Math 189 Fall 2023, Test II (solutions)

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The exam will begin at 130 and officially end at 245; what will actually happen at 245 is that I will give a ten minute warning and collect all papers at 255 sharp.

You are allowed a single sheet of notes and a non-graphing calculator, as on the first exam.

1. The *Lucas numbers* are defined by the recursive definition $L_1 = 1; L_2 = 3; L_{n+2} = L_{n+1} + L_n$. Compute L_8. (Of course, this will involve computing all of the first eight terms; I am not suggesting that you find the closed form!)

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8
1	3	4	7	11	18	29	47

answer

2. Arithmetic and geometric sequences. Do both parts.

(a) The sequence $\{a_i\}$ whose first few terms are $1, 4, 7, 10, 13\ldots$ is either arithmetic or geometric. Tell me which and give me a closed form formula for the nth term (the first term is a_1). Compute a_{50}. Compute the sum of the first 50 terms without actually adding them all up, and indicate your method.

arithmetic

$$a_n = 3n-2 = 1 + 3(n-1)$$

$$a_{50} = 148$$

$$\sum_{i=1}^{50} a_i = \frac{1+148}{2} \cdot 50 = 3725$$

(b) The sequence $\{b_i\}$ whose first few terms are $2, 6, 18, 54\ldots$ is either arithmetic or geometric. Tell me which and give me a closed form formula for the nth term (the first term is b_0). Compute b_{10}. Compute the sum of the first 11 terms (b_0 to b_{10}) without actually adding them all up, and indicate your method.

$$b_i = 2 \cdot 3^i$$

$$b_{10} = 2 \cdot 3^{10} = 118098$$

$$\sum_{i=1}^{10} b_i = \frac{3 \cdot 2^{11} - 3 \cdot 2^0}{3-1} = 177146$$

3. The sequence $\{c_i\}$ is given with the first few terms $3, 2, 9, 24, 47, 78 \ldots$. The first term is c_0.

   Apply the method of differences to it to construct enough sequences to tell me what the degree of the polynomial defining c_i is.

   Determine the closed form of c_i as a polynomial. You may use Levin's polynomial fitting method or my binomial coefficient method; your answer should be in the form of a fully simplified polynomial, whichever method you use.

$3, 2, 9, 24, 47, 78$

$-1 \quad 7 \quad 15 \quad 23 \quad 31$

$8 \quad 8 \quad 8 \quad 8$

is a second degree polynomial

$c_i = Ax^2 + Bx + C$

$3 = c_0 = A \cdot 0^2 + B \cdot 0^2 + C$ so $C = 3$

$2 = c_1 = A \cdot 1 + B \cdot 1 + 3$ so $A + B = -1$

$9 = c_2 = A \cdot 2^2 + B \cdot 2 + 3$ so $4A + 2B = 6$

$4A + 4B = -4$

$-2B = 10$

$B = -5$

$A = 4$

$c_i = 4i^2 - 5i + 3$

4. The sequence $\{d_i\}$ is defined by $d_0 = 3; d_1 = 5; d_{n+2} = 2d_{n+1} + 3d_n$.

   Compute up to d_6 using the recurrence relation.

   Use the method of characteristic equations to find a closed form formula for this sequence. Show all work.

3	5	19	53	163	485	1459
0	1	2	3	4	5	6

$d_6 = 1459$

char poly is $x^2 - 2x - 3 = 0$

$(x-3)(x+1)$

$d_i = A \cdot 3^i + B \cdot (-1)^i$

$A + B = 3$

$3A - B = 5$

$4A = 8$

$A = 2$

$\boxed{d_i = 2 \cdot 3^i + (-1)^i}$

5. Do one of the following proofs by mathematical induction. In your proof, be sure to clearly identify the basis step, the induction hypothesis, and the induction goal, and to highlight where in your proof the induction hypothesis is used.

   If you work on both proofs, your best work will count.

   (a) Prove that the sum $1 + 3 + 5 + \ldots + (2n - 1)$ of the first n odd numbers is equal to n^2, by mathematical induction.

