Math 189 Fall 2022: Final Examination

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The final exam is officially given Wednesday, Dec 14 2:30–4:30 pm. Unless someone raises an objection, I will actually collect papers at 4:45 pm. You may use a plain scientific calculator without graphing or symbolic computation features.

There are some notes about logical rules attached, which you may tear off. They support problem 7.

1. Fill in each sentence with $\in$ or $\subseteq$ in such a way as to make it true. If both work, say both, if neither work, say neither.

   (a) $\{a, b\}\_\_\_\{b, \{\{a, b\}\}, a\}$

   (b) $\{-3\}\_\_\_\mathbb{Z}$

   (c) $\emptyset\_\_\_\{d, e, \emptyset\}$

   (d) $x\_\_\_\{x, y\}$

2. In a sophomore class of 23 students at a small school, every student takes at least one of English, Math, French. 15 take English, 11 take Math and 14 take French. 5 take English and Math, 8 take Math and French, and 8 take English and French. How many brave students are taking all three subjects?

3. Each of these four questions about k choices from n alternatives is answered in a different way, because of different combinations of conditions: in some we are allowed to repeat choices, and in some we are not; in some the order in which we make choice matters and in others it does not. Briefly answer each question, and **include calculations and brief explanation of the conditions which apply.**

   (a) A committee with 10 members wants to choose a three member executive committee. In how many ways can this be done?

   (b) How many five letter "words" (they don't need to be in the dictionary or even pronouncable) are possible for you to make, assuming that that you have at least 5 of each letter in your bag of letter tiles?

(c) You go to the florist and order a bunch of a dozen roses. There are pink, white, and red roses. How many bunches of a dozen are possible?

(d) 7 scrabble tiles with different letters on them are on the table in front of you. You idly make a 4 letter "word" using these tiles (no requirement that it be in the dictionary or even possible to pronounce). How many ways can you do this?

4. Euclidean algorithm; find a modular reciprocal and solve a modular equation.

   The three tasks are all connected!

   (a) Find integers x and y such that $225x + 121y = \texttt{gcd}(225, 121)$. Show all work. This should include the usual table and should also make it clear that you know what x is, what y is and what $\texttt{gcd}(225, 121)$ is.

   (b) Find the reciprocal of 121 in mod 225 arithmetic.

   (c) Solve the equation $121z \equiv_{225} 5$ for z. Your answer should be a remainder mod 225.

5. Number theory 2 Chinese remainder theorem

   Solve the system of equations

   $x \equiv_{225} 46$

   $x \equiv_{121} 86$

   Give the smallest positive solution and the general solution.

6. Eulerian walks and trails

   Explain why a graph with an Eulerian walk (a path which visits each edge in the graph exactly once) will have either zero or two vertices of odd degree.

   Two graphs are pictured. In one there is an Eulerian walk (a walk which visits each edge in the graph exactly once); in the other there is not. Present the walk in the graph which has one as a sequence of vertices (vertices can be repeated, of course); explain briefly why the graph which does not have one cannot have one.

7. For each statement given, write a formal proof using our rules of propoisitional logic. The one you do better on will count for 7 points out of 10 and the one you do worse on for 3. A summary of our formal rules is attached to the test (not the sample test, you can use the manual for that).

   If you wish, you may replace one of the parts with a truth table demonstration that the rule of *modus tollens* is valid. An extra page is supplied for this. If you do all three, your best work will be used.

   (a) Prove $((A \land B) \lor (B \land C)) \to B$ (hint: use proof by cases)

(b) Prove $((A \vee \neg B) \wedge (\neg A \vee C)) \to (B \to C)$ (hint: use disjunctive syllogism)

Intentional extra page

8. Give a proof in the style discussed in class of the statement "The product of two odd numbers is odd". First write the sentence out with appropriate use of variables and of quantifiers (English "for all" or "for some") is fine), making it clear that there is an implication in the statement, then prove it.

   The definition of "odd" is taken to be: An integer x is odd iff there is an integer k such that $x = 2k + 1$.

9. Do both parts. Proofs by mathematical induction are expected. The part on which you do better will count 70 percent and the part you do worse on 30 percent.

   In both parts, be sure to clearly identify the basis step, the induction hypothesis, the induction goal, and show where the induction hypothesis is used in the proof of the induction goal.

   I give an extra part because it is a practice exam.

   (a) Prove that the sum of the first n odd integers is n^2. State the theorem using summation notation, then prove it by mathematical induction.

(b) Prove using mathematical induction that $10^n - 1$ is divisible by 9 for each natural number n.

1 Proof strategies from the manual of logical style (for problem 7)

1.1 Conjunction

In this section we give rules for handling "and". These are so simple that we barely notice that they exist!

1.2 Proving a conjunction

To prove a statement of the form $A \wedge B$, first prove A, then prove B.

This strategy can actually be presented as a rule of inference:

$$\begin{array}{l} A \\ B \\ \hline A \wedge B \end{array}$$

If we have hypotheses A and B, we can draw the conclusion $A \wedge B$: so a strategy for proving $A \wedge B$ is to first prove A then prove B. This gives a proof in two parts, but notice that there are no assumptions being introduced in the two parts: they are not separate cases.

