Mathematical Reasoning

Writing and Proof
Version 2.1

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For printings of Version 2.1 dated May 26, 2020 or later, the notation $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$ (for each natural number n) introduced in Section 6.2 has been changed to $R_n = \{0, 1, 2, \ldots, n-1\}$. This was done so this set will not be confused with the mathematically correct definition of $\mathbb{Z}_n$, the integers modulo n, in Section 7.4. This change has also been made in Version 3 of this textbook.

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Cover Photograph: The Little Mac Bridge on the campus of Grand Valley State University in Allendale, Michigan.

Contents

Note to Students

This book may be different than other mathematics textbooks you have used since one of the main goals of this book is to help you to develop the ability to construct and write mathematical proofs. So this book is not just about mathematical content but is also about the process of doing mathematics. Along the way, you will also learn some important mathematical topics that will help you in your future study of mathematics.

This book is designed not to be just casually read but rather to be *engaged*. It may seem like a cliché (because it is in almost every mathematics book now) but there is truth in the statement that *mathematics is not a spectator sport*. To learn and understand mathematics, you must *engage* in the process of doing mathematics. So you must actively read and study the book, which means to have a pencil and paper with you and be willing to follow along and fill in missing details. This type of engagement is not easy and is often frustrating, but if you do so, you will learn a great deal about mathematics and more importantly, about doing mathematics.

Recognizing that actively studying a mathematics book is often not easy, several features of the textbook have been designed to help you become more engaged as you study the material. Some of the features are:

- **Preview Activities.** With the exception of Sections 1.1 and 3.6, each section has exactly two preview activities. Some preview activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other preview activities will introduce new concepts and definitions that will be used when that section is discussed in class. It is very important that you work on these preview activities before starting the rest of the section. Please note that answers to these preview activities are not included in the text. This book is designed to be used for a course and it is left up to the discretion of each individual instructor as to how to distribute the answers to the preview activities.

- **Progress Checks.** Several progress checks are included in each section. These are either short exercises or short activities designed to help you determine if you are understanding the material as you are studying the material in the section. As such, it is important to work through these progress checks to test your understanding, and if necessary, study the material again before proceeding further. So it is important to attempt these progress checks before checking the answers, which are are provided in Appendix B.
- **Chapter Summaries.** To assist you with studying the material in the text, there is a summary at the end of each chapter. The summaries usually list the important definitions introduced in the chapter and the important results proven in the chapter. If appropriate, the summary also describes the important proof techniques discussed in the chapter.
- **Answers for Selected Exercises.** Answers or hints for several exercises are included in an Appendix C. Those exercises with an answer or a hint in the appendix are preceded by a star ($\star$). The main change in Version 2.0 of this textbook from the previous versions is the addition of more exercises with answers or hints in the appendix.

Although not part of the textbook, there are now 107 online videos with about 14 hours of content that span the first seven chapters of this book. These videos are freely available online at Grand Valley's Department of Mathematics YouTube channel on this playlist:

`http://gvsu.edu/s/0l1`

These online videos were created and developed by Dr. Robert Talbert of Grand Valley State University.

There is also a website for the textbook. For this website, go to

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and click on the TEXTBOOKS button in the upper right corner.

You may find some things there that could be of help. For example, there currently is a link to study guides for the sections of this textbook. Good luck with your study of mathematics and please make use of the online videos and the resources available in the textbook and at the website for the textbook. If there are things that you think would be good additions to the book or the web site, please feel free to send me a message at mathreasoning@gmail.com.

Preface

Mathematical Reasoning: Writing and Proof is designed to be a text for the first course in the college mathematics curriculum that introduces students to the processes of constructing and writing proofs and focuses on the formal development of mathematics. The primary goals of the text are to help students:

- Develop logical thinking skills and to develop the ability to think more abstractly in a proof oriented setting.
- Develop the ability to construct and write mathematical proofs using standard methods of mathematical proof including direct proofs, proof by contradiction, mathematical induction, case analysis, and counterexamples.
- Develop the ability to read and understand written mathematical proofs.
- Develop talents for creative thinking and problem solving.
- Improve their quality of communication in mathematics. This includes improving writing techniques, reading comprehension, and oral communication in mathematics.
- Better understand the nature of mathematics and its language.

Another important goal of this text is to provide students with material that will be needed for their further study of mathematics.

This type of course has now become a standard part of the mathematics major at many colleges and universities. It is often referred to as a "transition course" from the calculus sequence to the upper-level courses in the major. The transition is from the problem-solving orientation of calculus to the more abstract and theoretical upper-level courses. This is needed today because many students complete their study of calculus without seeing a formal proof or having constructed a proof of their own. This is in contrast to many upper-level mathematics courses, where the emphasis is on the formal development of abstract mathematical ideas, and the expectations are that students will be able to read and understand proofs and be able to construct and write coherent, understandable mathematical proofs. Students should be able to use this text with a background of one semester of calculus.

Important Features of the Book

Following are some of the important features of this text that will help with the transition from calculus to upper-level mathematics courses.

1. **Emphasis on Writing in Mathematics**

   Issues dealing with writing mathematical exposition are addressed throughout the book. Guidelines for writing mathematical proofs are incorporated into the book. These guidelines are introduced as needed and begin in Section 1.2. Appendix A contains a summary of all the guidelines for writing mathematical proofs that are introduced throughout the text. In addition, every attempt has been made to ensure that every completed proof presented in this text is written according to these guidelines. This provides students with examples of well-written proofs.

   One of the motivating factors for writing this book was to develop a textbook for the course "Communicating in Mathematics" at Grand Valley State University. This course is part of the university's Supplemental Writing Skills Program, and there was no text that dealt with writing issues in mathematics that was suitable for this course. This is why some of the writing guidelines in the text deal with the use of LaTeXor a word processor that is capable of producing the appropriate mathematical symbols and equations. However, the writing guidelines can easily be implemented for courses where students do not have access to this type of word processing.

2. **Instruction in the Process of Constructing Proofs**

   One of the primary goals of this book is to develop students' abilities to construct mathematical proofs. Another goal is to develop their abilities to write the proof in a coherent manner that conveys an understanding of the proof to the reader. These are two distinct skills.

   Instruction on how to write proofs begins in Section 1.2 and is developed further in Chapter 3. In addition, Chapter 4 is devoted to developing students' abilities to construct proofs using mathematical induction.

Students are introduced to a method to organize their thought processes when attempting to construct a proof that uses a so-called know-show table. (See Section 1.2 and Section 3.1.) Students use this table to work backward from what it is they are trying to prove while at the same time working forward from the assumptions of the problem. The know-show tables are used quite extensively in Chapters 1 and 3. However, the explicit use of know-show tables is gradually reduced and these tables are rarely used in the later chapters. One reason for this is that these tables may work well when there appears to be only one way of proving a certain result. As the proofs become more complicated or other methods of proof (such as proofs using cases) are used, these know-show tables become less useful.

So the know-show tables are not to be considered an absolute necessity in using the text. However, they are useful for students beginning to learn how to construct and write proofs. They provide a convenient way for students to organize their work. More importantly, they introduce students to a way of thinking about a problem. Instead of immediately trying to write a complete proof, the know-show table forces students to stop, think, and ask questions such as

- Just exactly what is it that I am trying to prove?
- How can I prove this?
- What methods do I have that may allow me to prove this?
- What are the assumptions?
- How can I use these assumptions to prove the result?

Being able to ask these questions is a big step in constructing a proof. The next task is to answer the questions and to use those answers to construct a proof.

3. Emphasis on Active Learning

One of the underlying premises of this text is that the best way to learn and understand mathematics is to be actively involved in the learning process. However, it is unlikely that students will learn all the mathematics in a given course on their own. Students actively involved in learning mathematics need appropriate materials that will provide guidance and support in their learning of mathematics. There are several ways this text promotes active learning.

- With the exception of Sections 1.1 and 3.6, each section has exactly two preview activities. These preview activities should be completed by the students prior to the classroom discussion of the section. The purpose of the preview activities is to prepare students to participate in the classroom discussion of the section. Some preview activities will review prior mathematical work that is necessary for the new section. This prior work may contain material from previous mathematical courses or it may contain material covered earlier in this text. Other preview activities will introduce new concepts and definitions that will be used when that section is discussed in class.
- Several progress checks are included in each section. These are either short exercises or short activities designed to help the students determine if they are understanding the material as it is presented. Some progress checks are also intended to prepare the student for the next topic in the section. Answers to the progress checks are provided in Appendix B.
- Explorations and activities are included at the end of the exercises of each section. These activities can be done individually or in a collaborative learning setting, where students work in groups to brainstorm, make conjectures, test each others' ideas, reach consensus, and, it is hoped, develop sound mathematical arguments to support their work. These activities can also be assigned as homework in addition to the other exercises at the end of each section.

