1 Homework 12, assigned 11/8/2023 – due Monday November 6

1. Compute the `gcd` of each pair of numbers and present the gcd as a linear combination of the original pair of numbers.

   You can get answers (with care) using the spreadsheet, but I strongly recommend doing these (and more examples) by hand. You will need the skill of constructing these tables and reading them correctly for later activities in this unit.

   (a) 55,34

   pasted in spreadsheet calculations – the gcd is 1 and 1 = (13)(54) + (-21)(34)

```
a= 55 1 0
b= 34 0 1
21 1 -1 1
13 -1 2 1
8 2 -3 1
5 -3 5 1
3 5 -8 1
2 -8 13 1
1 13 -21 1
```

(b) 337,216

$\texttt{gcd}(337, 216) = 1 = (25)(337) - (39)(216)$

```
a= 337 1 0
b= 216 0 1
121 1 -1 1
95 -1 2 1
26 2 -3 1
17 -7 11 3
9 9 -14 1
8 -16 25 1
1 25 -39 1
```

(c) 12076, -8976. You need to think about how to handle the fact that b is negative. The spreadsheet does **not** work correctly with negative b. Hint: I would do a calculation with positive values then fix it.

Spreadsheet calculation below.

$$\texttt{gcd}(12076, -8976) = \texttt{gcd}(12076, 8976) =$$

$$(-1025)(12076)+(1379)(8976) = (-1025)(12076)+(-1379)(-8976)$$

Notice that the inconvenient minus sign is easily finessed.

```
a= 12076 1 0
b= 8976 0 1
3100 1 -1 1
2776 -2 3 2
324 3 -4 1
184 -26 35 8
140 29 -39 1
44 -55 74 1
8 194 -261 3
4 -1025 1379 5
0 2244 -3019 2
```

2. Write out a proof that if $d|a$ and $d|b$, then $d|(a-b)$. This is very similar to something I did in class.

   Suppose that d, a, b are integers. Suppose that $d|a$ and $d|b$ hold. This means that there are integers x, y such that $dx = a$ and $dy = b$. We are looking for an integer z such that $dz = a - b$. Observe that $a - b = dx = dy = d(x - y)$. Setting $z = x - y$ and observing that the integers are closed under subtraction, we have found an integer z such that $dz = d(x - y) = a - b$, so we have shown that $d|(a - b)$ follows for any integers a, b such that $d|a$ and $d|b$.

3. Suppose that each of us have a large supply of 115 pound notes and a large supply of 389 pound notes. How can I pay you one pound?

```
a= 389 1 0
b= 115 0 1
44 1 -3 3
27 -2 7 2
17 3 -10 1
10 -5 17 1
7 8 -27 1
3 -13 44 1
1 34 -115 2
```

we see that $(389)(34) - (115)(115) = 1$, so if I pay you 34 389-pound notes and you pay me 115 115-pound notes, I have paid you one pound.

4. Suppose that in my mad scientist lab, recently devastated by a monster I unwittingly created, I have a balance, but can only find a large supply of 651 gram weights and a large supply of 133 gram weights.

   ```
   a= 651 1 0
   b= 133 0 1
   119 1 -4 4
   14 -1 5 1
   7 9 -44 8
   ```

   Can I verify the weight of an object that is supposed to weigh 28 grams?

   Yes. Put 36 651-gram weights in the left pan, put 176 133-gram weights in the right pan, and putting 28 grams in the right pan should balance it (I multiplied all the numbers for seven grams by four).

   What is the smallest weight I can check with these weights (remember that I can put my known weights in either pan of the balance).

   Seven grams. Put 9 651-gram weights in the left pan, put 44 133-gram weights in the right pan, and putting seven grams in the right pan should balance it. This is smallest because any weight which is an integer linear combination of the two given weights will be a multiple of the gcd of the two given weights.

5. (threatened extra induction problem): Prove by mathematical induction that the recursively defined sequence $\{a_i\}_{i\in\mathbb{N}}$ defined by $a_0 = 1, a_1 = 3, a_{n+2} = 6a_{n+1} - 8a_n$ satisfies $a_n = \frac{2^n+4^n}{2}$. This does not involve the methods of 2.4 (no characteristic polynomial is involved, though of course one would use it to find this formula). [Typos should all be fixed: I don't know how I did that, I knew exactly what sequence I meant and I did computational checks to make sure the formula was right!]

   Basis steps: $1 = a_0 = \frac{2^0+4^0}{2} = \frac{2}{2} = 1$ check.

   $3 = a_1 = \frac{2^1+4^1}{2} = \frac{6}{2} = 3$ check.

   Now suppose for an arbitrarily chosen k we have for all $m \leq k$ that $a_k = \frac{2^k+4^k}{2}$. Our goal is to show

$$a_{k+1} = \frac{2^{k+1}+4^{k+1}}{2}.$$

   If $k = 1$ we have already shown this. If $k > 1$ we know that $a_k = \frac{2^k+4^k}{2}$ and

$$a_{k-1} = \frac{2^{k-1}+4^{k-1}}{2},$$

   by inductive hypothesis.

   Then $a_{k+1} = 6a_k - 8a_{k-1}$ by the recurrence relation so $= 6(\frac{2^k+4^k}{2}) - 8(\frac{2^{k-1}+4^{k-1}}{2})$ by the inductive hypothesis $= 3\cdot 2^k + 3\cdot 4^k - 4\cdot 2^{k-1} - 4\cdot 4^{k-1} = 3\cdot 2^k + 3\cdot 4^k - 2\cdot 2^k - 4^k = 2^k + 2\cdot 4^k = \frac{2^{k+1}+4^{k+1}}{2}$.