \supset :. (y) :. (\exists x) : p \lor \phi x . \supset . q \lor \phi y$ (3)

$[(3).(*9{\cdot}08)] \quad \vdash :: p \supset q . \supset :. (\exists x) . \sim(p \lor \phi x) . \lor . (y) . q \lor \phi y$ (4)

$[(4).(*9{\cdot}01)] \quad \vdash :: p \supset q . \supset :. (x) . p \lor \phi x . \supset . (y) . q \lor \phi y$ (5)

$[(5).(*9{\cdot}04)] \quad \vdash :: p \supset q . \supset :. p . \lor . (x) . \phi x : \supset : q . \lor . (y) . \phi y$

*9·501. $\vdash :: p \supset q . \supset :. p . \lor . (\exists x) . \phi x : \supset : q . \lor . (\exists x) . \phi x$ [As above]

*9·51. $\vdash :: p . \supset . (x) . \phi x : \supset :. p \lor r . \supset : (x) . \phi x . \lor . r$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. p \supset \phi x . \supset : p \lor r . \supset . \phi x \lor r$ (1)

$\vdash . (1) . *9{\cdot}13{\cdot}21 . \quad \supset \vdash :: (x) . p \supset \phi x . \supset :. (x) : p \lor r . \supset . \phi x \lor r$ (2)

$\vdash . (2) . (*9{\cdot}03{\cdot}04) . \supset \vdash . \text{Prop}$

*9·511. $\vdash :: p . \supset . (\exists x) . \phi x : \supset :. p \lor r . \supset : (\exists x) . \phi x . \lor . r$ [As above]

*9·52. $\vdash :: (x) . \phi x . \supset . q : \supset :. (x) . \phi x . \lor . r : \supset . q \lor r$

Dem.

$\vdash . *1{\cdot}6 . \quad \supset \vdash :. \phi x \supset q . \supset : \phi x \lor r . \supset . q \lor r$ (1)

$\vdash . (1) . *9{\cdot}13{\cdot}22 . \quad \supset \vdash :: (\exists x) . \phi x \supset q . \supset :. (\exists x) : \phi x \lor r . \supset . q \lor r$ (2)

$\vdash . (2) . (*9{\cdot}05{\cdot}01) . \supset \vdash :: (x) . \phi x . \supset . q : \supset :. (x) . \phi x \lor r . \supset . q \lor r$ (3)

$\vdash . (3) . (*9{\cdot}03) . \quad \supset \vdash . \text{Prop}$

*9·521. $\vdash :: (\exists x) . \phi x . \supset . q : \supset :. (\exists x) . \phi x . \lor . r : \supset . q \lor r$ [As above]

*9·6. $(x) . \phi x$, $\sim(x) . \phi x$, $(\exists x) . \phi x$ and $\sim(\exists x) . \phi x$ are of the same type. [*9·131, (7) and (8)]

*9·61. If $\phi \hat{x}$ and $\psi \hat{x}$ are elementary functions of the same type, there is a function $\phi \hat{x} \lor \psi \hat{x}$.

Dem.

By *9·14·15, there is an a for which "ψa," and therefore "ϕa," are significant, and therefore so is "$\phi a \lor \psi a$," by the primitive idea of disjunction. Hence the result by *9·15.

The same proof holds for functions of any number of variables.

∗9·62. If $\phi(\hat{x}, \hat{y})$ and $\psi\hat{z}$ are elementary functions, and the x-argument to ϕ is of the same type as the argument to ψ, there are functions

$$(y) . \phi(\hat{x}, y) . \mathbf{v} . \psi\hat{x}, \quad (\exists y) . \phi(\hat{x}, y) . \mathbf{v} . \psi\hat{x}.$$

Dem.

By ∗9·15, there are propositions $\phi(x, b)$ and ψa, where by hypothesis x and a are of the same type. Hence by ∗9·14 there is a proposition $\phi(a, b)$, and therefore, by the primitive idea of disjunction, there is a proposition $\phi(a, b) \mathbf{v} \psi a$, and therefore, by ∗9·15 and ∗9·03, there is a proposition $(y) . \phi(a, y) . \mathbf{v} . \psi a$. Similarly there is a proposition $(\exists y) . \phi(a, y) . \mathbf{v} . \psi a$. Hence the result, by ∗9·15.

∗9·63. If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ are elementary functions of the same type, there are functions $(y) . \phi(\hat{x}, y) . \mathbf{v} . (z) . \psi(\hat{x}, z)$, etc. [Proof as above]

We have now completed the proof that, in the primitive propositions of ∗1, any *one* of the propositions that occur may be replaced by $(x) . \phi x$ or $(\exists x) . \phi x$. It follows that, by merely repeating the proofs, we can show that any other of the propositions that occur in these propositions can be simultaneously replaced by $(x) . \psi x$ or $(\exists x) . \psi x$. Thus all the primitive propositions of ∗1, and therefore all the propositions of ∗2—∗5, hold equally when some or all of the propositions concerned are of one of the forms $(x) . \phi x$, $(\exists x) . \phi x$, which was to be proved.

It follows, by mere repetition of the proofs, that the propositions of ∗1—∗5 hold when p, q, r are replaced by propositions containing any number of apparent variables.

$*10$. THEORY OF PROPOSITIONS CONTAINING ONE APPARENT VARIABLE

Summary of $*10$.

The chief purpose of the propositions of this number is to extend to formal implications (*i.e.* to propositions of the form $(x) . \phi x \supset \psi x$) as many as possible of the propositions proved previously for material implications, *i.e.* for propositions of the form $p \supset q$. Thus *e.g.* we have proved in $*3{\cdot}33$ that

$$p \supset q . q \supset r . \supset . p \supset r.$$

Put
$$p = \text{Socrates is a Greek},$$
$$q = \text{Socrates is a man},$$
$$r = \text{Socrates is a mortal}.$$

Then we have "if 'Socrates is a Greek' implies 'Socrates is a man,' and 'Socrates is a man' implies 'Socrates is a mortal,' it follows that 'Socrates is a Greek' implies 'Socrates is a mortal.'" But this does not of itself prove that if all Greeks are men, and all men are mortals, then all Greeks are mortals.

Putting
$$\phi x . = . x \text{ is a Greek},$$
$$\psi x . = . x \text{ is a man},$$
$$\chi x . = . x \text{ is a mortal},$$

we have to prove

$$(x) . \phi x \supset \psi x : (x) . \psi x \supset \chi x : \supset : (x) . \phi x \supset \chi x.$$

It is such propositions that have to be proved in the present number. It will be seen that formal implication ($(x) . \phi x \supset \psi x$) is a relation of two functions $\phi \hat{x}$ and $\psi \hat{x}$. Many of the formal properties of this relation are analogous to properties of the relation "$p \supset q$" which expresses material implication; it is such analogues that are to be proved in this number.

We shall assume in this number, what has been proved in $*9$, that the propositions of $*1$—$*5$ can be applied to such propositions as $(x) . \phi x$ and $(\exists x) . \phi x$. Instead of the method adopted in $*9$, it is possible to take negation and disjunction as new primitive ideas, as applied to propositions containing apparent variables, and to assume that, with the new meanings of negation and disjunction, the primitive propositions of $*1$ still hold. If this method is adopted, we need not take $(\exists x) . \phi x$ as a primitive idea, but may put

$*10{\cdot}01$. $(\exists x) . \phi x . = . \sim (x) . \sim \phi x$ Df

In order to make it clear how this alternative method can be developed, we shall, in the present number, assume nothing of what has been proved in $*9$ except certain propositions which, in the alternative method, will be primitive propositions, and (what in part characterizes the alternative method)

the applicability to propositions containing apparent variables of analogues of the primitive ideas and propositions of $*1$, and therefore of their consequences as set forth in $*2$—$*5$.

The two following definitions merely serve to introduce a notation which is often more convenient than the notation $(x) . \phi x \supset \psi x$ or $(x) . \phi x \equiv \psi x$.

$*10{\cdot}02$. $\phi x \supset_x \psi x . = . (x) . \phi x \supset \psi x$ Df

$*10{\cdot}03$. $\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x$ Df

The first of these notations is due to Peano, who, however, has no notation for $(x) . \phi x$ except in the special case of a formal implication.

The following propositions ($*10{\cdot}1{\cdot}11{\cdot}12{\cdot}121{\cdot}122$) have already been given in $*9$. $*10{\cdot}1$ is $*9{\cdot}2$, $*10{\cdot}11$ is $*9{\cdot}13$, $*10{\cdot}12$ is $*9{\cdot}25$, $*10{\cdot}121$ is $*9{\cdot}14$, and $*10{\cdot}122$ is $*9{\cdot}15$. These five propositions must all be taken as primitive propositions in the alternative method; on the other hand, $*9{\cdot}1$ and $*9{\cdot}11$ are not required as primitive propositions in the alternative method.

The propositions of the present number are very much used throughout the rest of the work. The propositions most used are the following:

$*10{\cdot}1$. $\vdash : (x) . \phi x . \supset . \phi y$

I.e. what is true in all cases is true in any one case.

$*10{\cdot}11$. If ϕy is true whatever possible argument y may be, then $(x) . \phi x$ is true. In other words, whenever the propositional function ϕy can be asserted, so can the proposition $(x) . \phi x$.

$*10{\cdot}21$. $\vdash :. (x) . p \supset \phi x . \equiv : p . \supset . (x) . \phi x$

$*10{\cdot}22$. $\vdash :. (x) . \phi x . \psi x . \equiv : (x) . \phi x : (x) . \psi x$

The conditions of significance in this proposition demand that ϕ and ψ should take arguments of the same type.

$*10{\cdot}23$. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

I.e. if ϕx always implies p, then if ϕx is ever true, p is true.

$*10{\cdot}24$. $\vdash : \phi y . \supset . (\exists x) . \phi x$

I.e. if ϕy is true, then there is an x for which ϕx is true. This is the sole method of proving existence-theorems.

$*10{\cdot}27$. $\vdash :. (z) . \phi z \supset \psi z . \supset : (z) . \phi z . \supset . (z) . \psi z$

I.e. if ϕz always implies ψz, then "ϕz always" implies "ψz always." The three following propositions, which are equally useful, are analogous to $*10{\cdot}27$.

$*10{\cdot}271$. $\vdash :. (z) . \phi z \equiv \psi z . \supset : (z) . \phi z . \equiv . (z) . \psi z$

$*10{\cdot}28$. $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

$*10{\cdot}281$. $\vdash :. (x) . \phi x \equiv \psi x . \supset : (\exists x) . \phi x . \equiv . (\exists x) . \psi x$

$*10{\cdot}35$. $\vdash :. (\exists x) . p . \phi x . \equiv : p : (\exists x) . \phi x$

$*10{\cdot}42$. $\vdash :. (\exists x) . \phi x . \vee . (\exists x) . \psi x : \equiv . (\exists x) . \phi x \vee \psi x$

$*10{\cdot}5$. $\vdash :. (\exists x) . \phi x . \psi x . \supset : (\exists x) . \phi x : (\exists x) . \psi x$

It should be noticed that whereas $*10{\cdot}42$ expresses an equivalence, $*10{\cdot}5$ only expresses an implication. This is the source of many subsequent differences between formulae concerning addition and formulae concerning multiplication.

$*10{\cdot}51$. $\vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : \phi x . \supset_x . \sim \psi x$

This proposition is analogous to

$$\vdash : \sim(p . q) . \equiv . p \supset \sim q$$

which results from $*4{\cdot}63$ by transposition.

Of the remaining propositions of this number, some are employed fairly often, while others are lemmas which are used only once or twice, sometimes at a much later stage.