4. Other Important Features of the Book

- Several sections of the text include exercises called Evaluation of Proofs. (The first such exercise appears in Section 3.1.) For these exercises, there is a proposed proof of a proposition. However, the proposition may be true or may be false. If a proposition is false, the proposed proof is, of course, incorrect, and the student is asked to find the error in the proof and then provide a counterexample showing that the proposition is false. However, if the proposition is true, the proof may be incorrect or not well written. In keeping with the emphasis on writing, students are then asked to correct the proof and/or provide a well-written proof according to the guidelines established in the book.
- To assist students with studying the material in the text, there is a summary at the end of each chapter. The summaries usually list the important definitions introduced in the chapter and the important results

proven in the chapter. If appropriate, the summary also describes the important proof techniques discussed in the chapter.

- Answers or hints for several exercises are included in an appendix. This was done in response to suggestions from many students at Grand Valley and some students from other institutions who were using the book. In addition, those exercises with an answer or a hint in the appendix are preceded by a star ($\star$).

Content and Organization

Mathematical content is needed as a vehicle for learning how to construct and write proofs. The mathematical content for this text is drawn primarily from elementary number theory, including congruence arithmetic; elementary set theory; functions, including injections, surjections, and the inverse of a function; relations and equivalence relations; further topics in number theory such as greatest common divisors and prime factorizations; and cardinality of sets, including countable and uncountable sets. This material was chosen because it can be used to illustrate a broad range of proof techniques and it is needed as a prerequisite for many upper-level mathematics courses.

The chapters in the text can roughly be divided into the following classes:

- Constructing and Writing Proofs: Chapters 1, 3, and 4
- Logic: Chapter 2
- Mathematical Content: Chapters 5, 6, 7, 8, and 9

The first chapter sets the stage for the rest of the book. It introduces students to the use of conditional statements in mathematics, begins instruction in the process of constructing a direct proof of a conditional statement, and introduces many of the writing guidelines that will be used throughout the rest of the book. This is not meant to be a thorough introduction to methods of proof. Before this is done, it is necessary to introduce the students to the parts of logic that are needed to aid in the construction of proofs. This is done in Chapter 2.

Students need to learn some logic and gain experience in the traditional language and proof methods used in mathematics. Since this is a text that deals with constructing and writing mathematical proofs, the logic that is presented in Chapter 2 is intended to aid in the construction of proofs. The goals are to provide students with a thorough understanding of conditional statements, quantifiers, and logical equivalencies. Emphasis is placed on writing correct and useful negations of statements, especially those involving quantifiers. The logical equivalencies that are presented provide the logical basis for some of the standard proof techniques, such as proof by contrapositive, proof by contradiction, and proof using cases.

The standard methods for mathematical proofs are discussed in detail in Chapter 3. The mathematical content that is introduced to illustrate these proof methods includes some elementary number theory, including congruence arithmetic. These concepts are used consistently throughout the text as a way to demonstrate ideas in direct proof, proof by contrapositive, proof by contradiction, proof using cases, and proofs using mathematical induction. This gives students a strong introduction to important mathematical ideas while providing the instructor a consistent reference point and an example of how mathematical notation can greatly simplify a concept.

The three sections of Chapter 4 are devoted to proofs using mathematical induction. Again, the emphasis is not only on understanding mathematical induction but also on developing the ability to construct and write proofs that use mathematical induction.

The last five chapters are considered "mathematical content" chapters. Concepts of set theory are introduced in Chapter 5, and the methods of proof studied in Chapter 3 are used to prove results about sets and operations on sets. The idea of an "element-chasing proof" is also introduced in Section 5.2.

Chapter 6 provides a thorough study of functions. Functions are studied before relations in order to begin with the more specific notion with which students have some familiarity and move toward the more general notion of a relation. The concept of a function is reviewed but with attention paid to being precise with terminology and is then extended to the general definition of a function. Various proof techniques are employed in the study of injections, surjections, composition of functions, inverses of functions, and functions acting on sets.

Chapter 7 introduces the concepts of relations and equivalence relations. Section 7.4 is included to provide a link between the concept of an equivalence relation and the number theory that has been discussed throughout the text.

Chapter 8 continues the study of number theory. The highlights include problems dealing with greatest common divisors, prime numbers, the Fundamental Theorem of Arithmetic, and linear Diophantine equations.

Finally, Chapter 9 deals with further topics in set theory, focusing on cardinality, finite sets, countable sets, and uncountable sets.

Designing a Course

Most instructors who use this text will design a course specifically suited to their needs and the needs of their institution. However, a standard one-semester course in constructing and writing proofs could cover the first six chapters of the text and at least one of Chapter 7, Chapter 8, or Chapter 9. Please note that Sections 4.3, 5.5, 6.6, 7.4, and 8.3 can be considered optional sections. These are interesting sections that contain important material, but the content of these sections is not essential to study the material in the rest of the book.

Supplementary Materials for the Instructor

Instructors for a course may obtain pdf files that contain the solutions for the preview activities and the solutions for the exercises.

To obtain these materials, send an email message to the author at mathreasoning@gmail.com, and please include the name of your institution (school, college, or university), the course for which you are considering using the text, and a link to a website that can be used to verify your position at your institution.

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You may find some things there that could be of help to your students. For example, there currently is a link to study guides for most of the sections of this textbook. If there are things that you think would be good additions to the book or the web site, please feel free to send me a message at mathreasoning@gmail.com.

Chapter 1

Introduction to Writing Proofs in Mathematics

1.1 Statements and Conditional Statements

Much of our work in mathematics deals with statements. In mathematics, a **statement** is a declarative sentence that is either true or false but not both. A statement is sometimes called a **proposition**. The key is that there must be no ambiguity. To be a statement, a sentence must be true or false, and it cannot be both. So a sentence such as "The sky is beautiful" is not a statement since whether the sentence is true or not is a matter of opinion. A question such as "Is it raining?" is not a statement because it is a question and is not declaring or asserting that something is true.

Some sentences that are mathematical in nature often are not statements because we may not know precisely what a variable represents. For example, the equation $2x + 5 = 10$ is not a statement since we do not know what x represents. If we substitute a specific value for x (such as $x = 3$), then the resulting equation, $2 \cdot 3 + 5 = 10$ is a statement (which is a false statement). Following are some more examples:

- There exists a real number x such that $2x + 5 = 10$.
  This is a statement because either such a real number exists or such a real number does not exist. In this case, this is a true statement since such a real number does exist, namely $x = 2.5$.

- For each real number x, $2x + 5 = 2\left(x + \frac{5}{2}\right)$.
  This is a statement since either the sentence $2x + 5 = 2\left(x + \frac{5}{2}\right)$ is true when any real number is substituted for x (in which case, the statement is true) or there is at least one real number that can be substituted for x and produce a false statement (in which case, the statement is false). In this case, the given statement is true.

- Solve the equation $x^2 - 7x + 10 = 0$.
  This is not a statement since it is a directive. It does not assert that something is true.

- $(a + b)^2 = a^2 + b^2$ is not a statement since it is not known what a and b represent. However, the sentence, "There exist real numbers a and b such that $(a+b)^2 = a^2+b^2$" is a statement. In fact, this is a true statement since there are such integers. For example, if $a = 1$ and $b = 0$, then $(a + b)^2 = a^2 + b^2$.

- Compare the statement in the previous item to the statement, "For all real numbers a and b, $(a + b)^2 = a^2 + b^2$." This is a false statement since there are values for a and b for which $(a + b)^2 \neq a^2 + b^2$. For example, if $a = 2$ and $b = 3$, then $(a + b)^2 = 5^2 = 25$ and $a^2 + b^2 = 2^2 + 3^2 = 13$.

Progress Check 1.1 (**Statements**)
Which of the following sentences are statements? Do not worry about determining whether a statement is true or false; just determine whether each sentence is a statement or not.

1. $3 + 4 = 8$.

2. $2 \cdot 7 + 8 = 22$.

3. $(x - 1) = \sqrt{x + 11}$.

4. $2x + 5y = 7$.

5. There are integers x and y such that $2x + 5y = 7$.

6. There are integers x and y such that $23x + 37y = 52$.

7. Given a line L and a point P not on that line and in the same plane, there is a unique line in that plane through P that does not intersect L.

8. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

9. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ for all real numbers a and b.

