Manual of Logical Style (fresh version 2021)

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1 Introduction

This is a fresh version of a document I have been working on with my classes at various levels for years. The idea that I am promoting is that the logical form of a statement we are trying to prove, or the logical form of statements we have proved or are assuming for the sake of argument, can be used to guide the writing of proofs, sometimes almost automatically...to a certain point where we have to think.

The document is organized by logical operation (the top level logical operation of the statement we are looking at) and under each logical operation as a heading we have to consider separately on the one hand how to prove a statement of that form, and on the other hand how to use a statement of that form which we have proved or assumed in the proof of another statement.

This document introduces fancy logical notation but the intention is that the reader should learn how to handle the same logical structures of statements when they occur in English, as they do in mathematical writing all the time.

Boneheaded errors in this document (or any posted handout) are likely, and I will award points to students who point them out to me. The first one to find an error gets the points (unless it is me, of course).

2 Arguments, valid or otherwise

An *argument* is a list of statements in the following format:

$$\frac{\begin{array}{c}P_1\\ \vdots\\ P_n\end{array}}{C}$$

where the P_i's are the *premises* and C is the *conclusion*. An argument is *valid* if every assignment of values to variables appearing in the argument which makes all the premises true also makes the conclusion true.

We will use argument format mostly to express formal rules of reasoning compactly. The Marcel theorem prover which we will introduce later manipulates arguments, as you will see.

A single line representation of an argument as $P_1, \ldots, P_n \vdash C$ may also be useful.

2.1 Using truth tables to determine whether (small) arguments are valid or invalid

We give an example of a valid argument. Some simple valid arguments (including this one) we will call *logical rules* because they are building blocks of general arguments for us.

The argument

$$\begin{array}{c} A \\ A \to B \\ \hline B \end{array}$$

is valid, and it should be intuitively convincing that it is: if we have that A is true, and that it is true that if A, then B, then we surely expect B to be true. We call it a logical rule and give it the traditional Latin nickname *modus ponens*.

We present a truth table verification of this.

	A	B	$A(\mathbf{P_1})$	$A \to B(\mathbf{P_2})$	$B(\mathbf{C})$
1	T	T	T	T	T
2	T	F	T	F	T
3	F	T	F	T	T
4	F	F	F	T	T

The only line in which both premises are true is line 1, and on this line the conclusion is true. So the argument presented is valid.

We present another valid argument, which is not a logical rule for us but is in other commonly used presentations of propositional logic, and as such has a name, *transitivity of implication*.

The argument is

$$\begin{array}{c} A \to B \\ B \to C \\ \hline A \to C \end{array}$$

Again, it should be quite intuitively convincing that this is valid.

	A	B	C	$A \to B(\mathbf{P_1})$	$B \to C(\mathbf{P_2})$	$A \to C(\mathbf{C})$
1	T	T	T	T	T	T
2	T	T	F	T	F	F
3	T	F	T	F	T	T
4	T	F	F	F	T	F
5	F	T	T	T	T	T
6	F	T	F	T	F	T
7	F	F	T	T	T	T
8	F	F	F	T	T	T

The premises are all true on lines 1,5,7,8, and in each of these lines the conclusion is also true, so the argument is valid.

We give an example of a common argument which is not valid. Such arguments may be called *fallacies* if they represent common errors in reasoning: I think this one is often called arguing to the converse.

The argument is

$$\frac{A \to B}{B \to A}$$

Here is the relevant truth table

	A	B	$A \to B(\mathbf{P_1})$	$B \to A(\mathbf{C})$
1	T	T	T	T
2	T	F	F	T
3	F	T	T	F
4	F	F	T	T

There is only one premise, and it is true (so all the premises are true) in lines 1,3,4. The argument is not valid, because in line 3, which has all premises true, the conclusion is false.

