SOLUTIONS

Exam I, Math 189, Fall 2024

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September 29, 2024

This exam will be given to the class on Monday, Sept 29, 2024 in the regular class period. At 1015 I will actually give a ten minute warning before collecting papers.

You are allowed one standard sized sheet of notebook paper with whatever notes you like written on it. You are allowed a standard scientific calculator without graphing or symbolic computation capabilities.

1. Verify the validity of the rule of modus tollens

$$\frac{\begin{array}{c} A \to B \\ \neg B \end{array}}{\neg A}$$

using a truth table. The rightmost columns of your truth table should be the premises and conclusion of the rule. Highlight rows relevant to the validity of the rule and give an explanation in English of why the table shows that the rule is valid.

	A	B	A→B	¬B	¬A
1	T	T	T	F	F
2	T	F	F	T	F
3	F	T	T	F	T
4	F	F	T	T	T

The only row in which the premises are both true is 4, and the conclusion is also true in that row

2. Prove

$$(P \to \neg R) \land (\neg Q \to R) \to (P \to Q)$$

using the formal rules taught in class. Hint: there is an application of modus tollens and of double negation elimination. Use rules actually listed in the appendix to the test: in particular, do not try to use the contrapositive theorem.

Prove $(P \to \neg R) \land (\neg Q \to R) \to P \to Q$

Assume (1) $(P \to \neg R) \land (\neg Q \to R)$

Goal $P \to Q$

Assume (2) P

Goal Q

(3) $P \to \neg R$ simp 1

(4) $\neg R$ mp 2,3

(5) $\neg Q \to R$ simp 1

(6) $\neg\neg Q$ m.t. 4,5

(7) Q dne 6

(8) $P \to Q$ deduction 2-7

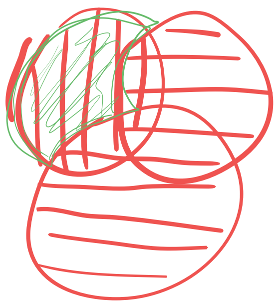
(9) the theorem ded 1-8

3. (a) Use Venn diagrams to verify the identity

$$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C).$$

Do this by drawing a Venn diagram showing how each side of the equation is computed (provide a key to any shadings you use and outlining the set which is the result of the calculation): the outlined part of the diagram for each side should be the same.

$A \setminus (B \cup C)$

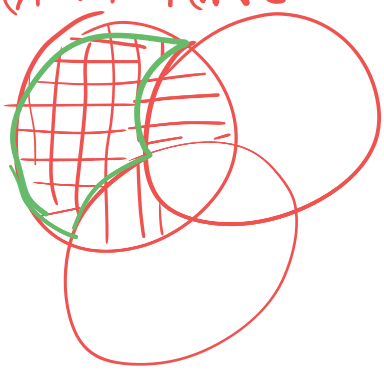


A |||

$B \cup C$ ≡

//// result set

$(A \setminus B) \cap (A \setminus C)$



$A \setminus B$ ||||

$A \setminus C$ ≡

□ result

(b) Fill in the appropriate relation between the indicated items ($\in$, $\subseteq$, both or neither)

We officially assume here that numbers are not sets.

i. $\emptyset$___ $\{\emptyset\}$ both
ii. $\{1, 2\}$___ $\{2, \{1, 2\}, 1\}$ both
iii. $\{2\}$___$\{1, 2, 3\}$ $\subseteq$
iv. 3___$\mathbb{N}$ $\in$

4. In each part, I give the domain, codomain, and graph of a relation (the graph in two line notation). In each case, tell me whether what you are given is a function (and explain, mentioning specific domain and/or codomain elements), and if it is a function, whether it is an injection (and explain, mentioning specific domain and/or codomain elements), and whether it is a surjection (and explain, mentioning specific domain and/or codomain elements). If it has an inverse function, write a representation of the inverse in the same notation, with the domain elements in the top row in standard order.

(a) domain $\{1, 2, 3\}$, codomain $\{a, b\}$, graph $\begin{pmatrix} 1 & 2 & 2 & 3 \\ b & a & b & a \end{pmatrix}$

not a function, $2 \to a$ and $2 \to b$

(b) domain $\{1, 2, 3\}$, codomain $\{a, b, c\}$, graph $\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix}$

bijection

inverse is $\begin{pmatrix} a & b & c \\ 1 & 3 & 2 \end{pmatrix}$

(c) domain $\{1, 2, 3\}$, codomain $\{a, b, c\}$, graph $\begin{pmatrix} 1 & 2 & 3 \\ b & a & a \end{pmatrix}$

function

not surjective c not used as a value

not injective because $f(2) = f(3) = a$

5. 1.1 A certain state has license plates consisting of three letters followed by three digits. In each part, give a numerical answer and the supporting calculation.

   (a) How many plates are possible if no additional restrictions are imposed?

   $26^3 \cdot 10^3 = 17576000$

   (b) How many plates are possible if no letter or digit can appear immediately following the same letter or digit?

