Extensionality in Zermelo - Fraenkel set theory.

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0. Introduction

Scott showed in [1] that ZF (Zermelo - Fraenkel set theory without foundation) is really weakened when one drops the axiom of extensionality (EXT), while Z (Zermelo set theory without foundation) is not ; here a theory T is said weaker than T′ iff there is an interpretation of T in T′. In [2] Gandy proposed a strengthened version of ZF, in some enlarged language, which is not weakened when EXT is dropped. Here we propose some versions of ZF, axiomatized in the language of ZF (primitive non-logical symbols : $\in$ and $=$) which are not weakened (for relative interpretability) by dropping EXT. We also introduce a generalized notion of rank, having some common properties with the rank - hierarchies studied by M. von Rimscha in [3].

1. Formalism

1.1 Z is the Zermelo set theory whose axioms are :

1) EXT : $\forall t\,(t\in x \Leftrightarrow t\in y) \Rightarrow x=y$
2) Empty set : $\exists x\, \forall t\; t\notin x$
3) Pairing : $\forall x,y\, \exists z\, \forall t\, [t\in z \Leftrightarrow (t=x \vee t=y)]$
4) Powerset : $\forall x\, \exists y\, \forall t\, (t\in y \Leftrightarrow t\subset x)$
5) Union : $\forall x\, \exists y\, \forall t\, (t\in y \Leftrightarrow \exists z\in x\; t\in z)$
6) Infinity : $\exists x\, \exists y\, (\forall t\; t\notin y \;\wedge\; y\in x \;\wedge\; \forall z\in x\;\; z\cup\{z\}\in x)$

7) Separationscheme:
if x does not appear free in the formula φ:

$$\forall a \; \exists x \; \forall t \; (t \in x \Leftrightarrow t \in a \wedge \varphi)$$

1.2 ZF is the Zermelo - Fraenkel set theory whose axioms are 1 - 7 together with
8) Replacementscheme:

$$\forall x \in a \; \exists y \; (\varphi(x,y) \wedge \forall z \, (\varphi(x,z) \Rightarrow y = z))$$

$$\Rightarrow \exists b \; \forall v \; (v \in b \Leftrightarrow \exists x \in a \; \varphi(x,v)).$$

1.3 Coll is the collectionscheme:

$$\forall x \in a \; \exists y \; \varphi(x,y) \Rightarrow$$
$$\exists b \; \forall u \in a \; \exists v \in b \; \varphi(u,v)$$

1.4 A model M for $\mathcal{L}_{ZF}$ (the language of ZF) is a structure $\langle A, E, \sim \rangle$ with $E \subset A \times A$ and $\sim \subset A \times A$; E is intended to interpret $\in$ and $\sim$ to interpret $=$; when $\sim$ is the "real" equality on A, the model $M = \langle A, E, = \rangle$ will simply be written $M = \langle A, E \rangle$.

1.5 The closure axiom Cl is:
$\forall x \; \exists t \; (t$ is a transitive set $\wedge \; x \in t)$

1.6 Z^- is Z without EXT
ZF^- is ZF without EXT.

③

1.7 If T and T' are two theories, $T \underset{i}{\subset} T'$ means that one can interpret T in T' and $T \underset{i}{=} T'$ means that $T \underset{i}{\subseteq} T'$ and $T' \underset{i}{\subseteq} T$.

1.8 When working in a theory with axiom EXT, the use of terms such as $\{x \mid \varphi(x, ...)\}$ is clear: $\{x \mid \varphi(x, ...)\}$ is the unique set y (when such a set exists) such that $\forall t \, (t \in y \leftrightarrow \varphi(t, ...))$. When EXT is dropped, the term in itself becomes meaningless, but can still be used in formulas to allow useful abbreviation. For example: $\{x \mid \varphi(x, ...)\} \in y$ should be understood as: $\exists z \in y \ \forall x \, (x \in z \leftrightarrow \varphi(x, ..))$. So, even when EXT is dropped, we will go on using terms as $\mathcal{P}x$ (powerset), $\cup x$ (union), ... to abbreviate our formulas.

2. Contractions

2.1 In [4] we showed that $Z^- \underset{i}{=} Z$ (first proved by Scott). The main idea used in our proof is that of "contraction". We give here some definitions and properties. The reader will find more in [4].

If $M = \langle A, E \rangle$ is a model for $\mathcal{L}_{ZF}$ and $\sim$ an equivalence relation on A, we define the operation $+$ by:

$x \sim^+ y \leftrightarrow (\forall t \, E x \ \exists t' E y \ t \sim t') \wedge (\forall z \, E y \ \exists z' E x \ z \sim z'$

④

Let Eq be the set of all equivalence on A, ordered by the relation $\subset$ (an equivalence being considered as a set of ordered pairs). It is easy to show that $+$ is a morphism on Eq : if $\sim_1 \subset \sim_2$, then $\sim_1^+ \subset \sim_2^+$. An element $\sim$ of Eq will be called <u>increasing</u> (<u>final</u> in [4]) iff $\sim \subset \sim^+$. The <u>contractions</u> on M are the fixed points of $+$: $\sim \in Eq$ is a contraction iff $\sim = \sim^+$. If for $\sim \in Eq$ we define $M/_\sim$ in the natural way:

$$M/_\sim = \langle A, E_\sim, \sim \rangle, \text{ where } x E_\sim y \iff \exists x' \sim x$$

$$\exists y' \sim y \quad x' E y'$$, it is easy to show that:

$\sim$ is increasing iff $x E_\sim y \iff \exists x' \sim x \ x' E y$

$\sim$ is a contraction iff $M/_\sim \models EXT$.