If we give this rule a name at all, we call it "conjunction".

1.2.1 Using a conjunction

If we are entitled to assume $A \wedge B$, we are further entitled to assume A and B. This can be summarized in two rules of inference:

$$\begin{array}{l} A \wedge B \\ \hline A \end{array}$$

$$\begin{array}{l} A \wedge B \\ \hline B \end{array}$$

This has the same flavor as the rule for proving a conjunction: a conjunction just breaks apart into its component parts.

If we give this rule a name at all, we call it "simplification".

1.3 Implication

In this section we give rules for implication. There is a single basic rule for implication in each subsection, and then some derived rules which also involve negation, based on the equivalence of an implication with its contrapositive. These are called derived rules because they can actually be justified in terms of the basic rules. We like the derived rules, though, because they allow us to write proofs more compactly.

1.3.1 Proving an implication

The basic strategy for proving an implication: To prove $A \to B$, add A to your list of assumptions and prove B; if you can do this, $A \to B$ follows without the additional assumption.

Stylistically, we indent the part of the proof consisting of statements depending on the additional assumption A: once we are done proving B under the assumption and thus proving $A \to B$ without the assumption, we discard the assumption and thus no longer regard the indented group of lines as proved.

This rule is called "deduction".

The indirect strategy for proving an implication: To prove $A \to B$, add $\neg B$ as a new assumption and prove $\neg A$: if you can do this, $A \to B$ follows without the additional assumption. Notice that this amounts to proving $\neg B \to \neg A$ using the basic strategy, which is why it works.

This rule is called "proof by contrapositive" or "indirect proof".

1.3.2 Using an implication

modus ponens: If you are entitled to assume A and you are entitled to assume $A \to B$, then you are also entitled to assume B. This can be written as a rule of inference:

$$\frac{\begin{array}{l} A \\ A \to B \end{array}}{B}$$

when you just have an implication: If you are entitled to assume $A \to B$, you may at any time adopt A as a new goal, for the sake of proving

B, and as soon as you have proved it, you also are entitled to assume B. Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus ponens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

modus tollens: If you are entitled to assume $\neg B$ and you are entitled to assume $A \to B$, then you are also entitled to assume $\neg A$. This can be written as a rule of inference:

$$\frac{\begin{array}{l} A \to B \\ \neg B \end{array}}{\neg A}$$

Notice that if we replace $A \to B$ with the equivalent contrapositive $\neg B \to \neg A$, then this becomes an example of modus ponens. This is why it works.

when you just have an implication: If you are entitled to assume $A \to B$, you may at any time adopt $\neg B$ as a new goal, for the sake of proving $\neg A$, and as soon as you have proved it, you also are entitled to assume $\neg A$. Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus tollens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

1.4 Absurdity

The symbol $\perp$ represents a convenient fixed false statement. The point of having this symbol is that it makes the rules for negation much cleaner.

1.4.1 Proving the absurd

We certainly hope we never do this except under assumptions! If we are entitled to assume A and we are entitled to assume $\neg A$, then we are entitled to assume $\perp$. Oops! This rule is called *contradiction*.

$$\frac{\begin{array}{c} A \\ \neg A \end{array}}{\perp}$$

1.4.2 Using the absurd

We hope we never really get to use it, but it is very useful. If we are entitled to assume $\perp$, we are further entitled to assume A (no matter what A is). From a false statement, anything follows. We can see that this is valid by considering the truth table for implication.

This rule is called "absurdity elimination".

1.5 Negation

The rules involving just negation are stated here. We have already seen derived rules of implication using negation, and we will see derived rules of disjunction using negation below.

1.5.1 Proving a negation

direct proof of a negation (basic): To prove $\neg A$, add A as an assumption and prove $\perp$. If you complete this proof of $\perp$ with the additional assumption, you are entitled to conclude $\neg A$ without the additional assumption (which of course you now want to drop like a hot potato!). This is the direct proof of a negative statement: proof by contradiction, which we describe next, is subtly different.

Call this rule "negation introduction".

proof by contradiction (derived): To prove a statement A of any logical form at all, assume $\neg A$ and prove $\perp$. If you can prove this under the additional assumption, then you can conclude A under no additional assumptions. Notice that the proof by contradiction of A is a direct proof of the statement $\neg\neg A$, which we know is logically equivalent to A; this is why this strategy works.

Call this rule "reductio ad absurdum".

1.5.2 Using a negation:

double negation (basic): If you are entitled to assume $\neg\neg A$, you are entitled to assume A. Call this rule "double negation elimination".

contradiction (basic): This is the same as the rule of contradiction stated above under proving the absurd: if you are entitled to assume A and you are entitled to assume $\neg A$, you are also entitled to assume $\perp$. You also feel deeply queasy.

$$\frac{\begin{array}{c} A \\ \neg A \end{array}}{\perp}$$

if you have just a negation: If you are entitled to assume $\neg A$, consider adopting A as a new goal: the point of this is that from $\neg A$ and A you would then be able to deduce $\perp$ from which you could further deduce whatever goal C you are currently working on. This is especially appealing as soon as the current goal to be proved becomes $\perp$, as the rule of contradiction is the only way there is to prove $\perp$.