$*10{\cdot}01$. $(\exists x) . \phi x . = . \sim(x) . \sim \phi x$ Df

This definition is only to be used when we discard the method of $*9$ in favour of the alternative method already explained. In either case we have

$$\vdash : (\exists x) . \phi x . \equiv . \sim(x) . \sim \phi x.$$

$*10{\cdot}02$. $\phi x \supset_x \psi x . = . (x) . \phi x \supset \psi x$ Df

$*10{\cdot}03$. $\phi x \equiv_x \psi x . = . (x) . \phi x \equiv \psi x$ Df

$*10{\cdot}1$. $\vdash : (x) . \phi x . \supset . \phi y$ [$*9{\cdot}2$]

$*10{\cdot}11$. If ϕy is true whatever possible argument y may be, then $(x) . \phi x$ is true. [$*9{\cdot}13$]

This proposition is, in a sense, the converse of $*10{\cdot}1$. $*10{\cdot}1$ may be stated: "What is true of all is true of any," while $*10{\cdot}11$ may be stated: "What is true of any, however chosen, is true of all."

$*10{\cdot}12$. $\vdash :. (x) . p \vee \phi x . \supset : p . \vee . (x) . \phi x$ [$*9{\cdot}25$]

According to the definitions in $*9$, this proposition is a mere example of "$q \supset q$," since by definition the two sides of the implication are different symbols for the same proposition. According to the alternative method, on the contrary, $*10{\cdot}12$ is a substantial proposition.

$*10{\cdot}121$. If "ϕx" is significant, then if a is of the same type as x, "ϕa" is significant, and vice versa. [$*9{\cdot}14$]

It follows from this proposition that two arguments to the same function must be of the same type; for if x and a are arguments to $\phi\hat{x}$, "ϕx" and "ϕa" are significant, and therefore x and a are of the same type. Thus the above primitive proposition embodies the outcome of our discussion of the vicious-circle paradoxes in Chapter II of the Introduction.

$*10{\cdot}122$. If, for some a, there is a proposition ϕa, then there is a function $\phi\hat{x}$, and vice versa. [$*9{\cdot}15$]

$*10{\cdot}13$. If $\phi\hat{x}$ and $\psi\hat{x}$ take arguments of the same type, and we have "$\vdash . \phi x$" and "$\vdash . \psi x$," we shall have "$\vdash . \phi x . \psi x$."

Dem.

By repeated use of $*9{\cdot}61{\cdot}62{\cdot}63{\cdot}131$ (3), there is a function $\sim \dot{\phi}\hat{x} \lor \sim \psi\hat{x}$.

Hence by $*2{\cdot}11$ and $*3{\cdot}01$,

$\vdash : \sim\phi x \lor \sim\psi x \,.\lor .\, \phi x \,.\, \psi x$ (1)

$\vdash .(1).*2{\cdot}32.(*1{\cdot}01). \supset \vdash :. \phi x . \supset : \psi x . \supset . \phi x . \psi x$ (2)

$\vdash .(2).*9{\cdot}12. \supset \vdash .$ Prop

$*10{\cdot}14.\ \vdash :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y$

This proposition is true whenever it is significant, but it is not always significant when its hypothesis is significant. For the thesis demands that ϕ and ψ should take arguments of the same type, while the hypothesis does not demand this. Hence, if it is to be applied when ϕ and ψ are given, or when ψ is given as a function of ϕ or vice versa, we must not argue from the hypothesis to the thesis unless, in the supposed case, ϕ and ψ take arguments of the same type.

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (x) . \phi x . \supset . \phi y$ (1)

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi y$ (2)

$\vdash .(1).(2). *10{\cdot}13 . \supset \vdash : (x) . \phi x . \supset . \phi y : (x) . \psi x . \supset . \psi y :$

$[*3{\cdot}47] \quad \supset \vdash :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y :. \supset \vdash .$ Prop

$*10{\cdot}2.\ \vdash :. (x) . p \lor \phi x . \equiv : p . \lor . (x) . \phi x$

Dem.

$\vdash . *10{\cdot}1 . *1{\cdot}6 . \supset \vdash :. p . \lor . (x) . \phi x : \supset . p \lor \phi y :.$

$[*10{\cdot}11] \quad \supset \vdash :. (y) :. p . \lor . (x) . \phi x : \supset . p \lor \phi y :.$

$[*10{\cdot}12] \quad \supset \vdash :. p . \lor . (x) . \phi x : \supset . (y) . p \lor \phi y$ (1)

$\vdash . *10{\cdot}12 . \quad \supset \vdash :. (y) . p \lor \phi y . \supset : p . \lor . (x) . \phi x$ (2)

$\vdash .(1).(2). \quad \supset \vdash .$ Prop

$*10{\cdot}21.\ \vdash :. (x) . p \supset \phi x . \equiv : p . \supset . (x) . \phi x \quad \left[*10{\cdot}2 \frac{\sim p}{p}\right]$

This proposition is much more used than $*10{\cdot}2$.

$*10{\cdot}22.\ \vdash :. (x) . \phi x . \psi x . \equiv : (x) . \phi x : (x) . \psi x$

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash : (x) . \phi x . \psi x . \supset . \phi y . \psi y .$ (1)

$[*3{\cdot}26] \quad \supset . \phi y :$

$[*10{\cdot}11] \quad \supset \vdash :. (y) : (x) . \phi x . \psi x . \supset . \phi y :.$

$[*10{\cdot}21] \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . (y) . \phi y$ (2)

$\vdash .(1). *3{\cdot}27 . \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . \psi z :.$

$[*10{\cdot}11] \quad \supset \vdash :. (z) : (x) . \phi x . \psi x . \supset . \psi z :.$

$[*10{\cdot}21] \quad \supset \vdash :. (x) . \phi x . \psi x . \supset . (z) . \psi z$ (3)

$\vdash .(2).(3).$ Comp $. \supset \vdash :. (x) . \phi x . \psi x . \supset : (y) . \phi y : (z) . \psi z$ (4)

$\vdash . *10{\cdot}14{\cdot}11 . \quad \supset \vdash :. (y) :. (x) . \phi x : (x) . \psi x : \supset . \phi y . \psi y :.$

$[*10{\cdot}21] \quad \supset \vdash :. (x) . \phi x : (x) . \psi x : \supset . (y) . \phi y . \psi y$ (5)

$\vdash .(4).(5). \quad \supset \vdash .$ Prop

The above proposition is true whenever it is significant; but, as was pointed out in connexion with $*10{\cdot}14$, it is not always significant when "$(x) . \phi x : (x) . \psi x$" is significant.

$*10{\cdot}221$. If ϕx contains a constituent $\chi(x, y, z, \ldots)$ and ψx contains a constituent $\chi(x, u, v, \ldots)$, where χ is an elementary function and $y, z, \ldots u, v, \ldots$ are either constants or apparent variables, then $\phi\hat{x}$ and $\psi\hat{x}$ take arguments of the same type. This can be proved in each particular case, though not generally, provided that, in obtaining ϕ and ψ from χ, χ is only submitted to negations, disjunctions and generalizations. The process may be illustrated by an example. Suppose ϕx is $(y) . \chi(x, y) . \supset . \theta x$, and ψx is $fx . \supset . (y) . \chi(x, y)$. By the definitions of $*9$, ϕx is $(\exists y) . \sim\chi(x, y) \vee \theta x$, and ψx is $(y) . \sim fx \vee \chi(x, y)$. Hence since the primitive ideas $(x) . Fx$ and $(\exists x) . Fx$ only apply to functions, there are functions $\sim\chi(\hat{x}, \hat{y}) \vee \theta\hat{x}$, $\sim f\hat{x} \vee \chi(\hat{x}, \hat{y})$. Hence there is a proposition $\sim\chi(a, b) \vee \theta a$. Hence, since "$p \vee q$" and "$\sim p$" are only significant when p and q are propositions, there is a proposition $\chi(a, b)$. Similarly, for some u and v, there are propositions $\sim fu \vee \chi(u, v)$ and $\chi(u, v)$. Hence by $*9{\cdot}14$, u and a, v and b are respectively of the same type, and (again by $*9{\cdot}14$) there is a proposition $\sim fa \vee \chi(a, b)$. Hence ($*9{\cdot}15$) there are functions $\sim\chi(a, \hat{y}) \vee \theta a$, $\sim fa \vee \chi(a, \hat{y})$, and therefore there are propositions

$$(\exists y) . \sim\chi(a, y) \vee \theta a, \quad (y) . \sim fa \vee \chi(a, y),$$

i.e. there are propositions ϕa, ψa, which was to be proved. This process can be applied similarly in any other instance.

$*10{\cdot}23$. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

Dem.

$\vdash . *4{\cdot}2 . (*9{\cdot}03) . \supset \vdash :. (x) . \sim\phi x \vee p . \equiv : (x) . \sim\phi x . \vee . p :$

$[(*9{\cdot}02)] \qquad \equiv . (\exists x) . \phi x . \supset . p \qquad (1)$

$\vdash . (1) . (*1{\cdot}01) . \supset \vdash . \text{Prop}$

In the above proof, we employ the definitions of $*9$. In the alternative method, in which $(\exists x) . \phi x$ is defined in accordance with $*10{\cdot}01$, the proof proceeds as follows.

$*10{\cdot}23$. $\vdash :. (x) . \phi x \supset p . \equiv : (\exists x) . \phi x . \supset . p$

Dem.

$\vdash . \text{Transp} . (*10{\cdot}01) . \supset \vdash :. (\exists x) . \phi x . \supset . p : \equiv : \sim p . \supset . (x) . \sim\phi x :$

$[*10{\cdot}21] \qquad \equiv : (x) : \sim p . \supset . \sim\phi x : \qquad (1)$

$[*10{\cdot}1] \qquad \supset : \sim p . \supset . \sim\phi x :$

$[\text{Transp}] \qquad \supset : \phi x . \supset . p :.$

$[*10{\cdot}11] \qquad \supset \vdash :. (x) :. (\exists x) . \phi x . \supset . p : \supset : \phi x . \supset . p :.$

$[*10{\cdot}21] \quad \supset \vdash :. (\exists x) . \phi x . \supset . p : \supset : (x) : \phi x . \supset . p$ (2)

$\vdash . *10{\cdot}1 . \supset \vdash :. (x) : \phi x . \supset . p : \supset : \phi x \supset p :$

$[\text{Transp}] \qquad \supset : \sim p . \supset . \sim \phi x :.$

$[*10{\cdot}11{\cdot}21] \quad \supset \vdash :. (x) : \phi x . \supset . p : \supset : (x) : \sim p . \supset . \sim \phi x :$

$[(1)] \qquad \supset : (\exists x) . \phi x . \supset . p$ (3)

$\vdash . (2) . (3) . \supset \vdash . \text{Prop}$

Whenever we have an asserted proposition of the form $p \supset \phi x$, we can pass by $*10{\cdot}11{\cdot}21$ to an asserted proposition $p . \supset . (x) . \phi x$. This passage is constantly required, as in the last line but one of the above proof. It will be indicated merely by the reference "$*10{\cdot}11{\cdot}21$," and the two steps which it requires will not be separately put down.

$*10{\cdot}24.\ \vdash : \phi y . \supset . (\exists x) . \phi x$

This is $*9{\cdot}1$. In the alternative method, the proof is as follows.

Dem.

