

No guarantees
that I haven't
made errors!

Exam I, Math 189, Fall 2023 (Solutions)

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This exam will be given to the class on Friday Oct 6 in the regular class period. At 245 I will give a ten minute warning before collecting papers.

You are allowed one standard sized sheet of notebook paper with whatever notes you like written on it. You are allowed a standard scientific calculation without graphing or symbolic computation capabilities.

1. Do two of the three parts. If you do all three, your best work will count. The formal rules from the manual of logical style are summarized in an appendix to the test, which you may tear off for convenience.
- (a) Set up and fill out a truth table verifying the validity of one of the forms of the rule of disjunctive syllogism,

$$\frac{A \vee \neg B \quad B}{A}$$

Provide an explanation in English which makes it clear that you understand why your table verifies this.

①

A	B	$A \vee \neg B$	B	A
T	T	F	T	T
T	F	T	F	T
F	T	F	T	F
F	F	T	F	F

In row 1, the only row where both premises are true, A is true. (the conclusion)

(b) Prove

$$((B \vee \neg A) \wedge (C \vee \neg B)) \rightarrow (A \rightarrow C)$$

using the rules in the manual. Hint: set it up using the standard method of proving an implication, then use disjunctive syllogism.

Prove $((B \vee \neg A) \wedge (C \vee \neg B)) \rightarrow A \rightarrow C$

Assume $(B \vee \neg A) \wedge (C \vee \neg B)$

Goal: $A \rightarrow C$

Assume A

Goal C

③ $B \vee \neg A$ simp 1

④ B d.s. 2,3

⑤ $C \vee \neg B$ simp 1

⑥ C d.s. 4,5

⑦ $A \rightarrow C$ deduction 2-6

8 the theorem deduction 1-7

(c) Verify the validity of the rule of constructive dilemma,

$$\frac{\begin{array}{l} P \vee Q \\ P \rightarrow R \\ Q \rightarrow S \end{array}}{R \vee S}$$

with a proof in the style of the manual. This question is NOT asking for a truth table.

① $P \vee Q$ premise
 ② $P \rightarrow R$ premise
 ③ $Q \rightarrow S$ premise
 Goal: $R \vee S$
 Assume ④ $\neg R$ (for alternative elimination)
 Goal: S
 ⑤ $\neg P$ m.t. 2,4
 ⑥ Q d.s. 1,5
 ⑦ S m.p. 6,3
 ⑧ $R \vee S$ alternative elimination 4-7

2. Define “ a is odd” as “There is an integer x such that $a = 2x + 1$ ”.

Prove that the product of two odd integers is an odd integer. Prove this in the style taught in class, starting by translating the statement into an implication.

Prove For all x, y if x, y are odd integers then $x \cdot y$ is odd
Let $x, y \in \mathbb{Z}$ be chosen arbitrarily
Assume x is odd

① y is odd

so ③ we can choose $a \in \mathbb{Z}$ s.t. $2a+1 = x$

④ we can choose $b \in \mathbb{Z}$ s.t. $2b+1 = y$

Restate the goal: Find $c \in \mathbb{Z}$ s.t. $2c+1 = xy$

$$xy = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 =$$

$$2(2ab + a + b) + 1$$

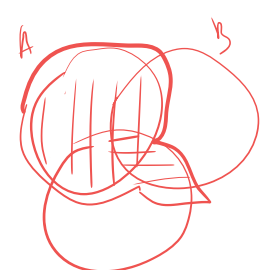
$2ab + b + 1$ is an integer by closure property,

Let $c = 2ab + b + 1$, an integer, and we have $2c+1 = xy$ ✓

3. (a) Use Venn diagrams to verify the identity

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Do this by drawing a Venn diagram showing how each side of the equation is computed (provide a key to any shadings you use and outlining the set which is the result of the calculation): the outlined part of the diagram for each side should be the same.



$$A \cup (B \cap C)$$

||| A \equiv B \cap C



$$(A \cup B) \cap (A \cup C)$$

||| A \cup B \equiv A \cup C

Same result set
outlined in
both cases,
equal.