   $26 \cdot 25 \cdot 25 \cdot 10 \cdot 9 \cdot 9 = 13162500$

(c) How many plates contain at least one 7 (with no other restrictions)?

$$\underbrace{26^3 \cdot 10^3}_{\text{all plates}} - \underbrace{26^3 \cdot 9^3}_{\text{the ones with no 7's}} = 4763096$$

(d) How many plates contain exactly one B?

$$\underbrace{3}_{\text{where is the B!}} \cdot \underbrace{25^2}_{\text{the other two letters}} \cdot \underbrace{10^3}_{\text{the digits}} = 1875000$$

6. Do both parts.

(a) Write out the eighth row of Pascal's triangle without constructing the rows above it. Your paper should show work supporting this using only multiplication and division (not binomial coefficients or factorials).

$$1 \cdot \frac{8}{1} = 8 \cdot \frac{7}{2} = 28 \cdot \frac{6}{3} = 56 \cdot \frac{5}{4} = 70 \cdot \frac{4}{5} = 56 \cdot \frac{3}{6} = 28 \cdot \frac{2}{7} = 8 \cdot \frac{1}{8} = 1$$

(b) How many anagrams are there of the word COLONOSCOPY? Give the supporting calculation and the numerical answer. The supporting calculation will involve factorials; you might want to do some fraction simplification and maybe some other clever moves before you compute the numerical answer, but you should be able to get it.

$$\frac{11! \leftarrow \text{all letters}}{\underset{\text{O's}}{\underset{\uparrow}{4!}}\ \underset{\text{C's}}{\underset{\uparrow}{2!}}} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{2} = 11\cdot 5\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5$$

831600

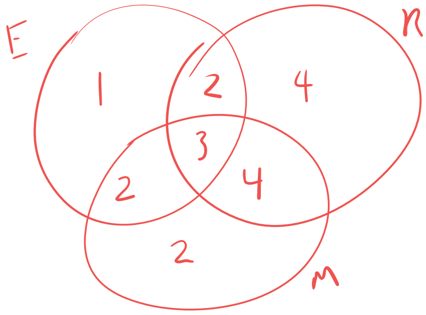
7. Some of the 23 students at my favorite exclusive prep school are each sweating through English, Russian, and/or Math.

   8 students take English, 13 take Russian, and 11 (the best!) take Math

   5 take English and Russian, 5 take English and Math, 7 take Russian and Math   3 take all three.

   How many students are having an easy semester taking other courses but none of these?

   Extra credit: How many students are taking just Math?



23 - 18 = 5 students
take none of the courses

2 take just math

$$18 = 8 + 13 + 11 - 5 - 5 - 7 + 3$$

8. How many functions are there from $\{1,2\}$ to $\{a,b,c\}$? How many of these do you expect to be injective? How many do you expect to be surjective. Give calculations or reasons for each of these numbers without using the list you make afterward to justify them.

There are $3^2 = 9$ functions

$3_2 = 3 \cdot 2 = 6$ of them will be injective

None of them will be surjective because $3 > 2$

List all of the functions from $\{1,2\}$ to $\{a,b,c\}$. You may give arrow diagrams or you may give their graphs in table notation or two line notation. No dashes are expected, though correct use of dashes will impress me. Identify each of the functions as injective, surjective, or neither.

$* \begin{pmatrix} 1 & 2 \\ a & a \end{pmatrix}$ $\quad + \begin{pmatrix} 1 & 2 \\ b & a \end{pmatrix}$ $\quad + \begin{pmatrix} 1 & 2 \\ c & a \end{pmatrix}$

$+ \begin{pmatrix} 1 & 2 \\ a & b \end{pmatrix}$ $\quad * \begin{pmatrix} 1 & 2 \\ b & b \end{pmatrix}$ $\quad + \begin{pmatrix} 1 & 2 \\ c & b \end{pmatrix}$

$+ \begin{pmatrix} 1 & 2 \\ a & c \end{pmatrix}$ $\quad + \begin{pmatrix} 1 & 2 \\ b & c \end{pmatrix}$ $\quad * \begin{pmatrix} 1 & 2 \\ c & c \end{pmatrix}$

none are surjective

$*$ = neither injective nor surjective

$+$ = injective

1 Proof strategies from the manual of logical style

1.1 Conjunction

In this section we give rules for handling "and". These are so simple that we barely notice that they exist!

1.2 Proving a conjunction

To prove a statement of the form $A \wedge B$, first prove A, then prove B.

This strategy can actually be presented as a rule of inference:

$$\begin{array}{l} A \\ B \\ \hline A \wedge B \end{array}$$

If we have hypotheses A and B, we can draw the conclusion $A \wedge B$: so a strategy for proving $A \wedge B$ is to first prove A then prove B. This gives a proof in two parts, but notice that there are no assumptions being introduced in the two parts: they are not separate cases.

This rule is called "conjunction".

1.2.1 Using a conjunction

If we are entitled to assume $A \wedge B$, we are further entitled to assume A and B. This can be summarized in two rules of inference:

$$\begin{array}{l} A \wedge B \\ \hline A \end{array}$$

$$\begin{array}{l} A \wedge B \\ \hline B \end{array}$$

This has the same flavor as the rule for proving a conjunction: a conjunction just breaks apart into its component parts.