$\vdash . *10{\cdot}1 . \supset \vdash : (x) . \sim \phi x . \supset . \sim \phi y :$

$[\text{Transp}] \supset \vdash : \phi y . \supset . \sim (x) . \sim \phi x :$

$[(*10{\cdot}01)] \supset \vdash . \text{Prop}$

$*10{\cdot}25.\ \vdash : (x) . \phi x . \supset . (\exists x) . \phi x \qquad [*10{\cdot}1{\cdot}24]$

$*10{\cdot}251.\ \vdash : (x) . \sim \phi x . \supset . \sim \{(x) . \phi x\} \qquad [*10{\cdot}25 . \text{Transp}]$

$*10{\cdot}252.\ \vdash : \sim \{(\exists x) . \phi x\} . \equiv . (x) . \sim \phi x \qquad [*4{\cdot}2 . (*9{\cdot}02)]$

$*10{\cdot}253.\ \vdash : \sim \{(x) . \phi x\} . \equiv . (\exists x) . \sim \phi x \qquad [*4{\cdot}2 . (*9{\cdot}01)]$

In the alternative method, in which $(\exists x) . \phi x$ is defined as in $*10{\cdot}01$, the proofs of $*10{\cdot}252{\cdot}253$ are as follows.

$*10{\cdot}252.\ \vdash : \sim \{(\exists x) . \phi x\} . \equiv . (x) . \sim \phi x \qquad [*4{\cdot}13 . (*10{\cdot}01)]$

$*10{\cdot}253.\ \vdash : \sim \{(x) . \phi x\} . \equiv . (\exists x) . \sim \phi x$

Dem.

$\vdash . *10{\cdot}1 . \supset \vdash : (x) . \phi x . \supset . \phi y .$

$[*2{\cdot}12] \qquad \supset . \sim (\sim \phi y) :$

$[*10{\cdot}11{\cdot}21] \supset \vdash : (x) . \phi x . \supset . (y) . \sim (\sim \phi y) :$

$[\text{Transp}] \supset \vdash : \sim \{(y) . \sim (\sim \phi y)\} . \supset . \sim \{(x) . \phi x\} :$

$[(*10{\cdot}01)] \supset \vdash : (\exists y) . \sim \phi y . \qquad \supset . \sim \{(x) . \phi x\}$ (1)

$\vdash . *10{\cdot}1 . \supset \vdash : (y) . \sim (\sim \phi y) . \qquad \supset . \sim (\sim \phi x) .$

$[*2{\cdot}14] \qquad \supset . \phi x :$

$[*10{\cdot}11{\cdot}21] \supset \vdash : (y) . \sim (\sim \phi y) . \qquad \supset . (x) . \phi x :$

$[\text{Transp}] \supset \vdash : \sim \{(x) . \phi x\} . \qquad \supset . \sim \{(y) . \sim (\sim \phi y)\} .$

$[(*10{\cdot}01)] \qquad \supset . (\exists y) . \sim \phi y$ (2)

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

***10·26.** $\vdash :. (z) . \phi z \supset \psi z : \phi x : \supset . \psi x$ [*10·1 . Imp]

This is one form of the syllogism in Barbara. *E.g.* put $\phi z . = . z$ is a man, $\psi z . = . z$ is mortal, $x =$ Socrates. Then the proposition becomes: "If all men are mortal, and Socrates is a man, then Socrates is mortal."

Another form of the syllogism in Barbara is given in *10·3. The two forms, formerly wrongly identified, were first distinguished by Peano and Frege.

***10·27.** $\vdash :. (z) . \phi z \supset \psi z . \supset : (z) . \phi z . \supset . (z) . \psi z$

This is *9·21. In the alternative method, the proof is as follows.

Dem.

$\vdash . *10{\cdot}14 . \quad \supset \vdash :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . \phi y \supset \psi y . \phi y .$
$\supset . \psi y :.$

[Ass] $\quad \supset \vdash :. (y) :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . \psi y :.$

[*10·1] $\quad \supset \vdash :. (z) . \phi z \supset \psi z : (z) . \phi z : \supset . (y) . \psi y \quad (1)$

[*10·21]

$\vdash . (1) . \text{Exp} . \supset \vdash . \text{Prop}$

***10·271.** $\vdash :. (z) . \phi z \equiv \psi z . \supset : (z) . \phi z . \equiv . (z) . \psi z$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \text{Hp} . \supset : (z) . \phi z \supset \psi z :$
[*10·27] $\quad \supset : (z) . \phi z . \supset . (z) . \psi z \quad (1)$

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. \text{Hp} . \supset : (z) . \psi z \supset \phi z :$
[*10·27] $\quad \supset : (z) . \psi z . \supset . (z) . \phi z \quad (2)$

$\vdash . (1) . (2) . \text{Comp} . \supset \vdash . \text{Prop}$

***10·28.** $\vdash :. (x) . \phi x \supset \psi x . \supset : (\exists x) . \phi x . \supset . (\exists x) . \psi x$

This is *9·22. In the alternative method, the proof is as follows.

Dem.

$\vdash . *10{\cdot}1 . \quad \supset \vdash :. (x) . \phi x \supset \psi x . \supset . \phi y \supset \psi y .$
[Transp] $\quad \supset . \sim \psi y \supset \sim \phi y :.$

[*10·11·21] $\supset \vdash :. (x) . \phi x \supset \psi x . \supset : (y) . \sim \psi y \supset \sim \phi y :$
[*10·27] $\quad \supset : (y) . \sim \psi y . \supset . (y) . \sim \phi y :$
[Transp] $\quad \supset : (\exists y) . \phi y . \supset . (\exists y) . \psi y :. \supset \vdash . \text{Prop}$

***10·281.** $\vdash :. (x) . \phi x \equiv \psi x . \supset : (\exists x) . \phi x . \equiv . (\exists x) . \psi x$ [*10·22·28 . Comp]

***10·29.** $\vdash :. (x) . \phi x \supset \psi x : (x) . \phi x \supset \chi x : \equiv : (x) : \phi x . \supset . \psi x . \chi x$

Dem.

$\vdash . *10{\cdot}22 . \quad \supset \vdash :. (x) . \phi x \supset \psi x : (x) . \phi x \supset \chi x :$
$\equiv : (x) : \phi x \supset \psi x . \phi x \supset \chi x \quad (1)$

$\vdash . *4{\cdot}76 . \quad \supset \vdash :. \phi x \supset \psi x . \phi x \supset \chi x . \equiv : \phi x . \supset . \psi x . \chi x :.$

[*10·11] $\quad \supset \vdash :. (x) :. \phi x \supset \psi x . \phi x \supset \chi x . \equiv : \phi x . \supset . \psi x . \chi x :.$

[*10·271] $\quad \supset \vdash :. (x) : \phi x \supset \psi x . \phi x \supset \chi x : \equiv : (x) : \phi x . \supset . \psi x . \chi x \quad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

This is an extension of the principle of composition.

$*10{\cdot}3$. $\vdash :. (x) . \phi x \supset \psi x : (x) . \psi x \supset \chi x : \supset . (x) . \phi x \supset \chi x$

This is the second form of the syllogism in Barbara.

Dem.

$$\vdash . *10{\cdot}22{\cdot}221 . \supset \vdash : \mathrm{Hp} . \supset . (x) . \phi x \supset \psi x . \psi x \supset \chi x .$$
$$[\mathrm{Syll}.*10{\cdot}27] \qquad \supset . (x) . \phi x \supset \chi x : \supset \vdash . \mathrm{Prop}$$

$*10{\cdot}301$. $\vdash :. (x) . \phi x \equiv \psi x : (x) . \psi x \equiv \chi x : \supset . (x) . \phi x \equiv \chi x$

Dem.

$$\vdash . *10{\cdot}22{\cdot}221 . \supset \vdash :. \mathrm{Hp} . \supset : (x) . \phi x \equiv \psi x . \psi x \equiv \chi x :$$
$$[*4{\cdot}22 . *10{\cdot}27] \qquad \supset : (x) . \phi x \equiv \chi x :. \supset \vdash . \mathrm{Prop}$$

In the second line of the proofs of $*10{\cdot}3$ and $*10{\cdot}301$, we abbreviate the process of proof in a way which is often convenient. In $*10{\cdot}3$, the full process would be as follows:

$$\vdash . \mathrm{Syll} . \supset \vdash : \phi x \supset \psi x . \psi x \supset \chi x . \supset . \phi x \supset \chi x :$$
$$[*10{\cdot}11] \supset \vdash : (x) : \phi x \supset \psi x . \psi x \supset \chi x . \supset . \phi x \supset \chi x :$$
$$[*10{\cdot}27] \supset \vdash : (x) . \phi x \supset \psi x . \psi x \supset \chi x . \supset . (x) . \phi x \supset \chi x$$

The above two propositions show that formal implication and formal equivalence are transitive relations between functions.

$*10{\cdot}31$. $\vdash :. (x) . \phi x \supset \psi x . \supset : (x) : \phi x . \chi x . \supset . \psi x . \chi x$

Dem.

$$\vdash . \mathrm{Fact} . *10{\cdot}11 . \supset \vdash :. (x) :. \phi x \supset \psi x . \supset : \phi x . \chi x . \supset . \psi x . \chi x \qquad (1)$$
$$\vdash . (1) . *10{\cdot}27 . \quad \supset \vdash . \mathrm{Prop}$$

$*10{\cdot}311$. $\vdash :. (x) . \phi x \equiv \psi x . \supset : (x) : \phi x . \chi x . \equiv . \psi x . \chi x$

Dem.

$$\vdash . *4{\cdot}36 . *10{\cdot}11 . \supset \vdash :. (x) :. \phi x \equiv \psi x . \supset : \phi x . \chi x . \equiv . \psi x . \chi x \qquad (1)$$
$$\vdash . (1) . *10{\cdot}27 . \quad \supset \vdash . \mathrm{Prop}$$

The above two propositions are extensions of the principle of the factor.

$*10{\cdot}32$. $\vdash : \phi x \equiv_x \psi x . \equiv . \psi x \equiv_x \phi x$

Dem.

$$\vdash . *10{\cdot}22 . \supset \vdash : \phi x \equiv_x \psi x . \equiv . \phi x \supset_x \psi x . \psi x \supset_x \phi x .$$
$$[*4{\cdot}3] \qquad \equiv . \psi x \supset_x \phi x . \phi x \supset_x \psi x .$$
$$[*10{\cdot}22] \qquad \equiv . \psi x \equiv_x \phi x : \supset \vdash . \mathrm{Prop}$$

This proposition shows that formal equivalence is symmetrical.

$*10{\cdot}321$. $\vdash : \phi x \equiv_x \psi x . \phi x \equiv_x \chi x . \supset . \psi x \equiv_x \chi x$

Dem.

$$\vdash . *10{\cdot}32 . \mathrm{Fact} . \supset \vdash : \mathrm{Hp} . \supset . \psi x \equiv_x \phi x . \phi x \equiv_x \chi x .$$
$$[*10{\cdot}301] \qquad \supset . \psi x \equiv_x \chi x : \supset \vdash . \mathrm{Prop}$$

$*10{\cdot}322$. $\vdash : \psi x \equiv_x \phi x . \chi x \equiv_x \phi x . \supset . \psi x \equiv_x \chi x$

Dem.

$$\vdash . *10{\cdot}32 . \supset \vdash : \mathrm{Hp} . \supset . \psi x \equiv_x \phi x . \phi x \equiv_x \chi x .$$
$$[*10{\cdot}301] \qquad \supset . \psi x \equiv_x \chi x : \supset \vdash . \mathrm{Prop}$$

***10·33.** $\vdash :. (x) : \phi x . p : \equiv : (x) . \phi x : p$

Dem.