10. The derivative of the sine function is the cosine function.

11. Does the equation $3x^2 - 5x - 7 = 0$ have two real number solutions?

12. If ABC is a right triangle with right angle at vertex B, and if D is the midpoint of the hypotenuse, then the line segment connecting vertex B to D is half the length of the hypotenuse.

13. There do not exist three integers x, y, and z such that $x^3 + y^3 = z^3$.

How Do We Decide If a Statement Is True or False?

In mathematics, we often establish that a statement is true by writing a mathematical proof. To establish that a statement is false, we often find a so-called counterexample. (These ideas will be explored later in this chapter.) So mathematicians must be able to discover and construct proofs. In addition, once the discovery has been made, the mathematician must be able to communicate this discovery to others who speak the language of mathematics. We will be dealing with these ideas throughout the text.

For now, we want to focus on what happens before we start a proof. One thing that mathematicians often do is to make a conjecture beforehand as to whether the statement is true or false. This is often done through exploration. The role of exploration in mathematics is often difficult because the goal is not to find a specific answer but simply to investigate. Following are some techniques of exploration that might be helpful.

Techniques of Exploration

- **Guesswork and conjectures**. Formulate and write down questions and conjectures. When we make a guess in mathematics, we usually call it a conjecture.

- **Examples**. **Constructing appropriate examples is extremely important**. Exploration often requires looking at lots of examples. In this way, we can gather information that provides evidence that a statement is true, or we might find an example that shows the statement is false. This type of example is called a **counterexample**.

  For example, if someone makes the conjecture that $\sin(2x) = 2\sin(x)$, for all real numbers x, we can test this conjecture by substituting specific values for x. One way to do this is to choose values of x for which $\sin(x)$ is known. Using $x = \dfrac{\pi}{4}$, we see that

$$\sin\left(2\left(\frac{\pi}{4}\right)\right) = \sin\left(\frac{\pi}{2}\right) = 1, \text{ and}$$
$$2\sin\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2}.$$

Since $1 \neq \sqrt{2}$, these calculations show that this conjecture is false. However, if we do not find a counterexample for a conjecture, we usually cannot claim the conjecture is true. The best we can say is that our examples indicate the conjecture is true. As an example, consider the conjecture that

If x and y are odd integers, then $x + y$ is an even integer.

We can do lots of calculations, such as $3 + 7 = 10$ and $5 + 11 = 16$, and find that every time we add two odd integers, the sum is an even integer. However, it is not possible to test every pair of odd integers, and so we can only say that the conjecture appears to be true. (We will prove that this statement is true in the next section.)

- **Use of prior knowledge**. This also is very important. We cannot start from square one every time we explore a statement. We must make use of our acquired mathematical knowledge. For the conjecture that $\sin(2x) = 2\sin(x)$, for all real numbers x, we might recall that there are trigonometric identities called "double angle identities." We may even remember the correct identity for $\sin(2x)$, but if we do not, we can always look it up. We should recall (or find) that

  for all real numbers x, $\sin(2x) = 2\sin(x)\cos(x)$.

  We could use this identity to argue that the conjecture "for all real numbers x, $\sin(2x) = 2\sin(x)$" is false, but if we do, it is still a good idea to give a specific counterexample as we did before.

- **Cooperation and brainstorming**. Working together is often more fruitful than working alone. When we work with someone else, we can compare notes and articulate our ideas. Thinking out loud is often a useful brainstorming method that helps generate new ideas.

Progress Check 1.2 (Explorations)
Use the techniques of exploration to investigate each of the following statements. Can you make a conjecture as to whether the statement is true or false? Can you determine whether it is true or false?

1. $(a+b)^2 = a^2 + b^2$, for all real numbers a and b.
2. There are integers x and y such that $2x + 5y = 41$.
3. If x is an even integer, then x^2 is an even integer.
4. If x and y are odd integers, then $x \cdot y$ is an odd integer.

Conditional Statements

One of the most frequently used types of statements in mathematics is the so-called conditional statement. Given statements P and Q, a statement of the form "If P then Q" is called a **conditional statement**. It seems reasonable that the truth value (true or false) of the conditional statement "If P then Q" depends on the truth values of P and Q. The statement "If P then Q" means that Q must be true whenever P is true. The statement P is called the **hypothesis** of the conditional statement, and the statement Q is called the **conclusion** of the conditional statement. Since conditional statements are probably the most important type of statement in mathematics, we give a more formal definition.

Definition. A **conditional statement** is a statement that can be written in the form "If P then Q," where P and Q are sentences. For this conditional statement, P is called the **hypothesis** and Q is called the **conclusion**.

Intuitively, "If P then Q" means that Q must be true whenever P is true. Because conditional statements are used so often, a symbolic shorthand notation is used to represent the conditional statement "If P then Q." We will use the notation $P \to Q$ to represent "If P then Q." When P and Q are statements, it seems reasonable that the truth value (true or false) of the conditional statement $P \to Q$ depends on the truth values of P and Q. There are four cases to consider:

- P is true and Q is true.
- P is true and Q is false.
- P is false and Q is true.
- P is false and Q is false.

The conditional statement $P \to Q$ means that Q is true whenever P is true. It says nothing about the truth value of Q when P is false. Using this as a guide, we define the conditional statement $P \to Q$ to be false only when P is true and Q is false, that is, only when the hypothesis is true and the conclusion is false. In all other cases, $P \to Q$ is true. This is summarized in Table 1.1, which is called a **truth table** for the conditional statement $P \to Q$. (In Table 1.1, T stands for "true" and F stands for "false.")

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.1: Truth Table for $P \to Q$

The important thing to remember is that the conditional statement $P \to Q$ has its own truth value. It is either true or false (and not both). Its truth value depends on the truth values for P and Q, but some find it a bit puzzling that the conditional statement is considered to be true when the hypothesis P is false. We will provide a justification for this through the use of an example.

Example 1.3 Suppose that I say

"If it is not raining, then Daisy is riding her bike."

We can represent this conditional statement as $P \to Q$ where P is the statement, "It is not raining" and Q is the statement, "Daisy is riding her bike."

Although it is not a perfect analogy, think of the statement $P \to Q$ as being *false* to mean that I lied and think of the statement $P \to Q$ as being *true* to mean that I did not lie. We will now check the truth value of $P \to Q$ based on the truth values of P and Q.

1. Suppose that both P and Q are true. That is, it is not raining and Daisy is riding her bike. In this case, it seems reasonable to say that I told the truth and that $P \to Q$ is true.

2. Suppose that P is true and Q is false or that it is not raining and Daisy is not riding her bike. It would appear that by making the statement, "If it is not

raining, then Daisy is riding her bike," that I have not told the truth. So in this case, the statement $P \to Q$ is false.

3. Now suppose that P is false and Q is true or that it is raining and Daisy is riding her bike. Did I make a false statement by stating that if it is not raining, then Daisy is riding her bike? The key is that I did not make any statement about what would happen if it was raining, and so I did not tell a lie. So we consider the conditional statement, "If it is not raining, then Daisy is riding her bike," to be true in the case where it is raining and Daisy is riding her bike.

4. Finally, suppose that both P and Q are false. That is, it is raining and Daisy is not riding her bike. As in the previous situation, since my statement was $P \to Q$, I made no claim about what would happen if it was raining, and so I did not tell a lie. So the statement $P \to Q$ cannot be false in this case and so we consider it to be true.

Progress Check 1.4 (**Explorations with Conditional Statements**)

1. Consider the following sentence:

   If x is a positive real number, then $x^2 + 8x$ is a positive real number.

   Although the hypothesis and conclusion of this conditional sentence are not statements, the conditional sentence itself can be considered to be a statement as long as we know what possible numbers may be used for the variable x. From the context of this sentence, it seems that we can substitute any positive real number for x. We can also substitute 0 for x or a negative real number for x provided that we are willing to work with a false hypothesis in the conditional statement. (In Chapter 2, we will learn how to be more careful and precise with these types of conditional statements.)

   (**a**) Notice that if $x = -3$, then $x^2 + 8x = -15$, which is negative. Does this mean that the given conditional statement is false?

   (**b**) Notice that if $x = 4$, then $x^2 + 8x = 48$, which is positive. Does this mean that the given conditional statement is true?

   (**c**) Do you think this conditional statement is true or false? Record the results for at least five different examples where the hypothesis of this conditional statement is true.

2. "If n is a positive integer, then $(n^2 - n + 41)$ is a prime number." (Remember that a prime number is a positive integer greater than 1 whose only positive factors are 1 and itself.)

   To explore whether or not this statement is true, try using (and recording your results) for $n = 1$, $n = 2$, $n = 3$, $n = 4$, $n = 5$, and $n = 10$. Then record the results for at least four other values of n. Does this conditional statement appear to be true?