The set of all increasing equivalences and the set of all contractions (on M) are complete lattices. In particular, this implies the existence, for each model M, of a least contraction (minimum contraction) and a greatest contraction (maximum contraction). Intuitively, the minimum contraction can be obtained by the following construction: $\sim_0$ is $=$, $\sim_{\alpha+1}$ is $(\sim_\alpha)^+$, $\sim_\gamma$ is $\bigcup_{\alpha<\gamma} \sim_\alpha$

if γ is a limit ordinal; $\sim_\alpha$ is an increasing chain: $\alpha < \beta \Rightarrow \sim_\alpha \subset \sim_\beta$; each $\sim_\alpha$ is an increasing equivalence: $\sim_\alpha \subset \sim_\alpha^+$. If δ is the first ordinal α

such that $\sim_{\alpha+1} = \sim_{\alpha}$, then $\sim_{\delta}$ is the minimum contraction on M (see [4], [5]). The same construction starting with $\sim_0 = A \times A$, gives a decreasing chain and allows to define the maximum contraction on M.

Other authors have introduced similar ideas: the reader will find them in [6] and [7].

2.2 The results concerning contractions can be proved in a metatheory as weak as Z^-: the use of ordinals is not essential for the construction of the minimum or maximum contraction on M; the proofs simply have to be written more carefully, as one is missing EXT (see [4]).

Let us now show briefly how to obtain an interpretation of Z in Z^-. First, remark that $Z^- + Cl \subseteq_i Z^-$: this interpretation is trivially given by the "structure" $\langle H, \in, = \rangle$ in Z^-, where H is the class $\{x \mid \exists t \, (t \text{ is a transitive set} \wedge x \in t)\}$. So it suffices now to give an interpretation of Z in $Z^- + Cl$. So define, in $Z^- + Cl$, the equivalence $\sim$ by: $x \sim y$ iff $\exists t \, [t$ is a transitive set $\wedge$ $\forall R$ (R is the minimum contraction on the structure $\langle t, \in \rangle \Rightarrow x R y$)]. In fact $\sim$ is the minimum (definable) contraction on the "structure" $\langle V, \in \rangle$, where V is the universal class: $V = \{x \mid x = x\}$. Remark that if one replaces, in the definition of $\sim$, "minimum contraction" by "maximum contraction", $\sim$ will be the maximum (definable) contraction on $\langle V, \in \rangle$.

It is now easy to see that, if $\sim$ is a definable contraction on $\langle V, \in \rangle$, the structure

$\langle V, \in_\sim, \sim \rangle$ is a model of $Z + Cl$
(remember that $\in_\sim$ is defined by:
$x \in_\sim y \iff \exists x' \sim x\ \exists y' \sim y\ \ x' \in y'$).
The verification is easy, knowing the two following essential facts:
let $\varphi_\sim$ be the formula obtained from φ by replacing $\in$ by $\in_\sim$ and $=$ by $\sim$;
trivially: $\langle V, \in \rangle \models \varphi_\sim \iff \langle V, \in_\sim, \sim \rangle \models \varphi$.

Then:

1) $x \sim x' \Rightarrow (\varphi_\sim(x, \vec{a}) \iff \varphi_\sim(x', \vec{a}))$

2) $x \in_\sim y \iff \exists x' \in y\ \ x' \sim x$.

2.3 The same structure $\langle V, \in_\sim, \sim \rangle$ considered in $ZF^- + Cl$ is not necessarily a model of the replacement scheme. Our idea is to strengthen somewhat ZF in such a way that in the strengthened version T, without ExT, $\langle V, \in_\sim, \sim \rangle$ is a model of T (with ExT).

3. Z with Coll

3.1 Z + Coll is an extension of ZF such as we are searching (2.3). It is well-known that, in Z with the axiom of foundation, Coll is equivalent to the Replacement scheme. But, when one drops the axiom of foundation, Coll is in fact stronger than the Replacement scheme.

⑦

So $Z + \mathrm{Coll}$ is a non-trivial strengthening of ZF. Though, one has: $Z + \mathrm{Coll} \underset{i}{=} ZF$.

3.2 <u>Theorem 1</u>: $Z^- + \mathrm{Coll} \underset{i}{=} Z + \mathrm{Coll}$.

<u>Proof</u>:

1) $Z^- + \mathrm{Coll} \vdash \mathrm{Cl}$. Indeed: let ω be (one of) the sets of natural numbers ($0 \equiv \emptyset$, $n+1 \equiv n \cup \{n\}$). Let $\varphi(x, y)$ be the formula (we suppose a fixed): y is a function f with domain x such that $f(0) = a$ and $\forall z\ f(z+1) = \bigcup f(z)$.

