Dr Holmes's notes on the Ruler Postulate

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These are some notes on the axiomatics in Venema. Venema has a wonderful book, but some things I think benefit from a different approach.

Primitive notions that we start with. There are *points*. There are *lines*, which are sets of points (a line is a set of points, but of course not all sets of points are lines). We will refer to the set of all points as the plane.

We have a primitive d, the *distance function*, which sends any pair of points to a real number. Note that the distance from point P to point Q, which we write $d(P, Q)$, is written just as PQ in Venema, which we find potentially confusing and so avoid.

More primitive notions will be introduced later.

The first two axioms are

Existence Postulate: There are at least two distinct points.

Incidence Postulate: For each pair of distinct points A, B there is exactly one line L such that $A \in L$ and $B \in L$: we denote this line by $\overleftrightarrow{AB}$.

The next axiom is best preceded by a definition.

Definition (coordinate function): Let L be a line. A *coordinate function for* L is defined as a function from L to the set $\mathbb{R}$ of real numbers which is

1. one-to-one (for any points P, Q, $f(P) = f(Q) \to P = Q$),
2. onto (for any real number r, there is a point $P \in L$ such that $f(P) = r$; we can also write $P = f^{-1}(r)$),
3. and distance preserving: for any points $P, Q \in L$, $d(P, Q) = |f(P) - f(Q)|$.

Ruler Postulate: For any line L, there is a coordinate function for L.

Semi-Metric Theorem: For any points A, B, $d(A,B) \geq 0$, $d(A,B) = d(B,A)$, and $d(A,B) = 0$ iff $A = B$.

Proof of Semi-Metric Theorem: Let A, B be points. Define a point P as B, in case B is distinct from A, and otherwise as some point distinct from A (there is such a point by the Existence Postulate).

Let L be the line $\overleftrightarrow{AP}$. Note that both A and B are on L, because B is either A or P. Note the use of the Incidence Postulate.

Let f be a coordinate function for L.

$d(A,B) = |f(A) - f(B)| = |f(B) - f(A)| = d(B,A)$.

$d(A,B) = |f(A) - f(B)| \geq 0$. Further, $|f(A) - f(B)| = 0$ if and only if $f(A) = f(B)$, and in turn this is true if and only if $A = B$, because f is one-to-one.

As we note briefly under one of the headings, a coordinate function f for L has an inverse f^{-1} such that for each real number r, $f^{-1}(r)$ is the unique point P on L such that $f(P) = r$.

The use of the Ruler Postulate depends on some facility with the notion of absolute value.

A line does not have a uniquely determined coordinate function. We give a complete account of what coordinate functions a line has.

Theorem: Let L be a line and let f be a coordinate function for L. For any real number c and any $\sigma = \pm 1$, the function g from L to $\mathbb{R}$ defined by $g(P) = \sigma f(P) + c$ is a coordinate function.

Proof: g is one-to-one: suppose $g(P) = g(Q)$. It follows by definition of g that $\sigma f(P) + c = \sigma f(Q) + c$, from which it follows by algebra that $f(P) = f(Q)$ from which it follows by the fact that f is a coordinate function and so one-to-one that $P = Q$, so we have shown that g is one-to-one.

g is onto: Let r be a real number. We want to find a point P on L such that $g(P) = r$. That is, we want to find P such that $\sigma f(P) + c = r$, for which we need $f(P) = \sigma(r - c)$. So let $P = f^{-1}(\sigma(r - c))$.

$g(P) = g(f^{-1}(\sigma(r-c))) = \sigma(f(f^{-1}(\sigma(r-c)))) + c = \sigma(\sigma(r-c)) + c = (r-c) + c = r$. Note the use of the fact that $\sigma = \pm 1$, so $\sigma^2 = 1$.

g is distance preserving: $|g(P) - g(Q)| = |(\sigma f(P) + c) - \sigma f(Q) + c| = |\sigma(f(P) - f(Q))| = |\sigma||f(P) - f(Q)| = |f(P) - f(Q)| = d(P, Q)$. Notice the use of the fact that $|\sigma| = 1$ and the fact that f is a coordinate function and so distance preserving.