- (b) Fill in the appropriate relation between the indicated items (\in , \subseteq , both or neither)

We officially assume here that numbers are not sets.

- i. $\emptyset_{\{\emptyset\}}$
- ii. $\{1, 2\}_{\{3, \{1, 2\}, 1\}}$
- iii. $2_{\{\{1, 2, 3\}\}}$
- iv. $\{3\}_{\mathbb{N}}$

$$\emptyset \in \{\emptyset\} \text{ und } \emptyset \subseteq \{\emptyset\}$$

$$\{1, 2\} \in \{3, \{1, 2\}, 1\}$$

$$2 \in \{1, 2, 3\}$$

$$\{3\} \subseteq \mathbb{N}$$

sen -
typos
here in
problem...

4. In each part, I give the domain, codomain, and graph of a relation (the graph in two line notation). In each case, tell me whether what you are given is a function (and explain, mentioning specific domain and/or codomain elements), and if it is a function, whether it is an injection (and explain, mentioning specific domain and/or codomain elements), and whether it is a surjection (and explain, mentioning specific domain and/or codomain elements). If it has an inverse function, write a representation of the inverse in the same notation, with the domain elements in the top row in standard order.

(a) domain $\{1, 2, 3\}$, codomain $\{a, b\}$, graph $\begin{pmatrix} 1 & 2 & 2 & 3 \\ b & a & c & b \end{pmatrix}$

not a function, 2 related to both a, c

(b) domain $\{1, 2, 3\}$, codomain $\{a, b, c\}$, graph $\begin{pmatrix} 1 & 2 & 3 \\ a & b & b \end{pmatrix}$

a function, not one-to-one

because 2, 3 both relate to b

not a surjection, c not used

(c) domain $\{1, 2, 3\}$, codomain $\{a, b, c\}$, graph $\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$

a function,

injection and surjection

inverse $\begin{pmatrix} a & b & c \\ 3 & 1 & 2 \end{pmatrix}$

5. 1.1 A certain state has license plates consisting of three letters followed by four digits. In each part, give a numerical answer and the supporting calculation.

(a) How many plates are possible if no additional restrictions are imposed?

$$26^3 \cdot 10^4$$

$$175760000$$

(b) How many plates are possible if no letter or digit can appear more than once on the plate?

$$26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 \cdot 7$$

$$78624000$$

- (c) How many plates contain at least one A (with no other restrictions)?

$$25^3 \cdot 10^4 \text{ contain no A's}$$

$$26^3 \cdot 10^4 - 25^3 \cdot 10^4 = 19510000$$

- (d) How many plates contain exactly one 8?

$$26^3 \cdot 4 \cdot 9^3 = 51251616$$

\uparrow \uparrow
 first the
 of 8 other
 digits

6. Do two of the three parts. If you do all three, your best work will count.

- (a) Write out the seventh row of Pascal's triangle without constructing the rows above it. Your paper should show work supporting this using only multiplication and division (not binomial coefficients or factorials).

$$\binom{1}{1} \cdot \frac{6}{1} = \binom{2}{1} \cdot \frac{5}{3} = \binom{3}{1} \cdot \frac{4}{4} = \binom{3}{5} \cdot \frac{3}{5} = \binom{2}{1} \cdot \frac{2}{6} = \binom{1}{1} \cdot \frac{1}{1}$$

- (b) How many ways are there to divide nine children into three teams of three, the Lions, the Tigers, and the Bears?

$$\binom{9}{3} \cdot \binom{6}{3} \cdot \binom{3}{3} = 1680$$

How many ways are there to divide nine children into three teams?
(this is not the same question!)

$$\frac{1680}{3!} = 280$$

- (c) How many anagrams are there of the word ABRACADABRA?

$$\frac{11!}{5! 2! 2!} = 83160$$

7. Four problems are presented. They are all of the form “make k choices from n alternatives”: in some cases order matters, in some it does not, in some cases repetitions are allowed, in some cases they are not. For each problem, do the (short) calculation of the answer and identify which kind of problem it is (order matters or not, repetitions allowed or not).