As we miss EXT, the structure $\langle \omega, \in \rangle$ is not well-ordered, but only well-founded.

So by "induction" on $x \in \omega$, one shows:

- $\varphi(x, f) \wedge \varphi(x, f') \Rightarrow \forall z \in x\ \ f(z) \sim_{EXT} f'(z)$

  where $u \sim_{EXT} v$ means: $\forall t\,(t \in u \Longleftrightarrow t \in v)$

- $\forall x \in \omega\ \exists y\ \varphi(x, y)$

By Coll, we conclude: $\exists b\ \forall x \in \omega\ \exists f \in b\ \varphi(x, f)$.
Take $c = \{f \in b \mid \exists x \in \omega\ \varphi(x, f)\}$; such a set exists by the Separationscheme. Take $d = \{z \mid \exists x \in \omega\ \exists f \in c\ \ z \in f(x)\}$. Then d is a transitive set and $a \subset d$. So $t = \mathcal{P}d$ is a transitive set and $a \in t$.

We proved (in $Z^- + \mathrm{Coll}$): $\forall a\ \exists t\ (t \text{ transitive} \wedge a \in t)$.

2) Let $\sim$ be the minimum (or the maximum) contraction definable on $\langle V, \in \rangle$ in $Z^- + \text{Coll}$ (as $Z^- + \text{Coll} \vdash \text{Cl}$ we know this exists; see 2.2) We already know that $\langle V, \in_\sim, \sim \rangle$ is a model of $Z + \text{Cl}$. Let us show that it is a model of Coll. Suppose ψ is the following axiom of Coll: $\forall a\, (\forall x \in a\, \exists y\, \varphi(x,y) \Rightarrow \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x,y))$

We have to prove $\psi_\sim$ in $Z^- + \text{Coll}$. So suppose $\forall x \in_\sim a\, \exists y\, \varphi_\sim(x,y)$. As $x \in a \Rightarrow x \in_\sim a$, we have: $\forall x \in a\, \exists y\, \varphi_\sim(x,y)$. As $\sim$ is definable, $\varphi_\sim$ is a formula of $\mathcal{L}_{ZF}$; so, by Coll:

$$\exists b\, \forall x \in a\, \exists y \in b\, \varphi_\sim(x,y). \qquad (*)$$

But, if $x \in_\sim a$, then $\exists x' \sim x\ \ x' \in a$. So, by $(*)$ $\exists y \in b\ \varphi_\sim(x', y)$. But $\varphi_\sim(x',y) \Leftrightarrow \varphi_\sim(x,y)$

(see 2.2). So: $\forall x \in_\sim a\ \exists y \in b\ \varphi_\sim(x,y)$,

and this implies: $\forall x \in_\sim a\ \exists y \in_\sim b\ \varphi_\sim(x,y)$.

4. Rank

4.1 Let $M = \langle A, E \rangle$ be a model for $\mathcal{L}_{ZF}$. $\Lambda(x)$ will be an abbreviation of: $\forall t\ \text{A}\ \neg t \in x$ (¬ is the negation symbol).

<u>Definition</u>:

(9)

If $R \subset A \times A$,

R is a rank on M iff

0) R is a pre-order (reflexive and transitive)

1) $E \subset R$

2) $\forall x, y \in A \quad (\Lambda(x) \wedge \Lambda(y) \Rightarrow x R y)$

3) $\overline{R} \subset R$

4) $(\Lambda(x) \wedge \neg\Lambda(y) \wedge x R y) \Rightarrow \exists k \in \mathbb{N} \; \exists x' \; (\Lambda(x') \wedge x' E^k y)$

5) $\overline{R} \supset R$

6) the restriction of R to the well-founded part of M corresponds to the ordinary rank definition.

where:

- $\overline{R}$ is defined by:

$$x \overline{R} y \Leftrightarrow [(\Lambda(x) \wedge x R y) \vee (\neg\Lambda(x) \wedge \forall t E x \; \exists z E y \; t R z)]$$

- $\mathbb{N}$ is the set of naturals numbers and E^k is $E \circ E \circ \ldots \circ E$ (k times)
- The ordinary rank is defined by:

$$B_0 = \{x \in A \mid \Lambda(x)\}$$
$$B_{\alpha+1} = \{x \in A \mid \forall t E x \quad t \in B_\alpha\}$$
$$B_\gamma = \bigcup_{\alpha<\gamma} B_\alpha \quad \text{if } \gamma \text{ is a limit ordinal.}$$

The well-founded part of M is $WF = \bigcup_\alpha B_\alpha$.

Condition 6 is exactly:

$\forall x, y \in WF \quad x R y \Leftrightarrow \forall \alpha \text{ ordinal } (x \in B_\alpha \Rightarrow y \in B_\alpha)$

4.2 Some facts

1) Define, in ZF with the axiom of foundation:

$x R y \Leftrightarrow \forall \alpha \text{ ordinal } (x \in R_\alpha \Rightarrow y \in R_\alpha)$

where $R_0 = \emptyset$, $R_{\alpha+1} = \mathcal{P} R_\alpha$, $R_\gamma = \bigcup_{\alpha<\gamma} R_\alpha$ if γ limit ordinal.

