$$\frac{\begin{matrix} A \\ A \to B \end{matrix}}{B}$$ modus ponens

$(A \to B \wedge C) \longleftrightarrow (A \to B) \wedge (A \to C)$

Part I Assume ① $A \to (B \wedge C)$

Goal: $A \to B \wedge A \wedge C$

Goal: $A \to B$

Assume ② A

Goal B

(you get $B \wedge C$ by modus ponens, and you can extract C by simp)

① $A \to B$

Goal: $A \to C$

Similarly

③ ??

$A \to C$

④ $(A \to B) \wedge (A \to C)$ conj ?, ??

⑤ 2A

Part II Assume: (9) $(A \to B) \land (A \to C)$

Goal: $A \to (B \land C)$

Assume (10) A

Goal: $B \land C$

(11) $A \to B$ simp 9

(12) $A \to C$ simp 9

(13) B m.p. 10, 11

(x) $A \to (B \land C$

The theorem: biconditional introduction 1-v, 9-x

$$\frac{A \land B}{A} \qquad \frac{A \land B}{B}$$

rule of simplification

Contrapositive Theorem

Prove $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$

Part $\underline{I}$: Assume (1) $P \to Q$

Goal: $\neg Q \to \neg P$

- Assume (2) $\neg Q$
  - Goal: $\neg P$
    - Assume (3) P
      - Goal: $\perp$
      - (4) Q m.p. 3, 1
      - (5) $\perp$ contradiction 2, 4
  - (6) $\neg P$ negation introduction 3-5

(7) $\neg Q \to \neg P$ ded 2-6 finished Part $\underline{I}$!

1. $P \to Q$
(2) $\neg Q$
3 P
4 Q

Part II: Assume (8) $\neg Q \to \neg P$

Goal: $P \to Q$

Assume (9) P

Goal: Q

Assume (10) $\neg Q$ for reductio

Goal: $\bot$

(11) $\neg P$ m.p. 10, 8

(12) $\bot$ contra 9, 11

(13) Q reductio ad absurdum 10-12

(14) $P \to Q$ ded. 9-13

(15) $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ biconditional introduction 1-7, 8-14

8. $\neg Q \to \neg P$

9. P

10 $\neg Q$

11 $\neg P$

The contrapositive theorem justifies new rules!

Modus Tollens

$$\frac{\begin{array}{c} P \to Q \\ \neg Q \end{array}}{\neg P}$$

$$\cancel{\frac{\begin{array}{c} P \to Q \\ Q \to \bot \end{array}}{P \to \bot}}$$

(1) $P \to Q$ premise

(2) $\neg Q$ premise

Goal: $\neg P$

(3) $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ contrapositive theorem

(4) $\neg Q \to \neg P$ b.i.m.p. 1, 3

(1) $P \to Q$

(2) $\neg Q$

(3) $\neg Q \to \neg P$

(4) $\neg P$

⑤ $\neg P$ m.p. 2,4

$$\frac{\begin{array}{l} P \to Q \\ \neg Q \end{array}}{\neg P}$$

The strategy of indirect proof

Goal: Prove $P \to Q$

| Assume ? $\neg Q$

| Goal: $\neg P$

| ⋮

| ?? $\neg P$

$P \to Q$ indirect proof ? – ??

| Assume (m) $\neg Q$

| Goal: $\neg P$

| ⋮

| (n) $\neg P$

(n+1) $\neg Q \to \neg P$ ded ~~?–??~~ m–n

(n+2) ~~$\neg Q \to$~~ $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ Contrapositive theorem

(n+1) (n+3) $P \to Q$ bmp n+1, n+2 indirect proof m–n

Rules combining Disjunction [or] and Negation [not]

$$\frac{P \vee Q \qquad \neg P}{Q} \qquad\qquad \frac{P \vee Q \qquad \neg Q}{P}$$

disjunctive syllogism.

$$\frac{\neg P \vee Q \qquad P}{Q} \qquad\qquad \frac{P \vee \neg Q \qquad Q}{P}$$

(1) $P \vee Q$ premise

(2) $\neg Q$ premise

Goal: P

Prove by cases on (1)

Case I Assume 1a P

Goal: P

2a P copy line 1a

Case II Assume 1b Q

Goal: P

2b $\perp$ 2, 1b contradiction

$P \vee Q$

$\neg Q$

Q

(3b) P ex falso 2b

(4) P proof by cases 1, 1a–2a, 1b–3b

Strategy for Proving a Disjunction

$A \vee B$ is equivalent to $(\neg A) \to B$

It is also equivalent to $(\neg B) \to A$

Goal: $A \vee B$

(1) Assume $\neg A$

Goal: B

⋮

(n) B

(n+1) $A \vee B$ alternative elimination 1–n

Goal: $A \vee B$

Assume $\neg B$

Goal: A

⋮

A

$A \vee B$

Example:

Factor Theorem If $ab=0$ then $a=0$ or $b=0$.

Suppose $ab=0$

Suppose $a \neq 0$.

Then $a \cdot a^{-1} = 1$.

so $b = 1b = a \cdot a^{-1} \cdot b = a \cdot b \cdot a^{-1} = 0$.

Proof $ab=0 \to (a=0 \vee b=0)$

Assume $ab=0$

Goal: $a=0 \vee b=0$

Assume $a \neq 0$.

Goal: $b=0$

$b = 1b = a^{-1}ab = a^{-1}0 = 0$ ✓

$a=0 \vee b=0$ a.e. by the block above.

Excluded Middle

$A \vee \neg A$

Goal: $A \vee \neg A$

Assume [1] $\neg A$

Goal: $\neg A$

[2] $\neg A$ copy 1

$A \vee \neg A$ a.e. 1-2

Goal: $A \vee \neg A$

Assume [1] $\neg\neg A$

Goal: A

[2] A d.ne. 1

$A \vee \neg A$ a.e. 1-2

Try to prove Excluded Middle directly.

Goal: $A \vee \neg A$

Assume ① $\neg(A \vee \neg A)$ Goal:

Goal: $\bot$

~~Try to prove $A \vee \neg A$ in order to get $\bot$~~

Try to prove A (from A, $A \vee \neg A$ follows and I get $\bot$)

Assume ② $\neg A$

Goal: $\bot$

③ $A \vee \neg A$ addition line 2

④ $\bot$ 1,3 contradiction

⑤ A ~~neg~~ reductio 2-4

⑥ $A \vee \neg A$ addition 5

⑦ $\bot$ 1,6 contradiction

⑧ $A \vee \neg A$ reductio 1-7