Math 189 Fall 2022: Draft Final Examination

Dr Holmes

December 11, 2022

The final exam is officially given Wednesday, Dec 14 2:30–4:30 pm. Unless someone raises an objection, I will actually collect papers at 4:45 pm.

Solutions to this sample exam will be distributed before final exam week starts. This test was constructed largely by adapting problems from the hour exams, with two new questions on recent material. You should probaby be sure you are ready for any question on Test I and Test II, in case I make some change in my design decisions.

1. Fill in each sentence with $\in$ or $\subseteq$ in such a way as to make it true. If both work, say both, if neither work, say neither.

    (a) $\{a, b\}$___$\{b, \{a, b\}, a\}$ both

    (b) $\emptyset$ $\subseteq$ $\{d, e, f\}$

    (c) -3 $\in$ $\mathbb{Z}$

    (d) x___$\{\{x, y\}\}$ neither

2. In a sophomore class of 22 students at a small school, every student takes at least one of English, Math, French. 16 take English, 10 take Math and 14 take French. 5 take English and Math, 7 take Math and French, and 9 take English and French. How many brave students are taking all three subjects?

$$|E \cup M \cup F| = |E| + |M| + |F| - |E \cap M| - |E \cap F| - |F \cap M| + |E \cap F \cap M|$$

so $22 = 16 + 10 + 14 - 5 - 9 - 7 + x$

$x = 22 - 16 - 10 - 14 + 5 + 9 + 7 = \boxed{3}$

3 students took all three classes

3. Each of these four questions about k choices from n alternatives is answered in a different way, because of different combinations of conditions: in some we are allowed to repeat choices, and in some we are not; in some the order in which we make choice matters and in others it does not. Briefly answer each question, and **include calculations and brief explanation of the conditions which apply.**

(a) A committee with 8 members wants to choose a three member executive committee. In how many ways can this be done?

$$\binom{8}{3} = \frac{8\cdot7\cdot\cancel{6}}{\cancel{3}\cdot2\cdot1} = 56$$

order of choices does not matter

repeated choices are not allowed

(b) 8 scrabble tiles with different letters on them are on the table in front of you. You idly make a 5 letter "word" using these tiles (no requirement that it be in the dictionary or even possible to pronounce). How many ways can you do this?

$\cancel{26\cdot25\cdot24\cdot23}\cdot22$

$8\cdot7\cdot6\cdot5\cdot4 = 6720$

its correct, didnt need to erase anything

order matters

repetition arent allowed

(c) How many four letter "words" (they don't need to be in the dictionary or even pronouncable) are possible for you to make, assuming that that you have at least 4 of each letter in your bag of letter tiles?

~~26~~ 26^4 456976

~~repeat~~ order matters,
repetitions allowed

(d) You go to the florist and order a bunch of a dozen roses. There are pink, white, red and exotic genetically engineered blue roses. How many bunches of a dozen are possible?

$$\binom{12+4-1}{12} = \binom{12+4-1}{3} = 455$$

order does not matter
repetitions are allowed

4. Euclidean algorithm; find a modular reciprocal and solve a modular equation.

The three tasks are all connected!

(a) Find integers x and y such that $211x + 121y = \texttt{gcd}(211, 121)$. Show all work. This should include the usual table and should also make it clear that you know what x is, what y is and what $\texttt{gcd}(211, 121)$ is.

	x	y	q
211	1	0	
121	0	1	
90	1	-1	1
31	-1	2	1
28	3	-5	2
3	-4	7	1
1	39	-68	9

$\boxed{(39)(211) + (-68)(121) = \gcd(211,121) = 1}$

(b) Find the reciprocal of 121 in mod 211 arithmetic.

The reciprocal is $211 - 68 = \boxed{143}$

(c) Solve the equation $121z \equiv_{211} 5$ for z. Your answer should be a remainder mod 211.

$121z \equiv_{211} 5$

$(121)(143)z \equiv_{211} (5)(143)$

because reciprocal $\rightarrow$ $1 \cdot z \equiv_{211} 715$

$z \equiv_{211} 715 - 3(211) = \boxed{82}$ ← the solution

check: $(121)(82) = 9922 - (47)(211) = 5$ as desired

5. Number theory 2 Chinese remainder theorem

Solve the system of equations

$x \equiv_{211} 179$

$x \equiv_{121} 4$

Give the smallest positive solution and the general solution.

the way we usually do it, with the larger modulus

$x = 179 + 211k$ for some integer k by the first equation

So

$179 + 211k \equiv_{121} 4$

$58 + 90k \equiv_{121} 4$ (subtract 121 from the large values on left)