Then the "relation" R is a rank on the "structure" $\langle V, \in \rangle$, in the sense defined in 4.1.
This shows that our definition generalizes the usual rank-notion.

2) Conditions 0 – 5 imply condition 6: trivial by "induction" over the well-founded part of M.
In particular, a well-founded structure admits only one rank, which corresponds to the usual rank.

3) If $M \models EXT$, condition 2 is trivially satisfied if Φ is.

4) $R \subset S \Rightarrow \overline{R} \subset \overline{S}$

5) R satisfies 0, 1, 2, 4 $\Rightarrow$ $\overline{R}$ satisfies 0, 1, 2, 4

6) R satisfies 3 $\Rightarrow$ $\overline{R}$ satisfies 3

7) R satisfies 5 $\Rightarrow$ $\overline{R}$ satisfies 5.

8) Let $R(0)$ be the smallest (for inclusion $\subset$) pre-order including E and $=$ and $\{\langle x, y\rangle \mid \Lambda(x) \wedge \Lambda(y)\}$. Then $R(0)$ satisfies conditions 0, 1, 2, 4, 5. Define $R(\alpha+1) = \overline{R(\alpha)}$ and $R(\gamma) = \bigcup_{\alpha<\gamma} R(\alpha)$ if γ is a limit ordinal.

The union of an increasing chain of relations satisfying 0, 1, 2, 4, 5 satisfies 0, 1, 2, 4, 5. So let δ be the first ordinal α such that $R(\alpha+1) = R(\alpha)$; $R(\delta)$ is a fixed point for the operation $\overline{}$ and so satisfies condition 3 too. So by fact 2, $R(\delta)$ is a rank, and in fact the minimum rank on M.
Notation: $R(\delta) \equiv \leq_{Min}$.

9) Take $S(0) = \{\langle x,y\rangle \mid \neg\Lambda(x) \wedge \neg\Lambda(y)\} \cup \{\langle x,y\rangle \mid \Lambda(x) \wedge \Lambda(y)\} \cup \{\langle x,y\rangle \mid \Lambda(x) \wedge \neg\Lambda(y) \wedge \exists k \in \mathbb{N}\ \exists x' (\Lambda(x') \wedge x' E^k y)\}$.
$S(0)$ satisfies conditions 0–4. Define (as in fact 8): $S(\alpha+1) = \overline{S(\alpha)}$, $S(\gamma) = \bigcap_{\alpha<\gamma} S(\gamma)$ if γ is a limit ordinal. One easily shows that the intersection of relations satisfying 0–4 satisfies 0–4. We obtain a decreasing chain $S(\alpha)$. Let δ be the first ordinal α such that $S(\alpha) = S(\alpha+1)$. Then $S(\delta)$ is a rank on M, and in fact the maximum rank (notation: $\leq_{Max}$): if R is any rank on M, then $R \subset \leq_{Max}$.

10) Let $\sim$ be an equivalence on A and suppose $R \subset A \times A$ satisfies condition 2; then: $\sim \subset R \Rightarrow \sim^+ \subset \overline{R}$.

11) Let $\sim_{Min}$ be the minimum contraction on M and $\sim_{Max}$ be the maximum contraction on M.

Then: $\sim_{Min} \subset \leq_{Min}$ and $\sim_{Max} \subset \leq_{Max}$.

Proof:

(1) Define: $\sim_0$ ~~is~~ $=$, $\sim_{\alpha+1}$ is $(\sim_\alpha)^+$, $\sim_\gamma$ is $\bigcup_{\alpha<\gamma} \sim_\alpha$ if γ is a limit ordinal.
Then one easily shows by induction on α and facts 8, 10: $\forall \alpha \quad \sim_\alpha \subset R(\alpha)$.
So clearly: $\sim_{Min} \subset \leq_{Min}$.

(2) Define: $\sim_0$ is $(A \times A)^+$, $\sim_{\alpha+1}$ is $(\sim_\alpha)^+$, $\sim_\gamma$ is $\bigcap_{\alpha<\gamma} \sim_\alpha$ if γ is a limit ordinal.
Then one shows by induction on α and facts 9, 10: $\forall \alpha \quad \sim_\alpha \subset S(\alpha)$.
So clearly: $\sim_{Max} \subset \leq_{Max}$

12) Take $\leq^* = \bigcap \{R \mid R \subset A \times A$ and R satisfies 0–3$\}$.
Then $\leq^*$ satisfies 0–3 (see fact 9) and is the minimum relation satisfying 0–3. So:
$\leq^* \subset \leq_{Min}$ (I). As $\leq^*$ satisfies 3:
$\overline{\leq^*} \subset \leq^*$ (II). As $\leq^*$ satisfies 0–3, $\overline{\leq^*}$ too satisfies 0–3, so by definition of $\leq^*$:
$\leq^* \subset \overline{\leq^*}$ (III). By II and III, we conclude:
$\leq^* = \overline{\leq^*}$. But as $R(0) \subset \leq^*$, this implies: $\forall \alpha \ R(\alpha) \subset \leq^*$. So $\leq_{Min} \subset \leq^*$ (IV).
By I and IV: $\leq^*$ is exactly $\leq_{Min}$.