Further Remarks about Conditional Statements

1. The conventions for the truth value of conditional statements may seem a bit strange, especially the fact that the conditional statement is true when the hypothesis of the conditional statement is false. The following example is meant to show that this makes sense.

   Suppose that Ed has exactly \$52 in his wallet. The following four statements will use the four possible truth combinations for the hypothesis and conclusion of a conditional statement.

   - If Ed has exactly \$52 in his wallet, then he has \$20 in his wallet. This is a true statement. Notice that both the hypothesis and the conclusion are true.
   - If Ed has exactly \$52 in his wallet, then he has \$100 in his wallet. This statement is false. Notice that the hypothesis is true and the conclusion is false.
   - If Ed has \$100 in his wallet, then he has at least \$50 in his wallet. This statement is true regardless of how much money he has in his wallet. In this case, the hypothesis is false and the conclusion is true.
   - If Ed has \$100 in his wallet, then he has at least \$80 in his wallet. This statement is true regardless of how much money he has in his wallet. In this case, the hypothesis is false and the conclusion is false.

   This is admittedly a contrived example but it does illustrate that the conventions for the truth value of a conditional statement make sense. The message is that in order to be complete in mathematics, we need to have conventions about when a conditional statement is true and when it is false.

2. The fact that there is only one case when a conditional statement is false often provides a method to show that a given conditional statement is false. In

Progress Check 1.4, you were asked if you thought the following conditional statement was true or false.

If n is a positive integer, then $\left(n^2 - n + 41\right)$ is a prime number.

Perhaps for all of the values you tried for n, $\left(n^2 - n + 41\right)$ turned out to be a prime number. However, if we try $n = 41$, we get

$$\begin{aligned} n^2 - n + 41 &= 41^2 - 41 + 41 \\ n^2 - n + 41 &= 41^2. \end{aligned}$$

So in the case where $n = 41$, the hypothesis is true (41 is a positive integer) and the conclusion is false $\left(41^2 \text{ is not prime}\right)$. Therefore, 41 is a counterexample for this conjecture and the conditional statement

"If n is a positive integer, then $\left(n^2 - n + 41\right)$ is a prime number"

is false. There are other counterexamples (such as $n = 42$, $n = 45$, and $n = 50$), but only one counterexample is needed to prove that the statement is false.

3. Although one example can be used to prove that a conditional statement is false, in most cases, we cannot use examples to prove that a conditional statement is true. For example, in Progress Check 1.4, we substituted values for x for the conditional statement "If x is a positive real number, then $x^2 + 8x$ is a positive real number." For every positive real number used for x, we saw that $x^2 + 8x$ was positive. However, this does not prove the conditional statement to be true because it is impossible to substitute every positive real number for x. So, although we may believe this statement is true, to be able to conclude it is true, we need to write a mathematical proof. Methods of proof will be discussed in Section 1.2 and Chapter 3.

Progress Check 1.5 (Working with a Conditional Statement)
Sometimes, we must be aware of conventions that are being used. In most calculus texts, the convention is that any function has a domain and a range that are subsets of the real numbers. In addition, when we say something like "the function f is differentiable at a", it is understood that a is a real number. With these conventions, the following statement is a true statement, which is proven in many calculus texts.

If the function f is differentiable at a, then the function f is continuous at a.

Using only this true statement, is it possible to make a conclusion about the function in each of the following cases?

1. It is known that the function f, where $f(x) = \sin x$, is differentiable at 0.
2. It is known that the function f, where $f(x) = \sqrt[3]{x}$, is not differentiable at 0.
3. It is known that the function f, where $f(x) = |x|$, is continuous at 0.
4. It is known that the function f, where $f(x) = \dfrac{|x|}{x}$ is not continuous at 0.

Closure Properties of Number Systems

The primary number system used in algebra and calculus is the **real number system**. We usually use the symbol $\mathbb{R}$ to stand for the set of all real numbers. The real numbers consist of the rational numbers and the irrational numbers. The **rational numbers** are those real numbers that can be written as a quotient of two integers (with a nonzero denominator), and the **irrational numbers** are those real numbers that cannot be written as a quotient of two integers. That is, a rational number can be written in the form of a fraction, and an irrational number cannot be written in the form of a fraction. Some common irrational numbers are $\sqrt{2}$, π, and e. We usually use the symbol $\mathbb{Q}$ to represent the set of all rational numbers. (The letter $\mathbb{Q}$ is used because rational numbers are quotients of integers.) There is no standard symbol for the set of all irrational numbers.

Perhaps the most basic number system used in mathematics is the set of **natural numbers**. The natural numbers consist of the positive whole numbers such as 1, 2, 3, 107, and 203. We will use the symbol $\mathbb{N}$ to stand for the set of natural numbers. Another basic number system that we will be working with is the set of **integers**. The integers consist of zero, the natural numbers, and the negatives of the natural numbers. If n is an integer, we can write $n = \dfrac{n}{1}$. So each integer is a rational number and hence also a real number.

We will use the letter $\mathbb{Z}$ to stand for the set of integers. (The letter $\mathbb{Z}$ is from the German word, *Zahlen*, for numbers.) Three of the basic properties of the integers are that the set $\mathbb{Z}$ is **closed under addition**, the set $\mathbb{Z}$ is **closed under multiplication**, and the set of integers is **closed under subtraction**. This means that

- If x and y are integers, then $x + y$ is an integer;

- If x and y are integers, then $x \cdot y$ is an integer; and
- If x and y are integers, then $x - y$ is an integer.

Notice that these so-called closure properties are defined in terms of conditional statements. This means that if we can find one instance where the hypothesis is true and the conclusion is false, then the conditional statement is false.

Example 1.6 (Closure)

1. In order for the set of natural numbers to be closed under subtraction, the following conditional statement would have to be true: If x and y are natural numbers, then $x - y$ is a natural number. However, since 5 and 8 are natural numbers, $5 - 8 = -3$, which is not a natural number, this conditional statement is false. Therefore, the set of natural numbers is not closed under subtraction.

2. We can use the rules for multiplying fractions and the closure rules for the integers to show that the rational numbers are closed under multiplication. If $\dfrac{a}{b}$ and $\dfrac{c}{d}$ are rational numbers (so a, b, c, and d are integers and b and d are not zero), then
$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$
Since the integers are closed under multiplication, we know that ac and bd are integers and since $b \neq 0$ and $d \neq 0$, $bd \neq 0$. Hence, $\dfrac{ac}{bd}$ is a rational number and this shows that the rational numbers are closed under multiplication.

Progress Check 1.7 (Closure Properties)
Answer each of the following questions.

1. Is the set of rational numbers closed under addition? Explain.
2. Is the set of integers closed under division? Explain.
3. Is the set of rational numbers closed under subtraction? Explain.

Exercises for Section 1.1

⋆ **1.** Which of the following sentences are statements?

(**a**) $3^2 + 4^2 = 5^2$.
(**b**) $a^2 + b^2 = c^2$.
(**c**) There exist integers a, b, and c such that $a^2 = b^2 + c^2$.
(**d**) If $x^2 = 4$, then $x = 2$.
(**e**) For each real number x, if $x^2 = 4$, then $x = 2$.
(**f**) For each real number t, $\sin^2 t + \cos^2 t = 1$.
(**g**) $\sin x < \sin\left(\dfrac{\pi}{4}\right)$.
(**h**) If n is a prime number, then n^2 has three positive factors.
(**i**) $1 + \tan^2 \theta = \sec^2 \theta$.
(**j**) Every rectangle is a parallelogram.
(**k**) Every even natural number greater than or equal to 4 is the sum of two prime numbers.

⋆ **2.** Identify the hypothesis and the conclusion for each of the following conditional statements.

(**a**) If n is a prime number, then n^2 has three positive factors.
(**b**) If a is an irrational number and b is an irrational number, then $a \cdot b$ is an irrational number.
(**c**) If p is a prime number, then $p = 2$ or p is an odd number.
(**d**) If p is a prime number and $p \neq 2$, then p is an odd number.
(**e**) If $p \neq 2$ and p is an even number, then p is not prime.

⋆ **3.** Determine whether each of the following conditional statements is true or false.

(**a**) If $10 < 7$, then $3 = 4$.
(**b**) If $7 < 10$, then $3 = 4$.
(**c**) If $10 < 7$, then $3 + 5 = 8$.
(**d**) If $7 < 10$, then $3 + 5 = 8$.

⋆ **4.** Determine the conditions under which each of the following conditional sentences will be a true statement.