1.6 Disjunction

In this section, we give basic rules for disjunction which do not involve negation, and derived rules which do. The derived rules can be said to be the default strategies for proving a disjunction, but they *can* be justified using the seemingly very weak basic rules (which are also very important rules, but often used in a "forward" way as rules of inference). The basic strategy for using an implication (proof by cases) is of course very often used and very important. The derived rules in this section are justified by the logical equivalence of $P \vee Q$ with both $\neg P \rightarrow Q$ and $\neg Q \rightarrow P$: if they look to you like rules of implication, that is because somewhere underneath they are.

1.6.1 Proving a disjunction

the basic rule for proving a disjunction (two forms): To prove $A \vee B$, prove A. Alternatively, to prove $A \vee B$, prove B. You do *not* need to prove both (you should not expect to be able to!)

This can also be presented as a rule of inference, called *addition*, which comes in two different versions.

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

the default rule for proving a disjunction (derived, two forms): To prove $A \vee B$, assume $\neg B$ and attempt to prove A. If A follows with the additional assumption, $A \vee B$ follows without it.

Alternatively (do not do both!): To prove $A \vee B$, assume $\neg A$ and attempt to prove B. If B follows with the additional assumption, $A \vee B$ follows without it.

Notice that the proofs obtained by these two methods are proofs of $\neg B \rightarrow A$ and $\neg A \rightarrow B$ respectively, and both of these are logically equivalent to $A \vee B$. This is why the rule works. Showing that this rule can be derived from the basic rules for disjunction is moderately hard.

Call both of these rules "disjunction introduction", or "alternative elimination".

1.6.2 Using a disjunction

proof by cases (basic): If you are entitled to assume $A \vee B$ and you are trying to prove C, first assume A and prove C (case 1); then assume B and attempt to prove C (case 2).

Notice that the two parts are proofs of $A \rightarrow C$ and $B \rightarrow C$, and notice that $(A \rightarrow C) \wedge (B \rightarrow C)$ is logically equivalent to $(A \vee B) \rightarrow C$ (this can be verified using a truth table).

This strategy is very important in practice.

disjunctive syllogism (derived, various forms): If you are entitled to assume $A \vee B$ and you are also entitled to assume $\neg B$, you are further entitled to assume A. Notice that replacing $A \vee B$ with the equivalent $\neg B \rightarrow A$ turns this into an example of modus ponens.

If you are entitled to assume $A \vee B$ and you are also entitled to assume $\neg A$, you are further entitled to assume B. Notice that replacing $A \vee B$ with the equivalent $\neg A \rightarrow B$ turns this into an example of modus ponens.

Combining this with double negation gives further forms: from B and $A \vee \neg B$ deduce A, for example.

Disjunctive syllogism in rule format:

$$\frac{\begin{array}{r} A \vee B \\ \neg B \end{array}}{A}$$

$$\frac{\begin{array}{r} A \vee B \\ \neg A \end{array}}{B}$$

Some other closely related forms which we also call "disjunctive syllogism":

$$\frac{\begin{array}{r} A \vee \neg B \\ B \end{array}}{A}$$

$$\frac{\begin{array}{r} \neg A \vee B \\ A \end{array}}{B}$$

1.7 Biconditional

Some of the rules for the biconditional are derived from the definition of $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$. There is a further very powerful rule allowing us to use biconditionals to justify replacements of one expression by another.

1.7.1 Proving biconditionals

the basic strategy for proving a biconditional: To prove $A \leftrightarrow B$, first assume A and prove B; then (finished with the first assumption) assume B and prove A. Notice that the first part is a proof of $A \rightarrow B$ and the second part is a proof of $B \rightarrow A$.

Call this rule "biconditional introduction".

derived forms: Replace one or both of the component proofs of implications with the contrapositive forms. For example one could first assume A and prove B, then assume $\neg A$ and prove $\neg B$ (changing part 2 to the contrapositive form).

1.7.2 Using biconditionals

The rules are all variations of modus ponens and modus tollens. Call them biconditional modus ponens (bimp) or biconditional modus tollens (bimt) as appropriate.

If you are entitled to assume A and $A \leftrightarrow B$, you are entitled to assume B.

If you are entitled to assume B and $A \leftrightarrow B$, you are entitled to assume A.

If you are entitled to assume $\neg A$ and $A \leftrightarrow B$, you are entitled to assume $\neg B$.

If you are entitled to assume $\neg B$ and $A \leftrightarrow B$, you are entitled to assume $\neg A$.

These all follow quite directly using modus ponens and modus tollens and one of these rules:

If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $A \rightarrow B$.

If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $B \rightarrow A$.

The validity of these rules is evident from the definition of a biconditional as a conjunction.