- (a) You have a huge bag of Scrabble tiles with at least five of each letter. In how many ways can you arrange five letters on your Scrabble tray (a fancy way of asking, in how many ways can you form five “words”?)

$$26^5 = 11881376$$

order matters
repetitions are allowed

- (b) Ten runners compete in a race. Prizes will be awarded for first second and third place. There will be no ties. How many possible outcomes are there?

$$10 \cdot 9 \cdot 8 = 720$$

order matters
repetitions are not allowed

- (c) If you have a large supply of pennies, nickels, dimes and quarters in a jar, reach in and pull out a handful of eight, how many possible outcomes are there?

$$\binom{8+4-1}{8} = 165$$

order
doesn't matter
reps allowed

00/00/00/00 8 coins 3 positions
 $\binom{11}{3}$ or $\binom{11}{8}$

- (d) Forty novelists are competing for the five book shortlist of a publisher. The publisher will list the shortlist in alphabetical order by last name of author. How many outcomes are possible?

order doesn't
matter,
reps not allowed

$$\binom{40}{5} = 658,008$$

8. Two parts:

- (a) The 24 students at my favorite exclusive prep school are each sweating through English, Russian, and/or Math.

13 students take English, 12 take Russian, and 11 (the best!) take Math

6 take English and Russian, 8 take English and Math, 5 take Russian and Math

How many students are suffering through all three?

7

How many students are taking just Math?

$$13 + 12 + 11 - 6 - 8 - 5 + x = 24$$

$$x = 9$$

solution
is
impossible
(this question was
scrubbed)

- (b) Write down an expression for the number of elements of the union of four sets A, B, C, D in terms of the sizes of the sets and their various intersections (to be lectured on Monday the 2nd)

$$\begin{aligned}
 |A \cup B \cup C \cup D| = & \\
 & |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| \\
 & - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| \\
 & + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|
 \end{aligned}$$

1 Proof strategies from the manual of logical style

1.1 Conjunction

In this section we give rules for handling “and”. These are so simple that we barely notice that they exist!

1.2 Proving a conjunction

To prove a statement of the form $A \wedge B$, first prove A , then prove B .

This strategy can actually be presented as a rule of inference:

$$\frac{\begin{array}{c} A \\ B \end{array}}{A \wedge B}$$

If we have hypotheses A and B , we can draw the conclusion $A \wedge B$: so a strategy for proving $A \wedge B$ is to first prove A then prove B . This gives a proof in two parts, but notice that there are no assumptions being introduced in the two parts: they are not separate cases.

This rule is called “conjunction”.

1.2.1 Using a conjunction

If we are entitled to assume $A \wedge B$, we are further entitled to assume A and B . This can be summarized in two rules of inference:

$$\frac{A \wedge B}{A}$$
$$\frac{A \wedge B}{B}$$

This has the same flavor as the rule for proving a conjunction: a conjunction just breaks apart into its component parts.

This rule is called “simplification”.

1.3 Implication

In this section we give rules for implication. There is a single basic rule for implication in each subsection, and then some derived rules which also involve negation, based on the equivalence of an implication with its contrapositive. These are called derived rules because they can actually be justified in terms of the basic rules. We like the derived rules, though, because they allow us to write proofs more compactly.

1.3.1 Proving an implication

The basic strategy for proving an implication: To prove $A \rightarrow B$, add A to your list of assumptions and prove B ; if you can do this, $A \rightarrow B$ follows without the additional assumption.

Stylistically, we indent the part of the proof consisting of statements depending on the additional assumption A : once we are done proving B under the assumption and thus proving $A \rightarrow B$ without the assumption, we discard the assumption and thus no longer regard the indented group of lines as proved.

This rule is called “deduction”.

The indirect strategy for proving an implication: To prove $A \rightarrow B$, add $\neg B$ as a new assumption and prove $\neg A$: if you can do this, $A \rightarrow B$ follows without the additional assumption. Notice that this amounts to proving $\neg B \rightarrow \neg A$ using the basic strategy, which is why it works.

This rule is called “proof by contrapositive” or “indirect proof”.

