Holmes

Math 189 section 3 Fall 2021: Test I

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October 4, 2021

You will have the period (12 pm – 1:15 pm) as the official time to take the test. At 1:15 pm I will actually give a five minute warning to get your paper to me.

Laws of propositional logic from the zybook section on calculation with logical equivalences are attached, as is a set of rules from the manual of logical style. You are welcome to detach these sections from the test to use them more easily.

You are allowed to bring one standard sized sheet of notebook paper to the test with whatever you like written or printed on it. There is no use for a calculator on this test (calculators will be allowed on future exams, with restrictions).

Each of the eight numbered problems on the test has the same weight as any other.

Lettered or numbered parts of problems have equal weight except that in problem 4 (manual of logical style proofs) and problem 8 (even/odd and divisibility proofs), each of which is a problem with two parts, the weight of the part you do best in will be much higher (70 percent) then the weight of the part you do worse on (30 percent).

18

1. Inverse converse and contrapositive

   I assert that if Poltroon wins the third race, I will eat my hat.

   (a) State in English the inverse of this conditional statement.

   If Poltroon does not win the third race, I will not eat my hat

   (b) State in English the converse of this conditional statement.

   If I eat my hat, Poltroon will win the third race

   (c) State in English the contrapositive of this conditional statement.

   If I do not eat my hat, Poltroon will not win the third race

   (d) We have four statements now (including the original one): indicate which ones are equivalent to each other.

   original ≡ d, b ≡ c

   (e) State the exact conditions under which the contrapositive of this statement will be true and its converse will be false, if this is possible.

   If Poltroon does not win the third race, and I do eat my hat.

2. Use the method of truth tables to show that $P \to (Q \to R)$ is logically equivalent to $(P \wedge Q) \to R$.

   Make sure that you put the table in the standard format with letters in alphabetical order, for ease of grading.

   Be sure to state in English the fact about rows or columns of the table which verifies the logical equivalence, highlighting all relevant rows or columns.

P	Q	R	$P \to (Q \to R)$					$(P \wedge Q) \to R$				
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	F	T	F	F	T	T	T	F	F
T	F	T	T	T	F	T	T	T	F	F	T	T
T	F	F	T	T	F	T	F	T	F	F	T	F
F	T	T	F	T	T	T	T	F	F	T	T	T
F	T	F	F	T	T	F	F	F	F	T	T	F
F	F	T	F	T	F	T	T	F	F	F	T	T
F	F	F	F	T	F	T	F	F	F	F	T	F

Because these columns are the same, the two statements are logically equivalent (they have the same truth value in every case)

3. One of the proposed logical rules

$$\frac{\begin{array}{c} P \\ Q \to P \end{array}}{Q}$$

and

$$\frac{\begin{array}{c} P \\ \neg P \vee Q \end{array}}{Q}$$ is valid and one is invalid.

Use truth table methods to show that the one is valid and the other is invalid.

Highlight relevant rows and/or columns and say in English why they establish validity or invalidity.

	P	Q	P (premises)	$Q \to P$ (premises)	Q (conclusion)
1	T	T	T	T	T
2	T	F	T	T	F
3	F	T	F	F	T
4	F	F	F	T	F

line 2 ~~is the only one in which the premises are~~ has the premises true and the conclusion false, so the argument is invalid

	P	Q	P (premises)	$\neg P \vee Q$ (premises)	conclusion
1	T	T	T	T	T
2	T	F	T	F	F
3	F	T	F	T	T
4	F	F	F	T	F

The only line in which both premises are true is 1, and on that line the conclusion is also true, so it is valid.

(extra page if wanted for problem 3)

method drop point do Harder

4. Prove two theorems using the mathod of the manual of logical style
Using the rules in the manual of logical style, prove the two theorems.

(a)

$$((P \to Q) \land (Q \to R)) \to (P \to R)$$

Assume, ① $(P \to Q) \land (Q \to R)$

Goal: $P \to R$

Assume ② P

Goal R

③ $P \to Q$ simp 1

④ $Q \to R$ simp 1

⑤ Q mp 2,3

⑥ R mp 4,5

7 $P \to R$ deduction 2-6

8 The theorem deduction 1-7

(b)

$$(P \vee \neg Q) \wedge (Q \vee R) \rightarrow (P \vee R)$$

Hint: use deduction, then alternative elimination, then applications of disjunctive syllogism.

Assume 1 $(P \vee \neg Q) \wedge (Q \vee R)$

Goal: $P \vee R$

Assume 2 $\neg P$

Goal: R

3 $P \vee \neg Q$ simp 1

4 $Q \vee R$ simp 1

5 $\neg Q$ d.s. 2, 3

6. R d.s, 4, 5

7 $P \vee R$ alternative elimination 2-6

8. the theorem deduction 1-7

5. Prove something using the laws of propositional logic

   Use the laws of propositional logic from section 1.5 (in the second appendix to the test) to show that $P \to (Q \to R)$ is logically equivalent to $(P \wedge Q) \to R$. You are welcome to use Boolean algebra notation in working this problem if you prefer it. Show all steps (make sure for example that you write all parentheses so that you are not skipping applications of the associative law). Name the rule you use at each step of the calculation.