Notice that verifying validity requires you to look at many rows of the table, as a rule: you need to verify that every row of the table which has T for each premise has T for the conclusion. To verify invalidity, it is always sufficient to find a single row in which all the premises are T and the conclusion is F.

3 Variables

In the first sections, we will use capital letters as *propositional variables*: these are variable *sentences* as it were.

4 Conjunction

This section is about "and". We use the notation $P \wedge Q$ to represent "P and Q".

The truth table for this operation is

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

This document is not about truth table reasoning, but truth tables are useful for making it clear what we mean.

The plan for proving a statement $A \wedge B$ is simplicity itself: first prove A, then prove B. We just prove one after the other, with no special local assumptions.

This is summarized in a logical rule (when we say that an argument is a logical rule, we are just saying that it is valid (and important as a basic form of valid argument)).

$$\frac{\begin{array}{c}A\\B\end{array}}{A \wedge B}$$

This is called the rule of *conjunction*.

The plan for using a conjunction $A \wedge B$ which we have proved or assumed is also simplicity itself. We can draw the conclusion A. We can draw the conclusion B. In other words, we can just pull it apart.

This is summarized in two rules:

$$\frac{A \wedge B}{A}$$

and

$$\frac{A \wedge B}{B}$$

Both of these rules are called rules of *simplification*.

5 Implication

This section is about "if..., then..." statements, which are also called *conditional statements*. We use the notation $A \to B$ to represent "If A then B".

The truth table for this operation is

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

The plan for proving an implication (conditional statement) $A \to B$ takes a little thought to understand. The strategy is to assume A for the sake of argument, then deduce B. When you finish this argument, you have *not* proved B (nor have you proved any of the intervening steps), because they all depend on the assumption A that you made for the sake of argument. But you have proved $A \to B$. Here is a format for this kind of argument:

Prove: $A \to B$

> **Assume (1):** A (we give any statement we can use a line number so we can refer to it, when we are being very formal).
>
> **Goal:** B (the goal is just a comment: notice that we do not give it a line number, since we cannot *use* it).
>
> **intervening proof steps:** $\vdots$
>
> **(n):** B

($n+1$): $A \to B$ rule of deduction, lines 1-n

The lines using the assumption A are indented, and we need to remember that once we have proved $A \to B$ we are no longer entitled to refer to anything in that block of statements, because all those lines depend (presumably) on the assumption A which we are no longer making.

A suggestion of how to make this more formal follows in tiny print.

A suggestion of how to formalize this rule completely is given by

$$\frac{A \vdash B}{A \to B}$$

or even

$$\frac{[P_1, \ldots, P_n,]A \vdash B}{[P_1, \ldots, P_n, \vdash]A \to B}$$

In the second version, the additional P_i's can be thought of as the hypotheses of larger indented boxes containing the one we introduced for the hypothesis A. The general idea is, if we can deduce B from A [and possibly some other hypotheses] then we can conclude $A \to B$ [without the hypothesis A but with the others if there are any].

Using an implication is different (and simpler). This can be expressed in a formal rule:

$$\frac{\begin{array}{c} A \\ A \to B \end{array}}{B}$$

If we know (or have assumed) A and we know (or have assumed) $A \to B$, then we can draw the conclusion B. This rule is called *modus ponens*.

There are some other rules for indirect reasoning with conditional statements, which we will introduce below after we discuss negation.

6 Biconditional (If and only if)

This section is about "if and only if", or *biconditional* statements. We use the notation $A \leftrightarrow B$ to represent A if and only if B, for which we also use the abbreviation "A iff B".

The truth table for this operation is:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

We can also *define* $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$.

The strategy for proving a statement $A \leftrightarrow B$ takes the form which the definition above suggests: first prove that A implies B, then prove that B implies A.