This rule is called "simplification".

1.3 Implication

In this section we give rules for implication. There is a single basic rule for implication in each subsection, and then some derived rules which also involve negation, based on the equivalence of an implication with its contrapositive. These are called derived rules because they can actually be justified in terms of the basic rules. We like the derived rules, though, because they allow us to write proofs more compactly.

1.3.1 Proving an implication

The basic strategy for proving an implication: To prove $A \to B$, add A to your list of assumptions and prove B; if you can do this, $A \to B$ follows without the additional assumption.

Stylistically, we indent the part of the proof consisting of statements depending on the additional assumption A: once we are done proving B under the assumption and thus proving $A \to B$ without the assumption, we discard the assumption and thus no longer regard the indented group of lines as proved.

This rule is called "deduction".

The indirect strategy for proving an implication: To prove $A \to B$, add $\neg B$ as a new assumption and prove $\neg A$: if you can do this, $A \to B$ follows without the additional assumption. Notice that this amounts to proving $\neg B \to \neg A$ using the basic strategy, which is why it works.

This rule is called "proof by contrapositive" or "indirect proof".

1.3.2 Using an implication

modus ponens: If you are entitled to assume A and you are entitled to assume $A \to B$, then you are also entitled to assume B. This can be written as a rule of inference:

$$\frac{\begin{array}{l} A \\ A \to B \end{array}}{B}$$

when you just have an implication: If you are entitled to assume $A \to B$, you may at any time adopt A as a new goal, for the sake of proving

B, and as soon as you have proved it, you also are entitled to assume B. Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus ponens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

modus tollens: If you are entitled to assume $\neg B$ and you are entitled to assume $A \to B$, then you are also entitled to assume $\neg A$. This can be written as a rule of inference:

$$\frac{\begin{array}{l} A \to B \\ \neg B \end{array}}{\neg A}$$

Notice that if we replace $A \to B$ with the equivalent contrapositive $\neg B \to \neg A$, then this becomes an example of modus ponens. This is why it works.

when you just have an implication: If you are entitled to assume $A \to B$, you may at any time adopt $\neg B$ as a new goal, for the sake of proving $\neg A$, and as soon as you have proved it, you also are entitled to assume $\neg A$. Notice that no assumptions are introduced by this strategy. This proof strategy is just a restatement of the rule of *modus tollens* which can be used to suggest the way to proceed when we have an implication without its hypothesis.

1.4 Absurdity

The symbol $\perp$ represents a convenient fixed false statement. The point of having this symbol is that it makes the rules for negation much cleaner.

1.4.1 Proving the absurd

We certainly hope we never do this except under assumptions! If we are entitled to assume A and we are entitled to assume $\neg A$, then we are entitled to assume $\perp$. Oops! This rule is called *contradiction*.

$$\frac{\begin{array}{c} A \\ \neg A \end{array}}{\perp}$$

1.4.2 Using the absurd

We hope we never really get to use it, but it is very useful. If we are entitled to assume $\bot$, we are further entitled to assume A (no matter what A is). From a false statement, anything follows. We can see that this is valid by considering the truth table for implication.

This rule is called "absurdity elimination" or "ex falso".

1.5 Negation

The rules involving just negation are stated here. We have already seen derived rules of implication using negation, and we will see derived rules of disjunction using negation below.

1.5.1 Proving a negation

direct proof of a negation (basic): To prove $\neg A$, add A as an assumption and prove $\bot$. If you complete this proof of $\bot$ with the additional assumption, you are entitled to conclude $\neg A$ without the additional assumption (which of course you now want to drop like a hot potato!). This is the direct proof of a negative statement: proof by contradiction, which we describe next, is subtly different.

Call this rule "negation introduction". You won't be marked off if you call it "reductio ad absurdum", but it is not quite the same thing.

proof by contradiction (derived): To prove a statement A of any logical form at all, assume $\neg A$ and prove $\bot$. If you can prove this under the additional assumption, then you can conclude A under no additional assumptions. Notice that the proof by contradiction of A is a direct proof of the statement $\neg\neg A$, which we know is logically equivalent to A; this is why this strategy works.

Call this rule "reductio ad absurdum".

1.5.2 Using a negation:

double negation (basic): If you are entitled to assume $\neg\neg A$, you are entitled to assume A. Call this rule "double negation elimination".

contradiction (basic): This is the same as the rule of contradiction stated above under proving the absurd: if you are entitled to assume A and you are entitled to assume $\neg A$, you are also entitled to assume $\perp$. You also feel deeply queasy.

$$\frac{\begin{array}{c} A \\ \neg A \end{array}}{\perp}$$

if you have just a negation: If you are entitled to assume $\neg A$, consider adopting A as a new goal: the point of this is that from $\neg A$ and A you would then be able to deduce $\perp$ from which you could further deduce whatever goal C you are currently working on. This is especially appealing as soon as the current goal to be proved becomes $\perp$, as the rule of contradiction is the only way there is to prove $\perp$.