$\vdash .$ *10·1 $.$ $\supset \vdash :. (x) : \phi x . p : \supset . \phi y . p .$

[*3·27] $\supset . p$ (1)

$\vdash . (1) .$ *3·26 $.$ $\supset \vdash :. (x) : \phi x . p : \supset . \phi y :$ (2)

[*10·11·21] $\supset \vdash :. (x) : \phi x . p : \supset . (y) . \phi y$

$\vdash . (2) . (3) .$ $\supset \vdash :. (x) : \phi x . p : \supset : (y) . \phi y : p$ (3)

$\vdash .$ *10·1 $.$ $\supset \vdash :. (y) . \phi y . \quad \supset . \phi x :.$ (4)

[Fact] $\supset \vdash :. (y) . \phi y : p : \supset . \phi x . p :.$

[*10·11·21] $\supset \vdash :. (y) . \phi y : p : \supset : (x) : \phi x . p$ (5)

$\vdash . (4) . (5) .$ $\supset \vdash .$ Prop

***10·34.** $\vdash :. (\exists x) . \phi x \supset p . \equiv : (x) . \phi x . \supset . p$

This follows immediately from *9·05·01 and *1·01. In the alternative method, the proof is as follows.

Dem.

$\vdash .$ *4·2 $.$ (*10·01) $. \supset$

$\vdash :. (\exists x) . \phi x \supset p . \quad \equiv : \sim \{(x) . \sim (\phi x \supset p)\} :$

[*4·61.*10·271] $\equiv : \sim \{(x) : \phi x . \sim p\} :$

[*10·33] $\equiv : \sim \{(x) . \phi x : \sim p\} :$

[*4·53] $\equiv : \sim \{(x) . \phi x\} . \lor . p :$

[*4·6] $\equiv : (x) . \phi x . \supset . p$

***10·35.** $\vdash :. (\exists x) . p . \phi x . \equiv : p : (\exists x) . \phi x$

Dem.

$\vdash .$ *3·26 $.$ $\supset \vdash : p . \phi x . \supset . p :$

[*10·11] $\supset \vdash : (x) : p . \phi x . \supset . p :$

[*10·23] $\supset \vdash : (\exists x) . p . \phi x . \supset . p$ (1)

$\vdash .$ *3·27 $.$ $\supset \vdash : p . \phi x . \supset . \phi x :$

[*10·11] $\supset \vdash : (x) : p . \phi x . \supset . \phi x :$

[*10·28] $\supset \vdash : (\exists x) . p . \phi x . \supset . (\exists x) . \phi x$ (2)

$\vdash .$ *3·2 $.$ $\supset \vdash :. p . \supset : \phi x . \supset . p . \phi x .$

[*10·11·21] $\supset \vdash :. p . \supset : (x) : \phi x . \supset . p . \phi x :$

[*10·28] $\supset : (\exists x) . \phi x . \supset . (\exists x) . p . \phi x$ (3)

$\vdash . (1) . (2) . (3) .$ Imp $. \supset \vdash .$ Prop

***10·36.** $\vdash :. (\exists x) . \phi x \lor p . \equiv : (\exists x) . \phi x . \lor . p$

This follows immediately from *9·05. In the alternative method, the proof is as follows.

Dem.

$\vdash .$ *4·64 $.$ $\supset \vdash : \phi x \lor p . \equiv . \sim \phi x \supset p :$

[*10·11] $\supset \vdash : (x) : \phi x \lor p . \equiv . \sim \phi x \supset p :$

[*10·281] $\supset \vdash :. (\exists x) . \phi x \lor p . \equiv : (\exists x) . \sim \phi x \supset p :$

[*10·34] $\equiv : (x) . \sim \phi x . \supset . p :$

[*4·6.(*10·01)] $\equiv : (\exists x) . \phi x . \lor . p :. \supset \vdash .$ Prop

The above proposition is only required in order to lead to the following:

$*10 \cdot 37.$ $\vdash :. (\exists x) . p \supset \phi x . \equiv : p . \supset . (\exists x) . \phi x$ $\left[*10 \cdot 36 \frac{\sim p}{p}\right]$

$*10 \cdot 39.$ $\vdash :. \phi x \supset_x \chi x : \psi x \supset_x \theta x : \supset : \phi x . \psi x . \supset_x . \chi x . \theta x$

Dem.

$\vdash . *10 \cdot 22 . \supset \vdash :. \text{Hp} . \supset : (x) : \phi x \supset \chi x . \psi x \supset \theta x :$

$[*3 \cdot 47 . *10 \cdot 27] \quad \supset : (x) : \phi x . \psi x . \supset . \chi x . \theta x :. \supset \vdash . \text{Prop}$

This proposition is only true when the conclusion is significant; the significance of the hypothesis does not insure that of the conclusion. On the conditions of significance, see the remarks on $*10 \cdot 4$, below.

$*10 \cdot 4.$ $\vdash :. \phi x \equiv_x \chi x . \psi x \equiv_x \theta x . \supset : \phi x . \psi x . \equiv_x . \chi x . \theta x$

Dem.

$\vdash . *10 \cdot 22 . \quad \supset \vdash :. \text{Hp} . \supset : \phi x \supset_x \chi x . \psi x \supset_x \theta x :$ (1)

$[*10 \cdot 39] \quad \supset : \phi x . \psi x . \supset_x . \chi x . \partial x$ (2)

Similarly $\quad \vdash :. \text{Hp} . \supset : \chi x . \theta x . \supset_x . \phi x . \psi x$

$\vdash . (1) . (2) . \text{Comp} . \supset \vdash :. \text{Hp} . \supset : \phi x . \psi x . \supset_x . \chi x . \theta x : \chi x . \theta x . \supset_x . \phi x . \psi x :$

$[*10 \cdot 22] \quad \supset : \phi x . \psi x . \equiv_x . \chi x . \theta x :. \supset \vdash . \text{Prop}$

In $*10 \cdot 4$ and many later propositions, as in $*10 \cdot 39$, the conclusion may be not significant when the hypothesis is true. Hence, in order that it may be legitimate to use $*10 \cdot 4$ in *inference, i.e.* to pass from the *assertion* of the hypothesis to the *assertion* of the conclusion, the functions ϕ, ψ, χ, θ must be such as to have overlapping ranges of significance. In virtue of $*10 \cdot 221$, this is secured if they are of the forms $F\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $f\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $G\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$, $g\{x, \chi(x, \hat{y}, \hat{z}, \ldots)\}$. It is also secured if ϕ and ψ or ϕ and θ or χ and ψ or χ and θ are of such forms, for ϕ and χ must have overlapping ranges of significance if the hypothesis is to be significant, and so must ψ and θ.

$*10 \cdot 41.$ $\vdash :. (x) . \phi x . \vee . (x) . \psi x : \supset . (x) . \phi x \vee \psi x$

Dem.

$\vdash . *10 \cdot 1 . \quad \supset \vdash : (x) . \phi x . \supset . \phi y .$

$[*2 \cdot 2] \quad \supset . \phi y \vee \psi y$ (1)

$\vdash . *10 \cdot 1 . \quad \supset \vdash : (x) . \psi x . \supset . \psi y .$

$[*1 \cdot 3] \quad \supset . \phi y \vee \psi y$ (2)

$\vdash . (1) . (2) . *10 \cdot 13 . \supset \vdash :. (x) . \phi x . \supset . \phi y \vee \psi y : (x) . \psi x . \supset . \phi y \vee \psi y :.$

$[*3 \cdot 44] \quad \supset \vdash :. (x) . \phi x . \vee . (x) . \psi x : \supset . \phi y \vee \psi y$

$[*10 \cdot 11 \cdot 21] \quad \supset \vdash :. (x) . \phi x . \vee . (x) . \psi x : \supset . (y) . \phi y \vee \psi y :. \supset \vdash . \text{Prop}$

Observe that in the above proof the uses of $*2 \cdot 2$ and $*1 \cdot 3$ are only legitimate if ϕy and ψy have overlapping ranges of significance, for otherwise, if y is such that there is a proposition ϕy, it is such that there is no proposition ψy, and conversely.

$*10{\cdot}411.\ \vdash :.\ \phi x \equiv_x \chi x \,.\, \psi x \equiv_x \theta x \,.\, \supset : \phi x \lor \psi x \,.\, \equiv_x \,.\, \chi x \lor \theta x$

Dem.

$$\begin{array}{ll} \vdash \,.\, *10{\cdot}14 \,.\, \supset \vdash :.\ \mathrm{Hp} \,.\, \supset : \phi x \equiv \chi x \,.\, \psi x \equiv \theta x : & \\ {[*4{\cdot}39]} \qquad \supset : \phi x \lor \psi x \,.\, \equiv \,.\, \chi x \lor \theta x & (1) \\ \vdash \,.\, (1) \,.\, *10{\cdot}11{\cdot}21 \,.\, \supset \vdash \,.\, \mathrm{Prop} & \end{array}$$

$*10{\cdot}412.\ \vdash : \phi x \equiv_x \psi x \,.\, \equiv \,.\, \sim\phi x \equiv_x \sim\psi x \quad [*4{\cdot}11 \,.\, *10{\cdot}11{\cdot}271]$

$*10{\cdot}413.\ \vdash :.\ \phi x \equiv_x \chi x \,.\, \psi x \equiv_x \theta x \,.\, \supset : \phi x \supset \psi x \,.\, \equiv_x \,.\, \chi x \supset \theta x$

Dem.

$$\begin{array}{l} \vdash \,.\, *10{\cdot}411{\cdot}412 \,.\, \supset \vdash :.\ \mathrm{Hp} \,.\, \supset : \sim\phi x \lor \psi x \,.\, \equiv_x \,.\, \sim\chi x \lor \theta x \\ {[(*1{\cdot}01)]} \qquad \supset : \phi x \supset \psi x \,.\, \equiv_x \,.\, \chi x \supset \theta x :. \supset \vdash \,.\, \mathrm{Prop} \end{array}$$

$*10{\cdot}414.\ \vdash :.\ \phi x \equiv_x \chi x \,.\, \psi x \equiv_x \theta x \,.\, \supset : \phi x \equiv \psi x \,.\, \equiv_x \,.\, \chi x \equiv \theta x$

Dem.

$$\begin{array}{ll} \vdash \,.\, *10{\cdot}413 \dfrac{\psi, \phi, \theta, \chi}{\phi, \psi, \chi, \theta} \,.\, *10{\cdot}32 \,.\, \supset \vdash :.\ \mathrm{Hp} \,.\, \supset : \psi x \supset \phi x \,.\, \equiv_x \,.\, \theta x \supset \chi x & (1) \\ \vdash \,.\, *10{\cdot}413 \,.\, (1) \,.\, *10{\cdot}4 \,.\, \qquad \supset \vdash \,.\, \mathrm{Prop} & \end{array}$$

The propositions $*10{\cdot}413{\cdot}414$ are chiefly used in cases where either χ is replaced by ϕ or θ is replaced by ψ, in which case half the hypothesis becomes superfluous, being true by $*4{\cdot}2$.

$*10{\cdot}42.\ \vdash :.\ (\exists x) \,.\, \phi x \,.\, \lor \,.\, (\exists x) \,.\, \psi x : \equiv \,.\, (\exists x) \,.\, \phi x \lor \psi x$

Dem.

$$\begin{array}{ll} \vdash \,.\, *10{\cdot}22 \,. & \supset \vdash :.\ (x) \,.\, \sim\phi x : (x) \,.\, \sim\psi x : \equiv \,.\, (x) \,.\, \sim\phi x \,.\, \sim\psi x :. \\ {[*4{\cdot}11]} & \supset \vdash :.\ \sim\{(x) \,.\, \sim\phi x : (x) \,.\, \sim\psi x\} \,.\, \equiv \,.\, \sim\{(x) \,.\, \sim\phi x \,.\, \sim\psi x\} :. \\ {[*4{\cdot}51{\cdot}56 . *10{\cdot}271]} & \supset \vdash :.\ \sim\{(x) \,.\, \sim\phi x\} \,.\, \lor \,.\, \sim\{(x) \,.\, \sim\psi x\} : \\ & \qquad \equiv \,.\, \sim\{(x) \,.\, \sim(\phi x \lor \psi x)\} :. \\ {[*10{\cdot}253]} & \supset \vdash :.\ (\exists x) \,.\, \phi x \,.\, \lor \,.\, (\exists x) \,.\, \psi x : \equiv \,.\, (\exists x) \,.\, \phi x \lor \psi x :. \\ & \qquad \supset \vdash \,.\, \mathrm{Prop} \end{array}$$

This proposition is very frequently used. It should be contrasted with $*10{\cdot}5$, in which we have only an implication, not an equivalence.

