$P \to Q$

$\neg P \lor Q$

$\overset{A}{(P \mid P)} \lor \overset{B}{Q}$

$((P \mid P) \mid Q) \mid ((P \mid P) \mid Q)$

$\neg P$ is $P \mid P$

$A \lor B = \neg(A \mid B)$

so $A \lor B = (A \mid B) \mid (A \mid B)$

$\neg P \quad P \downarrow P$

$\neg(P \lor Q) \quad P \downarrow Q$

$\neg(\neg P \land \neg Q) \quad \neg(P \land Q)$

$(P \to Q) \land (Q \to P)$

$P \lor Q$

$[(P \land Q) \lor (\neg P \land \neg Q)$

Implication and Negation

$$\frac{P \to Q \quad \neg Q}{\neg P}$$ modus tollens

Proof: $P \to Q$

Assume $\neg Q$

Goal: $\neg P$

indirect proof

disjunctive syllogism

a method for using disjunctions

$$\frac{P \vee Q \quad \neg P}{Q} \qquad \frac{P \vee Q \quad \neg Q}{P}$$

$$\frac{\neg P \vee Q \quad P}{Q} \qquad \frac{P \vee \neg Q \quad Q}{P}$$

} disjunctive syllogism

① $P \vee Q$ premise

② $\neg P$ premise

Goal: Q

prove by cases on ①

Case I: Assume (1a) P Goal: Q

(2a) $\bot$ contra 2, 1a

(3a) Q ex falso 1b

Case II: Assume (1b) Q Goal: Q

(2b) Q copy 1b

(4) Q proof by cases 1, 1a-3a, 1b-2b

$P \vee Q$

$\neg P$

P

Alternate Elimination The general method of proving a disjunction

Pre $A \vee B$

| $\neg A$ assume (m)
| Goal: B
| .
| .
| .
| B (n)

$\neg A \to B$ deduction

(m,n) $A \vee B$ alternate elimination m-n

A	B	$A \vee B$			$\neg A$		$\neg A \to B$	
T	T	T	T	T	F	T	T	T
T	F	T	T	F	F	T	T	F
F	T	F	T	T	T	F	T	T
F	F	F	F	F	T	F	F	F

Theorem: $(A \vee B) \leftrightarrow (\neg A \to B)$

Part I: Assume ① $A \vee B$

Goal: $\neg A \to B$

Assume ② $\neg A$

Goal B

Proof by cases on ①

Case I Assume ①a A

②a $\bot$ contra 2, 1a

③a B ex falso 2a

Case II Assume ①b B

②b B copy 1b

④ proof by cases 1, 1a-3a, 1b-2b

⑤ $\neg A \to B$ deduction 2-4

Part II: Assume ⑥ $\neg A \to B$

Goal: $A \vee B$

Assume ⑦ $\neg(A \vee B)$ for the sake of a contradiction

Goal: $\bot$

Goal: A

Assume ⑧ $\neg A$ for reductio

Goal: $\bot$

⑨ B m.p. 8, 6

⑩ $A \vee B$ addition 9

⑪ $\bot$ contra 10, 7

⑫ A reductio 8-11

⑬ $A \vee B$ addition 12

⑭ $\bot$ contra 7, 13

⑮ $A \vee B$ reductio 7-14

Prove: $((A \wedge B) \vee (B \wedge C)) \to B$

Assume ① $(A \wedge B) \vee (B \wedge C)$

Goal: B

Proof by cases on 1

Case I: Assume ⑭ $A \wedge B$

Goal: B

⑳ B simpl a

Case II: Assume ⑮ $B \wedge C$

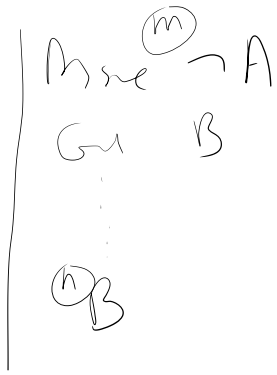
Goal B

㉕ C simpl a

③ B proof by cases 1, 14-20, 15-25

$((A \wedge B) \vee (B \wedge C)) \to B$ ded. 1-3

Prove $A \vee B$:



(nn) $\neg A \to B$ ded m-n

(nn2) $(A \vee B) \leftrightarrow (\neg A \to B)$ theorem

(nn3) (nn) $A \vee B$ b.mp nn1, nn2 alternate elimination m-n

$P \vee Q$
$P \to R$
$Q \to S$

$R \vee S$

Constructive dilemma

$P \vee Q$
$P \to R$
$Q \to R$

R

proof by cases

① $P \vee Q$ premise
② $P \to R$ premise
③ $Q \to S$ premise

Goal: $R \vee S$

proof by cases on ①:

Case I Assume (1a) P
Goal: $R \vee S$
(2a) R mp 1a, 2
(3a) $R \vee S$ add 2a

Case II Assume (1b) Q
Goal $R \vee S$
(2b) S mp. 1b, 3
(3b) $R \vee S$ add 2b

$R \vee S$ proof by cases 1, 1a-3a, 1b-3b

① $P \vee Q$ prem

② $P \to R$ prem

③ $Q \to S$ prem

Goal: $R \vee S$

Assume ④ $\neg R$ [for classical elimination]

Goal: S

⑤ $\neg P$ m.t. 2, 4

⑥ Q d.s. 5, 1

⑦ S m.p. 3, 6

Excluded middle: Prove ① $A \vee \neg A$

Assume ② $\neg \neg A$ for a.e.

Goal A

③ A d.no. on 2

④ $A \vee \neg A$ a.e. 2-3

Theorem: $\neg (P \to Q) \leftrightarrow (P \wedge \neg Q)$

Part 1:

Assume ① $\neg (P \to Q)$

Goal: $P \wedge \neg Q$

Goal 1: P

Assume ② $\neg P$ for reductio

Goal $\bot$

Goal ③ $P \to Q$

Assume ④ P
Goal: Q
⑤ ⊥ contra 2, 4
⑥ Q ex falso
⑦ P → Q ded 4-6
⑧ ⊥ contra 1, 7
⑨ P reductio 2-8

Goal 2: ¬Q
Assume ⑩ Q
Goal: ⊥
Goal: P → Q
Assume ⑬ P
Goal: Q
⑭ Q copy 12
⑮ P → Q deduction 13-14
⑯ ⊥ 1, 15
⑰ ¬Q 10-16
⑱ P ∧ ¬Q conj 9, 17

Part II: Assume ⑲ P ∧ ¬Q
Goal: * ¬(P → Q)
Assume ⑳ P → Q
Goal: ⊥
㉑ P simp 19

| (22) Q imp. 20,21
| (23) $\neg Q$ simp 19
| (24) $\bot$ 22,23 contra

(25) $\neg(P \to Q)$ neg into 20-24