Homework 3 Solutions

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They are supposed to do six of these, best six count if they do more.

1.1.1, If $x < 0$ and $y < z$, then $xy > xz$

They are supposed to prove this from the given properties of order. Look at the actual axioms when deciding whether what they do makes sense.

$y < z$, so $y + (-y) < z + (-y)$ (1.17i), that is, $0 < z + (-y)$, and $z + (-y) > 0$.

$x < 0$ so $x + (-x) < 0 + (-x)$ (1.17i) so $0 < -x$ so $-x > 0$.

It follows that $(-x)(z + (-y)) > 0$ (1.17ii) and $(-x)(z + (-y)) = xy - xz > 0$ so $(xy - xz) + xz > 0 + xz$ that is, $xy > xz$.

Feel free to ask me if you see other patterns of reasoning how much I would award for them. Reasoning from given sets of axioms about things they know perfectly well is always tricky to mark.

1.1.3, Suppose $0 < x < y$. Show that $x^2 < y^2$.

We have $y > 0$, so $y + x > 0 + x$ (add x to both sides) so $y + x > 0$ (transitivity).

We have $x < y$ so $y > x$ so $y + (-x) > x + (-x) = 0$.

Thus we have $y + x$ and $y - x$ positive, so $(y + x)(y - x) > 0$, so $y^2 - x^2 > 0$, so (add x^2 to both sides) $y^2 > x^2$ so $x^2 < y^2$.

Notice in 1.1.1 and 1.1.3 I am using equational algebra pretty freely but I stick to the exact basic properties of order that the section gives, which are in definition 1.1.1 and definition 1.1.7.

1.1.4 (tricky, I may offer advice if you ask: this needs to be proved from the defini

I did this more concretely than the book asks for, talking about sets of real numbers where they are talking about an abstract ordered set S in which all the specified sups and infs exist. The argument is the same.

Let A be a nonempty subset of B, a nonempty set of reals which is bounded above and below. Show that $\texttt{inf}(B) \leq \texttt{inf}(A) \leq \texttt{sup}(A) \leq \texttt{sup}(B)$.

let m be a lower bound for B and let M be an upper bound for B: we are given that there are such numbers.

For any element a of A, $a \in B$ because $A \subseteq B$, so $m \leq a$. Thus m is a lower bound for A, so $m \leq \texttt{inf}(A)$ because $\texttt{inf}(A)$ is the greatest lower bound of A (which exists by the completeness property). This is true for *any* lower bound of B, and $\texttt{inf}(B)$ is a lower bound of B, existing by the completeness property, so we have $\texttt{inf}(B) \leq \texttt{inf}(A)$, the first inequality to be proved.

A is nonempty: let a be an element of A. $\texttt{inf}(A)$ exists by the completeness property and is a lower bound for A, so $\texttt{inf}(A) \leq a$. $\texttt{sup}(A)$ exists by the completeness property and is an upper bound for A, so $a \leq \texttt{sup}(A)$. By transitivity, $\texttt{inf}(A) \leq \texttt{sup}(A)$, the second inequality to be proved.

For any element a of A, $a \in B$ because $A \subseteq B$, sp $M \geq a$. Thus M is an upper bound for A, so $M \geq \texttt{sup}(A)$ because $\texttt{sup}(A)$ is the least upper bound of A (which exists by the completeness property). This is true for *any* upper bound of B, and $\texttt{sup}(B)$ is an upper bound of B, existing by the completeness property, so we have $\texttt{sup}(B) \geq \texttt{sup}(A)$, so $\texttt{sup}(A) \leq \texttt{sup}(B)$, the third inequality to be proved.

1.1.5 (I've commented on this), Suppose that S is an ordered set, $A \subseteq S$, and $b \in A$ is an upper bound for A. Show that $b = \sup(A)$.

We suppose that b is an upper bound for A. Suppose $c < b$: c cannot be an upper bound for A because we would have to have $b \leq c$, contrary to hypothesis about c. So b is the least upper bound for A, so $b = \sup(A)$ by definition of sup.

1.1.8, They should come up with the addition and multiplication tables of mod 3 arithmetic.

In the addition table, all sums involving 0 are handled by the identity property. $1 + 1 = 1$ cannot be true, because adding -1 to both sides would give $1 = 0$. So $1 + 1 = 2$. 1 has to have an additive inverse, it isn't 1 itself, so it is 2, so $1 + 2 = 2 + 1 = 0$. $2 + 2 = 2 + (1 + 1) = (2 + 1) + 1 = 0 + 1 = 1$.

In the multiplication table, every entry except $2 \cdot 2$ follows from the identity or zero properties of multiplication. 2 has to have a multiplicative inverse and it cannot be 0 or 1, so it must be 2 itself, so $2 \cdot 2 = 1$.

I'd be interested to see reasoning, but if they get mod 3 arithmetic give them the benefit of the doubt.

It can't be an ordered field: in any ordered field, we have $0 < 1$. Add 1 to both sides and we get $1 < 2$. Add 1 to both sides and we get $2 < 0$. Transitivity gives $1 < 0$ among other absurdities.