(**a**) If $a + 2 = 5$, then $8 < 5$.

(**b**) If $5 < 8$, then $a + 2 = 5$.

5. Let P be the statement "Student X passed every assignment in Calculus I," and let Q be the statement "Student X received a grade of C or better in Calculus I."

(**a**) What does it mean for P to be true? What does it mean for Q to be true?

(**b**) Suppose that Student X passed every assignment in Calculus I and received a grade of B−, and that the instructor made the statement $P \to Q$. Would you say that the instructor lied or told the truth?

(**c**) Suppose that Student X passed every assignment in Calculus I and received a grade of C−, and that the instructor made the statement $P \to Q$. Would you say that the instructor lied or told the truth?

(**d**) Now suppose that Student X did not pass two assignments in Calculus I and received a grade of D, and that the instructor made the statement $P \to Q$. Would you say that the instructor lied or told the truth?

(**e**) How are Parts (5b), (5c), and (5d) related to the truth table for $P \to Q$?

6. Following is a statement of a theorem which can be proven using calculus or precalculus mathematics. For this theorem, a, b, and c are real numbers.

Theorem. If f is a quadratic function of the form $f(x) = ax^2 + bx + c$ and $a < 0$, then the function f has a maximum value when $x = \dfrac{-b}{2a}$.

Using **only** this theorem, what can be concluded about the functions given by the following formulas?

⋆ (**a**) $g(x) = -8x^2 + 5x - 2$

⋆ (**b**) $h(x) = -\dfrac{1}{3}x^2 + 3x$

⋆ (**c**) $k(x) = 8x^2 - 5x - 7$

(**d**) $j(x) = -\dfrac{71}{99}x^2 + 210$

(**e**) $f(x) = -4x^2 - 3x + 7$

(**f**) $F(x) = -x^4 + x^3 + 9$

7. Following is a statement of a theorem which can be proven using the quadratic formula. For this theorem, a, b, and c are real numbers.

Theorem If f is a quadratic function of the form $f(x) = ax^2 + bx + c$ and $ac < 0$, then the function f has two x-intercepts.

Using **only** this theorem, what can be concluded about the functions given by the following formulas?

(**a**) $g(x) = -8x^2 + 5x - 2$

(**b**) $h(x) = -\frac{1}{3}x^2 + 3x$

(**c**) $k(x) = 8x^2 - 5x - 7$

(**d**) $j(x) = -\frac{71}{99}x^2 + 210$

(**e**) $f(x) = -4x^2 - 3x + 7$

(**f**) $F(x) = -x^4 + x^3 + 9$

8. Following is a statement of a theorem about certain cubic equations. For this theorem, b represents a real number.

Theorem A. If f is a cubic function of the form $f(x) = x^3 - x + b$ and $b > 1$, then the function f has exactly one x-intercept.

Following is another theorem about x-intercepts of functions:

Theorem B. If f and g are functions with $g(x) = k \cdot f(x)$, where k is a nonzero real number, then f and g have exactly the same x-intercepts.

Using only these two theorems and some simple algebraic manipulations, what can be concluded about the functions given by the following formulas?

(**a**) $f(x) = x^3 - x + 7$

(**b**) $g(x) = x^3 + x + 7$

(**c**) $h(x) = -x^3 + x - 5$

(**d**) $k(x) = 2x^3 + 2x + 3$

(**e**) $r(x) = x^4 - x + 11$

(**f**) $F(x) = 2x^3 - 2x + 7$

* **9.** (**a**) Is the set of natural numbers closed under division?

(**b**) Is the set of rational numbers closed under division?

(**c**) Is the set of nonzero rational numbers closed under division?

(**d**) Is the set of positive rational numbers closed under division?

(**e**) Is the set of positive real numbers closed under subtraction?

(**f**) Is the set of negative rational numbers closed under division?

(**g**) Is the set of negative integers closed under addition?

Explorations and Activities

10. Exploring Propositions. In Progress Check 1.2, we used exploration to show that certain statements were false and to make conjectures that certain statements were true. We can also use exploration to formulate a conjecture that we believe to be true. For example, if we calculate successive powers of 2 $\left(2^1, 2^2, 2^3, 2^4, 2^5, \ldots\right)$ and examine the units digits of these numbers, we could make the following conjectures (among others):

- If n is a natural number, then the units digit of 2^n must be 2, 4, 6, or 8.
- The units digits of the successive powers of 2 repeat according to the pattern "2, 4, 8, 6."

(a) Is it possible to formulate a conjecture about the units digits of successive powers of 4 $\left(4^1, 4^2, 4^3, 4^4, 4^5, \ldots\right)$? If so, formulate at least one conjecture.

(b) Is it possible to formulate a conjecture about the units digit of numbers of the form $7^n - 2^n$, where n is a natural number? If so, formulate a conjecture in the form of a conditional statement in the form "If n is a natural number, then . . . ."

(c) Let $f(x) = e^{2x}$. Determine the first eight derivatives of this function. What do you observe? Formulate a conjecture that appears to be true. The conjecture should be written as a conditional statement in the form, "If n is a natural number, then . . . ."

1.2 Constructing Direct Proofs

Preview Activity 1 (Definition of Even and Odd Integers)
Definitions play a very important role in mathematics. A direct proof of a proposition in mathematics is often a demonstration that the proposition follows logically from certain definitions and previously proven propositions. A **definition** is an agreement that a particular word or phrase will stand for some object, property, or other concept that we expect to refer to often. In many elementary proofs, the answer to the question, "How do we prove a certain proposition?", is often answered by means of a definition. For example, in Progress Check 1.2 on page 5, all of the examples should have indicated that the following conditional statement is true:

If x and y are odd integers, then $x \cdot y$ is an odd integer.

In order to construct a mathematical proof of this conditional statement, we need a precise definition of what it means to say that an integer is an even integer and what it means to say that an integer is an odd integer.

> **Definition**. An integer a is an **even integer** provided that there exists an integer n such that $a = 2n$. An integer a is an **odd integer** provided there exists an integer n such that $a = 2n + 1$.

Using this definition, we can conclude that the integer 16 is an even integer since $16 = 2 \cdot 8$ and 8 is an integer. By answering the following questions, you should obtain a better understanding of these definitions. These questions are not here just to have questions in the textbook. Constructing and answering such questions is a way in which many mathematicians will try to gain a better understanding of a definition.

1. Use the definitions given above to
   (a) Explain why 28, -42, 24, and 0 are even integers.
   (b) Explain why 51, -11, 1, and -1 are odd integers.

It is important to realize that mathematical definitions are not made randomly. In most cases, they are motivated by a mathematical concept that occurs frequently.

2. Are the definitions of even integers and odd integers consistent with your previous ideas about even and odd integers?

Preview Activity 2 (Thinking about a Proof)
Consider the following proposition:

Proposition. If x and y are odd integers, then $x \cdot y$ is an odd integer.

Think about how you might go about proving this proposition. A **direct proof** of a conditional statement is a demonstration that the conclusion of the conditional statement follows logically from the hypothesis of the conditional statement. Definitions and previously proven propositions are used to justify each step in the proof. To help get started in proving this proposition, answer the following questions:

1. The proposition is a conditional statement. What is the hypothesis of this conditional statement? What is the conclusion of this conditional statement?
2. If $x = 2$ and $y = 3$, then $x \cdot y = 6$, and 6 is an even integer. Does this example prove that the proposition is false? Explain.
3. If $x = 5$ and $y = 3$, then $x \cdot y = 15$. Does this example prove that the proposition is true? Explain.

In order to prove this proposition, we need to prove that whenever both x and y are odd integers, $x \cdot y$ is an odd integer. Since we cannot explore all possible pairs of integer values for x and y, we will use the definition of an odd integer to help us construct a proof.

4. To start a proof of this proposition, we will assume that the hypothesis of the conditional statement is true. So in this case, we assume that both x and y are odd integers. We can then use the definition of an odd integer to conclude that there exists an integer m such that $x = 2m + 1$. Now use the definition of an odd integer to make a conclusion about the integer y.

Note: The definition of an odd integer says that a certain other integer exists. This definition may be applied to both x and y. However, do not use the same letter in both cases. To do so would imply that $x = y$ and we have not made that assumption. To be more specific, if $x = 2m + 1$ and $y = 2m + 1$, then $x = y$.

5. We need to prove that if the hypothesis is true, then the conclusion is true. So, in this case, we need to prove that $x \cdot y$ is an odd integer. At this point, we usually ask ourselves a so-called **backward question**. In this case, we ask, "Under what conditions can we conclude that $x \cdot y$ is an odd integer?" Use the definition of an odd integer to answer this question.