Observation about absolute values: For any real number x, there is $\sigma = \pm 1$ such that $|x| = \sigma x$, and for any τ, if $\tau = \pm 1$ and $\tau x \geq 0$, $\tau x = |x|$.

Lemma: If r, s, x, y are real numbers, $r \neq s$, and $|x - r| = |y - r|$ and $|x - s| = |y - s|$ then $x = y$. In a geometric manner, we can say that if r and s are distinct real numbers, and x and y have the same distances from r and s respectively, then $x = y$: if we know the distance of a real number from both r and s, we have exactly determined that number.

Proof: Let $r \neq s$. Let $|x - r| = |y - r| = d_1$ and let $|x - s| = |y - s| = d_2$.

For any z, T and d, $|z - t| = d$ implies that there is $\sigma = \pm 1$ such that $z = t + \sigma d$.

It follows from this that there is $\sigma_1 = \pm 1$ such that $x = r + \sigma_1 d_1$, and if $y \neq x$, it follows that $y = r - \sigma_1 d_1$. Similarly, there is $\sigma_2 = \pm 1$ such that $x = s + \sigma_2 d_2$ and if $y \neq x$, it follows that $y = s - \sigma_2 d_2$.

It then follows that $x + y = (r + \sigma_1 d_1) + (r - \sigma_1 d_1) = 2r$ and $x + y = (s + \sigma_2 d_2) + (s - \sigma_2 d_2) = 2s$, so $2r = 2s$, so $r = s$, which is a contradiction, so our assumption that $y \neq x$ is shown to be false.

I enjoy the elimination of case analysis by the use of variables equal to 1 or -1 in this presentation.

Corollary: If f and g are coordinate functions for the same line L, and $P \neq Q$ are distinct points on L, and $f(P) = g(P)$ and $f(Q) = g(Q)$, we have $f = g$. Coordinate functions need only agree at two distinct points to be known to be equal.

Proof: Let R be an arbitarily chosen point on L.

We have $d(R, P) = |g(R) - g(P)|$ and $d(R, P) = |f(R) - f(P)|$. But also $d(R, P) = |g(R) - g(P)| = |g(R) - f(P)|$.

We have $d(R,Q) = |g(R) - g(Q)|$ and $d(R,P) = |f(R) - f(Q)|$. But also $d(R,Q) = |g(R) - g(Q)| = |g(R) - f(Q)|$.

Now apply the previous lemma with $x = f(R), y = g(R), r = f(P), s = f(Q)$ to conclude that $f(R) = g(R)$ for every $R \in L$, so $f = g$.

Theorem: If L is a line with coordinate function f, and we use R as an independent variable ranging over L, every coordinate function g is if the form $g(R) = c + sf(R)$ where c is a real number and $s = \pm 1$.

Proof: Let L be a line. Let f be a coordinate function for L. Let g be a coordinate function for L.

Let P, Q be two distinct points on L. Define h, a function from L to the real numbers, by $h(R) = g(P) + \frac{g(Q)-g(P)}{f(Q)-f(P)}(f(R) - f(P))$.

h is a coordinate function because $h(R) = c + sf(R)$ where c is a real number and $s = \pm 1$. (c being $g(P) - \frac{g(Q)-g(P)}{f(Q)-f(P)}(f(P))$, and s being $\frac{g(Q)-g(P)}{f(Q)-f(P)}$)

$h(P) = g(P) + \frac{g(Q)-g(P)}{f(Q)-f(P)}(f(P) - f(P)) = g(P)$

$h(Q) = g(P) + \frac{g(Q)-g(P)}{f(Q)-f(P)}(f(Q) - f(P)) = g(P) + g(Q) - g(P) = g(Q)$

so by the previous corollary, h is the same coordinate function as g, since they agree at two distinct points, and h is of the form $h(R) = c + sf(R)$ where c is a real number and $s = \pm 1$, establishing that g is of this form.

Now we introduce the notions of betweenness, segments, congruence, and rays.

Definition: We say that three points A, B, C are *collinear* iff $A \neq B$, $A \neq C$, $B \neq C$, and there is a line L such that $A \in L$, $B \in L$, and $C \in L$, i.e., the three points are distinct, and they all lie on the same line.

