Math 189, Fall 2022, Homework 7 Solutions

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September 29, 2022

Again, I am only going to grade selected problems, which I will mark with a star.

Homework 7 on sections 2.3 and 2.4 (you are free to use my technique using binor

section 2.3

problems 2*, The difference sequences

1,1,1

1,2,3,4

1,2,4,7, 11

1,2, 4, 8, 15, 26

(n choose 0) + (n choose 1) + (n choose 2) + (n choose 3)

$\frac{n^3}{6} + \frac{5n}{6} + 1$

3, 1,1

3,4,5

3,6,10,15

2,5,11,21,36

2(n choose 0) + 3(n choose 1) + 3(n choose 2) + n choose 3

$\frac{n^3}{6} + n^2 + \frac{11n}{6} + 2$

4 (I know I did 4 in class! I want to see full calculations, using either the book's n

1,1,1

3,4,5,6

3,6,10,15,21

3,6,12,22,37,58

3(n choose 0) + 3(n choose 1) + 3(n choose 2) + 1(n choose 3)

$\frac{n^3}{6} + n^2 + \frac{11n}{6} + 3$

9*, $(n^2 + 3n + 4) - ((n-1)^2 + 3(n-1) + 4)$ simplifies to $2n + 2$, so the difference sequence is arithmetic.

12, The difference sequence is 1,3,6,10,15 and the first term is 1

so the sequence is

1,4,10,20,35

method of differences:

1,1

2,3,4,5

1,3,6,10,15

0,1,4,10,20,35

This turns out to be $\frac{n(n+1)(n+2)}{6}$ which you may notice is $((n+2)$ choose 3) [so it is found in the triangle]

indexing needs to start at 0, so I added the obvious n=0 value

There is a way to fix it if you use just the sequence of values given, ask me if you are interested.

(n choose 1) + 2(n choose 2) +(n choose 3)

There are 680 cannonballs if the pyramid has 15 layers.

and section 2.4 problems

5, The characteristic polynomial is $r^2 - 3 - 4 = (x-4)(x+1)$ so we will have $a_n = A4^n + B(-1)^n$.

so $A + B = 2 = a_0$ and $4A - B = 3 = a_1$

Adding these equations, $5A = 5$ so $A = 1$, so $B = (2 - A) = 1$.

The solution is $a_n = 4^n + (-1)^n$.

6*, Like 5, except that

$A + B = 5$ and $4A - B = 8$.

adding these equations we get $5A = 13$, $A = \frac{13}{5}$ and $B = 5 - A = \frac{12}{5}$

so $a_n = \frac{13}{5} \cdot 4^n + \frac{12}{5} \cdot (-1)^n$.

8, We are given that $r^n = \alpha r^{n-1} + \beta^{n-2}$ for any n and also that $q^n = \alpha q^{n-1} + \beta^{n-2}$ for any n: this is what it means for these sequences to satisfy this recurrence relation.

We want to show that $a_n = cr^n + dq^n$ satisfies the same relation $a_n = \alpha a_{n-1} + \beta a_{n-2}$.

The verifying calculation:

$\alpha a_{n-1} + \beta a_{n-2} = \alpha(cr^{n-1} + dq^{n-1}) + \beta(cr^{n-2} + dq^{n-2}) = c(\alpha r^{n-1} + \beta r^{n-2}) + d(\alpha q^{n-1} + \beta q^{n-2}) = cr^n + dq^n = a_n$.

9, I didn't intend to assign 9, and I'm not marking it...

10*, $a_n = 4a_{n-1} + 5a_{n-2}$: one of 4 length 1 tiles followed by n-1 tiles + one of 5 length 2 tiles followed by n-2 tiles.

$a_1 = 4$, $a_2 = 4 \cdot 4 + 5 = 21$

4,21,104,521,2604,13021

characteristic polynomial $r^2 - 4r - 5 = (r-5)(r+1)$

so $a_n = A5^n + B(-1)^n$

where 5A-B = 4, 25A+B = 21

so 30A = 25, A = $\frac{5}{6}$, B = 5A-4 = $\frac{1}{6}$

so $a_n = \frac{5}{6}5^n + \frac{1}{6}(-1)^n$.

13 (again, the book has answers: for full credit you need to show convincing work) The characteristic polynomial is $r^2 - 4r + 4 = (r-2)^2$. This gives a possible solution 2^n and the weird alternative $n2^n$.

In general, we have $a_n = A2^n + Bn2^n$

suppose $a_0 = 1 = A2^0 + B02^0 = 1$. Then $A = 1$.

This applies to both sets of initial values.

If $a_1 = A2^1 + B12^1 = 2$ then $2 + 2B = 2$ so $B = 0$
and $a_n = 2^n$.
If $a_1 = A2^1 + B12^1 = 8$ then $2 + 2B = 8$ so $B = 3$
and $a_n = 2^n + 3n2^n$.