Math 515 Exam 2 Study Guide

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Contents

1 2C Measures and Their Properties

1.1 Theorem 2.57: Measure preserves order; measure of a set difference

Suppose (X, S, μ) is a measure space and $D, E \in S$ are such that $D \subseteq E$. Then

(a) $\mu(D) \leq \mu(E)$

(b) $\mu(E \setminus D) = \mu(E) - \mu(D)$ provided that $\mu(D) < \infty$

1.2 Theorem 2.58: Countable subadditivity

Suppose (X, S, μ) is a measure space and $E_1, E_2, ... \in S$. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leq \sum_{k=1}^{\infty} \mu(E_k)$$

1.3 Theorem 2.59: Measure of an increasing union

Suppose (X, S, μ) is a measure space and $E_1 \subseteq E_2 \subseteq ...$ is an increasing sequence of sets in S. Then

$$\mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k)$$

1.4 Theorem 2.60: Measure of a decreasing intersection

Suppose (X, S, μ) is a measure space and $E_1 \supseteq E_2 \supseteq ...$ is a decreasing sequence of sets in S with $\mu(E_1) < \infty$. Then

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) = \lim_{k\to\infty} \mu(E_k)$$

1.5 2C Problems

Exercise 1. (1) Explain why there does not exist a measure space (X, S, μ) with the property that $\{\mu(E) : E \in S\} = [0, 1)$

Exercise 2. (9) Suppose μ and v are measures on a measurable space (X, S). Prove that $\mu + v$ is a measure on (X, S). [Here, $\mu + v$ is the usual sum of two functions: if $E \in S$, then $(\mu + v(E) = \mu(E) + v(E)$.]

Exercise 3. (10) Give an example of a measure space (X, S, μ) and a deceasing sequence $E_1 \supseteq E_2 \supseteq \dots$ of sets in S such that

$$\mu\Big(\bigcap_{k=1}^{\infty} E_k\Big) \neq \lim_{k\to\infty} \mu(E_k)$$

2 2D Lebesgue Measure

2.1 Theorem 2.62: additivity of outer measure if one of the sets is open

Suppose A and G are disjoint subsets of $\mathbb{R}$ and G is open. Then

$$|A \cup G| = |A| + |G|$$

2.2 Theorem 2.63: additivity of outer measure if one of the sets is closed

Suppose A and F are disjoint subsets of $\mathbb{R}$ and F is closed. Then

$$|A \cup F| = |A| + |F|$$

2.3 Theorem 2.66: additivity of outer measure if one of the sets is a Borel set

Suppose A and B are disjoint subsets of $\mathbb{R}$ and B is a Borel set, Then

$$|A \cup B| = |A| + |B|$$

2.4 Theorem 2.76b: the Cantor set has measure 0

(b) The Cantor set has Lebesgue measure 0.

2.5 2D Problems

Exercise 4. (5) Prove that if $A \subseteq \mathbb{R}$ is Lebesgue measurable, then there exists an increasing sequence $F_1 \subseteq F_2 \subseteq ...$ of closed sets contained in A such that

$$\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0.$$

Exercise 5. (17) Let C denote the Cantor set. Prove that $\{\frac{1}{2}x + \frac{1}{2}y : x, y \in C\} = [0, 1]$.

3 2E Convergence of Measurable Functions

3.1 Theorem 2.84: uniform limit of continuous functions is continuous

Suppose $B \subseteq \mathbb{R}$ and $f_1, f_2, ...$ is a sequence of functions from B to $\mathbb{R}$ that converges uniformly on B to a function $f : B \to \mathbb{R}$. Suppose $b \in B$ and f_k is continuous at b for each $k \in \mathbb{Z}^+$. Then f is continuous at b.

3.2 Theorem 2.89: approximation by simple functions

Suppose (X, S) is a measurable space and $f : X \to [-\infty, \infty]$ is S-measurable. Then there exists a sequence $f_1, f_2, ...$ of functions from X to $\mathbb{R}$ such that

(a) each f_k is a simple S-measurable function;

(b) $|f_k(x)| \leq |f_{k+1}(x)| \leq |f(x)|$ for all $k \in \mathbb{Z}^+$ and all $x \in X$;

(c) $\lim_{k\to\infty} f_k(x) = f(x)$ for every $x \in X$;

(d) $f_1, f_2, ...$ converges uniformly on X to f if f is bounded.

