Solutions

Math 275, Fall 2020 Practice Exam

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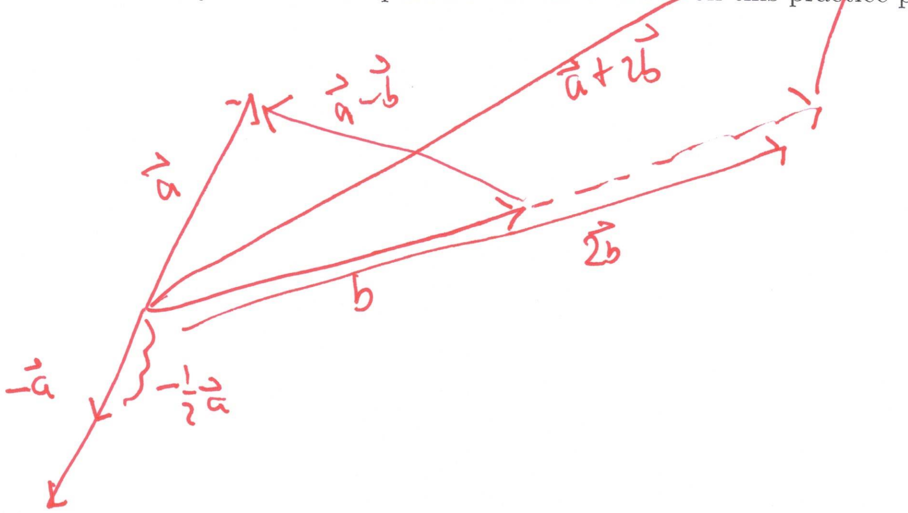
This might be longer than your actual test; the problems, mostly taken from old tests of mine, are similar to what you will see.

Where approximate calculations with your calculator are needed, please give answers to two decimal places.

1. Arrows representing two vectors **a** and **b** are shown. Draw arrows representing **a-b**, **a+2b**, and $-\frac{1}{2}\mathbf{a}$, in each case drawing additional copies of **a** and/or **b** as appropriate to make it clear what you are doing.

   Draw your own non-parallel vectors **a** and **b** on this practice paper.



2. Determine the scalar component of $\langle 1,-2,1\rangle$ in the direction of $\langle 1,1,2\rangle$, and determine the vector projection of $\langle 1,-2,1\rangle$ onto $\langle 1,1,2\rangle$. Hint: this will be a vector parallel to $\langle 1,1,2\rangle$!

For a couple of points, use your work above to express $\langle 1,-2,1\rangle$ as the sum of a vector parallel to $\langle 1,1,2\rangle$ and a vector perpendicular to $\langle 1,1,2\rangle$. There is an advantage to this: it should give you an easy check of your work.

$\vec{b} = \langle 1,-2,1\rangle$

$\vec{a} = \langle 1,1,2\rangle$

scalar component $= \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} = \dfrac{1\cdot 1 - 2\cdot 1 + 1\cdot 2}{\sqrt{1^2+1^2+2^2}} = \dfrac{1}{\sqrt{6}}$

vector projection $= \dfrac{1}{\sqrt{6}}\left\langle \dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}} \right\rangle = \left\langle \dfrac{1}{6}, \dfrac{1}{6}, \dfrac{1}{3}\right\rangle$

$\dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|} \quad \dfrac{\vec{a}}{\|\vec{a}\|}$

$\langle 1,-2,1\rangle = \left\langle \dfrac{1}{6}, \dfrac{1}{6}, \dfrac{1}{3}\right\rangle + \left\langle \dfrac{5}{6}, -\dfrac{13}{6}, \dfrac{2}{3}\right\rangle$

$\dfrac{5}{6} - \dfrac{13}{6} + \dfrac{4}{3}$

check: $-\dfrac{5}{6}\cdot\dfrac{1}{6} + \dfrac{13}{6}\cdot\dfrac{1}{6}$

$\left\langle \dfrac{5}{6}, -\dfrac{13}{6}, \dfrac{2}{3}\right\rangle \cdot \langle 1,1,2\rangle =$

$\dfrac{5}{6} - \dfrac{13}{6} + \dfrac{8}{6} = 0$ ✓

3. Determine the measure in degrees of the angle $\angle BAC$ (the angle at the vertex A) in the triangle with vertices $A = (1,1,1), B = (2,3,4), C = (5,-3,1)$ using an appropriate vector operation.

Determine the area of the triangle using another appropriate vector operation (remember that it's a triangle).

$\vec{AB} = \langle 1, 2, 3 \rangle$

$\vec{AC} = \langle +4, -4, 0 \rangle$

$$\cos(\theta) = \frac{1\cdot(+4) + 2(-4) + 3\cdot 0}{\sqrt{1^2+2^2+3^2}\,\sqrt{4^2+4^2+0^2}} = \frac{-4}{\sqrt{14}\sqrt{32}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{14\cdot 32}}\right) \approx 100.89^\circ$$

$$\begin{matrix} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 2 & 3 & 1 & 2 \\ 4 & -4 & 0 & 4 & -4 \end{matrix}$$

Cross product is $12\vec{i} + 12\vec{j} - 12\vec{k}$

4. Give a vector parametric equation for the line AB in the previous problem. $\vec{r}(t) = \langle 1, 1, 1 \rangle + t\langle 1, 2, 3 \rangle$

Give an equation for the plane passing through the three points of the previous problem in the form $ax + by + cz = d$; you might want to check that the three points satisfy the equation. Hint: you should have done more than half the work needed for this already in the previous problem!

the plane - $\langle 12, 12, -12 \rangle$ is the normal vector

$12x + 12y - 12z = 12$

$x + y - z = 1$

5. Write scalar parametric equations for the line L_1 with vector parametric equation $\langle 1, 1, 1\rangle + t\langle 2, -1, 3\rangle$ and the line L_2 with vector parametric equation $\langle 4, -3, 4\rangle + s\langle 1, 2, 3\rangle$.