1.3.2 Using an implication

modus ponens: If you are entitled to assume A and you are entitled to assume $A \rightarrow B$, then you are also entitled to assume B . This can be written as a rule of inference:

$$\frac{A \quad A \rightarrow B}{B}$$

when you just have an implication: If you are entitled to assume $A \rightarrow B$, you may at any time adopt A as a new goal, for the sake of proving

B , and as soon as you have proved it, you also are entitled to assume B . Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus ponens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

modus tollens: If you are entitled to assume $\neg B$ and you are entitled to assume $A \rightarrow B$, then you are also entitled to assume $\neg A$. This can be written as a rule of inference:

$$\frac{A \rightarrow B \quad \neg B}{\neg A}$$

Notice that if we replace $A \rightarrow B$ with the equivalent contrapositive $\neg B \rightarrow \neg A$, then this becomes an example of *modus ponens*. This is why it works.

when you just have an implication: If you are entitled to assume $A \rightarrow B$, you may at any time adopt $\neg B$ as a new goal, for the sake of proving $\neg A$, and as soon as you have proved it, you also are entitled to assume $\neg A$. Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus tollens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

1.4 Absurdity

The symbol \perp represents a convenient fixed false statement. The point of having this symbol is that it makes the rules for negation much cleaner.

1.4.1 Proving the absurd

We certainly hope we never do this except under assumptions! If we are entitled to assume A and we are entitled to assume $\neg A$, then we are entitled to assume \perp . Oops! This rule is called *contradiction*.

$$\frac{A \quad \neg A}{\perp}$$

1.4.2 Using the absurd

We hope we never really get to use it, but it is very useful. If we are entitled to assume \perp , we are further entitled to assume A (no matter what A is). From a false statement, anything follows. We can see that this is valid by considering the truth table for implication.

This rule is called “absurdity elimination” or “ex falso”.

1.5 Negation

The rules involving just negation are stated here. We have already seen derived rules of implication using negation, and we will see derived rules of disjunction using negation below.

1.5.1 Proving a negation

direct proof of a negation (basic): To prove $\neg A$, add A as an assumption and prove \perp . If you complete this proof of \perp with the additional assumption, you are entitled to conclude $\neg A$ without the additional assumption (which of course you now want to drop like a hot potato!). This is the direct proof of a negative statement: proof by contradiction, which we describe next, is subtly different.

Call this rule “negation introduction”. You won’t be marked off if you call it “reductio ad absurdum”, but it is not quite the same thing.

proof by contradiction (derived): To prove a statement A of any logical form at all, assume $\neg A$ and prove \perp . If you can prove this under the additional assumption, then you can conclude A under no additional assumptions. Notice that the proof by contradiction of A is a direct proof of the statement $\neg\neg A$, which we know is logically equivalent to A ; this is why this strategy works.

Call this rule “reductio ad absurdum”.

1.5.2 Using a negation:

double negation (basic): If you are entitled to assume $\neg\neg A$, you are entitled to assume A . Call this rule “double negation elimination”.

contradiction (basic): This is the same as the rule of contradiction stated above under proving the absurd: if you are entitled to assume A and you are entitled to assume $\neg A$, you are also entitled to assume \perp . You also feel deeply queasy.

$$\frac{A \quad \neg A}{\perp}$$

if you have just a negation: If you are entitled to assume $\neg A$, consider adopting A as a new goal: the point of this is that from $\neg A$ and A you would then be able to deduce \perp from which you could further deduce whatever goal C you are currently working on. This is especially appealing as soon as the current goal to be proved becomes \perp , as the rule of contradiction is the only way there is to prove \perp .

1.6 Disjunction

In this section, we give basic rules for disjunction which do not involve negation, and derived rules which do. The derived rules can be said to be the default strategies for proving a disjunction, but they *can* be justified using the seemingly very weak basic rules (which are also very important rules, but often used in a “forward” way as rules of inference). The basic strategy for using an implication (proof by cases) is of course very often used and very important. The derived rules in this section are justified by the logical equivalence of $P \vee Q$ with both $\neg P \rightarrow Q$ and $\neg Q \rightarrow P$: if they look to you like rules of implication, that is because somewhere underneath they are.