Properties of Number Systems

At the end of Section 1.1, we introduced notations for the standard number systems we use in mathematics. We also discussed some closure properties of the standard number systems. For this text, it is assumed that the reader is familiar with these closure properties and the basic rules of algebra that apply to all real numbers. That is, it is assumed the reader is familiar with the properties of the real numbers shown in Table 1.2.

Constructing a Proof of a Conditional Statement

In order to prove that a conditional statement $P \to Q$ is true, we only need to prove that Q is true whenever P is true. This is because the conditional statement is true whenever the hypothesis is false. So in a direct proof of $P \to Q$, we assume that P is true, and using this assumption, we proceed through a logical sequence of steps to arrive at the conclusion that Q is true.

Unfortunately, it is often not easy to discover how to start this logical sequence of steps or how to get to the conclusion that Q is true. We will describe a method of exploration that often can help in discovering the steps of a proof. This method will involve working forward from the hypothesis, P, and backward from the conclusion, Q. We will use a device called the **"know-show table"** to help organize

For all real numbers x, y, and z

Identity Properties	$x + 0 = x$ and $x \cdot 1 = x$
Inverse Properties	$x + (-x) = 0$ and if $x \neq 0$, then $x \cdot \dfrac{1}{x} = 1$.
Commutative Properties	$x + y = y + x$ and $xy = yx$
Associative Properties	$(x + y) + z = x + (y + z)$ and $(xy)\,z = x\,(yz)$
Distributive Properties	$x\,(y + z) = xy + xz$ and $(y + z)\,x = yx + zx$

Table 1.2: Properties of the Real Numbers

our thoughts and the steps of the proof. This will be illustrated with the proposition from Preview Activity 2.

Proposition. *If x and y are odd integers, then $x \cdot y$ is an odd integer.*

The first step is to identify the hypothesis, P, and the conclusion, Q, of the conditional statement. In this case, we have the following:

P: x and y are odd integers. Q: $x \cdot y$ is an odd integer.

We now treat P as what we know (we have assumed it to be true) and treat Q as what we want to show (that is, the goal). So we organize this by using P as the first step in the know portion of the table and Q as the last step in the show portion of the table. We will put the know portion of the table at the top and the show portion of the table at the bottom.

Step	**Know**	**Reason**
P	x and y are odd integers.	Hypothesis
$P1$		
$\vdots$	$\vdots$	$\vdots$
$Q1$		
Q	$x \cdot y$ is an odd integer.	?
Step	**Show**	**Reason**

We have not yet filled in the reason for the last step because we do not yet know how we will reach the goal. The idea now is to ask ourselves questions about what we know and what we are trying to prove. We usually start with the conclusion that we are trying to prove by asking a so-called **backward question.** The basic form of the question is, "Under what conditions can we conclude that Q is true?" How we ask the question is crucial since we must be able to answer it. We should first try to ask and answer the question in an abstract manner and then apply it to the particular form of statement Q.

In this case, we are trying to prove that some integer is an odd integer. So our backward question could be, "How do we prove that an integer is odd?" At this time, the only way we have of answering this question is to use the definition of an odd integer. So our answer could be, "We need to prove that there exists an integer q such that the integer equals $2q + 1$." We apply this answer to statement Q and insert it as the next to last line in the know-show table.

Step	**Know**	**Reason**
P	x and y are odd integers.	Hypothesis
$P1$		
$\vdots$	$\vdots$	$\vdots$
$Q1$	There exists an integer q such that $xy = 2q + 1$.	
Q	$x \cdot y$ is an odd integer.	Definition of an odd integer
Step	**Show**	**Reason**

We now focus our effort on proving statement $Q1$ since we know that if we can prove $Q1$, then we can conclude that Q is true. We ask a backward question about $Q1$ such as, "How can we prove that there exists an integer q such that $x \cdot y = 2q + 1$?" We may not have a ready answer for this question, and so we look at the know portion of the table and try to connect the know portion to the show portion. To do this, we work forward from step P, and this involves asking a **forward question.** The basic form of this type of question is, "What can we conclude from the fact that P is true?" In this case, we can use the definition of an odd integer to conclude that there exist integers m and n such that $x = 2m + 1$ and $y = 2n + 1$. We will call this Step $P1$ in the know-show table. It is important to notice that we were careful not to use the letter q to denote these integers. If we had used q again, we would be claiming that the same integer that gives $x \cdot y = 2q + 1$ also gives $x = 2q + 1$. This is why we used m and n for the integers x and y since there is no guarantee that x equals y. The basic rule of thumb is to use a different symbol for each new object we introduce in a proof. So at this point, we have:

- Step $P1$. We know that there exist integers m and n such that $x = 2m + 1$ and $y = 2n + 1$.

- Step $Q1$. We need to prove that there exists an integer q such that $x \cdot y = 2q + 1$.

We must always be looking for a way to link the "know part" to the "show part". There are conclusions we can make from $P1$, but as we proceed, we must always keep in mind the form of statement in $Q1$. The next forward question is, "What can we conclude about $x \cdot y$ from what we know?" One way to answer this is to use our prior knowledge of algebra. That is, we can first use substitution to write $x \cdot y = (2m + 1)(2n + 1)$. Although this equation does not prove that $x \cdot y$ is odd, we can use algebra to try to rewrite the right side of this equation $(2m + 1)(2n + 1)$ in the form of an odd integer so that we can arrive at step $Q1$. We first expand the right side of the equation to obtain

$$\begin{aligned} x \cdot y &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \end{aligned}$$

Now compare the right side of the last equation to the right side of the equation in step $Q1$. Sometimes the difficult part at this point is the realization that q stands for some integer and that we only have to show that $x \cdot y$ equals two times some integer plus one. Can we now make that conclusion? The answer is yes because we can factor a 2 from the first three terms on the right side of the equation and obtain

$$\begin{aligned} x \cdot y &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

We can now complete the table showing the outline of the proof as follows:

Step	**Know**	**Reason**
P	x and y are odd integers.	Hypothesis
$P1$	There exist integers m and n such that $x = 2m + 1$ and $y = 2n + 1$.	Definition of an odd integer.
$P2$	$xy = (2m + 1)(2n + 1)$	Substitution
$P3$	$xy = 4mn + 2m + 2n + 1$	Algebra
$P4$	$xy = 2(2mn + m + n) + 1$	Algebra
$P5$	$(2mn + m + n)$ is an integer.	Closure properties of the integers
$Q1$	There exists an integer q such that $xy = 2q + 1$.	Use $q = (2mn + m + n)$
Q	$x \cdot y$ is an odd integer.	Definition of an odd integer

It is very important to realize that we have only constructed an outline of a proof. Mathematical proofs are not written in table form. They are written in narrative form using complete sentences and correct paragraph structure, and they follow certain conventions used in writing mathematics. In addition, most proofs are written only from the forward perspective. That is, although the use of the backward process was essential in discovering the proof, when we write the proof in narrative form, we use the forward process described in the preceding table. A completed proof follows.

Theorem 1.8. *If x and y are odd integers, then $x \cdot y$ is an odd integer.*

Proof. We assume that x and y are odd integers and will prove that $x \cdot y$ is an odd integer. Since x and y are odd, there exist integers m and n such that

$$x = 2m + 1 \text{ and } y = 2n + 1.$$

Using algebra, we obtain

$$\begin{aligned} x \cdot y &= (2m + 1)(2n + 1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1. \end{aligned}$$

Since m and n are integers and the integers are closed under addition and multiplication, we conclude that $(2mn + m + n)$ is an integer. This means that $x \cdot y$ has been written in the form $(2q + 1)$ for some integer q, and hence, $x \cdot y$ is an odd integer. Consequently, it has been proven that if x and y are odd integers, then $x \cdot y$ is an odd integer. ■

Writing Guidelines for Mathematics Proofs

At the risk of oversimplification, doing mathematics can be considered to have two distinct stages. The first stage is to convince yourself that you have solved the problem or proved a conjecture. This stage is a creative one and is quite often how mathematics is actually done. The second equally important stage is to convince other people that you have solved the problem or proved the conjecture. This second stage often has little in common with the first stage in the sense that it does not really communicate the process by which you solved the problem or proved the conjecture. However, it is an important part of the process of communicating mathematical results to a wider audience.

A **mathematical proof** is a convincing argument (within the accepted standards of the mathematical community) that a certain mathematical statement is necessarily true. A proof generally uses deductive reasoning and logic but also contains some amount of ordinary language (such as English). A mathematical proof that you write should convince an appropriate audience that the result you are proving is in fact true. So we do not consider a proof complete until there is a well-written proof. So it is important to introduce some writing guidelines. The preceding proof was written according to the following basic guidelines for writing proofs. More writing guidelines will be given in Chapter 3.