Prove: $A \leftrightarrow B$

Part I:

> **Assume (1):** A
>
> **Goal:** B
>
> **intervening proof steps:** $\vdots$
>
> **(**n**):** B

($n+1$**):** $A \rightarrow B$ by deduction lines $1 - n$ [optional]

Part II:

> **Assume (**$(n+2)$**):** B
>
> **Goal:** A
>
> **intervening proof steps:** $\vdots$
>
> **(**m**):** A

($m+1$**):** $B \rightarrow A$ by deduction lines $(n+2) - m$ [optiona1]

$(m+2)$: $A \leftrightarrow B$ by biconditional introduction, lines $n+1$, $m+1$ [if these implication lines are supplied] or by 1-n, $(n+2)$-m; you may either explicitly prove the two implications and use them as references for the proof of the biconditional, or leave them out and use the two blocks of statements as the reference for the proof of the biconditional. Either style is allowed.

A biconditional $A \leftrightarrow B$ can be used in a proof in a way similar to the way a conditional is used, but in either direction:

$$\frac{\begin{array}{c} A \\ A \leftrightarrow B \end{array}}{B}$$

$$\frac{\begin{array}{c} B \\ A \leftrightarrow B \end{array}}{A}$$

are both valid rules, which we might call *biconditional modus ponens*.

There is a more general way of using a statement $A \leftrightarrow B$ which we will *not* as a rule use in our formal logic exercises, but which you should be aware of. If we have proved $A \leftrightarrow B$, we have shown that A and B are effectively saying the same thing, so we may freely substitute A for B and B for A. For example, because the de Morgan law $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ is a theorem, it is valid to replace $\neg(A \wedge B)$ with $\neg A \vee \neg B$ or vice versa in a more complicated logical expression as a step in a proof. We note that this is valid, but we do *not* allow use of this in exercises unless we specifically say so (for example, we might want to prove the de Morgan laws using our regular rules; certainly you are not allowed to use this substitution strategy to short-circuit the exercise).

7 A short sample proof

Prove: $((A \to B) \land (B \to C)) \to (A \to C)$

We start by applying the proof strategy for implication.

Assume (1): $(A \to B) \land (B \to C)$

Goal: $A \to C$

We again apply the proof strategy for implication.

Assume (2): A

Goal: C

We can't do anything more with this goal, as it is as simple as possible. We use simplification to unpack assumption 1.

(3): $A \to B$ simplification, line 1

(4): $B \to C$ simplification, line 1

(5): B modus ponens, lines 2,3

(6): C modus ponens, lines 4,5

(7): $A \to C$ deduction lines 2-6

(8): $((A \to B) \land (B \to C)) \to (A \to C)$ deduction lines 1-7

8 A longer sample proof

This was done in class.

Theorem: $(A \to (B \to C)) \leftrightarrow ((A \wedge B) \to C)$

This is a biconditional statement, which determines our initial setup.

Part I:

Assume(1): $A \to (B \to C)$

Goal: $(A \wedge B) \to C$

This goal is an implication, which again determines our setup.

Assume(2): $A \wedge B$

Goal: C

The goal C is as simple as possible, so it is time for us to look into *using* the numbered lines we have.

Notice that we could apply modus ponens with line 1 if we had the hypothesis A of line 1 as a line, and we can get A as a line by unpacking line 2.

(3): A simp line 2

(4): $B \to C$ m.p. lines 1,3

Now we can apply modus ponens if we can get B as a line, and once again we can get B as a line from line 2.

(5): B simp line 2

(6): C modus ponens lines 4,5

This is our goal above, so it is time for closing lines.

(7): $(A \wedge B) \to C$ deduction lines 2-6.

This completes Part I. We continue with Part II on the next page.

Part II: **Assume(8):** $(A \wedge B) \to C$

Goal: $A \to (B \to C)$

Assume(9): A

Goal: $B \to C$

Assume(10): B

Goal: C

This is the limit of the goal unpacking process: now we need to start using our numbered lines. Notice that we would like to have $A \wedge B$ as a line in order to apply m.p. with line 8. And we can get $A \wedge B$ from lines 9 and 10.