$90k \equiv_{121} 121 - 58 + 4 = 67$

reciprocal of 90 and 121 is 39 and this can be read from the table in the previous problem! the reciprocal of 211 and 121 is 39, and $211 \equiv_{121} 90$! You can set up a table to check of course

$\underbrace{(39)(90)}_{\equiv_{121} 1} k \equiv_{121} (67)(39) \equiv_{121} 2613 - (21)(121) \equiv_{121} \boxed{72}$

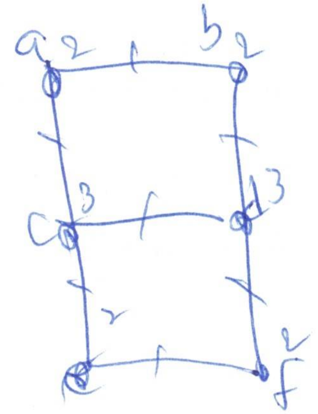
$x = 179 + (211)(72) = \boxed{15371}$ smallest solution

General solution is $15371 + 25531n$ for some integer n

25531 being $(211)(121)$

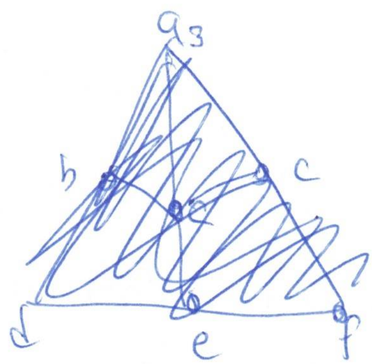
6. Eulerian walks and trails

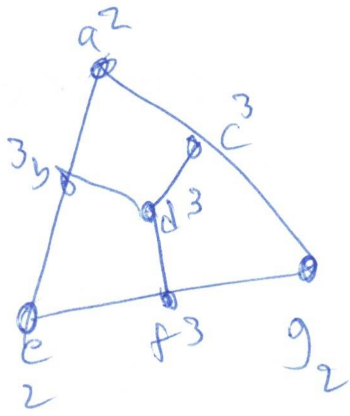
Two graphs are pictured. In one there is an Eulerian walk (a walk which visits each edge in the graph exactly once); in the other there is not. Present the walk in the graph which has one as a sequence of vertices (vertices can be repeated, of course); explain briefly why the graph which does not have one cannot have one.



we must start at one vertex of odd degree and end at the other.

c, a, b, d, c, e, f, d
for example





This one has 4 vertices of degree 3. A graph with an Eulerian walk can have no more than two ~~od~~ vertices of odd degree.

7. For each statement given, write a formal proof using our rules of propositional logic. The one you do better on will count for 7 points out of 10 and the one you do worse on for 3. A summary of our formal rules is attached to the test (not the sample test, you can use the manual for that).

I'm giving a third one because it is a practice test.

(a) Prove $((A \to B) \land (\neg C \to \neg B)) \to (A \to C)$

This one is tricky for a reason that won't happen on the actual test.

Prove $((A \to B) \land (\neg C \to \neg B)) \to A \to C$

Assume ① $(A \to B) \land (\neg C \to \neg B)$

Goal: ~~As~~ $A \to C$

Assume ② A

Goal C

③ $A \to B$ simp 1

④ B mp

sorry about that ⑤ $\neg\neg B$ dni, (double negation intro)

~~⑥ ... m.t. ...~~

⑥ $\neg C \to \neg B$ simp 1

⑦ C m.t. 6,5

8 $A \to C$ deduction 1-7

9 The theorem ded 1-8

I ought to allow

~~$A \to B$~~

~~B~~

~~A~~

~~as a form~~

$A \to \neg B$

B

A

as a form of modus tollens

I'll also list d.n.i as an allowable rule

(b) Prove $((A \wedge B) \vee (B \wedge C)) \rightarrow B$ (hint: use proof by cases)

Prove $((A \wedge B) \vee (B \wedge C)) \rightarrow B$

Assume 1 $(A \wedge B) \vee (B \wedge C)$

Goal B

Prove by cases on 1

Case 1 Assume 1a $A \wedge B$

Goal B

2a B simp 1

Case 2 Assume 1b $B \wedge C$

Goal: B

2b B simp 1b

3 B proof by cases 1, 1a–2a, 1b–2b

4 $((A \wedge B) \vee (B \wedge C)) \rightarrow B$ deduction 1–3

(c) Prove $((A \vee \neg B) \wedge (\neg A \vee C)) \to (B \to C)$ (hint: use disjunctive syllogism)