$P \to (Q \to R) \equiv$ cond. identity

$\neg P \vee (Q \to R) \equiv$ cond identity

$\neg P \vee (\neg Q \vee R) \equiv$ assoc $\vee$

$(\neg P \vee \neg Q) \vee R \equiv$ de Morgan

$\neg(P \wedge Q) \vee R \equiv$ cond. identity

$P \wedge Q \to R$

6. Read some statements about numbers in quantifier language and class as true or false; write some statements about numbers in quantifier language

Follow the instructions in each part.

(a) The statement $(\forall xy : (\exists z : x + z = y))$ is true if the domain of the quantifiers is the set of real numbers. How would you describe z in terms of x and y?

$z = y - x$

Does it remain true if the domain is the set of integers? If it is false, give a counterexample.

It remains true: integers are closed under subtraction

Does it remain true if the domain is the set of positive real numbers? If it is false, give a counterexample.

It is false: $x = 2$ $y = 1$ is counterexample ($1-2$ is negative)

(b) One of the statements $(\forall x : (\exists y : y = x^2)$ and $(\forall x : (\exists y : y^2 = x))$ is true if the quantifiers range over all real numbers, and one is false.

Translate each sentence into a natural statement in mathematical English (no variables!). Say which one is true, which one is false, and give a counterexample to indicate why the false one is false.

How could the domain of the variables be restricted to make both statements true?

Every real number has a square true

Every real number has a square root

false: if $x < 0$ there is no real y s.t. $y^2 = x$. So $x = -1$ is a counterexample.

The domain could be restricted to positive reals, making the statements both true.

(c) Write the statement which we express in English as "There is no largest integer" using nothing but <, letters, and logical symbols. Hint: think of it as saying, for every integer, there is a larger one.

$(\forall x \in \mathbb{Z} : \exists y \in \mathbb{Z} : x < y)$

7. de Morgan's law for quantifiers

(a) How would you express the negation of the sentence

$$(\forall x : \exists y : P(x,y) \rightarrow P(y,x))$$

in a form in which all negations are directly in front of $P(x,y)$ or $P(y,x)$? (you need to remember how to negate an implication).

$(\exists x : \forall y : P(x,y) \wedge \neg P(y,x))$

Extra credit: without knowing anything about P, this sentence actually must be true and its negation must be false. If you can briefly state why, extra points might be awarded.

For each x, take $y = x$ and we have $P(x,y) \rightarrow P(y,x)$ actually means $P(x,x) \rightarrow P(x,x)$, which is true.

~~$y = x$~~

$x = y$ gives counterexamples to the negation.

(b) Suppose we are in a context in which people who don't love each other, necessarily hate each other. We use $x\,L\,y$ for "x loves y". Write the following symbolic expressions and their negations as natural English sentences about love or hate (with no variables).

i. $(\forall x : (\exists y : x\,L\,y))$

Everybody loves somebody

The negation is

Someone hates everyone

ii. $(\exists y : (\forall x : x\,L\,y))$

Someone is loved by every one

Everyone is hated by someone

one could do symbolic calculation to check the results for the negations.

8. Write a proof about even and odd numbers or divisibility

Prove the two theorems in the style presented in the zybook.

(a) The square of any odd integer is odd.
The definition of "x is odd" is "There is an integer k such that $x = 2k + 1$".

Let x be an arbitrary chosen integer.
Assume that x is odd.
By assumption, we can choose an integer k s.t. $x = 2k+1$.
Our goal is to find an integer n such that
$2n+1 = x^2$.

Plugging in, we get
$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
Now by choice $2k^2 + 2k$ is an integer, so
$n := 2k^2 + 2k$ gives ~~[illegible]~~ $x^2 = 2n+1$, n an integer, so x^2 is odd.
and we have shown this for an arbitrary x so
($\forall x$: x is odd $\rightarrow$ x^2 is odd).

(b) For any integers x, y, z, if $x|y$ and $x|z$ then $x|(y+z)$.
The definition of $x|y$ ("x goes into y') is "There is an integer k such that $y = kx$".

Let x, y, z be arbitrary integers.

Suppose ① $x | y$ and ② $x | z$.

Then ③ we can choose an integer k such that $y = kx$

and ④ we can choose an integer l such that $z = lx$

so (plugging in from 3, 4)

~~x~~ $y + z = kx + lx = (k+l)x.$

$k+l$ is an integer by closure property so if we define n an integer as $k+l$ we have ~~x~~ $z = nx$, and

$x | z$ follows by definition.

We have shown that for any x, y, z, if $x|y$ and ~~y~~ $x|z$ we have $x | (y+z)$.