Definition: Let A, B, C be points. We define $A * B * C$, read "B is between A and C" as meaning "A, B, and C are collinear and

$$d(A,B) + d(B,C) = d(A,C).$$

Theorem: Let L be a line and let A, B, C be three distinct points on L. Let f be a coordinate function for L. Then $A * B * C$ holds if and only if either $f(A) < f(B) < f(C)$ or $f(C) < f(B) < f(A)$.

Proof: $d(A, B) = \sigma_1(f(B) - f(A))$, where $\sigma_1 = \pm 1$ and $\sigma_1(f(B) - f(A)) = \sigma_1 f(B) - \sigma_1 f(A) > 0$ (definition of absolute value).

$d(B, C) = \sigma_2(f(C) - f(B))$, where $\sigma_2 = \pm 1$ and $\sigma_2(f(C) - f(B)) = \sigma_2 f(C) - \sigma_2 f(B) > 0$ (definition of absolute value).

There are two cases: either $\sigma_1 = \sigma_2$ or $\sigma_1 \neq \sigma_2$.

Case 1 ($\sigma_1 = \sigma_2$): If $\sigma_1 = \sigma_2$, then $d(A, B) + d(B, C) = \sigma_1 f(B) - \sigma_1 f(A) + \sigma_2 f(C) - \sigma_2 f(A) = \sigma_1 f(B) - \sigma_1 f(A) + \sigma_1 f(C) - \sigma_1 f(B) = \sigma_1 f(C) - \sigma_1 f(A) = \sigma_1(f(C) - f(A)) = |f(C) - f(A)|$ (because $\sigma_1 = \pm 1$ and this is the sum of two nonnegative (in fact positive) quantities and so certainly nonnegative) $= d(A, C)$, so $A * B * C$ holds.

We also have $\sigma_1 f(A) < \sigma_1 f(B) < \sigma_1 f(C)$, so either $f(A) < f(B) < f(C)$ or $f(C) > f(B) > f(A)$, so in this case we have $A * B * C$ if and only if either $f(A) < f(B) < f(C)$ or $f(C) < f(B) < f(A)$, because both are true.

Case 2 ($\sigma_1 \neq \sigma_2$): If $\sigma_1 \neq \sigma_2$ then $\sigma_2 = -\sigma_1$ and $d(A, B) + d(B, C) = \sigma_1 f(B) - \sigma_1 f(A) + \sigma_2 f(C) - \sigma_2 f(A) = \sigma_1 f(B) - \sigma_1 f(A) - \sigma_1 f(C) + \sigma_1 f(B) = 2\sigma_1 f(B) - \sigma_1 f(A) - \sigma_1 f(C)$. We show that this is greater than either $\sigma_1(f(C) - f(A))$ or $-\sigma_1(f(C) - f(A))$ and so is greater than $d(A, C)$ (which is equal to whichever of these is positive).

$(2\sigma_1 f(B) - \sigma_1 f(A) - \sigma_1 f(C)) - \sigma_1(f(C) - f(A)) = 2\sigma_1(B) - 2\sigma_1(C) = 2\sigma_2(f(C) - f(B)) > 0$

$(2\sigma_1 f(B) - \sigma_1 f(A) - \sigma_1 f(C)) - (-\sigma_1(f(C) - f(A))) = 2\sigma_1(B) - 2\sigma_1(A) = 2\sigma_1(f(B) - f(A)) > 0$

This establishes the two inequalities. So if $\sigma_1 \neq \sigma_2$, we have $d(A, B) + d(B, C) > d(A, C)$, so $A * B * C$ does not hold.

We cannot have $f(A) < f(B) < f(C)$ or $f(C) < f(B) < f(A)$ in this case, because either of these inequalities implies $\sigma_1 = \sigma_2$ and forces us into the other case. So in this case we also have $A * B * C$ if and only if we have either $f(A) < f(B) < f(C)$ or $f(C) < f(B) < f(A)$, because both are false.