3.3 2E Problems

Exercise 6. (1) Suppose X is a finite set. Explain why a sequence of functions from X to $\mathbb{R}$ that converges pointwise on X also converges uniformly on X.

Exercise 7. (9) Suppose $F_1, ..., F_n$ are disjoint closed subsets of $\mathbb{R}$. Prove that if

$$g : F_1 \cup \cdots \cup F_n \to \mathbb{R}$$

is a function such that $g|_{F_k}$ is a continuous function for each $k \in \{1, ..., n\}$, then g is a continuous function.

4 3A Integration with Respect to a Measure

4.1 Theorem 3.4: integral of a characteristic function

Suppose (X, S, μ) is a measure space and $E \in S$. Then

$$\int \chi_E d\mu = \mu(E)$$

4.2 Theorem 3.20: integration is homogeneous

Suppose (X, S, μ) is a measure space and $f : X \to [-\infty, \infty]$ is a function such that $\int f d\mu$ is defined. If $c \in \mathbb{R}$ then

$$\int cf d\mu = c \int f d\mu$$

4.3 Theorem 3.21: additivity of integration

Suppose (X, S, μ) is a measure space and $f, g : X \to \mathbb{R}$ are S-measurable functions such that $\int |f| d\mu < \infty$ and $\int |g| d\mu < \infty$. Then

$$\int (f+g) d\mu = \int f d\mu + \int g d\mu$$

4.4 Theorem 3.22: integration is order preserving

Suppose (X, S, μ) is a measure space and $f, g : X \to \mathbb{R}$ are S-measurable functions such that $\int f d\mu$ and $\int g d\mu$ are defined. Suppose also that $f(x) \leq g(x)$ for all $x \in X$. Then

$$\int f d\mu \leq \int g d\mu$$

4.5 3A Problems

Exercise 8. (1) Suppose (X, S, μ) is a measure space and $f : X \to [0, \infty]$ is an S-measurable function such that $\int f d\mu < \infty$. Explain why

$$\inf_E f = 0$$

for each set $E \in S$ with $\mu(E) = \infty$.

Exercise 9. (3) Suppose (X, S, μ) is a measure space and $f : X \to [0, \infty]$ is an S-measurable function. Prove that

$$\int f d\mu > 0 \text{ if and only if } \mu(\{x \in X : f(x) > 0\}) > 0$$

Exercise 10. (8) Suppose λ denotes Lebesgue measure on $\mathbb{R}$. Give an example of a sequence $f_1, f_2, ...$ of simple Borel measurable functions from $\mathbb{R}$ to $[0, \infty)$ such that $\lim_{k\to\infty} f_k(x) = 0$ for every $x \in \mathbb{R}$ but $\lim_{k\to\infty} \int f_k d\lambda = 1$.

5 EXTRA! (part 1) 3B Limits of Integrals & Integrals of Limits

5.1 Theorem 3.28: integrals on small sets are small

Suppose (X, S, μ) is a measure space and $g : X \to [0, \infty]$ is S-measurable, and $\int g d\mu < \infty$.

Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\int_B g d\mu < \varepsilon$$

for every set $B \in S$ such that $\mu(B) < \delta$.

5.2 Theorem 3.29: integrable functions live mostly on sets of finite measure

Suppose (X, S, μ) is a measure space and$g : X \to [0, \infty]$ is S-measurable, and $\int g d\mu < \infty$.

Then for every $\varepsilon > 0$ there exists $E \in S$ such that $\mu(E) < \infty$ and

$$\int_{X \setminus E} g d\mu < \varepsilon$$

5.3 EXTRA! Problems

Exercise 11. (6) Suppose (X, S, μ) is a measure space and $f : X \to [0, \infty]$ is S-measurable, and P and P' are S-partitiions of X such that each set in P' is contained in some set in P. Prove that $\mathcal{L}(f, P) \leq \mathcal{L}(f, P')$.

6 EXTRA! (part 2) 4A Hardly-Littlewood Maximal Function

6.1 Theorem 4.1: Markov's Inequality

Suppose (X, S, μ) is a measure space and $h \in \mathcal{L}^1(\mu)$. Then

$$\mu(\{x \in X : |h(x)| \geq c\}) \leq \frac{1}{c}||h||_1$$

for every $c > 0$.

6.2 "Holmes's Theorem"

For any set A there is a Borel set B such that $A \subseteq B$ and $|A| = |B|$.