Pre ~~Assume~~ $((A \vee \neg B) \wedge (\neg A \vee C)) \to (B \to C)$

Assume ① $(A \vee \neg B) \wedge (\neg A \vee C)$

Goal: $B \to C$

Assume ② B

Goal: C

③ $A \vee \neg B$ simp 1

④ A d.s. 1,3

⑤ $\neg A \vee C$ simp 1

⑥ C d.s. 4,5

⑦ $A \to C$ ded. 2-6

⑧ the theorem deduction 1-7

Definition:
An integer n is odd iff there is an integer k s.t. $n = 2k+1$

an integer n is even iff there is an integer k s.t. $n = 2k$

8. Give a proof in the style discussed in class of the statement "The sum of two odd numbers is even". First write the sentence out with appropriate use of variables and of quantifiers (English "for all" or "for some") is fine), making it clear that there is an implication in the statement, then prove it.

Prove: The sum of two odd numbers is even

Restate: For any integers x, y, if x is odd and y is odd, then $x+y$ is even

Let $x, y \in \mathbb{Z}$ be chosen arbitrarily

Assume ① x ~~is~~ is odd

② y is odd

Goal: $x+y$ is even

by ① we can choose $k \in \mathbb{Z}$ s.t. ③ $x = 2k+1$

by ② we can choose $l \in \mathbb{Z}$ st. ④ $y = 2l+1$

Now our goal is "$x+y$ is even"
which we can rewrite as: Find an integer m s.t. $2m = x+y$

$$x+y \overset{3,4}{=} (2k+1)+(2l+1) \overset{\text{algebra}}{=} 2k+2l+2 \overset{\text{algebra}}{=} 2(k+l+1)$$

Let m be $k+l+1$, which is an integer (by closure properties of the integers)

then we have $2m = x+y$ and m is an integer,

so $x+y$ is even.

This completes the proof.

9. Do both parts. Proofs by mathematical induction are expected. The part on which you do better will count 70 percent and the part you do worse on 30 percent.

   In both parts, be sure to clearly identify the basis step, the induction hypothesis, the induction goal, and show where the induction hypothesis is used in the proof of the induction goal.

   I give an extra part because it is a practice exam.

   (a) Prove that the sum of the first n square integers is $\frac{n(n+1)(2n+1)}{6}$. State the theorem using summation notation, then prove it by mathematical induction.

~~Ba~~ Rewrite in summation form:

Prove that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

Basis: $\sum_{i=1}^{1} i^2 = 1^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$ ✓

Induction Step:

Let $k \geq 1$ be chosen arbitrarily

Assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

Goal: $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\sum_{i=1}^{k+1} i^2 = \left(\sum_{i=1}^{k} i^2\right) + (k+1)^2 \overset{\boxed{\text{ind hyp!}}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \overset{\text{algebra-steps omitted (which you would be expected to show)}}{=} \frac{(k+1)(k+2)(2k+3)}{6}$$

(b) Prove using mathematical induction that $n^3 + 5n$ is divisible by 3 for each natural number n.

(c) The Fibonacci numbers are defined by $F_1 = 1, F_2 = 1, F_i + F_{i+1} = F_{i+2}$.
Prove by mathematical induction that the sum of the first n Fibonacci numbers is $F_{n+2} - 1$.

Prove $3 \mid n^3 + 5n$ for any $n \geq 0$

Basis: $3 \mid 0^3 + 5n = 0$ is true.

Ind Step. Let $k \geq 0$ be chosen arbitrarily.
Assume $3 \mid (k^3 + 5k)$.
Goal: $3 \mid (k+1)^3 + 5(k+1)$

$$(k+1)^3 + 5(k+1) = k^3 + 3k^2 + 3k + 1 + 5k + 5$$
$$= (k^3 + 5k) + (3k^2 + 3k + 6)$$ which is divisible by 3.

divisible by 3, $\boxed{\text{ind hyp}}$ — divisible by 3 by inspection

---

Prove by math induction that for all $n \geq 1$, $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

Basis: $\sum_{i=1}^{1} F_i = F_1 = 1 = F_3 - 1 = 2 - 1$ ✓

Induction step: Let $k \geq 1$ be chosen arbitrarily.
Assume $\sum_{i=1}^{k} F_i = F_{k+2} - 1$
Goal: $\sum_{i=1}^{k+1} F_i = F_{k+3} - 1$

$$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1} \overset{\text{ind hyp}}{=} (F_{k+2} - 1) + F_{k+1} = (F_{k+1} + F_{k+2}) - 1$$
$$= F_{k+3} - 1$$
def of F_n!