1. **Begin with a carefully worded statement of the theorem or result to be proven.** This should be a simple declarative statement of the theorem or result. Do not simply rewrite the problem as stated in the textbook or given on a handout. Problems often begin with phrases such as "Show that" or "Prove that." This should be reworded as a simple declarative statement of the theorem. Then skip a line and write "Proof" in italics or boldface font (when using a word processor). Begin the proof on the same line. Make sure that all paragraphs can be easily identified. Skipping a line between paragraphs or indenting each paragraph can accomplish this.

   As an example, an exercise in a text might read, "Prove that if x is an odd integer, then x^2 is an odd integer." This could be started as follows:

   Theorem. If x is an odd integer, then x^2 is an odd integer.

   Proof: We assume that x is an odd integer . . .

2. **Begin the proof with a statement of your assumptions.** Follow the statement of your assumptions with a statement of what you will prove.

   Theorem. If x is an odd integer, then x^2 is an odd integer.

   Proof. We assume that x is an odd integer and will prove that x^2 is an odd integer.

3. **Use the pronoun "we."** If a pronoun is used in a proof, the usual convention is to use "we" instead of "I." The idea is to stress that you and the reader are doing the mathematics together. It will help encourage the reader to continue working through the mathematics. Notice that we started the proof of Theorem 1.8 with "We assume that . . . ."

4. **Use italics for variables when using a word processor.** When using a word processor to write mathematics, the word processor needs to be capable of producing the appropriate mathematical symbols and equations. The

mathematics that is written with a word processor should look like typeset mathematics. This means that italics is used for variables, boldface font is used for vectors, and regular font is used for mathematical terms such as the names of the trigonometric and logarithmic functions.

For example, we do not write sin (x) or *sin (x)*. The proper way to typeset this is $\sin(x)$.

5. **Display important equations and mathematical expressions.** Equations and manipulations are often an integral part of mathematical exposition. Do not write equations, algebraic manipulations, or formulas in one column with reasons given in another column. Important equations and manipulations should be displayed. This means that they should be centered with blank lines before and after the equation or manipulations, and if the left side of the equations does not change, it should not be repeated. For example,

   Using algebra, we obtain

$$\begin{aligned} x \cdot y &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1. \end{aligned}$$

   Since m and n are integers, we conclude that . . . .

6. **Tell the reader when the proof has been completed.** Perhaps the best way to do this is to simply write, "This completes the proof." Although it may seem repetitive, a good alternative is to finish a proof with a sentence that states precisely what has been proven. In any case, it is usually good practice to use some "end of proof symbol" such as $\blacksquare$.

Progress Check 1.9 (Proving Propositions)
Construct a know-show table for each of the following propositions and then write a formal proof for one of the propositions.

1. If x is an even integer and y is an even integer, then $x + y$ is an even integer.

2. If x is an even integer and y is an odd integer, then $x + y$ is an odd integer.

3. If x is an odd integer and y is an odd integer, then $x + y$ is an even integer.

Some Comments about Constructing Direct Proofs

1. When we constructed the know-show table prior to writing a proof for Theorem 1.8, we had only one answer for the backward question and one answer for the forward question. Often, there can be more than one answer for these questions. For example, consider the following statement:

   If x is an odd integer, then x^2 is an odd integer.

   The backward question for this could be, "How do I prove that an integer is an odd integer?" One way to answer this is to use the definition of an odd integer, but another way is to use the result of Theorem 1.8. That is, we can prove an integer is odd by proving that it is a product of two odd integers.

   The difficulty then is deciding which answer to use. Sometimes we can tell by carefully watching the interplay between the forward process and the backward process. Other times, we may have to work with more than one possible answer.

2. Sometimes we can use previously proven results to answer a forward question or a backward question. This was the case in the example given in Comment (1), where Theorem 1.8 was used to answer a backward question.

3. Although we start with two separate processes (forward and backward), the key to constructing a proof is to find a way to link these two processes. This can be difficult. One way to proceed is to use the know portion of the table to motivate answers to backward questions and to use the show portion of the table to motivate answers to forward questions.

4. Answering a backward question can sometimes be tricky. If the goal is the statement Q, we must construct the know-show table so that if we know that $Q1$ is true, then we can conclude that Q is true. It is sometimes easy to answer this in a way that if it is known that Q is true, then we can conclude that $Q1$ is true. For example, suppose the goal is to prove

   $$y^2 = 4,$$

   where y is a real number. A backward question could be, "How do we prove the square of a real number equals four?" One possible answer is to prove that the real number equals 2. Another way is to prove that the real number equals -2. This is an appropriate backward question, and these are appropriate answers.

However, if the goal is to prove

$$y = 2,$$

where y is a real number, we could ask, "How do we prove a real number equals 2?" It is not appropriate to answer this question with "prove that the square of the real number equals 4." This is because if $y^2 = 4$, then it is not necessarily true that $y = 2$.

5. Finally, it is very important to realize that not every proof can be constructed by the use of a simple know-show table. Proofs will get more complicated than the ones that are in this section. The main point of this section is not the know-show table itself, but the way of thinking about a proof that is indicated by a know-show table. In most proofs, it is very important to specify carefully what it is that is being assumed and what it is that we are trying to prove. The process of asking the "backward questions" and the "forward questions" is the important part of the know-show table. It is very important to get into the "habit of mind" of working backward from what it is we are trying to prove and working forward from what it is we are assuming. Instead of immediately trying to write a complete proof, we need to stop, think, and ask questions such as

- Just exactly what is it that I am trying to prove?
- How can I prove this?
- What methods do I have that may allow me to prove this?
- What are the assumptions?
- How can I use these assumptions to prove the result?

Progress Check 1.10 (Exploring a Proposition)
Construct a table of values for $\left(3m^2 + 4m + 6\right)$ using at least six different integers for m. Make one-half of the values for m even integers and the other half odd integers. Is the following proposition true or false?

If m is an odd integer, then $\left(3m^2 + 4m + 6\right)$ is an odd integer.

Justify your conclusion. This means that if the proposition is true, then you should write a proof of the proposition. If the proposition is false, you need to provide an example of an odd integer for which $\left(3m^2 + 4m + 6\right)$ is an even integer.

Progress Check 1.11 (Constructing and Writing a Proof)
The **Pythagorean Theorem** for right triangles states that if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$. For example, if $a = 5$ and $b = 12$ are the lengths of the two sides of a right triangle and if c is the length of the hypotenuse, then the $c^2 = 5^2 + 12^2$ and so $c^2 = 169$. Since c is a length and must be positive, we conclude that $c = 13$.

Construct and provide a well-written proof for the following proposition.

Proposition. If m is a real number and m, $m + 1$, and $m + 2$ are the lengths of the three sides of a right triangle, then $m = 3$.

Although this proposition uses different mathematical concepts than the one used in this section, the process of constructing a proof for this proposition is the same forward-backward method that was used to construct a proof for Theorem 1.8. However, the backward question, "How do we prove that $m = 3$?" is simple but may be difficult to answer. The basic idea is to develop an equation from the forward process and show that $m = 3$ is a solution of that equation.

Exercises for Section 1.2

1. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.

   ⋆ **(a)** If m is an even integer, then $m + 1$ is an odd integer.

   (b) If m is an odd integer, then $m + 1$ is an even integer.

2. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.

   (a) If x is an even integer and y is an even integer, then $x + y$ is an even integer.

   (b) If x is an even integer and y is an odd integer, then $x + y$ is an odd integer.

   ⋆ **(c)** If x is an odd integer and y is an odd integer, then $x + y$ is an even integer.

3. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.

⋆ **(a)** If m is an even integer and n is an integer, then $m \cdot n$ is an even integer.

⋆ **(b)** If n is an even integer, then n^2 is an even integer.

(c) If n is an odd integer, then n^2 is an odd integer.

4. Construct a know-show table and write a complete proof for each of the following statements:

⋆ **(a)** If m is an even integer, then $5m + 7$ is an odd integer.

(b) If m is an odd integer, then $5m + 7$ is an even integer.

(c) If m and n are odd integers, then $mn + 7$ is an even integer.

5. Construct a know-show table and write a complete proof for each of the following statements:

(a) If m is an even integer, then $3m^2 + 2m + 3$ is an odd integer.

⋆ **(b)** If m is an odd integer, then $3m^2 + 7m + 12$ is an even integer.