$*10{\cdot}43.\ \vdash : \phi z \equiv_z \psi z \,.\, \phi x \,.\, \equiv \,.\, \phi z \equiv_z \psi z \,.\, \psi x$

Dem.

$$\begin{array}{ll} \vdash \,.\, *10{\cdot}1 \,. \qquad \supset \vdash : \phi z \equiv_z \psi z \,.\, \supset \,.\, \phi x \equiv \psi x & (1) \\ \vdash \,.\, (1) \,.\, *5{\cdot}32 \,.\, \supset \vdash \,.\, \mathrm{Prop} & \end{array}$$

$*10{\cdot}5.\ \vdash :.\ (\exists x) \,.\, \phi x \,.\, \psi x \,.\, \supset : (\exists x) \,.\, \phi x : (\exists x) \,.\, \psi x$

Dem.

$$\begin{array}{ll} \vdash \,.\, *3{\cdot}26 \,.\, *10{\cdot}11 \,.\, \supset \vdash : (x) : \phi x \,.\, \psi x \,.\, \supset \,.\, \phi x : & \\ {[*10{\cdot}28]} \qquad \supset \vdash : (\exists x) \,.\, \phi x \,.\, \psi x \,.\, \supset \,.\, (\exists x) \,.\, \phi x & (1) \\ \vdash \,.\, *3{\cdot}27 \,.\, *10{\cdot}11 \,.\, \supset \vdash :.\ (x) : \phi x \,.\, \psi x \,.\, \supset \,.\, \psi x : & \\ {[*10{\cdot}28]} \qquad \supset \vdash : (\exists x) \,.\, \phi x \,.\, \psi x \,.\, \supset \,.\, (\exists x) \,.\, \psi x & (2) \\ \vdash \,.\, (1) \,.\, (2) \,.\, \mathrm{Comp} \,.\, \supset \vdash :.\ \mathrm{Prop} & \end{array}$$

The converse of the above proposition is false. The fact that this proposition states an implication, while $*10{\cdot}42$ states an equivalence, is the source of many subsequent differences between formulae concerning logical addition and formulae concerning logical multiplication.

$$*10{\cdot}51.\quad \vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : \phi x . \supset_x . \sim\psi x$$

Dem.

$$\vdash . *10{\cdot}252 . \supset \vdash :. \sim\{(\exists x) . \phi x . \psi x\} . \equiv : (x) . \sim(\phi x . \psi x) :$$
$$[*4{\cdot}51{\cdot}62 . *10{\cdot}271] \qquad \equiv : (x) : \phi x . \supset . \sim\psi x :. \supset \vdash . \text{Prop}$$

$$*10{\cdot}52.\quad \vdash :. (\exists x) . \phi x . \supset : (x) . \phi x \supset p . \equiv . p$$

Dem.

$$\vdash . *5{\cdot}5 . \supset \vdash :: \text{Hp} . \supset :. p . \equiv : (\exists x) . \phi x . \supset . p :$$
$$[*10{\cdot}23] \qquad \equiv : (x) . \phi x \supset p :: \supset \vdash . \text{Prop}$$

$$*10{\cdot}53.\quad \vdash :. \sim(\exists x) . \phi x . \supset : \phi x . \supset_x . \psi x$$

Dem.

$$\vdash . *2{\cdot}21 . *10{\cdot}11 . \supset$$
$$\vdash :. (x) :. \sim\phi x . \supset : \phi x . \supset . \psi x :.$$
$$[*10{\cdot}27] \quad \supset \vdash :. (x) . \sim\phi x . \supset : (x) : \phi x . \supset . \psi x :.$$
$$[*10{\cdot}252] \supset \vdash :. \sim(\exists x) . \phi x . \supset : (x) : \phi x . \supset . \psi x :. \supset \vdash . \text{Prop}$$

$$*10{\cdot}541.\quad \vdash :. \phi y . \supset_y . p \lor \psi y : \equiv : p . \lor . \phi y \supset_y \psi y$$

Dem.

$$\vdash . *4{\cdot}2 . (*1{\cdot}01) . \supset \vdash :. \phi y . \supset_y . p \lor \psi y : \equiv : (y) . \sim\phi y \lor p \lor \psi y :$$
$$[\text{Assoc}. *10{\cdot}271] \qquad \equiv : (y) . p \lor \sim\phi y \lor \psi y :$$
$$[*10{\cdot}2] \qquad \equiv : p . \lor . (y) . \sim\phi y \lor \psi y :$$
$$[(*1{\cdot}01)] \qquad \equiv : p . \lor . \phi y \supset_y \psi y :. \supset \vdash . \text{Prop}$$

The above proposition is only needed in order to lead to the following:

$$10{\cdot}542.\quad \vdash :. \phi y . \supset_y . p \supset \psi y : \equiv : p . \supset . \phi y \supset_y \psi y \qquad \left[*10{\cdot}541 \frac{\sim p}{p}\right]$$

This proposition is a lemma for $*84{\cdot}43$.

$$*10{\cdot}55.\quad \vdash :. (\exists x) . \phi x . \psi x : \phi x \supset_x \psi x : \equiv : (\exists x) . \phi x : \phi x \supset_x \psi x$$

Dem.

$$\vdash . *4{\cdot}71 . \supset \vdash :. \phi x \supset \psi x . \supset : \phi x . \psi x . \equiv . \phi x \qquad (1)$$
$$\vdash . (1) . *10{\cdot}11{\cdot}27 . \supset$$
$$\vdash :. \phi x \supset_x \psi x . \supset : (x) : \phi x . \psi x . \equiv . \phi x :$$
$$[*10{\cdot}281] \qquad \supset : (\exists x) . \phi x . \psi x . \equiv . (\exists x) . \phi x \qquad (2)$$
$$\vdash . (2) . *5{\cdot}32 . \supset \vdash . \text{Prop}$$

This proposition is a lemma for $*117{\cdot}12{\cdot}121$.

***10·56.** $\vdash :. \phi x . \supset_x . \psi x : (\exists x) . \phi x . \chi x : \supset . (\exists x) . \psi x . \chi x$

Dem.

$\vdash . *10{\cdot}31 . \supset \vdash :. \phi x . \supset_x . \psi x : \supset : \phi x . \chi x . \supset_x . \psi x . \chi x :$

[*10·28] $\supset : (\exists x) . \phi x . \chi x . \supset . (\exists x) . \psi x . \chi x$ (1)

$\vdash . (1) . \mathrm{Imp} . \supset \vdash .$ Prop

This proposition and *10·57 are used in the theory of series (Part V).

***10·57.** $\vdash :. \phi x . \supset_x . \psi x \vee \chi x : \supset : \phi x \supset_x \psi x . \vee . (\exists x) . \phi x . \chi x$

Dem.

$\vdash . *10{\cdot}51 . \mathrm{Fact} . \supset$

$\vdash :. \phi x . \supset_x . \psi x \vee \chi x : \sim(\exists x) . \phi x . \chi x : \supset : \phi x . \supset_x . \psi x \vee \chi x : \phi x . \supset_x . \sim\chi x :$

[*10·29] $\supset : \phi x . \supset_x . \psi x \vee \chi x . \sim\chi x :$

[*5·61] $\supset : \phi x . \supset_x . \psi x$ (1)

$\vdash . (1) . *5{\cdot}6 . \supset \vdash .$ Prop

$*11$. THEORY OF TWO APPARENT VARIABLES

Summary of $*11$.

In this number, the propositions proved for one variable in $*10$ are to be extended to two variables, with the addition of a few propositions having no analogues for one variable, such as $*11 \cdot 2 \cdot 21 \cdot 23 \cdot 24$ and $*11 \cdot 53 \cdot 55 \cdot 6 \cdot 7$. "$\phi(x, y)$" stands for a proposition containing x and containing y; when x and y are unassigned, $\phi(x, y)$ is a propositional function of x and y. The definition $*11 \cdot 01$ shows that "the truth of all values of $\phi(x, y)$" does not need to be taken as a new primitive idea, but is definable in terms of "the truth of all values of ψx." The reason is that, when x is assigned, $\phi(x, y)$ becomes a function of one variable, namely y, whence it follows that, for every possible value of x, "$(y) . \phi(x, y)$" embodies merely the primitive idea introduced in $*9$. But "$(y) . \phi(x, y)$" is again only a function of one variable, namely x, since y has here become an apparent variable. Hence the definition $*11 \cdot 01$ below is legitimate. We put:

$*11 \cdot 01$. $(x, y) . \phi(x, y) . = : (x) : (y) . \phi(x, y)$ Df

$*11 \cdot 02$. $(x, y, z) . \phi(x, y, z) . = : (x) : (y, z) . \phi(x, y, z)$ Df

$*11 \cdot 03$. $(\exists x, y) . \phi(x, y) . = : (\exists x) : (\exists y) . \phi(x, y)$ Df

$*11 \cdot 04$. $(\exists x, y, z) . \phi(x, y, z) . = : (\exists x) : (\exists y, z) . \phi(x, y, z)$ Df

$*11 \cdot 05$. $\phi(x, y) . \supset_{x, y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \supset . \psi(x, y)$ Df

$*11 \cdot 06$. $\phi(x, y) . \equiv_{x, y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \equiv . \psi(x, y)$ Df

All the above definitions are supposed extended to any number of variables that may occur.

The propositions of this section can all be extended to any finite number of variables; as the analogy is exact, it is not necessary to carry the process beyond two variables in our proofs.

In addition to the definition $*11 \cdot 01$, we need the primitive proposition that "whatever possible argument x may be, $\phi(x, y)$ is true whatever possible argument y may be" implies the corresponding statement with x and y interchanged except in "$\phi(x, y)$". Either may be taken as the meaning of "$\phi(x, y)$ is true whatever possible arguments x and y may be."

The propositions of the present number are somewhat less used than those of $*10$, but some of them are used frequently. Such are the following:

$*11 \cdot 1$. $\vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w)$

$*11 \cdot 11$. If $\phi(z, w)$ is true whatever possible arguments z and w may be, then $(x, y) . \phi(x, y)$ is true

These two propositions are the analogues of $*10 \cdot 1 \cdot 11$.

$*11\cdot2$. $\vdash : (x, y) . \phi(x, y) . \equiv . (y, x) . \phi(x, y)$

I.e. to say that "for all possible values of x, $\phi(x, y)$ is true for all possible values of y" is equivalent to saying "for all possible values of y, $\phi(x, y)$ is true for all possible values of x."

$*11\cdot3$. $\vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x, y) : p . \supset . \phi(x, y)$

This is the analogue of $*10\cdot21$.

$*11\cdot32$. $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \supset . (x, y) . \psi(x, y)$

I.e. "if $\phi(x, y)$ always implies $\psi(x, y)$, then '$\phi(x, y)$ always' implies '$\psi(x, y)$ always.'" This is the analogue of $*10\cdot27$. $*11\cdot33\cdot34\cdot341$ are respectively the analogues of $*10\cdot271\cdot28\cdot281$, and are also much used.

$*11\cdot35$. $\vdash :. (x, y) : \phi(x, y) . \supset . p : \equiv : (\exists x, y) . \phi(x, y) . \supset . p$

I.e. if $\phi(x, y)$ always implies p, then if $\phi(x, y)$ is ever true, p is true, and vice versa. This is the analogue of $*10\cdot23$.