13) Suppose $B \subset A$ and B is transitive in the sense of $M = \langle A, E \rangle$: $x E y \wedge y \in B \Rightarrow x \in B$.
Then:

(1) each rank R on $\langle B, E \cap (B \times B) \rangle$ can be extended to a rank S on M.

(2) if R is a rank on M, $R \cap (B \times B)$ is a rank on $\langle B, E \cap (B \times B) \rangle$

(3) if $\leq$ is the minimum rank on M (resp: maximum) then $\leq \cap (B \times B)$ is the minimum (resp: maximum) rank on $\langle B, E \cap (B \times B) \rangle$.

14) In $Z^- + C\ell$ one can define $\sim_{Max}$, $\sim_{Min}$, $\leq_{Max}$, $\leq_{Min}$ on the "structure" $\langle V, \in \rangle$.
For example: $\leq_{Max}$ on $\langle V, \in \rangle$ is defined by:

$x \leq_{Max} y \iff \exists t$ (t is a transitive set $\wedge$ $x, y \in t$ $\wedge$ $x R y$, where R is the maximum rank on $\langle t, \in \rangle$).

The proofs using ordinals can be re-written without any use of ordinals; however the versions with ordinals presented here are more intuitive and allow to see clearly how the proof works. The metatheory necessary to prove facts 2 – 13 is very weak: $Z^- + Ce$ is largely sufficient. The proof of fact 14 makes an intensive use of fact 13.

15) If $\leq$ is a rank on M and $\sim$ is a contraction on M such that $\sim \subset \leq$, then $\leq$ is a rank on $M/_\sim$.

Conversely, if $\sim$ is a contraction on M and $\leq$ a rank on $M/_\sim$, then $\leq$ is a rank on M.

(easy proof)

16) From fact 15 one concludes easily: if $\leq$ is the minimum (resp: maximum) rank on M and $\sim$ is a contraction on M such that $\sim \subset \leq$; then $\leq$ is the minimum (resp: maximum) rank on $M/_\sim$.

17) The rank introduced by M. von Rimscha in [8] has properties 0 – 3 but does not satisfy 4 – 6.

5. Other interesting extensions of ZF

5.1 We suppose $\tilde{\approx}$ is a definable equivalence. Then $Coll(\tilde{\approx})$ is the scheme defined by:

$$\forall x \in a\ \exists y\ (\varphi(x,y) \wedge \forall y'\ (\varphi(x,y') \Rightarrow y \tilde{\approx} y'))$$
$$\Rightarrow \exists b\ \forall x \in a\ \exists y \in b\ \varphi(x,y).$$

Remarks:

1) In Z, the replacement scheme is exactly $Coll(=)$ and $Coll$ is equivalent to $Coll(V \times V)$.
2) If $\tilde{\approx}_1 \subset \tilde{\approx}_2$ (as classes of ordered pairs), then $Coll(\tilde{\approx}_2) \Rightarrow Coll(\tilde{\approx}_1)$.

By fact 14 (4.2) we know that in $Z^- + Cl$ one can define $\sim_{Min}$, $\sim_{Max}$ (minimum and maximum definable contraction on $\langle V, \in \rangle$) and $\leq_{Min}$, $\leq_{Max}$ (minimum and maximum definable rank on $\langle V, \in \rangle$).
Let $\tilde{\approx}_{Min}$ be the following equivalence:
$x \tilde{\approx}_{Min} y \iff x \leq_{Min} y \wedge y \leq_{Min} x$. Define $\tilde{\approx}_{Max}$ in a similar way.
Sext will be the strong extensionality axiom (see [3]): ($\exists f$ isomorphism $: Cl(x) \to Cl(y)$ such that $f(x) = y$) $\Rightarrow x = y$. Here $Cl(x)$ is $\bigcap \{t \mid x \in t \wedge t$ is a transitive set$\}$.

5.2 We are now able to show:

Theorem 2:

I) $Z^- + Coll \ \underset{?}{\supset}\ Z + Coll + Sext$

II) $Z^- + Cl + Coll(\tilde{\approx}_{Max}) \ \underset{?}{\supset}\ Z + Cl + Coll(\tilde{\approx}_{Max}) + Sext$

III) $Z^- + Cl + Coll(\tilde{\approx}_{Min}) \ \underset{?}{\supset}\ Z + Cl + Coll(\tilde{\approx}_{Min})$.