(11): $A \wedge B$ conjunction lines 9,10

(12): C m.p. lines 8,11

(13): $B \to C$ deduction lines 10-12

(14): $A \to (B \to C)$ deduction lines 8-13

(15): the Theorem, by biconditional introduction, lines 1-7, 8-14.

9 Disjunction

This section is about “or” (more precisely about “and/or”). We use the notation $P \vee Q$ to represent “P or Q (or both)”.

The truth table for this operation is

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

The basic rules for proving disjunctions may seem very limited. A more powerful rule for proving disjunctions will be developed in a later section.

These rules are embodied in the two rules of *addition*:

$$\frac{A}{A \vee B}$$

and

$$\frac{B}{A \vee B}$$

To prove $A \vee B$, either prove A or prove B. This might not seem like enough, but we will see that it is.

The rule for using an “or” statement which is been proved or assumed is *proof by cases*. This can be packaged as a rule:

$$\frac{\begin{array}{c} P \vee Q \\ P \rightarrow C \\ Q \rightarrow C \end{array}}{C}$$

or as a strategy with indented blocks of the kind we have been presenting:

(1): $P \vee Q$

possible intervening proof lines: $\vdots$

Goal: C

Case 1:

> **Assume (1a):** P
>
> **Goal:** C
>
> **intervening proof steps:** $\vdots$
>
> **(n):** C

Case 2:

> **Assume (1b):** Q
>
> **Goal:** C
>
> **intervening proof steps:** $\vdots$
>
> **(m):** C

(m+1): C proof by cases by 1, 1a-n, 1b-m

Proofs by cases are ubiquitous in mathematical reasoning. Notice that in the block format, each of the cases could be recast as a proof of an implication by deduction, then the rule of proof cases in the argument form given above could be used to justify the conclusion.

We give an example of a proof by cases done in class.

Theorem: $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

> This statement looks almost like a statement of the rule of proof by cases, so it is not surprising that we use the rule to prove it.
>
> **Assume(1):** $(A \to C) \wedge (B \to C)$
>
> **Goal:** $(A \vee B) \to C$
>
> > **Assume(2):** $A \vee B$
> >
> > Once we have a numbered line which is an or statement, we should expect a proof by cases as at least a possibility.
> >
> > **Goal:** C
> >
> > and indeed we start a proof by cases, which starts on the next page.

Case I (from line 2): Assume (2a): A

Goal: C

Now we need to look at our numbered lines. If we unpack line 1, we get something useful.

(3a): $A \to C$ simp line 1

(4a): C m.p. lines 2a,3a

which is our goal.

Case II (from line 2): Assume (2b): B

Goal: C

Now we need to look at our numbered lines. If we unpack line 1, we get something useful.

(3b): $B \to C$ simp line 1

(4b): C m.p. lines 2a,3a

which is our goal.

(5): C proof by cases, 2, 2a-4a, 2b-4b

Proofs by cases have our most complicated line references, the line for the "or" assumption and a block for the proof of each case.

(6): $(A \vee B) \to C$ deduction 2-5

(7): the Theorem, deduction lines 1-6.

Here is a short proof to illustrate the use of the rule of addition.

Prove: $(A \to B) \to (A \to (B \vee C)))$

Assume(1): $A \to B$

Goal: $A \to (B \vee C)$

Assume(2): A

Goal: $B \vee C$

(3): B m.p. 1,2

(4): $B \vee C$ addition line 3

(5): $A \to (B \vee C)$ deduction lines 2-4

(6): $(A \to B) \to (A \to (B \vee C)))$ (the theorem) by deduction lines 1-5.

10 Exercises

Write formal proofs in the style given above of each theorem.