$*11\cdot45$. $\vdash :. (\exists x, y) : p . \phi(x, y) : \equiv : p : (\exists x, y) . \phi(x, y)$

This is the analogue of $*10\cdot35$.

$*11\cdot54$. $\vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) . \phi x : (\exists y) . \psi y$

This proposition is useful because it analyses a proposition containing two apparent variables into two propositions which each contain only one. "$\phi x . \psi y$" is a function of two variables, but is compounded of two functions of one variable each. Such a function is like a conic which is two straight lines: it may be called an "analysable" function.

$*11\cdot55$. $\vdash :. (\exists x, y) . \phi x . \psi(x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi(x, y)$

I.e. to say "there are values of x and y for which $\phi x . \psi(x, y)$ is true" is equivalent to saying "there is a value of x for which ϕx is true and for which there is a value of y such that $\psi(x, y)$ is true."

$*11\cdot6$. $\vdash :: (\exists x) :. (\exists y) . \phi(x, y) . \psi y : \chi x :. \equiv :. (\exists y) :. (\exists x) . \phi(x, y) . \chi x : \psi y$

This gives a transformation which is useful in many proofs.

$*11\cdot62$. $\vdash :: \phi x . \psi(x, y) . \supset_{x, y} . \chi(x, y) : \equiv :. \phi x . \supset_x : \psi(x, y) . \supset_y . \chi(x, y)$

This transformation also is often useful.

$*11\cdot01$. $(x, y) . \phi(x, y) . = : (x) : (y) . \phi(x, y)$ Df

$*11\cdot02$. $(x, y, z) . \phi(x, y, z) . = : (x) : (y, z) . \phi(x, y, z)$ Df

$*11\cdot03$. $(\exists x, y) . \phi(x, y) . = : (\exists x) : (\exists y) . \phi(x, y)$ Df

$*11\cdot04$. $(\exists x, y, z) . \phi(x, y, z) . = : (\exists x) : (\exists y, z) . \phi(x, y, z)$ Df

$*11\cdot05$. $\phi(x, y) . \supset_{x, y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \supset . \psi(x, y)$ Df

$*11\cdot06$. $\phi(x, y) . \equiv_{x, y} . \psi(x, y) : = : (x, y) : \phi(x, y) . \equiv . \psi(x, y)$ Df

with similar definitions for any number of variables.

$*11\cdot07$. "Whatever possible argument x may be, $\phi(x, y)$ is true whatever possible argument y may be" implies the corresponding statement with x and y interchanged except in "$\phi(x, y)$". Pp.

$*11\cdot1$. $\vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w)$

Dem.

$\vdash . *10\cdot1 . \supset \vdash : \mathrm{Hp} . \supset . (y) . \phi(z, y) .$
[$*10\cdot1$] $\supset . \phi(z, w) : \supset \vdash . \mathrm{Prop}$

$*11\cdot11$. If $\phi(z, w)$ is true whatever possible arguments z and w may be, then $(x, y) . \phi(x, y)$ is true.

Dem.

By $*10\cdot11$, the hypothesis implies that $(y) . \phi(z, y)$ is true whatever possible argument z may be; and this, by $*10\cdot11$, implies $(x, y) . \phi(x, y)$.

$*11\cdot12$. $\vdash :. (x, y) . p \vee \phi(x, y) . \supset : p . \vee . (x, y) . \phi(x, y)$

Dem.

$\vdash . *10\cdot12 . \supset \vdash :. (y) . p \vee \phi(x, y) . \supset : p . \vee . (y) . \phi(x, y) :.$
[$*10\cdot11\cdot27$] $\supset \vdash :. (x, y) . p \vee \phi(x, y) . \supset : (x) : p . \vee . (y) . \phi(x, y) :$
[$*10\cdot12$] $\supset : p . \vee . (x, y) . \phi(x, y) :. \supset \vdash . \mathrm{Prop}$

This proposition is only used for proving $*11\cdot2$.

$*11\cdot13$. If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ take their first and second arguments respectively of the same type, and we have "$\vdash . \phi(x, y)$" and "$\vdash . \psi(x, y)$," we shall have "$\vdash . \phi(x, y) . \psi(x, y)$." [Proof as in $*10\cdot13$]

$*11\cdot14$. $\vdash :. (x, y) . \phi(x, y) : (x, y) . \psi(x, y) : \supset : \phi(z, w) . \psi(z, w)$

Dem.

$\vdash . *10\cdot14 . \supset \vdash :. \mathrm{Hp} . \supset : (y) . \phi(z, y) : (y) . \psi(z, y)$
[$*10\cdot14$] $\supset : \phi(z, w) . \psi(z, w) :. \supset \vdash . \mathrm{Prop}$

This proposition, like $*10\cdot14$, is not always significant when its hypothesis is true. $*11\cdot13$, on the contrary, is always significant when its hypothesis is true. For this reason, $*11\cdot13$ may always be safely used in *inference*, whereas $*11\cdot14$ can only be used in *inference* (*i.e.* for the actual assertion of the conclusion when the hypothesis is asserted) if it is known that the conclusion is significant.

$*11\cdot2$. $\vdash : (x, y) . \phi(x, y) . \equiv . (y, x) . \phi(x, y)$

Dem.

$\vdash . *11\cdot1 . \quad \supset \vdash : (x, y) . \phi(x, y) . \supset . \phi(z, w)$ (1)

$\vdash . (1) . *11\cdot07\cdot11 . \supset \vdash :. (w, z) : (x, y) . \phi(x, y) . \supset . \phi(z, w)$ (2)

$\vdash . (2) . *11\cdot12 \dfrac{\sim\{(x, y) . \phi(x, y)\}}{p} . \supset$

$\vdash :. (x, y) . \phi(x, y) . \supset . (w, z) . \phi(z, w)$ (3)

Similarly $\vdash :. (w, z) . \phi(z, w) . \supset . (x, y) . \phi(x, y)$ (4)

$\vdash . (3) . (4) . \quad \supset \vdash . \mathrm{Prop}$

Note that "$(w, z) . \phi(z, w)$" is the same proposition as "$(y, x) . \phi(x, y)$"; a proposition is not a function of any apparent variable which occurs in it.

$*11\cdot21.\ \vdash : (x, y, z) . \phi(x, y, z) . \equiv . (y, z, x) . \phi(x, y, z)$

Dem.

$[(*11\cdot01\cdot02)] \vdash :: (x, y, z) . \phi(x, y, z) . \equiv :. (x) :. (y) : (z) . \phi(x, y, z) :.$

$[*11\cdot2] \qquad \equiv :. (y) :. (x) : (z) . \phi(x, y, z) :.$

$[*11\cdot2 . *10\cdot271] \qquad \equiv :. (y) :. (z) : (x) . \phi(x, y, z) :.$

$[(*11\cdot01\cdot02)] \qquad \equiv :. (y, z, x) . \phi(x, y, z) :: \supset \vdash . \text{Prop}$

$*11\cdot22.\ \vdash : (\exists x, y) . \phi(x, y) . \equiv . \sim\{(x, y) . \sim\phi(x, y)\}$

Dem.

$\vdash . *10\cdot252 . \text{Transp} . (*11\cdot03) . \supset$

$\vdash : (\exists x, y) . \phi(x, y) . \qquad \equiv . \sim\{(x) : \sim(\exists y) . \phi(x, y)\} .$

$[*10\cdot252\cdot271] \qquad \equiv . \sim\{(x) : (y) . \sim\phi(x, y)\} .$

$[(*11\cdot01)] \qquad \equiv . \sim\{(x, y) . \sim\phi(x, y)\} : \supset \vdash . \text{Prop}$

$*11\cdot23.\ \vdash : (\exists x, y) . \phi(x, y) . \equiv . (\exists y, x) . \phi(x, y)$

Dem.

$\vdash . *11\cdot22 . \supset \vdash : (\exists x, y) . \phi(x, y) . \equiv . \sim\{(x, y) . \sim\phi(x, y)\} .$

$[*11\cdot2 . \text{Transp}] \qquad \equiv . \sim\{(y, x) . \sim\phi(x, y)\} .$

$[*11\cdot22] \qquad \equiv . (\exists y, x) . \phi(x, y) : \supset \vdash . \text{Prop}$

$*11\cdot24.\ \vdash : (\exists x, y, z) . \phi(x, y, z) . \equiv . (\exists y, z, x) . \phi(x, y, z)$

Dem.

$[(*11\cdot03\cdot04)] \vdash :: (\exists x, y, z) . \phi(x, y, z) . \equiv :. (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z) :.$

$[*11\cdot23] \qquad \equiv :. (\exists y) :. (\exists x) : (\exists z) . \phi(x, y, z) :.$

$[*11\cdot23 . *10\cdot281] \qquad \equiv :. (\exists y) :. (\exists z) : (\exists x) . \phi(x, y, z) :.$

$[(*11\cdot03\cdot04)] \qquad \equiv :. (\exists y, z, x) . \phi(x, y, z) :: \supset \vdash . \text{Prop}$

$*11\cdot25.\ \vdash : \sim\{(\exists x, y) . \phi(x, y)\} . \equiv . (x, y) . \sim\phi(x, y) \quad [*11\cdot22 . \text{Transp}]$

$*11\cdot26.\ \vdash :. (\exists x) : (y) . \phi(x, y) : \supset : (y) : (\exists x) . \phi(x, y)$

Dem.

$\vdash . *10\cdot1\cdot28 . \supset \vdash :. (\exists x) : (y) . \phi(x, y) : \supset : (\exists x) . \phi(x, y) \qquad (1)$

$\vdash . (1) . *10\cdot11\cdot21 . \supset \vdash . \text{Prop}$

Note that the converse of this proposition is false. *E.g.* let $\phi(x, y)$ be the propositional function "if y is a proper fraction, then x is a proper fraction greater than y." Then for all values of y we have $(\exists x) . \phi(x, y)$, so that $(y) : (\exists x) . \phi(x, y)$ is satisfied. In fact "$(y) : (\exists x) . \phi(x, y)$" expresses the proposition: "If y is a proper fraction, then there is always a proper fraction greater than y." But "$(\exists x) : (y) . \phi(x, y)$" expresses the proposition: "There is a proper fraction which is greater than any proper fraction," which is false.

$*11\cdot27.\ \vdash :. (\exists x, y) : (\exists z) . \phi(x, y, z) : \equiv : (\exists x) : (\exists y, z) . \phi(x, y, z) :$
$\equiv : (\exists x, y, z) . \phi(x, y, z)$

Dem.

$\vdash . *4\cdot2 . (*11\cdot03) . \supset$

$\vdash :: (\exists x, y) : (\exists z) . \phi(x, y, z) : \equiv :. (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z)$ (1)

$\vdash . *4\cdot2 . (*11\cdot03) . \supset$

$\vdash :. (\exists y) : (\exists z) . \phi(x, y, z) : \equiv : (\exists y, z) . \phi(x, y, z)$ (2)

$\vdash . (2) . *10\cdot11\cdot281 . \supset$

$\vdash :: (\exists x) :. (\exists y) : (\exists z) . \phi(x, y, z) :. \equiv :. (\exists x) : (\exists y, z) . \phi(x, y, z)$ (3)

$\vdash . (1) . (3) . (*11\cdot04) . \supset \vdash . \mathrm{Prop}$

All the propositions of $*10$ have analogues which hold for two or more variables. The more important of these are proved in what follows.

$*11\cdot3.$ $\vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x, y) : p . \supset . \phi(x, y)$

Dem.

$\vdash . *10\cdot21 . \supset \vdash :. p . \supset . (x, y) . \phi(x, y) : \equiv : (x) : p . \supset . (y) . \phi(x, y) :$
$\equiv : (x, y) : p . \supset . \phi(x, y) :. \supset \vdash . \mathrm{Prop}$

$*11\cdot31.$ $\vdash :. (x, y) . \phi(x, y) : (x, y) . \psi(x, y) : \equiv : (x, y) : \phi(x, y) . \psi(x, y)$

Here the conditions of significance on the right-hand side require that ϕ and ψ should take arguments of the same types.