Proof:

I) We showed in part 3 that, if $\sim$ is a definable contraction on the structure $\langle V, \in \rangle$ in $Z^- + Coll$, then $\langle V, \in_\sim, \sim \rangle$ is a model of $Z + Coll$. If we take $\sim_{Max}$, then $\langle V, \in_{\sim_{Max}}, \sim_{Max} \rangle$ is a model of Sext. Indeed: (here $\sim$ means $\sim_{Max}$) if $\langle V, \in, \sim \rangle \models (\exists f$ isomorphism $: Cl(x) \to Cl(y)$ such that $f(x) = y)$, then is an increasing equivalence $\sim'$ on $\langle V, \in \rangle$ such that $x \sim' y$; this $\sim'$ is defined by: $a \sim' b \iff \langle V, \in_\sim, \sim \rangle \models$ $[(a, b \in Cl(x) \cup Cl(y) \wedge (f(a) = b \vee f(b) = a)) \vee (a, b \notin Cl(x) \cup Cl(y) \wedge a = b)]$. Clearly we have : $x \sim' y$.

But it is known (see [4]) that for each increasing equivalence $\sim'$ there is a contraction $\sim''$ such that $\sim' \subset \sim''$. As $\sim'' \subset \sim_{Max}$, we have : $x \sim_{Max} y$. So we conclude : $\langle V, \in_\sim, \sim \rangle \models x = y$.

II) In $Z^- + Cl + Coll(\approx_{Max})$, consider $\langle V, \in_\sim, \sim \rangle$ where $\sim$ is $\sim_{Max}$. We know already that this is a model of $Z + Cl +$ Sext (see part 3 and proof of I). To complete the proof, let us show that $\langle V, \in_\sim, \sim \rangle$ is a model of $Coll(\approx_{Max})$. Consider the formula ψ :

$$\forall x \in a \; \exists y (\varphi(x,y) \wedge \forall y' (\varphi(x,y') \Rightarrow y \approx_{Max} y')) \Rightarrow \exists b \; \forall x \in a \; \exists y \in b \;\; \varphi(x,y).$$

$\psi_\sim$ is :
$$[\forall x \in_\sim a \; \exists y (\varphi_\sim(x,y) \wedge \forall y' (\varphi_\sim(x,y') \Rightarrow (y \approx_{Max} y')_\sim))] \Rightarrow \exists b \; \forall x \in_\sim a \; \exists y \in_\sim b \; \varphi_\sim(x,y)$$

By fact 16 (4.2), $\leq_{Max}$ on $\langle V, \in \rangle$ is exactly $\leq_{Max}$ on $\langle V, \in_\sim, \sim \rangle$: so $(y \approx_{Max} y')_\sim$ is equivalent to $y \approx_{Max} y'$.

So suppose we have:

$\forall x \in_\sim a\ \exists y\, (\varphi_\sim(x,y) \wedge \forall y'\, (\varphi_\sim(x,y') \Rightarrow y \simeq_{max} y'))$.

It is easy to see that this implies:

$\forall x \in a\ \exists y\, (\varphi_\sim(x,y) \wedge \forall y'\, (\varphi_\sim(x,y') \Rightarrow y \simeq_{max} y'))$.

As we assume $Coll(\simeq_{max})$, this implies:

$\exists b\ \forall x \in a\ \exists y \in b\ \varphi_\sim(x,y)$.

But if $x \in_\sim a$, then $\exists x' \sim x\ x' \in a$. So $\exists y \in b\ \varphi_\sim(x',y)$.

As $(y \in b \Rightarrow y \in_\sim b)$ and $(\varphi_\sim(x',y) \Leftrightarrow \varphi_\sim(x,y))$,

we conclude: $\forall x \in_\sim a\ \exists y \in_\sim b\ \varphi_\sim(x,y)$.

III) This proof is similar to the preceding: it suffices to replace $\simeq_{max}$ by $\simeq_{min}$ and $\sim_{max}$ by $\sim_{min}$ to obtain the model $< V, \in_\sim, \sim >$ of $Z + Cl + Coll(\simeq_{min})$. However, this time there is no reason that $< V, \in_\sim, \sim >$ should be a model of Sext.

6. Preservation under contractions.

6.1 Consider $M = < A, E >$, $\sim$ an equivalence on M and φ a formula in $\mathcal{L}_{ZF}$.

φ is preserved by $\sim$ means:

$\forall \vec{v} \in M\ (M \models \varphi(\vec{v}) \Rightarrow M/_\sim \models \varphi(\vec{v}))$.

Some preservation results are mentioned in [4].

Here we give some further information.

Lemmas:

1) If $\sim$ is an increasing equivalence on M and φ has the property:

$\forall t, t', \vec{a} \in M\ (M/_\sim \models \varphi(t,\vec{a}) \Leftrightarrow \exists t' \sim t\ M \models \varphi(t',\vec{a}))$

then the formula $x = \{t \mid \varphi(t, \vec{a})\}$ is preserved by $\sim$.

<u>Proof</u>: as $\sim$ is increasing, we have: $x \, E_\sim \, y \iff \exists x' \sim x \;\; x' E y$. Take $x, \vec{a} \in M$ such that $M \models (x = \{t \mid \varphi(t, \vec{a})\})$. If $t E_\sim x$, then $\exists t' \sim t \;\; t' E x$; so we have: $\exists t' \sim t \;\; \varphi(t', \vec{a})$; by our hypothesis: $M/_\sim \models \varphi(t, \vec{a})$.
If $M/_\sim \models \varphi(t, \vec{a})$, then by our hypothesis: $\exists t' \sim t \;\; M \models \varphi(t', \vec{a})$; so $t' \sim t \wedge t' E x$, and this implies $t E_\sim x$. So we proved:

$$M/_\sim \models \forall t \, (t \in x \iff \varphi(t, \vec{a})).$$

2) If $\sim$ is an increasing equivalence on M, and φ, ψ are preserved by $\sim$, then: $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall x \, \varphi$, $\exists x \, \varphi$, $\forall x \in y \, \varphi$, $\exists x \in y \, \psi$ are preserved by $\sim$.
(Remember that the atomic formulas are preserved).