1.1.9, Let S be an ordered set and suppose $A \subset S$ and $\sup(A)$ exists.

Suppose $B \subseteq A$ and for any $x \in A$ there is $y \in B$ such that $y > x$.

Show that $\sup(B)$ exists and $\sup(B) = \sup(A)$.

Proof: for any $b \in B$, we have $b \in A$, so $b \leq \sup(A)$: so $\sup(A)$ is an upper bound for B.

Suppose that $c < \sup(A)$ is an upper bound for B. Since $c < \sup(A)$ we have that c is not an upper bound for A, so for some $a \in A$ we have $c < a$, and by hypotheses there is $d \in B$ such that $a < d$, so since $c < d \in B$ c is not an upper bound for B, so in fact $\sup(A)$ is the smallest upper bound for B, so $\sup(A) = \sup(B)$.

It is crucial to use the fact that $c < \sup(A)$ implies that something in A is above c; certainly you can't assume $c \in A$, which I can imagine a student doing.

1.1.11, Suppose $x \leq y$ and $z \leq w$ and deduce $x + z \leq y + w$.

Proof: $x \leq y$ implies (1) $x + z \leq y + z$ by adding z to both sides (which works for equations and less than statements). $z \leq w$ implies (2) $y + z \leq y + w$ by adding y to both sides. $x + z \leq y + w$ follows by transitivity of $\leq$ (which they may assume but which is easy: $a \leq b$ and $b \leq c$ obviously imply $a \leq c$ if $a = b$ or $b = c$, and otherwise apply transitivity of $<$).

if $x < y$ and $z \leq w$ show that $x + z < y + w$: if $z = w$ then $x < y$ implies $x + z < y + z$ which implies $x + z < y + w$ by substitution.

if $z < w$ then $x < y$ implies $x + z < y + z$, $z < w$ implies $y + z < y + w$, and $x + z < y + w$ follows by transitivity.

1.2.1, Prove that if $t > 0$ there is an $n \in \mathbb{N}$ such that $\frac{1}{n^2} < t$.

This is equivalent to $\frac{1}{t} < n^2$. Note that $\frac{1}{t} < n$ will work because $n \leq n^2$ will be true for $n \in \mathbb{N}$.

So, let $t > 0$ be chosen arbitrarily. $\frac{1}{t} > 0$ follows. By the Archimedean property there is $n \in \mathbb{N}$ such that $\frac{1}{t} < n$. We then have $\frac{1}{t} < n \leq n^2$, and multiplying through by $\frac{t}{n^2}$ we get $\frac{1}{n^2} < t$.

1.2.2, Prove that if $t \geq 0$ there is $n \in \mathbb{N}$ such that $n - 1 \leq t < n$.

Consider the set S of all natural numbers greater than t. It is nonempty by the Archimedean property (if $t > 0$; if $t = 0$ it is clearly nonempty). and so has a smallest element n by the well-ordering property of $\mathbb{N}$. $n > t$ by the definition of S. $n - 1 \notin S$ by choice of n as the smallest element of S, so $n - 1 \leq t$. So $n - 1 \leq t < n$.

1.2.13 (hint: binomial theorem), I don't think my binomial theorem hint works. The problem is that the conditions allow $x < 0$.

Prove it by induction on n.

$1 + x \leq 1 + x$, basis.

Suppose $1 + kx \leq (1 + x)^k$. [ind hyp]

So $(1 + kx)(1 + x) \leq (1 + x)^{k+1}$ [this uses the fact that $1 + x \geq 0$).

So $1 + (k + 1)x + kx^2 \leq (1 + x)^{k+1}$

and certainly $1 + (k + 1)x \leq 1 + (k + 1)x + kx^2 \leq (1 + x)^{k+1}$.

1.2.15 Show that for any real number y, the supremum of the set of rationals less than y is y:

Suppose otherwise. Let $z < y$ be the supremum of the set of rational numbers less than y. Then the interval (z, y) is of positive length and contains no rational number. This contradicts theorem 1.2.4ii.

By the way, the set of rational numbers less than y needs to be seen to be nonempty to do this (though I wouldnt fault a student for not noticing: this is easy, as there is a natural number N greater than $|y|$ (Archimedean property) and certainly $-N < -|y| \leq y$.

Define a Dedekind cut as a set of rationals which is downward closed as a subset of the rationals, bounded above, and has no largest element [this is the same as what Lebl says]: show that if D is a Dedekind cut, then $D = \{x \in \mathbb{Q} : x < y\}$ for some real number y: proof: let $y = \sup(D)$. Obviously any element of D is a rational less than y [since D has no largest element it cannot have its sup y as an element]. We also need to show that any rational less than y is in D: if $r < y$ is rational, then there is an element s of D with $r < s < D$ (because r cannot be an upper bound for D) and D is a downward closed subset of the rationals, so $r \in D$ because r is a rational less than s.

Show that there is a bijection between $\mathbb{R}$ and Dedekind cuts: this is really just a remark after the previous part is proved: the correspondence between reals y and the sets $\{x \in \mathbb{Q} : x < y\}$, which are the Dedekind cuts, clearly is a one to one correspondence.