Math 314 Whiteboard, 1/17/2024

TEST

$\lim_{x \to 5} x^2 = 25$

$\lim_{x \to 5} x^2 = 25$

$\lim_{x \to a} f(x) = L$ MEANS

for each $\varepsilon > 0$, there is a $\delta > 0$ such that for any x if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

$\lim_{x \to 5} x^2 = 25$ MEANS

For each $\varepsilon > 0$ there is $\delta > 0$ such that for any x ← this is what we want to prove.
if $0 < |x-5| < \delta$ then $|x^2 - 25| < \varepsilon$

Proof:

Let ε be an arbitrarily chosen positive real number.

Let $\delta = \min(1, \frac{\varepsilon}{11})$.

Let x be chosen arbitrarily
so suppose $0 < |x-5| < \delta$.

our goal is to show $|x^2 - 25| < \varepsilon$.

$|x-5| < 1$
so
$-1 < x-5 < 1$
so $9 < x+5 = |x+5| < 11$

$|x^2 - 25| = |x-5||x+5| < |x-5| \cdot 11 < \frac{\varepsilon}{11} \cdot 11 = \varepsilon$

because $|x+5| < 11$

because $|x-5| < \delta \le \frac{\varepsilon}{11}$

and we are done.

Scratch work to find δ [not officially part of the proof]

we need to enforce $|x^2 - 25| < \varepsilon$ by a condition $|x-5| < \delta$.

note that we are working backward from a desired conclusion to a hypothesis which will force it to be true.

$$|x^2 - 25| < \varepsilon$$

$(x-5)(x+5) = x^2 - 25$ (high school)

$$|x-5||x+5| < \varepsilon$$

$|a||b| = |ab|$

$$|x-5| < \frac{\varepsilon}{|x+5|}$$

$\delta = \frac{\varepsilon}{|x+5|}$ might tempt us, but δ cannot depend on x.

if we put an upper bound on x, (and so on $|x+5|$) notice that we put a lower bound on $\frac{\varepsilon}{|x+5|}$.

Let's put a preliminary upper bound on x:

suppose $0 < |x-5| < 1$.

then $-1 < x-5 < 1$

$$4 < x < 6$$

and $9 < x+5 = |x+5| < 11$

so $\frac{\varepsilon}{11} < \frac{\varepsilon}{|x+5|} < \frac{\varepsilon}{9}$

so if $0 < |x-5| < \min\left(1, \frac{\varepsilon}{11}\right)$

everything should work.

Now I'm going to prove one of the major limit theorems.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$] if all the lims are defined

problem with the way this is stated:

$$\underbrace{\lim_{x \to 0} \left(\frac{1}{x} + -\frac{1}{x}\right)}_{0} \stackrel{?}{=} \underbrace{\lim_{x \to 0} \frac{1}{x}}_{\text{undefined}} + \underbrace{\lim_{x \to 0} \left(-\frac{1}{x}\right)}_{\text{undefined}}$$

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

then $\lim_{x \to a} [f(x) + g(x)] = L + M$.

what does this MEAN?

if for any $\varepsilon_1 > 0$ there is a $\delta_1 > 0$ such that for any x
if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon_1$] $\lim_{x \to a} f(x) = L$

and for any $\varepsilon_2 > 0$ there is $\delta_2 > 0$ such that for any x
if $0 < |x-a| < \delta_1$ then $|g(x) - M| < \varepsilon_2$] $\lim_{x \to a} g(x) = M$

THEN for any $\varepsilon_3 > 0$ there is $\delta_3 > 0$ s.t. for any x
if $0 < |x-a| < \delta_3$ then $|(f(x) + g(x)) - (L+M)| < \varepsilon_3$.

THE PROOF BEGINS drumroll...

Let ε_3 be an arbitrarily chosen positive real number.

Let $\delta_3 = \min(\delta_1, \delta_2)$

where if $0 < |x-a| < \delta_1$ we have $|f(x) - L| < \frac{\varepsilon_3}{2}$
and if $0 < |x-a| < \delta_2$ we have $|g(x) - M| < \frac{\varepsilon_3}{3}$ } the existence of δ_1 and δ_2 follows from hypotheses

proof to continue

SCRATCH WORK:

I need to end up with $|(f(x)+g(x))-(L+M)| < \varepsilon_3$.

$$|(f(x)+g(x)) - (L+M)| = |(f(x)-L)+(g(x)-M)|$$
$$\leq |f(x)-L| + |g(x)-M| < \frac{\varepsilon_3}{2} + \frac{\varepsilon_3}{2} = \varepsilon_3$$

I want this to be be $< \frac{\varepsilon_3}{2}$

I want this to be $< \frac{\varepsilon_3}{2}$

Where do these come from!

I can find δ_1 such that if $0<|x-a|<\delta_1$, then $|f(x)-L| < \frac{\varepsilon_3}{2}$

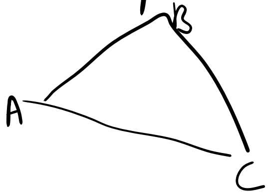
I can find δ_2 such that if $0<|x-a|<\delta_2$ then $|g(x)-M| < \frac{\varepsilon_3}{2}$

I want $\delta_3 = \min(\delta_1, \delta_2)$

TRIANGLE INEQUALITY:

The fact that we need is $|x+y| \le |x|+|y|$

usually called the triangle inequality.



$$d(A,B)+d(B,C) \ge d(A,C)$$

The connection: in the real line, $d(a,b) = |a-b|$.

$$|x+y| \le |x|+|y|$$

$$|x-(-y)| = d(x,-y) \qquad |x-0| + |0-(-y)|$$

$$d(x,0) + d(0,-y)$$

main proof continues

Let x be chosen arbitrarily.

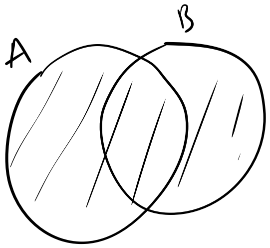
Suppose $0 < |x-a| < \delta_3 = \min(\delta_1, \delta_2)$.

Then $0 < |x-a| < \delta_1$, so $|f(x) - L| < \frac{\varepsilon_3}{2}$

and $0 < |x-a| < \delta_2$ so $|g(x) - M| < \frac{\varepsilon_3}{2}$

Now $|(f(x) + g(x)) - (L+M)| = |(f(x) - L) + (g(x) - M)|$

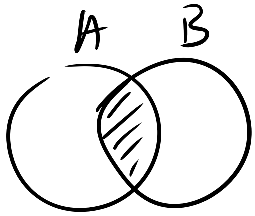
$\overset{\Delta}{\leq} |f(x) - L| + |g(x) - M| < \frac{\varepsilon_3}{2} + \frac{\varepsilon_3}{2} = \varepsilon$ and we are done.



$A \cup B$

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

$\{x : x \in A \wedge x \in B\}$



Mother to son:

If $\underbrace{\text{you clean your room}}_{P}$ then $\underbrace{\text{we will go out for pizza}}_{Q}$

P	Q	P → Q?	
T	T	T	
T	F	F	fink
F	T	T	chump
F	F	T	sigh