1. $(A \wedge B) \rightarrow (B \wedge A)$
2. $(A \wedge (A \rightarrow B)) \rightarrow B$
3. $(A \rightarrow (B \wedge C)) \rightarrow (A \rightarrow C)$
4. $((A \rightarrow B) \wedge (B \rightarrow C) \wedge (C \rightarrow D)) \rightarrow (A \rightarrow D)$
5. $(A \vee B) \rightarrow (B \vee A)$ Hint: you will want to prove this by cases on the hypothesis $A \vee B$. Notice that we do have rules which allow deduction of $B \vee A$ from A and from B (the rule of addition).

11 Negation

This section is about "not". We introduce the notation $\neg P$ for "It is not the case that P", or equivalently "P is false". To make it easier to state the rules for negation, we also introduce the symbol $\perp$ for a fixed false statement. We will refer to $\perp$ as "the absurd" on occasion.

The truth table for this operation is

P	$\neg P$
T	F
F	T

To prove a statement $\neg P$, assume P and reason to $\perp$.

Prove: $\neg P$

> **Assume (1):** P
>
> **Goal:** $\perp$ (we can express this by saying that our goal is a contradiction)
>
> **intervening proof steps:** $\vdots$
>
> **(n):** $\perp$

($n+1$): $\neg P$ by negation introduction, lines 1-n

There are two rules for using negative statements:

$$\frac{\begin{array}{c} P \\ \neg P \end{array}}{\perp}$$

This is the *rule of contradiction*, the only way to prove the absurd. We hope this never happens except under (unfortunate) assumptions made for the sake of argument!

$$\frac{\neg\neg P}{P}$$

This is the rule of *double negation elimination*.

One might think that the strategy we call "negation introduction" is the famous strategy usually called "proof by contradiction" or "reductio ad absurdum" (we prefer the Latin name since we have another rule called "contradiction"). It isn't quite: it is the direct strategy for proving a negative statement. Reductio ad absurdum is more general (it can be applied to prove statements of any form).

This proof strategy for a statement of any form P

Prove: P

> **Assume (1):** $\neg P$
>
> **Goal:** $\bot$
>
> **intervening proof steps:** $\vdots$
>
> **(n):** $\bot$

(n+1): $\neg\neg P$ negation introduction, lines 1-n

(n+2): P d.n.e, line $n+1$.

can be packaged more compactly as

Prove: P

> **Assume (1):** $\neg P$
>
> **Goal:** $\bot$
>
> **intervening proof steps:** $\vdots$
>
> **(n):** $\bot$

(n+1): P, reductio ad absurdum, lines 1-n.

The rule of "proof by contradiction" or "reduction ad absurdum" is thus seen to be a derived rule of our system, justified by negation introduction and double negation elimination. It is a very useful rule: a point of proof strategy is to attempt proof by contradiction whenever you cannot see what else to do.

We give some other useful derived rules.

We state and justify the rule of *double negation introduction*.

$$\frac{P}{\neg\neg P}$$

states the rule. The justification follows.

premise(1): P

Goal: $\neg\neg P$

> **Assume (2):** $\neg P$
>
> **Goal:** $\perp$
>
> **(3):** $\perp$ contradiction 1,2

(4): $\neg\neg P$ negation introduction, 2-3.

You are allowed to freely use the derived rule of double negation introduction.

It is a feature of the truth table definition of implication that anything follows from a false statement. We prove this in our format.

premise(1): $\perp$

Goal: P

> **Assume(2):** $\neg P$
>
> **Goal:** $\perp$
>
> **(3):** $\perp$ copied from 1

(4): P reductio ad absurdum 2-3.

We have verified that

$$\frac{\perp}{P}$$

is a valid argument. Any conditions which made all premises of this argument true (there are no such conditions) also make its conclusion true! We allow this to be used as a rule and call it "absurdity".

12 Implication and negation

In this section, we prove the Contrapositive Theorem then use it to justify the strategy of indirect proof for proving implications and the rule of modus tollens for using implications. In general terms, we present useful rules and strategies which combine the operations of implication and negation.

