Review of “Finsler Set Theory: Platonism and Circularity”, David Booth and Renatus Ziegler, eds.

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This book is a collection of translations of papers by Paul Finsler on set theory, with some additional material prepared by the editors.

In the translated papers, Finsler expresses his views on philosophy of set theory and philosophy of mathematics generally. He discusses the paradoxes of self-reference from a philosophical standpoint. He also analyzes the paradoxes of set theory from a more mathematical standpoint, with the aim of identifying the error of reasoning which led to the paradoxes.

Finsler proposes a set theory of his own which he believes will avoid the paradoxes. The consensus of modern set theorists is that Finsler's set theory is incoherent; the reviewer concurs, and thereby takes issue not only with Finsler himself but with the editors of the volume. We state Finsler's axioms and discuss them:

“We consider a system of things, which we call *sets*, and a relation, which we symbolize by β. The exact and complete description is achieved by means of the following axioms.

I. Axiom of Relation: For arbitrary sets M and N it is always uniquely determined whether M possesses the relation β to N or not.

II. Axiom of Identity: Isomorphic sets are identical.

III. Axiom of Completeness: The sets form a system of things which, by strict adherence to the axioms I and II, is no longer capable of extension.

That is, it is not possible to adjoin further things in such a way that the axioms I and II are satisfied.”

The relation β is the converse of the usual membership relation; it might be read “contains”. Axiom I asserts that the extensions of sets must be well-defined. Axiom II needs explanation (it is explained in Finsler's paper some time after it is stated). The idea is that the identity of a set should be completely determined by the isomorphism class of the relation β restricted to the transitive closure of the set (i.e., the collection of its elements, the elements of its elements, and so forth). A consequence of this axiom is the usual axiom of extensionality (sets with the same elements are the same) but axiom II is stronger than extensionality: for example, it implies that two sets which are their own sole elements must be equal. It is an anti-foundation axiom of the kind studied by Peter Aczel in his book *Non-well-founded sets*: Aczel includes Finsler's axiom among those that he considers.

The problem with Finsler's theory is his Axiom III. Literally, what it says is that the universe of sets is a maximal structure satisfying axioms I and II. The only way that a structure satisfying axioms I and II can be maximal is for its domain to be the universal class; otherwise, as Baer observed, one could take a new object and assign to it the extension of the Russell class of the purported maximal structure, thus extending it.

Finsler reads his axiom as implying that all consistent set definitions will be satisfied in his theory. But this is not possible, as Specker pointed out: the sets $\{x \mid x = x\}$ (the universe) and $\{x \mid x \notin x$ and for some $y, x \in y\}$ (the class of all elements which are not elements of themselves) can each be satisfied in a model of axioms I and II, but they cannot both be present in a single model of axioms I and II. The second set cannot be an element, and so cannot coexist with the set of all sets, which must have every set as an element.

Both Finsler and the editors attempt objections to Specker's refutation, but I find these objections unintelligible.

Further, the model theory of axioms I and II does not support Finsler's assertions about the consequences of axioms I-III. Finsler claims that the existence of the universal set is a consequence of his axioms, but the reviewer has shown that any maximal model of axioms I-II with a universal set can be converted to a maximal model of axioms I-II without a universal set.

Finsler goes on to introduce a further concept, which he does not support using reasoning based on axioms I-III. He argues convincingly that the problem with the set-theoretical paradoxes is that the definitions of the paradoxical sets are in some sense "circular". The new concept he defines is that of a "circle-free" set. He points out that a "circle-free" set certainly will not appear in its own transitive closure (it will not itself be one of the components from which it is constructed). But this is not enough; the collection of all sets which do not appear in their own transitive closures is itself paradoxical and so "circular" (think about whether it is an element of itself!) Finsler concluded that the correct definition of "circle-free" is

Definition: A set is *circle-free* if it does not belong to its own transitive closure and its definition does not refer to the concept "circle-free".

This is a "circular" definition, of course. He proceeds to develop the consequences of this definition, very largely independently of axioms I-III. The fascinating thing is that the consequences that he develops are basically those of the set theory of Ackermann (for a full description of this theory the reader is referred to the article by Azriel Levy referenced below). Ackermann's theory has sets and classes. The comprehension principle given for classes is that an arbitrary condition serves to define a class *of sets* (though there may and indeed must be classes which have non-set elements). The comprehension principle for sets is that any class of sets which can be defined without reference to the concept of sethood (and without non-set parameters) is a set. The similarity to the definition of the circle-free sets should be clear. The similarity extends to detail; every axiom of Ackermann's theory is reflected by a conclusion of Finsler's about circle-free sets, and Finsler anticipated proofs of Ackermann in detail, including the (perhaps surprising) proof of the axiom of infinity. Ackermann's set theory is consistent iff *ZFC* is consistent, so this much of Finsler's work can be put on a sound foudnation.

Our conclusion about Finsler's set theory is that, while the entire theory is untenable (and difficult to understand) the ideas behind axiom II and the notion of "circle-free" sets prove to be sound. Finsler appears to have had good intuition. Space forbids discussion of Finsler's other contributions evident from the papers in this volume, except to say that we find his thoughts about paradoxes of self-reference to be interesting, his thoughts about the distinction between sets and classes to be important, and his defense of mathematical Platonism to be admirable. We think that making these papers available in translation is a service to scholarship.

We found the supporting materials prepared by the editors to be unsatisfactory in most cases, with the notable exception of calling attention to the relation to the set theory of Ackermann. This material called for the services of editors with a clearer understanding of the mathematical issues involved. The editors attempt to defend the coherence of Finsler's axiom III against the objections of Baer, Specker, and others; no such defense is possible. The paper by Ziegler on paradoxes of self-reference presents a "solution" to paradoxes of self-reference which makes it impossible for two letters in an algebraic expression to refer to the same object! (Finsler's approach is much more sensible.) The editors should have presented an explanation of the real relation of Finsler's derivation of a "formally undecidable proposition" to the later work of Gödel; they do not. (Finsler does not actually succeed in presenting a formally undecidable proposition, though he is on the right track; his "proposition" would need a truth predicate, forbidden by Tarski's theorem, to actually be expressible in his language).

The combination of the fascinating but ultimately untenable set theoretical claims of Finsler and the unsatisfactory support provided to the reader by the editors led the reviewer to write an extended review of the book, which can be found on his Web page, `http://math.idbsu.edu/faculty/holmes.html`

Aczel, Peter, *Non-well-founded sets*, CSLI, Stanford, 1988.

Booth, David and Ziegler, Renatus, *Finsler Set Theory: Platonism and Cir-*

cularity, Birkäuser-Verlag, Basel, 1996.

Holmes, M. Randall, "Review of "Finsler Set Theory: Platonism and Circularity", David Booth and Renatus Ziegler, eds.", unpublished, available at `http://math.idbsu.edu/faculty/holmes.html`

Levy, Azriel, "The role of classes in set theory", in Müller, Gert, ed., *Sets and Classes*, North Holland, Amsterdam, 1976. See pp. 207-212 on Ackermann's theory.