<u>Proof</u>: trivial.

3) The following formulas are preserved by increasing equivalences on M:
$x = \{a\}$; $x = \{a, b\}$; $x = \langle a, b \rangle$ (the ordered pair of Kuratowski) ; $x = \cup a$; $x \subset a$; $x = a \times b$.

<u>Proof</u>: trivial using lemma 2.

4) The formula "$x = \mathcal{P} a$" is preserved by definable contractions on models M in which the separationscheme holds.

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<u>Proof</u>:

Suppose $M \models (x = \mathcal{P}a)$. This means:

$M \models [(\forall t \in x \; t \subset a) \wedge \forall t (\forall z \in t \; z \in a \Rightarrow t \in x)]$.

The formula $\forall t \in x \; t \subset a$ is preserved (lemmas 2, 3).

So let us show that: $M/_\sim \models \forall t \; (\forall z \in t \; z \in a \Rightarrow t \in x)$.

Suppose $\forall z \in_\sim t \; z \in_\sim a$. This means: $\forall z \in t \; \exists z' \in a \; z \sim z'$.

Take $t' \in M$ such that $M \models (t' = \{z' \in a \mid \exists z \in t \; z \sim z'\})$.

Such a t' exists because we assumed the separation scheme and we supposed $\sim$ is definable in M. As $\sim$ is a contraction and $M/_\sim \models \forall z \; (z \in t \Leftrightarrow z \in t')$, we have: $t \sim t'$. As $M \models t' \subset a$, we have: $t' \in x$.

So $t \in_\sim x$.

5) If M is a model of Z^- (resp. $Z^- + Cl$) and $\sim$ is a definable contraction on M, then $M/_\sim$ is a model of Z (resp. $Z + Cl$)

<u>Proof</u>: trivial using the preceding lemmas.

6) Coll is preserved by definable increasing equivalences.

<u>Proof</u>: see our theorem 1.

7) Suppose M is a model of the pairing axiom (axiom 3). Let $\langle x,y\rangle_M$ designate a $t \in A$ such that $M \models (t = \langle x,y\rangle)$. If $\sim$ is a contraction on M, then:

$$\langle x,y\rangle_M \sim \langle x',y'\rangle_M \Leftrightarrow x' \sim x \wedge y' \sim y$$

and $M/_\sim \models (a = \langle x,y\rangle) \Leftrightarrow a \sim \langle x,y\rangle_M$.

Proof: easy.

8) Suppose $a \in A$ has the property:
$\forall x \, E \, a \; \forall x' \in A \; (x \, r \, x' \rightarrow x = x')$.
Suppose r is a contraction on M and M satisfies the pairing axiom.
Then the formula "f is a surjective function: $a \rightarrow b$" is preserved by r.

Proof: this formula is exactly $\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4$ where:

$\theta_1 \equiv f \subset a \times b$
$\theta_2 \equiv \forall y \in b \; \exists x \in a \; \exists u \in f \quad u = \langle x, y \rangle$
$\theta_3 \equiv \forall x \in a \; \exists y \in b \quad \langle x, y \rangle \in f$
$\theta_4 \equiv \forall x \in a \; \forall y, z \in b \; (\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f \Rightarrow y = z)$.

Formulas $\theta_1, \theta_2, \theta_3$ are clearly preserved (lemmas 2, 3).
So suppose $M \models \theta_4$. We have to prove $M/_r \models \theta_4$.
If $M/_r \models (\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f)$, then (lemma 7) $\langle x, y \rangle_M \, E_r \, f \wedge \langle x, z \rangle_M \, E_r \, f$ and we have:

$\exists x' r x \; \exists y' r y \; \exists x'' r x \; \exists z' r z \; (\langle x', y' \rangle E f \wedge \langle x'', z' \rangle E f)$.

But by our hypothesis on a: $x = x' = x''$.
So $\langle x, y' \rangle E f \wedge \langle x, z' \rangle E f$. As f is a function in M: $y' = z'$. So we have: $M/_r \models (y = z)$.

9) Suppose $M = \langle A, E \rangle$ is a model of $\mathcal{L}_{ZF}$, $\sim$ is a definable contraction on M and $\theta(x)$ is a formula of $\mathcal{L}_{ZF}$. For our facility, we call X the class defined by $\theta(x)$: so $x \in X$ is only a notation for $\theta(x)$.
If X has the properties:

(1) $\forall x, y \in A \quad M \models (x \in X \wedge x \sim y \Rightarrow x = y)$

(2) $\forall x \in A \quad (M \models \theta(x) \Longleftrightarrow M/_{\sim} \models \theta(x))$

the class X is said to be <u>$\sim$-fixed</u>.