Theorem: $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$

Part I:

> **Assume(1):** $P \to Q$
>
> **Goal:** $\neg Q \to \neg P$
>
> > **Assume(2):** $\neg Q$
> >
> > **Goal:** $\neg P$
> >
> > > **Assume(3):** P
> > >
> > > **Goal:** $\perp$
> > >
> > > **(4):** Q m.p. 1,3
> > >
> > > **(5):** $\perp$ contradiction 2,4
> >
> > **(6):** $\neg P$ negation introduction, lines 3-5
>
> **(7):** $\neg Q \to \neg P$ deduction lines 2-6

Part II:

> **Assume(8):** $\neg Q \to \neg P$
>
> **Goal:** $P \to Q$
>
> > **Assume(9):** P
> >
> > **Goal:** Q
> >
> > > We prove this goal by reductio ad absurdum (we cannot see any other line of attack!)
> > >
> > > **Assume(10):** $\neg Q$
> > >
> > > **Goal:** $\perp$
> > >
> > > **(11):** $\neg P$ deduction 8,10
> > >
> > > **(12);** $\perp$ contradiction 9,11
> >
> > **(13):** Q reductio ad absurdum 10-12
>
> **(14):** $P \to Q$ deduction 9-13

(15): The Theorem, by biconditional introduction, 1-7, 8-14

The Contrapositive Theorem justifies some new strategies and rules. The first is indirect proof of an implication.

Prove: $P \to Q$

> **Assume(1):** $\neg Q$
>
> **Goal:** $\neg P$
>
> **intervening proof steps:** $\vdots$
>
> **(n):** $\neg P$

(n+1): $P \to Q$ indirect proof, 1-n

This can be seen to be justified in terms of rules and theorems above by adding some extra steps:

Prove: $P \to Q$

Prove: $\neg Q \to \neg P$

> **Assume(1):** $\neg Q$
>
> **Goal:** $\neg P$
>
> **intervening proof steps:** $\vdots$
>
> **(n):** $\neg P$

(n+1): $\neg Q \to \neg P$ deduction 1-n

(n+2): $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ contrapositive theorem

(n+3): $P \to Q$ biconditional m.p., n+1-n+2

Similarly, we could use the contrapositive theorem to justify a new rule for using implications:

$$\frac{\begin{array}{c} P \to Q \\ \neg Q \end{array}}{\neg P}$$

We give a formal justification of this rule using our rules and theorems from above. This rule is called *modus tollens*.

premise(1): $P \to Q$

premise(2): $\neg Q$

Goal: $\neg P$

> **Assume(3):** P
>
> **Goal:** $\perp$
>
> **(4):** Q m.p. 1,3
>
> **(5):** $\perp$ contradiction 2,4

(6): $\neg P$ negation introduction 3-5

which establishes the validity of the rule, without actually using the contrapositive theorem, though it should be easy to see that the contrapositive theorem could be used for this.

13 Disjunction and negation

In this section, we prove the validity of a definition of disjunction in terms of negation and implication and use it to motivate a more powerful rule of *alternative elimination* for proving disjunctions, and we introduce the rule of *disjunctive syllogism* for using disjunctions. In general terms, we present useful rules and strategies which combine the operations of disjunction and negation.

We note (you can verify this using truth tables) that $P \vee Q$ is logically equivalent to $\neg P \rightarrow Q$ and also to $\neg Q \rightarrow P$.

This suggests the following strategy (really a pair of strategies) for proving $P \vee Q$.

Prove: $P \vee Q$

> **Assume(1):** $\neg P$
>
> **Goal:** Q
>
> **intervening proof steps:** $\vdots$
>
> **(n):** Q

(n+1): $P \vee Q$ alternative elimination 1-n

The form below is just as good. You should not give proofs of both forms for the same statement (this would be redundant).

Prove: $P \vee Q$

> **Assume(1):** $\neg Q$
>
> **Goal:** P
>
> **intervening proof steps:** $\vdots$
>
> **(n):** P

(n+1): $P \vee Q$ alternative elimination 1-n

We present a justification for the first style of alternative elimination (a justification for the second would be very similar).

