Axiomatics of the Reals

we have a set $\mathbb{R}$ with binary operations $+$ and $\cdot$

field axiom - ninth grade algebra

$x+y = y+x \qquad xy = yx$ commutative laws

$(x+y)+z = x+(y+z) \quad (xy)z = x(yz)$ associative laws

we can regroup and reorder terms in a complicated sum or factors in a complicated product

This means we can write $x+y+z$, xyz

we have specific elements 0 and 1 of $\mathbb{R}$

$0+x = x+0 = x \qquad 1 \cdot x = x \cdot 1 = x$ identity laws

$x+(-x) = 0 \qquad x \cdot x^{-1} = 1$ (where $x \neq 0$) inverse laws

$$0 \neq 1$$

$x \cdot (y+z) = xy + xz \quad (x+y) \cdot z = xz + yz$ distributive laws

A system satisfying these axioms is called a field

$\mathbb{Q}$ is a field

$\mathbb{R}$ is a field

$\mathbb{Z}_p$ arithmetic mod p where p is prime is a field for each p.

We add some secret sauce to make $\mathbb{Z}_p$ go away.

Add an order relation

1. dichotomy for any $x, y \in \mathbb{R}$, exactly one of $x<y$, $x=y$, $y<x$ holds
2. transitivity

   for any $x, y, z \in \mathbb{R}$, $x<y \wedge y<z \rightarrow x<z$

$x \leq y$ is defined as $x<y \cup x=y$

$x>y$ is defined as $y<x$

$x \geq y$ is defined as $y \leq x$

3. monotonicity

   for any $x, y, z \in \mathbb{R}$, $x<y \rightarrow x+z<y+z$

   Assume ~~$x+z+-<$~~

   $x+z<y+z$ then $x+z+(-z)<y+z+(-z)$, so $x<y$.

$x-y$ can be defined as $x+(-y)$, and $\frac{x}{y}$ can be defined as xy^{-1} $(y \neq 0)$

4. positivity

   If $x>0$ and $y>0$, then $xy>0$.

If $x < y$ and $z > 0$ then $xz < yz$. $\quad 2 < 3$ but $-2 > -3$

If $x < y$ and $z < 0$ then $xz > yz$

If $x < y$ and $z > 0$ then

$y > x$, so $y - x > 0$

so $(y-x)z > 0$ by positivity

so $yz - xz > 0$ so $yz > xz$ so $xz < yz$.

"obvious facts"

$x \cdot 0 = 0$

$$x \cdot 0 \overset{\text{identity}}{=} x \cdot 0 + 0 \overset{\text{inverse}}{=} x \cdot 0 + \left(\underline{x \cdot 0 + -(x \cdot 0)}\right)$$

$$= (x \cdot 0 + x \cdot 0) + -(x \cdot 0)$$

$$= x \cdot (0 + 0) + -(x \cdot 0)$$

$$= x \cdot 0 + -(x \cdot 0)$$

$$= 0$$

$x \cdot (-y) = -(x \cdot y)$

Lemma: if $x+y=0$
then $y=-x$

Proof: $x+y=0 \to -x+(x+y) = -x+0$
$\Rightarrow y=-x$

$$x\cdot y + x\cdot(-y) = x\cdot(y+-y) = x\cdot 0 = 0$$

so $x\cdot(-y) = -(x\cdot y)$ by the lemma

$$(x-y)\cdot z = (x+(-y))\cdot z = x\cdot z + (-y)\cdot z$$
$$= xz + -(y\cdot z)$$
$$= xz - yz$$

if $x<y$ and $z<0$ then $xz > yz$.

$x<y \to \cancel{y} \; 0<y-x \to y-x>0.$

$z<0 \to 0<-z$ so $-z>0$

so $(y-x)(-z)>0$ so $\cancel{yz}$
$y(-z) - x(-z) > 0$
$-yz + -(-xz)$
$\cancel{xz > y} \; xz - yz > 0$
so $xz > yz$.

$-(-x) = x$ if $x+y=0$ then $y=-x$
$x+-x=0$ so $x = -(-x)$

$-x$ is not (necessarily) a negative number.

$(-x)(-y) = -((-x)\cdot y) = -(-(x\cdot y)) = xy$

not the same as

$x<0$ and $y<0 \Rightarrow xy>0$.

~~If~~ Proof:

if $x<0$ then $0<-x$ so $-x>0$

$y<0$ so $0<-y$ so $-y>0$

so (positivity) $(-x)(-y)>0$ ~~but~~

$xy=(-x)(-y)>0$.

Let $x,y \in \mathbb{R}$. if $xy>0$ then either $x>0$ and $y>0$

or $x<0$ and $y<0$.

Assume $xy>0$.

By trichotomy, $x>0$ or $x=0$ or $x<0$.

$x=0$ can be excluded: if $x=0$, $xy=0$ X

if $x>0$ (case 1)

$y>0$ ~~so~~ $x>0$ and $y>0$ so $\left(\begin{array}{l} x>0 \text{ and } y>0 \\ \text{or } x<0 \text{ and } y<0 \end{array}\right)$

or

$y=0$ impossible $xy=0$ X

or

$y<0$ if $x>0$ and $y<0$ then $xy<x0$ so $\overset{xy<0}{\cancel{x<0}}$

If $x<0$ case 2

$y>0 \rightarrow xy<0y$, $xy<0$ 'X'

or

$y=0$ impossible

or

$y<0$ ~~$x<0$ and $y<0 \rightarrow xy>0$~~

so $x<0$ and $y<0$ so

($x>0$ and $y>0$) or ($x<0$ and $y<0$)

Complex numbers make up a field but not an ordered field.

In an ordered field we can prove $1>0$

so $-1<0$

and we can prove (exercise) $x^2 \geq 0$

But $i^2<0$ so $\mathbb{C}$ cannot be an ordered field.

For example $\mathbb{Z}_3$ is not an ordered field

$1+1+1=0$ holds in $\mathbb{Z}_3$

$1>0$

$1+1>1$

$1+1+1>1+1$

$1+1+1>0$ hmm drawing

...tion: for any set $A \subseteq \mathbb{R}$ which is nonempty and bounded above, $\sup(A)$ exists, that is A has a least upper bound.

Theorem: for any set $A \subseteq \mathbb{R}$ which is nonempty and bounded below, $\inf(A)$ exists (A has a greatest lower bound)

Suppose $A \subseteq \mathbb{R}$ and A is nonempty and

$(\forall a \in A : b \leq a)$ {b is a lower bound}

show that A has a greatest lower bound.

Consider the set $-A$ defined as $\{-a : a \in A\}$.

$-A$ is nonempty (obvious)

For any x in $-A$, $-x \in A$ so $b \leq -x$

so $x \leq -b$

so $-b$ is an upper bound for $-A$

so $-A$ has a least upper bound c.

We claim that $-c$ is the greatest lower bound of A.

1. $-c$ is a lower bound for A:

If $a \in A$ then $-a \in -A$ and $-a \leq c$

so $a \leq -c$. so $-c$ is an upper bound for A

2. $-c$ is the greatest lower bound of A.

Suppose b is a lower bound of A.

We claim that $-c \geq b$.

~~Since b is a lower bound for A~~

Suppose $x \in -A$. Then $-x \in A$

so $-x \geq b$

so $x \leq -b$

so $-b$ is an upper bound for $-A$

so $-b \leq c$

so ~~...~~

$-c \leq b$.