Homework 8 solutions, Math 189, Fall 2022

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September 30, 2022

I'm planning to check off this assignment for full credit if you turned it in, but I am making solutions available so you can see how you did.

I am planning to revisit induction proofs and do harder ones as part of the logic and formal proof unit, which I think I will do last this time. So you will see more of this. You will also see induction used as a tool in the number theory and graph theory sections of the course, and questions about such induction proofs may be on the next exam.

section 2.5 problems 2, The statement you want to prove true for all $n \geq 0$ is $\sum_{i=0}^{n} 2^i = 2^{i+1} - 1$

Basis (n=0): $\sum_{i=0}^{0} 2^i = 2^{0+1} - 1$ is the statement to be proved in the basis step.

$\sum_{i=0}^{0} 2^i$

$= 2^0$ basic property of summations

$= 1$

$= 2 - 1$

$= 2^{0+1} - 1$

Induction step: Let $k \geq 0$ be chosen arbitrarily.

Suppose that $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$ (inductive hypothesis = ind hyp)

Our goal is to show that $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$ follows.

$\sum_{i=0}^{k+1} 2^i$

$= [\sum_{i=0}^{k} 2^i] + 2^{k+1}$ pull last term out of a summation: a basic property of summations

$= 2^{k+1} - 1 + 2^{k+1}$ IND HYP (fireworks go off)

$= 2^{k+2} - 1$ algebra

5, Prove that $7^n - 1$ is a multiple of 6 for all n

Basis (n=0): $7^0 - 1 = 0$ is indeed a multiple of 6.

Induction step: Let $k \geq 0$ be chosen arbitrarily.

Assume that $7^k - 1$ is divisible by 6.

Goal: show that $7^{k+1} - 1$ is divisible by 6.

Proof of induction step (this involves pulling a rabbit out of a hat in a way which you should have encountered in other proofs):

$7^{k+1} - 1$

$= (7^{k+1} - 7^k) + (7^k - 1)$ add and subtract the same thing

$= 6 \cdot 7^k + (7^k - 1)$: this is the sum of two terms, the first, $6 \cdot 7^k$, obviously divisible by 6 and the second, $(7^k - 1)$, divisible by six by IND HYP (fireworks go off); the sum of two numbers divisible by 6 is divisible by 6, so we have proved the induction goal.

6, Prove that $2^n < n!$ for all $n \geq 4$

Basis (n=4): $2^4 = 16 < 24 = 4!$. Check.

Induction step: Let $k \geq 4$ be chosen arbitrarily.

Suppose that $2^k < k!$. IND HYP

Our goal is $2^{k+1} < (k+1)!$.

Multiply both sides of the ind hyp (this is where it is used) by 2 to get $2^{k+1} < k!(2)$.

Since $k \geq 4$, we have $2 < k + 1$, so $k!(2) < k!(k+1) = (k+1)!$.

So we have shown $2^{k+1} < k!(2) < (k+1)!$, and the proof is complete by transitivity of $<$.

10 (we did it in class, no reason you shouldn't do your own writeup though),

Prove that the sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$, that is

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ is true for $n \geq 1$.

Basis (n=1): $\sum_{i=1}^{1} i^2 = 1^2$ (by basic property of summations) $= 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$. Check.

Induction step: Let $k \geq 1$ be chosen arbitrarily.

Suppose that $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

The goal is to prove that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$ follows.

The proof of the induction goal:

$\sum_{i=1}^{k+1} i^2$

$= [\sum_{i=1}^{k} i^2] + (k+1)^2$ pull last term out of a summation

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ IND HYP (20 gun salute)

$= \frac{k)(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6}$

$= \frac{(k+1)(k+2)(2k+3}{6}$, establishing the induction goal.

17, A number is even iff it is two times an integer.

Prove by induction that for any $n \geq 0$, $n^2 + n$ is even.

Basis (n=0): $0^2 + 0 = 0$ is even. Check.

Induction step: let $k \geq 0$ be chosen arbitrarily.

Assume (IND HYP) that $k^2 + k$ is even.

The induction goal is to show that $(k+1)^2 + (k+1)$ is even.

Proof of the induction goal: Since $k^2 + k$ is even (IND HYP) there is an integer m such that $k^2 + k = 2m$.

$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + (2k + 2) = 2m + 2(k+1) = 2(m + k + 1)$ which is even because $m + k + 1$ is an integer.

23, Show that the sum of the row of Pascal's triangle with second number n is 2^n.

The statement to be proved is that $\sum_{i=0}^{n}(n \text{ choose } i) = 2^n$

The hint tells you to use the identity (n choose k-1) + (n choose k) = (n+1 choose k).

I also use some clever manipulations of summations which I would like you to understand.

Basis: $\sum_{i=0}^{0}(0 \text{ choose } i) = (0 \text{ choose } 0) = 1 = 2^0$, check

Induction step: Let $k \geq 0$ be chosen arbitrarily.

Assume $\sum_{i=0}^{k}(k \text{ choose } i) = 2^k$ (IND HYP)

Goal: $\sum_{i=0}^{k+1}(k+1 \text{ choose } i) = 2^{k+1}$

Proof: I will work from right to left.

$2^{k+1} = 2^k + 2^k = \sum_{i=0}^{k}(k \text{ choose } i) + \sum_{i=0}^{k}(k \text{ choose } i)$ by IND HYP (gong is sounded), which is equal to $\sum_{i=0}^{k}(k \text{ choose } i) + \sum_{i=1}^{k+1}(k \text{ choose } i-1)$ by changing the indexing in the second copy of the sum. Pull out the first term from the left summation and the last term from the right summation to get $(k \text{ choose } 0) + \sum_{i=1}^{k}(k \text{ choose } i) + \sum_{i=1}^{k}(k \text{ choose } i-1) + (k \text{ choose } k)$.

The middle two sums now have the same range and can be added term by term.

$(k \text{ choose } 0) + [\sum_{i=1}^{k}((k \text{ choose } i) + (k \text{ choose } i-1))] + (k \text{ choose } k)$, and we apply the identity

$(k \text{ choose } 0) + [\sum_{i=1}^{k}(k+1 \text{ choose } i)] + (k \text{ choose } k)$

and observe that we can replace (k choose 0) with (k+1 choose 0) (both are 1) and similarly replace (k choose k) with (k+1 choose k+1) and see that we actually have $\sum_{i=0}^{k+1}(k+1 \text{ choose } i)$.

29 (read the hint). This problem is visual and I'll do it in class on request.