Prove: $P \vee Q$

[[**Prove:** $\neg P \to Q$]]

Assume(1): $\neg P$

Goal: Q

intervening proof steps: $\vdots$

(n): Q

[[**(n+1):** $\neg P \to Q$ deduction 1-n

Assume(n+2): $\neg(P \vee Q)$ for the sake of a contradiction.

Goal: P in order to show $P \vee Q$ by addition and get a contradiction

[**Assume(n+3):** $\neg P$
(n+4): Q mp n+1,n+3
(n+5): $P \vee Q$ addition n+4
(n+6): $\perp$ contradiction n+5, n+2

(n+7): P reductio ad absurdum n+3-n+6

(n+8): $P \vee Q$ addition n+7

(n+9): $\perp$ contradiction n+2,n+7]]

(n+10): $P \vee Q$ reductio ad absurdum, n+1-n+9

The double brackets indicate parts of this proof to be dropped to give the form of the alternative elimination rule, which is revealed by this proof to be a valid strategy to prove disjunctions. Of course one would change the justification of the last line to "alternative elimination" if one dropped the bracketed material. We give this for completeness: really, for this class I just want you to know the alternative elimination rule.

A useful rule for using disjunctions takes the following four forms, which are all called *disjunctive syllogism.*

$$\begin{array}{c} P \vee Q \\ \neg P \\ \hline Q \end{array}$$

$$\frac{\begin{array}{c} P \vee Q \\ \neg Q \end{array}}{P}$$

$$\frac{\begin{array}{c} \neg P \vee Q \\ P \end{array}}{Q}$$

$$\frac{\begin{array}{c} P \vee \neg Q \\ Q \end{array}}{P}$$

Each of these can be motivated by using truth table equivalences: for example the first follows from modus ponens combined with the equivalence of $P \vee Q$ to $\neg P \rightarrow Q$.

We give a proof of the first form using our other rules. We use the same device of double brackets to indicate parts to be deleted to give the disjunctive syllogism form.

(1): $P \vee Q$ premise

(2): $\neg P$ premise

Goal: Q

We will prove the goal by cases on (1).

Case 1:

Assume(1a): P

Goal: Q

(2a): $\perp$ contradiction 1a,2

(3a): Q absurdity, 2 (we showed above that we can deduce anything from $\perp$).

Case 2:

Assume(1b): Q

Goal: Q

(3): Q proof by cases 1, 1a-3a, 1b-1b

You do not need to know how to prove the rule of disjunctive syllogism, just how to use it. But we give the justification for the sake of completeness. More examples of the use of the various rules are found in the next section.

14 More examples

An example of a positive proof (using rules that do not involve negation).

Theorem: $((A \wedge B) \vee (B \wedge C)) \rightarrow B$

> **Assume(1):** $(A \wedge B) \vee (B \wedge C)$
>
> **Goal:** B
>
> > We prove this by cases on (1). This is basically the only thing we can do, if we are confined to the positive rules!
>
> **Case 1:**
>
> > **Assume(1a):** $A \wedge B$
> >
> > **Goal:** B
> >
> > **(2a):** B simp line 1a
>
> **Case 2: Assume(1b):** $B \wedge C$
>
> > **Goal:** B
> >
> > **(2b):** B simp line 1b
>
> **(3):** B proof by cases 1,1a-2a,1b-2b

(4): $((A \wedge B) \vee (B \wedge C)) \rightarrow B$ deduction 1-3

Next, we verify the classical rule of *constructive dilemma* in two different ways.

The rule to be verified is

$$\begin{array}{c} P \vee Q \\ P \rightarrow R \\ Q \rightarrow S \\ \hline R \vee S \end{array}$$

This should be extremely believable!