Dem.

$\vdash . *10\cdot22 . \supset \vdash :: (x, y) . \phi(x, y) : (x, y) . \psi(x, y) :$
$\equiv :. (x) :. (y) . \phi(x, y) : (y) . \psi(x, y) :.$
$\equiv :. (x, y) : \phi(x, y) . \psi(x, y) :: \supset \vdash . \mathrm{Prop}$

$[*10\cdot22\cdot271]$

The proofs of most of the following propositions are conducted exactly as those of $*11\cdot3\cdot31$ are conducted: the analogous proposition in $*10$ is used twice, together with $*10\cdot27$ or $*10\cdot271$ or $*10\cdot28$ or $*10\cdot281$ as the case may be. When proofs conform to this pattern we shall merely give references to the propositions used.

$*11\cdot311.$ If $\phi(\hat{x}, \hat{y})$, $\psi(\hat{x}, \hat{y})$ take arguments of the same type, and we have "$\vdash . \phi(x, y)$" and "$\vdash . \psi(x, y)$," we shall have "$\vdash . \phi(x, y) . \psi(x, y)$." [Proof as in $*10\cdot13$.]

$*11\cdot32.$ $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \supset . (x, y) . \psi(x, y)$
$[*10\cdot27]$

$*11\cdot33.$ $\vdash :. (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset : (x, y) . \phi(x, y) . \equiv . (x, y) . \psi(x, y)$
$[*10\cdot271]$

$*11\cdot34.$ $\vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \supset :$
$(\exists x, y) . \phi(x, y) . \supset . (\exists x, y) . \psi(x, y)$ $[*10\cdot27\cdot28]$

$*11\cdot341.$ $\vdash :. (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset :$
$(\exists x, y) . \phi(x, y) . \equiv . (\exists x, y) . \psi(x, y)$ $[*10\cdot271\cdot281]$

$*11\cdot35.$ $\vdash :. (x, y) : \phi(x, y) . \supset . p : \equiv : (\exists x, y) . \phi(x, y) . \supset . p$ $[*10\cdot23\cdot271]$

$*11\cdot36.$ $\vdash : \phi(z, w) . \supset . (\exists x, y) . \phi(x, y)$

Dem.

$\vdash . *11\cdot1 . \supset \vdash : (x, y) . \sim\phi(x, y) . \supset . \sim\phi(z, w)$ (1)

$\vdash . (1) . \mathrm{Transp} . \supset \vdash . \mathrm{Prop}$

∗11·37. $\vdash :: (x, y) : \phi(x, y) . \supset . \psi(x, y) :. (x, y) : \psi(x, y) . \supset . \chi(x, y) :.$
$\supset : (x, y) : \phi(x, y) . \supset . \chi(x, y)$

Dem.

In the following demonstration, "Hp" means the hypothesis of the proposition to be proved. We shall employ this abbreviation, whenever convenient, in all cases where the proposition to be proved is a hypothetical, *i.e.* is of the form "$p \supset q$." Similarly "Hp (1)" will mean "the hypothesis of (1)," and so on.

$$\vdash . *11\cdot31 . \supset \vdash :: \mathrm{Hp} . \supset :. (x, y) :. \phi(x, y) . \supset . \psi(x, y) : \psi(x, y) . \supset . \chi(x, y) \quad (1)$$

$$\vdash . \mathrm{Syll} . *11\cdot11 . \supset \vdash :. (x, y) :. \phi(x, y) . \supset . \psi(x, y) : \psi(x, y) . \supset . \chi(x, y) : \supset : \phi(x, y) . \supset . \chi(x, y) :.$$

$$[*11\cdot32] \quad \supset \vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \psi(x, y) . \supset . \chi(x, y) : \supset : (x, y) : \phi(x, y) . \supset . \chi(x, y) \quad (2)$$

$$\vdash . (1) . (2) . \mathrm{Syll} . \supset \vdash . \mathrm{Prop}$$

The above is a type of proof which recurs frequently in what follows. Proofs conforming to this pattern will be indicated only by the numbers of the propositions used.

∗11·371. $\vdash :: (x, y) : \phi(x, y) . \equiv . \psi(x, y) :. (x, y) : \psi(x, y) . \equiv . \chi(x, y) :.$
$\supset :. (x, y) : \phi(x, y) . \equiv . \chi(x, y)$ [∗11·31·11·33]

∗11·38. $\vdash :: (x, y) : \phi(x, y) . \supset . \psi(x, y) :. \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \supset . \psi(x, y) . \chi(x, y)$ [Fact . ∗11·11·32]

∗11·39. $\vdash :: (x, y) : \phi(x, y) . \supset . \psi(x, y) :. (x, y) : \chi(x, y) . \supset . \theta(x, y) :. \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \supset . \psi(x, y) . \theta(x, y)$ [∗3·47 . ∗11·11·32]

∗11·391. $\vdash :: (x, y) : \phi(x, y) . \supset . \psi(x, y) :. (x, y) : \phi(x, y) . \supset . \chi(x, y) :.$
$\equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y) . \chi(x, y)$

Dem.

$$\vdash . *4\cdot76 . \quad \supset \vdash :. \phi(x, y) . \supset . \psi(x, y) : \phi(x, y) . \supset . \chi(x, y) : \equiv : \phi(x, y) . \supset . \psi(x, y) . \chi(x, y) :.$$

$$[*11\cdot11\cdot33] \quad \supset \vdash :. (x, y) : \phi(x, y) . \supset . \psi(x, y) : \phi(x, y) . \supset . \chi(x, y) : \equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y) . \chi(x, y) ::$$

$$[*11\cdot31] \quad \supset \vdash :: (x, y) : \phi(x, y) . \supset . \psi(x, y) :. (x, y) : \phi(x, y) . \supset . \chi(x, y) :. \equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y) . \chi(x, y) ::$$

$$\supset \vdash . \mathrm{Prop}$$

∗11·4. $\vdash :: (x, y) : \phi(x, y) . \equiv . \psi(x, y) :. (x, y) : \chi(x, y) . \equiv . \theta(x, y) :. \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \equiv . \psi(x, y) . \theta(x, y)$

Dem.

$$\vdash . *11\cdot31 . \supset \vdash :: \mathrm{Hp} . \supset :. (x, y) :. \phi(x, y) . \equiv . \psi(x, y) : \chi(x, y) . \equiv . \theta(x, y) :.$$

$$[*4\cdot38 . *11\cdot11\cdot32] \quad \supset :. (x, y) : \phi(x, y) . \chi(x, y) . \equiv . \psi(x, y) . \theta(x, y) ::$$

$$\supset \vdash . \mathrm{Prop}$$

*11·401. $\vdash :: (x, y) : \phi(x, y) . \equiv . \psi(x, y) : \supset :.$
$(x, y) : \phi(x, y) . \chi(x, y) . \equiv . \psi(x, y) . \chi(x, y)$ $\left[*11{\cdot}4 \frac{\chi}{\theta} . \mathrm{Id}\right]$

*11·41. $\vdash :. (\exists x, y) . \phi(x, y) : \lor : (\exists x, y) . \psi(x, y) :$
$\equiv : (\exists x, y) : \phi(x, y) . \lor . \psi(x, y)$ [*10·42·281]

*11·42. $\vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \supset : (\exists x, y) . \phi(x, y) : (\exists x, y) . \psi(x, y)$ [*10·5]

*11·421. $\vdash :. (x, y) . \phi(x, y) . \lor . (x, y) . \psi(x, y) : \supset : (x, y) : \phi(x, y) . \lor . \psi(x, y)$
$\left[*11{\cdot}42 \frac{\sim\phi, \sim\psi}{\phi, \quad \psi} . \mathrm{Transp} . *4{\cdot}56\right]$

*11·43. $\vdash :. (\exists x, y) : \phi(x, y) . \supset . p : \equiv : (x, y) . \phi(x, y) . \supset . p$ [*10·34·281]

*11·44. $\vdash :. (x, y) : \phi(x, y) . \lor . p : \equiv : (x, y) . \phi(x, y) . \lor . p$ [*10·2·271]

*11·45. $\vdash :. (\exists x, y) : p . \phi(x, y) : \equiv : p : (\exists x, y) . \phi(x, y)$ [*10·35·281]

*11·46. $\vdash :. (\exists x, y) : p . \supset . \phi(x, y) : \equiv : p . \supset . (\exists x, y) . \phi(x, y)$ [*10·37·281]

*11·47. $\vdash :. (x, y) : p . \phi(x, y) : \equiv : p : (x, y) . \phi(x, y)$ [*10·33·271]

*11·5. $\vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x, y) . \phi(x, y)\} : \equiv : (\exists x, y) . \sim\phi(x, y)$

Dem.

$\vdash . *10{\cdot}253 . \supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : \sim\{(x) : (y) . \phi(x, y)\} :$
[(*11·01)] $\equiv : \sim\{(x, y) . \phi(x, y)\}$ (1)
$\vdash . *10{\cdot}253 . \supset \vdash : \sim\{(y) . \phi(x, y)\} . \equiv . (\exists y) . \sim\phi(x, y) :$
[*10·11·281] $\supset \vdash :. (\exists x) : \sim\{(y) . \phi(x, y)\} : \equiv : (\exists x) : (\exists y) . \sim\phi(x, y) :$
[(*11·03)] $\equiv : (\exists x, y) . \sim\phi(x, y)$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

*11·51. $\vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\}$

Dem.

$\vdash . *10{\cdot}252 . \mathrm{Transp} . \supset \vdash :. (\exists x) : (y) . \phi(x, y) : \equiv : \sim[(x) : \sim(y) . \phi(x, y)]$ (1)
$\vdash . *10{\cdot}253 . \supset \vdash :. \sim(y) . \phi(x, y) . \equiv : (\exists y) . \sim\phi(x, y) :.$
[*10·11·271] $\supset \vdash :. (x) : \sim(y) . \phi(x, y) : \equiv : (x) : (\exists y) . \sim\phi(x, y) :.$
[Transp] $\supset \vdash :. \sim[(x) : \sim\{(y) . \phi(x, y)\}] . \equiv : \sim\{(x) : (\exists y) . \sim\phi(x, y)\}$ (2)
$\vdash . (1) . (2) . \supset \vdash .$ Prop

*11·52. $\vdash :. (\exists x, y) . \phi(x, y) . \psi(x, y) . \equiv . \sim\{(x, y) : \phi(x, y) . \supset . \sim\psi(x, y)\}$

Dem.

$\vdash . *4{\cdot}51{\cdot}62 . \supset$
$\vdash :. \sim\{\phi(x, y) . \psi(x, y)\} . \equiv : \phi(x, y) . \supset . \sim\psi(x, y)$ (1)
$\vdash . (1) . *11{\cdot}11{\cdot}33 . \supset$
$\vdash :. (x, y) . \sim\{\phi(x, y) . \psi(x, y)\} : \equiv : (x, y) : \phi(x, y) . \supset . \sim\psi(x, y)$ (2)
$\vdash : (2) . \mathrm{Transp} . *11{\cdot}22 . \supset \vdash .$ Prop

*11·521. $\vdash :. \sim(\exists x, y) . \phi(x, y) . \sim\psi(x, y) . \equiv : (x, y) : \phi(x, y) . \supset . \psi(x, y)$
$\left[*11{\cdot}52 . \mathrm{Transp} . \frac{\sim\psi(x, y)}{\psi(x, y)}\right]$

$*11{\cdot}53.\quad \vdash :. (x, y) . \phi x \supset \psi y . \equiv : (\exists x) . \phi x . \supset . (y) . \psi y$

Dem.