The <u>partition property</u> is the following:

(3) $\forall a \subset X \quad \forall p$ (a partition of a) $\exists z \subset X \quad p \# z$
(here $p \# z$ means there exists a 1-1 function $p \to z$

We are able now to show the following <u>lemma</u>:
If $M \models Z$, $\sim$ is a definable contraction on M, X is a $\sim$-fixed definable class having the partition property and $M \models \forall b \, \exists a \subset X \; a \# b$
Then $M/_{\sim} \models \forall b \, \exists a \subset X \quad a \# b$.

<u>Proof</u>:
As X is $\sim$-fixed, X has the partition property in $M/_{\sim}$ too.

It is easy to show that, if X has the partition property, the sentence $\forall b \, \exists a \subset X \; a \# b$ is equivalent to $\forall b \, \exists a \subset X \; \exists f$ surjection $a \to b$. Indeed, one of the implications is trivial. The other is clear using the natural partition on a induced by the surjection f: $p = \{ y \subset a \mid \forall z, z' \in y \;\; f(z) = f(z')$

What preceeds shows that we have only to prove that the formula $\varphi \equiv \forall b\ \exists a \subset X\ \exists f$ surjection $a \to b$ is preserved by $\sim$. So suppose $M \models \varphi$. If $b \in A$, we have: $M \models \exists a \subset X\ \exists f$ surjection $a \to b$. By our hypothesis on X, a has the property: $\forall x \in a\ \forall x' \in A\ (x \sim x' \Rightarrow x = x')$. So by lemma 8, we have (with the same a and f): $M/_{\sim} \models f$ is a surjection $a \to b$.

10) Suppose $M \models Z$. "On" is the class of all ordinals and "Wf" the class of all well-founded sets ($Wf = \bigcup_{\alpha \in On} R_\alpha$). Then On and Wf are $\sim$-fixed for each definable contraction $\sim$ on M and have the partition property.

<u>Proof</u>: Let X be On or Wf. By induction on $\in$ one proves: $(x \in X \wedge y \sim x) \Rightarrow y = x$. As X is transitive, it is easy to see that $\forall x \in A$ $(M \models x \in X \iff M/_{\sim} \models x \in X)$.
X trivially has the partition property.

6.2 <u>Theorem 3</u>

Let T be one of the following theories: $Z + Cl$, $Z + Coll$, $Z + Cl + AC$, $Z + Cl + AC'$, $Z + Coll + AC$, $Z + Coll + AC'$ (where AC is the usual axiom of choice: $\forall b\ \exists a \subset On\ a \# b$; AC' is the weak

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axiom of choice "R" studied by M. von Rimscha in [8] : $\forall b \exists a \subset Wf \ a \# b$).

If M is a model of T and $\sim$ is an increasing definable equivalence on M,
Then there exists a unique definable contraction $\sim^*$ on M such that:

1) $M/_{\sim^*} \models T$

2) $\sim \subset \sim^*$ and for each definable contraction $\sim'$ on M : $\sim \subset \sim' \Rightarrow \sim^* \subset \sim'$
($\sim^*$ is the minimum definable contraction greater than $\sim$).

3) $M \models (x \sim^* y \iff \exists \alpha \in On \ \exists t$ (transitive set) $x \in t \wedge y \in t \wedge x(\sim_\alpha[t])y)$
where $\sim_\alpha[t]$ is defined in M by:

$$\begin{cases} \sim_0[t] \text{ is } \sim \cap (t \times t) \\ \sim_{\alpha+1}[t] \text{ is } (\sim_\alpha[t])^+ \text{ on the structure } \langle t, \in \rangle \\ \sim_\gamma[t] \text{ is } \bigcup_{\alpha<\gamma} (\sim_\alpha[t]) \text{ if } \gamma \text{ is a limit ordinal} \end{cases}$$

<u>Proof</u>: By induction on $\alpha \in On$, one proves: $\forall \alpha \ \sim_\alpha[t] \cap \sim_\alpha[t'] = \sim_\alpha[t \cap t']$ and $\forall \alpha \ \sim_\alpha[t] \subset \sim_{\alpha+1}[t]$. Then, defining $\sim^*$ as in part 3 of this theorem, one easily shows that $\sim^*$ is indeed a contraction satisfying part 2 of this theorem: the proof is similar ~~t~~o the proof of the existence of a least

contraction on a structure; the only difference is that here we start with "r_0 is r" instead of "r_0 is $=$" (see [4] and 2.1)

That $M/_{r^*}$ is a model of T if M is results from
- lemma 5 if T is $Z + Cl$
- lemmas 5, 6 if T is $Z + Coll$
- lemmas 5, 10 if T is $Z + Cl + AC$ or $Z + Cl + AC'$
- lemmas 5, 6, 10 if T is $Z + Coll + AC$ or $Z + Coll + AC'$

<u>Corollary</u>: If T is one of the theories considered in theorem 3, and r is a definable contraction on M

Then $(M \models T \Rightarrow M/_r \models T)$.

<u>Proof</u>: in this case r^* is r itself.

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