The Riemann integral

Randall Holmes

August 31, 2023

Let $a < b$ be real numbers.

We define $[a, b]$ as the set $\{x \in \mathbb{R} : a \leq x \leq b\}$, the closed interval from a to b.

A partition P is a finite sequence $\{x_i\}$ of elements of $[a, b]$ which is strictly increasing ($x_i < x_{i+1}$ where both are defined) and has $x_0 = a$ and $x_n = b$. The terminology here is strange because if P is the sequence, why do we write x_i instead of P_i?

I will write $\mathbb{P}[a, b]$ for the set of all partitions of $[a, b]$.

The partition determines a subdivision of $[a, b]$ into intervals, the subdivision being the set $\{[x_{i-1}, x_i] : i \in \mathbb{Z} \wedge 0 < i \leq n\}$. The union of this set is $[a, b]$ but it is not a partition of $[a, b]$ in the usual sense because the sets into which $[a, b]$ is subdvided are not pairwise disjoint (they share endpoints).

We define notation for special infima and suprema involving functions. Where f is a bounded function on a set A, we define $\mathtt{sup}_A f$ as the least upper bound of $\{f(x) : x \in A\}$ and define $\mathtt{inf}_A f$ as the greates lower bound of $\{f(x) : x \in A\}$.

We can now define the upper and lower sums for a bounded function f on $[a, b]$ determined by a partition P. He writes $L(f, P, [a, b])$ and $U(f, p, [a, b])$. I will just write $L(f, P)$ and $U(f, P)$.

We define $L(f, P)$ as $\Sigma_{i=1}^{n}(x_i - x_{i-1})\mathtt{inf}_{[x_{i-1}, x_i]} f$.

We define $U(f, P)$ as $\Sigma_{i=1}^{n}(x_i - x_{i-1})\mathtt{sup}_{[x_{i-1}, x_i]} f$.

We prove three inequalities involving these notions.

Let P' be a partition $\{y_i\}$ of $[a, b]$ such that the range of P' includes the range of P (that is, for every i such that $0 \leq i \leq n$, there is a j such that $y_j = x_i$). In particular, there is m such that $y_m = b$.

The first inequality is $L(f, P) \leq U(f, P)$, the easiest, I think.

$$L(f,P) = \Sigma_{i=1}^{n}(x_i - x_{i-1})\texttt{inf}_{[x_{i-1},x_i]}f$$

$$\leq \Sigma_{i=1}^{n}(x_i - x_{i-1})\texttt{sup}_{[x_{i-1},x_i]}f$$

simply because for each i, $\texttt{inf}_{[x_{i-1},x_i]}f \leq \texttt{sup}_{[x_{i-1},x_i]}f$, so term by term elements of the first finite series are less than elements of the second

$$= U(f,P)$$

.

The second is $L(f,P) \leq L(f,P')$: making partitions finer will leave the value of a lower sum fixed or increase it. Let $k(i)$ be defined by $x_i = y_{k(i)}$.

$$L(f,P) = \Sigma_{i=1}^{n}(x_i - x_{i-1})\texttt{inf}_{[x_{i-1},x_i]}f$$

$$= \Sigma_{i=1}^{n}(\Sigma_{j=k(i-1)+1}^{k(i)}(y_j - y_{j-1})\texttt{inf}_{[x_{i-1},x_i]}f)$$

which changes the subdivision to the one determined by P' without changing the function values

$$\leq \Sigma_{i=1}^{n}(\Sigma_{j=k(i-1)+1}^{k(i)}(y_j - y_{j-1})\texttt{inf}_{[y_{j-1},y_j]}f)$$

because the infimum of the function on a subinterval $[y_j-1, y_j]$ of $[x_{i-1}, x_i]$ will be greater than or equal to the infimum of the function on the whole interval: it is a general fact that the infimum of a function on a set $B \subseteq A$ is $\geq$ the infimum of the same function on A.

$$= \Sigma_{j=1}^{m}(y_j - y_{j-1})\texttt{inf}_{[y_{i-1},y_j]}f = L(f,P')$$

basically by the associative law of addition: it is exactly the same sum with the subgrouping removed.

The proof of the third inequality $U(f,P') \leq U(f,P)$ [making a partition finer might fix an upper sum but will usually decrease it] is done by analogy with the previous proof: prove the equivalent $U(f,P) \geq U(f,P')$ in the same way that the previous inequality was proved, but replacing $\leq$ in the proof above with $\geq$ and inf with sup, and use the fact that the supremum of a function on $B \subseteq A$ is less than or equal to the supremum of the same function on A.

Don't forget that supremum and infimum are just fancy Latin for "least upper bound" and "greatest lower bound".

Now we have the theorem that for any bounded function f on $[a, b]$ and partitions P, Q we have $L(f, P) \leq U(f, Q)$.

We prove this. Let V be the partition whose range is the union of the range of P and the range of Q. Note that for purposes of the inequalities above, V is both a P' and a Q': it is finer than both P and Q (which have no relationship to each other assumed).

The inequality theorems above then establish that $L(f, P) \leq L(f, V) \leq U(f, V) \leq U(f, Q)$.

A corollary of this is that the set of lower sums $\{L(f, P) : P \in \mathbb{P}[a, b]\}$ has an upper bound (any $U(f, Q)$ is an upper bound) and so has a least upper bound which we will call $L(f, [a, b])$, and similarly the set of upper sums $\{U(f, P) : P \in \mathbb{P}[a, b]\}$ has a lower bound (any $L(f, Q)$ is a lower bound) and so has a greatest lower bound which we will call $U(f, [a, b])$.

We can then define the Riemann integral $\int_a^b f$ as equal to $L(f, [a, b])$ and to $U(f, [a, b])$ if these numbers are equal to each other, and otherwise undefined.