$\vdash . *10{\cdot}21{\cdot}271 . \supset \vdash :. (x, y) . \phi x \supset \psi y . \equiv : (x) : \phi x . \supset . (y) . \psi y :$

$[*10{\cdot}23] \qquad \equiv : (\exists x) . \phi x . \supset . (y) . \psi y :. \supset \vdash . \text{Prop}$

$*11{\cdot}54.\quad \vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) . \phi x : (\exists y) . \psi y$

Dem.

$\vdash . *10{\cdot}35 . \supset \vdash :. (\exists y) . \phi x . \psi y . \quad \equiv : \phi x : (\exists y) . \psi y :.$

$[*10{\cdot}11{\cdot}281] \supset \vdash :. (\exists x, y) . \phi x . \psi y . \equiv : (\exists x) : \phi x : (\exists y) . \psi y :$

$[*10{\cdot}35] \qquad \equiv : (\exists x) . \phi x : (\exists y) . \psi y :. \supset \vdash . \text{Prop}$

This proposition is very often used.

$*11{\cdot}55.\quad \vdash :. (\exists x, y) . \phi x . \psi (x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi (x, y)$

Dem.

$\vdash . *10{\cdot}35 . \supset \vdash :. (\exists y) . \phi x . \psi (x, y) . \quad \equiv : \phi x : (\exists y) . \psi (x, y) :.$

$[*10{\cdot}11] \quad \supset \vdash :. (x) :. (\exists y) . \phi x . \psi (x, y) . \equiv : \phi x : (\exists y) . \psi (x, y) :.$

$[*10{\cdot}281] \supset \vdash :. (\exists x) : (\exists y) . \phi x . \psi (x, y) . \equiv : (\exists x) : \phi x : (\exists y) . \psi (x, y) :. \supset \vdash . \text{Prop}$

This proposition is very often used.

$*11{\cdot}56.\quad \vdash :. (x) . \phi x : (y) . \psi y : \equiv : (x, y) . \phi x . \psi y$

Dem.

$\vdash . *10{\cdot}33 . \supset \vdash :: (x) . \phi x : (y) . \psi y : \ \equiv :. (x) :. \phi x : (y) . \psi y \qquad (1)$

$\vdash . *10{\cdot}33 . \supset \vdash :. \qquad \phi x : (y) . \psi y : \ \equiv : (y) . \phi x . \psi y :.$

$[*10{\cdot}11] \quad \supset \vdash :. (x) :. \phi x : (y) . \psi y : \ \equiv : (y) . \phi x . \psi y :.$

$[*10{\cdot}271] \quad \supset \vdash :: (x) :. \phi x : (y) . \psi y :. \equiv : (x) : (y) . \phi x . \psi y :$

$[(*11{\cdot}01)] \qquad \equiv : (x, y) . \phi x . \psi y \qquad (2)$

$\vdash . (1) . (2) . \supset \vdash . \text{Prop}$

$*11{\cdot}57.\quad \vdash : (x) . \phi x . \equiv . (x, y) . \phi x . \phi y \quad [*11{\cdot}56 . *4{\cdot}24]$

The use of $*4{\cdot}24$ here depends upon the fact that $(x) . \phi x$ and $(y) . \phi y$ are the same proposition.

$*11{\cdot}58.\quad \vdash : (\exists x) . \phi x . \equiv . (\exists x, y) . \phi x . \phi y \quad [*11{\cdot}54 . *4{\cdot}24]$

$*11{\cdot}59.\quad \vdash :. \phi x . \supset_x . \psi x : \equiv : \phi x . \phi y . \supset_{x,y} . \psi x . \psi y$

Dem.

$\vdash . *11{\cdot}57 . \supset \vdash :. \phi x . \supset_x . \psi x : \equiv : (x, y) : \phi x . \supset . \psi x : \phi y . \supset . \psi y :$

$[*3{\cdot}47 . *11{\cdot}32] \qquad \supset : (x, y) : \phi x . \phi y . \supset . \psi x . \psi y \qquad (1)$

$\vdash . *11{\cdot}1 . \supset \vdash :. (x, y) : \phi x . \phi y . \supset . \psi x . \psi y : \supset : \phi x . \phi y . \supset . \psi x . \psi y \qquad (2)$

$\vdash . (2) \frac{x}{y} . *4{\cdot}24 . \supset \vdash :. \text{Hp} (2) . \supset : \phi x . \supset . \psi x \qquad (3)$

$\vdash . (3) . *10{\cdot}11{\cdot}21 . \supset$

$\qquad \vdash :. (x, y) : \phi x . \phi y . \supset . \psi x . \psi y : \supset : \phi x . \supset_x . \psi x \qquad (4)$

$\vdash . (1) . (4) . \supset \vdash . \text{Prop}$

$*11\cdot6.$ $\vdash :: (\exists x) :. (\exists y) . \phi(x,y) . \psi y : \chi x :. \equiv :. (\exists y) :. (\exists x) . \phi(x,y) . \chi x : \psi y$

This proposition is very frequently employed in subsequent proofs.

Dem.

$\vdash . *10\cdot35 . \supset \vdash :. (\exists y) . \phi(x,y) . \psi y : \chi x : \equiv : (\exists y) : \phi(x,y) . \psi y . \chi x :.$

$[*10\cdot11\cdot281] \supset \vdash :: (\exists x) :. (\exists y) . \phi(x,y) . \psi y : \chi x :$

$\equiv :. (\exists x) :. (\exists y) . \phi(x,y) . \psi y . \chi x :.$

$[*11\cdot23]$ $\equiv :. (\exists y) :. (\exists x) . \phi(x,y) . \psi y . \chi x :.$

$[*11\cdot341.\mathrm{Perm}]$ $\equiv :. (\exists y) :. (\exists x) . \phi(x,y) . \chi x . \psi y :.$

$[*10\cdot35\cdot281]$ $\equiv :. (\exists y) :. (\exists x) . \phi(x,y) . \chi x : \psi y :: \supset \vdash . \mathrm{Prop}$

$*11\cdot61.$ $\vdash :. (\exists y) : \phi x . \supset_x . \psi(x,y) : \supset : \phi x . \supset_x . (\exists y) . \psi(x,y)$

Dem.

$\vdash . *11\cdot26 . \supset \vdash :: \mathrm{Hp} . \supset :. (x) :. (\exists y) : \phi x . \supset . \psi(x,y)$ (1)

$\vdash . *10\cdot37 . \supset \vdash :. (\exists y) : \phi x . \supset . \psi(x,y) : \supset : \phi x . \supset . (\exists y) . \psi(x,y) :.$

$[*10\cdot11\cdot27] \supset \vdash :: (x) :. (\exists y) : \phi x . \supset . \psi(x,y) :. \supset :. (x) : \phi x . \supset . (\exists y) . \psi(x,y)$ (2)

$\vdash . (1) . (2) . \supset \vdash . \mathrm{Prop}$

$*11\cdot62.$ $\vdash :: \phi x . \psi(x,y) . \supset_{x,y} . \chi(x,y) : \equiv :. \phi x . \supset_x : \psi(x,y) . \supset_y . \chi(x,y)$

Dem.

$\vdash . *4\cdot87 . *11\cdot11\cdot33 . \supset$

$\vdash :: \phi x . \psi(x,y) . \supset_{x,y} . \chi(x,y) : \equiv :. (x,y) :. \phi x . \supset : \psi(x,y) . \supset . \chi(x,y) ::$

$[*10\cdot21\cdot11\cdot271]$ $\equiv :. (x) :. \phi x . \supset : (y) : \psi(x,y) . \supset . \chi(x,y) ::$ $\supset \vdash . \mathrm{Prop}$

$*11\cdot63.$ $\vdash :. \sim(\exists x, y) . \phi(x,y) . \supset : \phi(x,y) . \supset_{x,y} . \psi(x,y)$

Dem.

$\vdash . *2\cdot21 . *11\cdot11 . \supset \vdash :. (x,y) :. \sim\phi(x,y) . \supset : \phi(x,y) . \supset . \psi(x,y) :.$

$[*11\cdot32]$ $\supset \vdash :. (x,y) . \sim\phi(x,y) . \supset : (x,y) : \phi(x,y) . \supset . \psi(x,y) :.$

$[*11\cdot25]$ $\supset \vdash :. \sim(\exists x, y) . \phi(x,y) . \supset : (x,y) : \phi(x,y) . \supset . \psi(x,y) :.$ $\supset \vdash . \mathrm{Prop}$

$*11\cdot7.$ $\vdash :. (\exists x, y) : \phi(x,y) . \lor . \phi(y,x) : \equiv . (\exists x, y) . \phi(x,y)$

Dem.

$\vdash . *11\cdot41 . \supset \vdash :. (\exists x, y) : \phi(x,y) . \lor . \phi(y,x) :$

$\equiv : (\exists x, y) . \phi(x,y) . \lor . (\exists x, y) . \phi(y,x) :$

$[*11\cdot23]$ $\equiv : (\exists x, y) . \phi(x,y) . \lor . (\exists y, x) . \phi(y,x) :$

$[*4\cdot25]$ $\equiv : (\exists x, y) . \phi(x,y) :. \supset \vdash . \mathrm{Prop}$

In the last line of the above proof, use is made of the fact that

$$(\exists x, y) . \phi(x,y) \text{ and } (\exists y, x) . \phi(y,x)$$

are the same proposition.

The first use of the following proposition occurs in the proof of $*234\cdot12$. Its utility lies in its enabling us to pass from a hypothesis

$$\phi z . \chi w . \supset_{z,w} . \psi z . \theta w,$$

containing two apparent variables, to the product of two hypotheses each containing only one.

$$*11\cdot71.\quad \vdash :: (\exists z) . \phi z : (\exists w) . \chi w : \supset :. \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w : \equiv : \phi z . \chi w . \supset_{z,w} . \psi z . \theta w$$

Dem.

$$\vdash . *10\cdot1 . *3\cdot47 . \supset \vdash :. \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w : \supset : \phi z . \chi w . \supset . \psi z . \theta w \quad (1)$$

$$\vdash . (1) . *11\cdot11\cdot3 . \supset \vdash :. \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w : \supset : \phi z . \chi w . \supset_{z,w} . \psi z . \theta w \quad (2)$$

$$\vdash . *10\cdot1 . \supset \vdash :: \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset :. \phi z . \chi w . \supset_w . \psi z . \theta w :.$$

$$[*10\cdot28] \quad \supset :. (\exists w) . \phi z . \chi w . \supset . (\exists w) . \psi z . \theta w :.$$

$$[*10\cdot35] \quad \supset :. \phi z : (\exists w) . \chi w : \supset : \psi z : (\exists w) . \theta w \quad (3)$$

$$\vdash . (3) . \mathrm{Comm} . *3\cdot26 . \supset \vdash :: (\exists w) . \chi w : \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset : \phi z . \supset . \psi z \quad (4)$$

$$\vdash . (4) . *10\cdot11\cdot21 . \supset \vdash :: (\exists w) . \chi w . \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset : \phi z . \supset_z . \psi z \quad (5)$$

Similarly

$$\vdash :: (\exists z) . \phi z . \supset :. \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset : \chi w . \supset_w . \theta w \quad (6)$$

$$\vdash . (5) . (6) . *3\cdot47 . \mathrm{Comp} . \supset$$

$$\vdash :: \mathrm{Hp} . \supset \therefore \phi z . \chi w . \supset_{z,w} . \psi z . \theta w : \supset : \phi z . \supset_z . \psi z : \chi w . \supset_w . \theta w \quad (7)$$

$$\vdash . (2) . (7) . \supset \vdash . \mathrm{Prop}$$