These two lines intersect. Find the point of intersection (this does not mean, just find the appropriate values of s and t!)

L_1: $x = 1+2t$

$y = 1-t$

$z = 1+3t$

L_2: $x = 4+s$

$y = -3+2s$

$z = 4+3s$

solve simultaneous equations for x and y

$1+2t = 4+s$ $\qquad 2t - s = 3$

$1-t = -3+2s$ $\qquad t + 2s = 4$ subtract

$\times 2 \quad 2t + 4s = 8$

$-5s = -5 \quad s = 1$

$2t - 1 = 3 \quad t = 2$

at $t=2$

$1+2t = 5$

$1-t = -1$

$1+3t = 7$

at $s=1$

$4+s = 5$

$-3+2s = -1$

$4+3s = 7$

$(5, -1, 7)$

is the point of intersection

6. The point (1,1,1) lies on the plane $x + 2y + 3z = 6$ and also on the plane $x - y + z = 1$. Find vector parametric equations for the line in which these planes intersect.

Determine the angle at which the two planes intersect.

take the cross product of the normal vectors to the two planes to get a vector parallel to their line of intersection

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 2 & 3 & 1 & 2 \\ 1 & -1 & 1 & 1 & -1 \end{array}$$

$$5\vec{i} + 2\vec{j} - 3\vec{k}$$

parametric equations for the line

$$\langle 1,1,1\rangle + t\langle 5,2,-3\rangle$$

check: for $t=1$, $(6,3,-2)$ is on this line.

Check that it is on both planes

$6 + 3 \cdot 2 + 3(-2) = 6$ ✓

$6 - 3 + (-2) = 1$ ✓

7. Find the tangent vector to the curve parameterized by $\langle t^2, t^3, t^4 \rangle$ at the point $(4, -8, 16)$. (Hint: what is the value of t at this point?).

For a small bit of additional credit, give a vector parameterization of the tangent line to that curve at that point.

$\vec{r}(t) = \langle t^2, t^3, t^4 \rangle$

$\vec{r}\,'(t) = \langle 2t, 3t^2, 4t^3 \rangle$

$t = -2$ gives $\vec{r}(t) = \langle 4, -8, 16 \rangle$

tangent vector: $\vec{r}\,'(-2) = \langle 2(-2), 3(-2)^2, 4(-2)^3 \rangle = \langle -4, 12, -32 \rangle$

vector parameterization of tangent line is

$\vec{L}(t) = \langle 4, -8, 16 \rangle + t\langle -4, 12, -32 \rangle$

8. The acceleration vector of a particle at time t is $\langle 1, t\rangle$. Its velocity vector at time 0 is $\langle 1, -1\rangle$. Its position at time 0 is $\langle 2, 3\rangle$. Find the position and velocity vectors for the particle at time t.

$\vec{a}(t) = \langle 1, t \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle t + c_1, \frac{t^2}{2} + c_2 \rangle$

$\vec{v}(0) = \langle 0 + c_1, \frac{0^2}{2} + c_2 \rangle = \langle 1, -1 \rangle$ so $c_1 = 1$ $c_2 = -1$

thi $\vec{v}(t) = \langle t + 1, \frac{t^2}{2} - 1 \rangle \leftarrow$ velocity

$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \frac{t^2}{2} + t + c_1, \frac{t^3}{6} - t + c_2 \rangle$

$\vec{r}(0) = \langle \frac{0^2}{2} + 0 + c_1, \frac{0^3}{6} - 0 + c_2 \rangle = \langle 2, 3 \rangle$

so $c_1 = 2$, $c_2 = 3$

so $\vec{r}(t) = \langle \frac{t^2}{2} + t + 2, \frac{t^3}{6} - t + 3 \rangle \leftarrow$ position

9. Set up and evaluate the integral for the length of the helix

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 2t \rangle$$

from the point (1,0,0) to the point $(0, 1, \pi)$. If you set it up correctly, it will be very easy to evaluate! Hint: you need to identify the appropriate values of t to serve as bounds of the integral.

$\vec{r}'(t) = \langle -\sin(t), \cos(t), 2 \rangle$

$t = 0$ at $(1,0,0)$

$t = \frac{\pi}{2}$ at $(0,1,\pi)$

$$\int_0^{\frac{\pi}{2}} \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 2^2}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{5}\, dt = \frac{\pi\sqrt{5}}{2}$$

10. Write parametric equations for the tangent line to the helix of the previous problem at the point $(0, 1, \pi)$.

$\vec{r}'(\frac{\pi}{2}) = \langle -1, 0, 2 \rangle$

so $\vec{L}(t) = \langle 0, 1, \pi \rangle + t\langle -1, 0, 2 \rangle$

11. Write parametric equations for a helix of radius 2 around the z-axis which rises 4 units in the course of three complete rotations.

$\langle a\cos(t), a(\sin(t)), ct\rangle$ is general form for a helix around the z-axis

$\langle 2\cos(t), 2\sin(t), ct\rangle$ gives us radius 2

when $t = 6\pi$ we want $ct = 4$

$c(6\pi) = 4 \quad c = \frac{4}{6\pi} = \frac{2}{3\pi}$

$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), \frac{2}{3\pi}t\rangle$