   Write out the statement to be proved in summation notation. I strongly recommend that you use properties of summation notation in your proof.

Prove that $\sum_{i=1}^{n}(2i-1) = n^2$

Basis: $\sum_{i=1}^{1} 2\cdot i - 1 = 2\cdot 1 - 1 = 1 = 1^2$ ✓

Suppose that $\sum_{i=1}^{k}(2i-1) = k^2$ (ind hyp)

Goal is to show that $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$

Proof: $\sum_{i=1}^{k+1} 2i-1 = \sum_{i=1}^{k}(2i-1) + 2(k+1)-1$

$\overset{\text{ind hyp!}}{=} k^2 + 2k + 2 - 1$

$= k^2 + 2k + 1 = (k+1)^2$ as required

(b) Prove by mathematical induction that $n^3 + 8n$ is divisible by 3 for any positive integer n.

Basis: $(n=1)$: $1^3 + 8 \cdot 1 = 9 = 3 \cdot 3$ so $3 \mid 9$.

Ind Step: Let k be chosen arbitrarily.

Assume (ind hyp) that $3 \mid (k^3 + 8k)$

so $\exists w \in \mathbb{Z} : 3w = k^3 + 8k$

Goal: show that $3 \mid (k+1)^3 + 8(k+1)$

$(k+1)^3 + 8(k+1) = k^3 + 3k^2 + 3k + 1 + 8k + 8$

$= (k^3 + 8k) + (3k^2 + 3k + 9)$

$= 3w + 3(k^2 + k + 3) = 3(w + k^2 + k + 3)$

which is divisible by 3.

6. Find the greatest common divisor of 312 and 66 and express it in the form $311x + 66y$. Your work should clearly identify what $\texttt{gcd}(311, 66)$ is and what x and y are. You should use my table method, but do not rely on me to read the answer off your table: state it clearly after you finish the table work.

312	1	0	
66	0	1	
48	1	-4	4
18	-1	5	1
12	3	-14	2
$\boxed{6}$	-4	19	1
0			2

$\gcd(312, 66) = 6 = (-4)(312) + (19)(66)$

7. (a) Give the multiplication and addition tables for mod 7 arithmetic. Make separate tables of additive and multiplicative inverses in mod 7 arithmetic.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

a	-a
0	0
1	6
2	5
3	4
4	3
5	2
6	1

*	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

a	a^{-1}
0	–
1	1
2	4
3	5
4	2
5	3
6	6

(b) Compute the multiplicative inverse of 65 in mod 137 arithmetic. Then solve the equation $65x \equiv_{137} 4$.

137	1	0
65	0	1
7	1	-2
2	-9	19
1	28	-59

$137 - 59 = \boxed{78}$

$65 \cdot 78 x \equiv_{137} 4 \cdot 78$

$x \equiv_{137} 4 \cdot 78 \equiv_{137} \boxed{38}$

8. Prove one of the listed theorems. If you work on both of them, your best work will count.

   (a) Prove that there are infinitely many primes.

Suppose that there were finitely many primes $p_1, \dots, p_n$

Let $P = \left(\prod_{i=1}^{n} p_i\right) + 1$, the product of all the primes in our list, plus one. P must have a prime factor q.

q must be p_j for some j.

Then $q \mid P$ and $q \mid \prod_{i=1}^{n} p_i$ so $q \mid P - \prod_{i=1}^{n} p_i = 1$ and $q \mid 1$ is absurd.

(b) (Euclid's Lemma) Prove that if a, b are integers and ab is divisible by p then either p goes into a or p goes into b.

where p is a prime. This was left out...

~~since~~ suppose $p \mid ab$ and $p \nmid a$.

then $\gcd(p, a) = 1$, we have $u, v \in \mathbb{Z}$ s.t.

$$pu + av = 1$$

$$b = bpu + bav = p(bu) + (ab)v$$

which is divisible by p

so if $p \nmid a$, we must have $p \mid b$ - either $p \mid a$ or $p \mid b$.