Theorem (properties of betweenness): From the previous theorem and basic properties of order, we get three essential properties of betweenness which can be stated without reference to numbers:

1. for any points A, B, C, $A * B * C$ if and only if $C * B * A$,
2. for any three distinct points A, B, C lying on the same line, exactly one of $A * B * C$, $A * C * B$, and $B * A * C$ holds,
3. and for any points A, B, C, D, if $A * B * C$ and $A * C * D$ hold then $A * B * D$ holds.

Proof: If $A * B * C$ than A, B, C are collinear and $d(A, B) + d(B, C) = d(A, C)$ [definition] so C, B, A are collinear [symmetry of definition of collinear] and $d(C, B) + d(B, A) = d(C, A)$ (symmetry of distance and commutativity of addition] so $C * B * A$ [definition of betweenness].

If A, B, C are distinct and lie on the same line, we can choose without loss of generality a coordinate function f for the line such that $f(A) < f(C)$ (otherwise we could replace it with $-f$), and then we have exactly one of $f(A) < f(B) < f(C)$, $f(A) < f(C) < f(B)$ and $f(B) < f(A) < f(C)$ by basic properties of order on the real line, so we have exactly one of $A*B*C$, $A*C*B$, and $B*A*C$ by the Betweenness Theorem.

If A, B, C, D are points and $A*B*C$ and $A*C*D$, then all four points lie on the same line by the incidence axiom. Choose a coordinate function f for this line such that $f(A) < f(B)$. We then have $f(A) < f(B) < f(C)$ ($A * B * C$, betweenness theorem and the given order relation) so $f(A) < f(C)$, so $f(A) < f(C) < f(D)$ ($A * C * D$, betweenness theorem, and the given order relation) so $f(A) < f(B) < f(D)$ (order properties of the reals), so $A * B * D$ (betweenness theorem).

Definition: Let A and B be two distinct points. The segment from A to B, written $\overline{AB}$ is defined as $\{P : P = A \vee P = B \vee A * P * B\}$.

Definition: Let $\overline{AB}$ be a segment. We say that the length of $\overline{AB}$ is $d(A, B)$. We say that segments $\overline{AB}$ and $\overline{CD}$ are *congruent*, written $\overline{AB} \cong \overline{CD}$ iff they have the same length, that is, $d(A, B) = d(C, D)$.

This definition requires verification. We need to establish that if $\overline{AB} = \overline{CD}$, we must have $d(A, B) = d(C, D)$. This is proved as the following:

Lemma: If $\overline{AB} = \overline{CD}$, we must have $d(A, B) = d(C, D)$. In fact, we must have either $A = C \wedge B = D$ or $A = D \wedge B = C$.

Proof of Lemma: Let A, B, C, D be points with $\overline{AB} = \overline{CD}$. Let f be a coordinate function for $\overleftrightarrow{AB}$. We may suppose without loss of generality that $f(A) < f(B)$ (because otherwise we could use $-f$ instead).

Since $C \in \overline{CD} = \overline{AB}$ we have either $C = A$ or $C = B$ or $f(A) < f(C) < f(B)$ (by the betweenness theorem, with the alternative $f(B) < f(C) < f(A)$ ruled out because $f(A) < f(B)$).

Suppose for the sake of a contradiction that $f(A) < f(C) < f(B)$. We then observe further that since A and B belong to $\overline{AB} = \overline{CD}$, we have either $f(C) < f(A) < f(B) \leq f(D)$ or $f(D) \leq f(A) < f(B) < f(C)$, by the betweenness theorem and the known order fact $f(A) < f(B)$. But both of these are incompatible with $f(A) < f(C) < f(B)$, so this is false.

It follows that $C = A$ or $C = B$. If $C = A$, then $D = B$, because C and D must be distinct (because there is a segment between them). If $C = B$ then $D = A$, for the same reason. So $d(A, B) = d(C, D)$ or $d(D, C)$, but $d(D, C) = d(C, D)$, so $d(A, B) = d(C, D)$ holds in either case.

Definition: If A, B are distinct points, we define $\overrightarrow{AB}$, the ray from A through B, as $\{P : P = A \vee A * P * B \vee P = B \vee A * B * P\}$. We note that another possible definition is $\overleftrightarrow{AB} - \{P : P * A * B\}$.