1.6.1 Proving a disjunction

the basic rule for proving a disjunction (two forms): To prove $A \vee B$, prove A . Alternatively, to prove $A \vee B$, prove B . You do *not* need to prove both (you should not expect to be able to!)

This can also be presented as a rule of inference, called *addition*, which comes in two different versions.

$$\frac{A}{A \vee B}$$

$$\frac{B}{A \vee B}$$

the default rule for proving a disjunction (derived, two forms): To prove $A \vee B$, assume $\neg B$ and attempt to prove A . If A follows with the additional assumption, $A \vee B$ follows without it.

Alternatively (do not do both!): To prove $A \vee B$, assume $\neg A$ and attempt to prove B . If B follows with the additional assumption, $A \vee B$ follows without it.

Notice that the proofs obtained by these two methods are proofs of $\neg B \rightarrow A$ and $\neg A \rightarrow B$ respectively, and both of these are logically equivalent to $A \vee B$. This is why the rule works. Showing that this rule can be derived from the basic rules for disjunction is moderately hard.

Call both of these rules “alternative elimination”.

1.6.2 Using a disjunction

proof by cases (basic): If you are entitled to assume $A \vee B$ and you are trying to prove C , first assume A and prove C (case 1); then assume B and attempt to prove C (case 2).

Notice that the two parts are proofs of $A \rightarrow C$ and $B \rightarrow C$, and notice that $(A \rightarrow C) \wedge (B \rightarrow C)$ is logically equivalent to $(A \vee B) \rightarrow C$ (this can be verified using a truth table).

This strategy is very important in practice.

disjunctive syllogism (derived, various forms): If you are entitled to assume $A \vee B$ and you are also entitled to assume $\neg B$, you are further entitled to assume A . Notice that replacing $A \vee B$ with the equivalent $\neg B \rightarrow A$ turns this into an example of modus ponens.

If you are entitled to assume $A \vee B$ and you are also entitled to assume $\neg A$, you are further entitled to assume B . Notice that replacing $A \vee B$ with the equivalent $\neg A \rightarrow B$ turns this into an example of modus ponens.

Combining this with double negation gives further forms: from B and $A \vee \neg B$ deduce A , for example.

Disjunctive syllogism in rule format:

$$\frac{A \vee B \quad \neg B}{A}$$

$$\frac{A \vee B \quad \neg A}{B}$$

Some other closely related forms which we also call “disjunctive syllogism”:

$$\frac{A \vee \neg B \quad B}{A}$$

$$\frac{\neg A \vee B \quad A}{B}$$

1.7 Biconditional

Some of the rules for the biconditional are derived from the definition of $A \leftrightarrow B$ as $(A \rightarrow B) \wedge (B \rightarrow A)$. There is a further very powerful rule allowing us to use biconditionals to justify replacements of one expression by another.

1.7.1 Proving biconditionals

the basic strategy for proving a biconditional: To prove $A \leftrightarrow B$, first assume A and prove B ; then (finished with the first assumption) assume B and prove A . Notice that the first part is a proof of $A \rightarrow B$ and the second part is a proof of $B \rightarrow A$.

Call this rule “biconditional introduction”.

derived forms: Replace one or both of the component proofs of implications with the contrapositive forms. For example one could first assume

A and prove B , then assume $\neg A$ and prove $\neg B$ (changing part 2 to the contrapositive form).

1.7.2 Using biconditionals

The rules are all variations of modus ponens and modus tollens. Call them biconditional modus ponens (bimp) or biconditional modus tollens (bimt) as appropriate.

If you are entitled to assume A and $A \leftrightarrow B$, you are entitled to assume B .

If you are entitled to assume B and $A \leftrightarrow B$, you are entitled to assume A .

If you are entitled to assume $\neg A$ and $A \leftrightarrow B$, you are entitled to assume $\neg B$.

If you are entitled to assume $\neg B$ and $A \leftrightarrow B$, you are entitled to assume $\neg A$.

These all follow quite directly using modus ponens and modus tollens and one of these rules:

If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $A \rightarrow B$.

If you are entitled to assume $A \leftrightarrow B$, you are entitled to assume $B \rightarrow A$.

The validity of these rules is evident from the definition of a biconditional as a conjunction.