The first proof:

(1): $P \vee Q$ premise

(2): $P \to R$ premise

(3): $Q \to S$ premise

Goal: $R \vee S$

We prove this by cases on (1).

Case 1:

Assume(1a): P

Goal: $R \vee S$

(2a): R m.p. lines 2,1a

(3a): $R \vee S$ addition line 2a

Case 2:

Assume(1b): Q

Goal: $R \vee S$

(2b): S m.p. lines 3,1b

(3b): $R \vee S$ addition line 2b

The second proof:

(1): $P \vee Q$ premise

(2): $P \to R$ premise

(3): $Q \to S$ premise

Goal: $R \vee S$

We prove this by alternative elimination.

Assume(4): $\neg R$

Goal: S

(5): $\neg P$ modus tollens 4,2

(6): Q disjunctive syllogism (d.s.) 1,5

(7): S m.p. 6,3

(8): $R \vee S$ alternative elimination 4-7

15 Exercises

This is the second set of exercises, assigned 1/29/2018 and due next Monday (yes, I am giving a little extra time).

1. Prove $((P \vee \neg Q) \wedge (\neg P \vee R)) \rightarrow (Q \rightarrow R)$

   Hint: this starts with the usual setup for an implication, then repeatedly uses disjunctive syllogism.

2. Verify the rule of *destructive dilemma*

$$\begin{array}{c} P \rightarrow R \\ Q \rightarrow S \\ \neg R \vee \neg S \\ \hline \neg P \vee \neg Q \end{array}$$

   I give you the first few lines

   (1): $P \rightarrow R$

   (2): $Q \rightarrow S$

   (3): $\neg R \vee \neg S$

   Goal: $\neg P \vee \neg Q$

   Hints: you could prove this either by cases or by alternative elimination. If you prove it by alternative elimination, do notice that the hypothesis will be $\neg\neg P$ not $\neg P$ (and you can get P from this right away). No matter which way you prove it, you will definitely want to use the rule of modus tollens one or more times. This is very similar to the constructive dilemma proof in either case, but just a little more indirect.

3. Prove

$$\neg(P \vee Q) \leftrightarrow (\neg P \wedge \neg Q)$$

   . You may *not* use the de Morgan laws here: that is what you are trying to prove. You need to use the proof strategies in this document. I will do the other de Morgan law on Wednesday to illustrate some points that will be needed to complete this proof.

16 Example (one of de Morgan's laws)

Theorem:

$$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$$

Part I:

Assume(1): $\neg(P \wedge Q)$

Goal: $\neg P \vee \neg Q$

We will use the new alternative elimination strategy to prove this disjunction.

Assume(2): $\neg\neg P$

Goal: $\neg Q$

Assume (3): Q

Goal: $\perp$

(4): P double negation elimination line 2

(5): $P \wedge Q$ conjunction lines 4,3

(6): $\perp$ contradiction lines 1,5

(7): $\neg Q$ neg intro, 3-6

(8): $\neg P \vee \neg Q$ alternative elimination, 2-7

Part II of the proof is on the next page.

Part II:

Assume(9): $\neg P \vee \neg Q$

Goal: $\neg(P \wedge Q)$

Assume(10): $P \wedge Q$

Goal: $\perp$

We have unpacked things as far as we can. The next thing to do is to try proof by cases on line 9.

Case 1 (from (9):

Assume(9a): $\neg P$

Goal: $\perp$

(10a): P simp 10

(11a): $\perp$ contradiction lines 9a,10a

Case 2 (from (9):

Assume(9b): $\neg Q$

Goal: $\perp$

(10b): Q simp 10

(11b): $\perp$ contradiction lines 9b,10b

(12): $\perp$ proof by cases, 9, 9a-11a, 9b-11b

(13): $\neg(P \wedge Q)$ neg intro 10-12

(14): The theorem, by biconditional introduction, 1-8, 9-13

I thought I was going to use *reductio ad absurdum* in this but I didn't have to.