6. In this section, it was noted that there is often more than one way to answer a backward question. For example, if the backward question is, "How can we prove that two real numbers are equal?", one possible answer is to prove that their difference equals 0. Another possible answer is to prove that the first is less than or equal to the second and that the second is less than or equal to the first.

⋆ **(a)** Give at least one more answer to the backward question, "How can we prove that two real numbers are equal?"

(b) List as many answers as you can for the backward question, "How can we prove that a real number is equal to zero?"

(c) List as many answers as you can for the backward question, "How can we prove that two lines are parallel?"

⋆ **(d)** List as many answers as you can for the backward question, "How can we prove that a triangle is isosceles?"

7. Are the following statements true or false? Justify your conclusions.

(a) If a, b and c are integers, then $ab + ac$ is an even integer.

(b) If b and c are odd integers and a is an integer, then $ab + ac$ is an even integer.

8. Is the following statement true or false? Justify your conclusion.

If a and b are nonnegative real numbers and $a + b = 0$, then $a = 0$.

Either give a counterexample to show that it is false or outline a proof by completing a know-show table.

9. An integer a is said to be a **type 0 integer** if there exists an integer n such that $a = 3n$. An integer a is said to be a **type 1 integer** if there exists an integer n such that $a = 3n + 1$. An integer a is said to be a **type 2 integer** if there exists an integer m such that $a = 3m + 2$.

⋆ **(a)** Give examples of at least four different integers that are type 1 integers.

(b) Give examples of at least four different integers that are type 2 integers.

⋆ **(c)** By multiplying pairs of integers from the list in Exercise (9a), does it appear that the following statement is true or false?

If a and b are both type 1 integers, then $a \cdot b$ is a type 1 integer.

10. Use the definitions in Exercise (9) to help write a proof for each of the following statements:

⋆ **(a)** If a and b are both type 1 integers, then $a + b$ is a type 2 integer.

(b) If a and b are both type 2 integers, then $a + b$ is a type 1 integer.

(c) If a is a type 1 integer and b is a type 2 integer, then $a \cdot b$ is a type 2 integer.

(d) If a and b are both type 2 integers, then $a \cdot b$ is type 1 integer.

11. Let a, b, and c be real numbers with $a \neq 0$. The solutions of the **quadratic equation** $ax^2 + bx + c = 0$ are given by the **quadratic formula**, which states that the solutions are x_1 and x_2, where

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

(a) Prove that the sum of the two solutions of the quadratic equation $ax^2 + bx + c = 0$ is equal to $-\dfrac{b}{a}$.

(b) Prove that the product of the two solutions of the quadratic equation $ax^2 + bx + c = 0$ is equal to $\dfrac{c}{a}$.

12. (**a**) See Exercise (11) for the quadratic formula, which gives the solutions to a quadratic equation. Let a, b, and c be real numbers with $a \neq 0$. The discriminant of the quadratic equation $ax^2+bx+c = 0$ is defined to be $b^2 - 4ac$. Explain how to use this discriminant to determine if the quadratic equation has two real number solutions, one real number solution, or no real number solutions.

(**b**) Prove that if a, b, and c are real numbers with $a > 0$ and $c < 0$, then one solutions of the quadratic equation $ax^2 + bx + c = 0$ is a positive real number.

(**c**) Prove that if a, b, and c are real numbers with $a \neq 0$, $b > 0$, and $b < 2\sqrt{ac}$, then the quadratic equation $ax^2 + bx + c = 0$ has no real number solutions.

Explorations and Activities

13. Pythagorean Triples. Three natural numbers a, b, and c with $a < b < c$ are said to form a **Pythagorean triple** provided that $a^2 + b^2 = c^2$. For example, 3, 4, and 5 form a Pythagorean triple since $3^2 + 4^2 = 5^2$. The study of Pythagorean triples began with the development of the **Pythagorean Theorem** for right triangles, which states that if a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$. For example, if the lengths of the legs of a right triangle are 4 and 7 units, then $c^2 = 4^2 + 7^2 = 63$, and the length of the hypotenuse must be $\sqrt{63}$ units (since the length must be a positive real number). Notice that 4, 7, and $\sqrt{63}$ are not a Pythagorean triple since $\sqrt{63}$ is not a natural number.

(**a**) Verify that each of the following triples of natural numbers form a Pythagorean triple.

- 3, 4, and 5
- 6, 8, and 10
- 8, 15, and 17
- 10, 24, and 26
- 12, 35, and 37
- 14, 48, and 50

(**b**) Does there exist a Pythagorean triple of the form m, $m + 7$, and $m + 8$, where m is a natural number? If the answer is yes, determine all such Pythagorean triples. If the answer is no, prove that no such Pythagorean triple exists.

(**c**) Does there exist a Pythagorean triple of the form m, $m+11$, and $m+12$, where m is a natural number? If the answer is yes, determine all such

Pythagorean triples. If the answer is no, prove that no such Pythagorean triple exists.

14. **More Work with Pythagorean Triples**. In Exercise (13), we verified that each of the following triples of natural numbers are Pythagorean triples:

- 3, 4, and 5
- 6, 8, and 10
- 8, 15, and 17
- 10, 24, and 26
- 12, 35, and 37
- 14, 48, and 50

(**a**) Focus on the least even natural number in each of these Pythagorean triples. Let n be this even number and find m so that $n = 2m$. Now try to write formulas for the other two numbers in the Pythagorean triple in terms of m. For example, for 3, 4, and 5, $n = 4$ and $m = 2$, and for 8, 15, and 17, $n = 8$ and $m = 4$. Once you think you have formulas, test your results with $m = 10$. That is, check to see that you have a Pythagorean triple whose smallest even number is 20.

(**b**) Write a proposition and then write a proof of the proposition. The proposition should be in the form: If m is a natural number and $m \geq 2$, then ......

1.3 Chapter 1 Summary

Important Definitions

- Statement, page 1
- Conditional statement, pages **??**, 5
- Even integer, page 15
- Odd integer, page 15
- Pythagorean triple, page 29

Important Number Systems and Their Properties

- The natural numbers, $\mathbb{N}$; the integers, $\mathbb{Z}$; the rational numbers, $\mathbb{Q}$; and the real numbers, $\mathbb{R}$. See page 10.
- Closure Properties of the Number Systems

Number System	**Closed Under**
Natural Numbers, $\mathbb{N}$	addition and multiplication
Integers, $\mathbb{Z}$	addition, subtraction, and multiplication
Rational Numbers, $\mathbb{Q}$	addition, subtraction, multiplication, and division by nonzero rational numbers
Real Numbers, $\mathbb{R}$	addition, subtraction, multiplication, and division by nonzero real numbers

- Inverse, commutative, associative, and distributive properties of the real numbers. See page 18.

Important Theorems and Results

- **Exercise (1), Section 1.2**
  If m is an even integer, then $m + 1$ is an odd integer.
  If m is an odd integer, then $m + 1$ is an even integer.

- **Exercise (2), Section 1.2**
  If x is an even integer and y is an even integer, then $x + y$ is an even integer.
  If x is an even integer and y is an odd integer, then $x + y$ is an odd integer.
  If x is an odd integer and y is an odd integer, then $x + y$ is an even integer.

- **Exercise (3), Section 1.2**. *If x is an even integer and y is an integer, then $x \cdot y$ is an even integer.*
- **Theorem 1.8**. *If x is an odd integer and y is an odd integer, then $x \cdot y$ is an odd integer.*
- The **Pythagorean Theorem**, page 26. *If a and b are the lengths of the legs of a right triangle and c is the length of the hypotenuse, then $a^2 + b^2 = c^2$.*

Chapter 2

Logical Reasoning

2.1 Statements and Logical Operators

Preview Activity 1 (Compound Statements)
Mathematicians often develop ways to construct new mathematical objects from existing mathematical objects. It is possible to form new statements from existing statements by connecting the statements with words such as "and" and "or" or by negating the statement. A **logical operator** (or **connective**) on mathematical statements is a word or combination of words that combines one or more mathematical statements to make a new mathematical statement. A **compound statement** is a statement that contains one or more operators. Because some operators are used so frequently in logic and mathematics, we give them names and use special symbols to represent them.

- The **conjunction** of the statements P and Q is the statement "P **and** Q" and its denoted by $P \wedge Q$. The statement $P \wedge Q$ is true only when both P and Q are true.
- The **disjunction** of the statements P and Q is the statement "P **or** Q" and its denoted by $P \vee Q$. The statement $P \vee Q$ is true only when at least one of P or Q is true.
- The **negation** of the statement P is the statement "**not** P" and is denoted by $\neg P$. The negation of P is true only when P is false, and $\neg P$ is false only when P is true.
- The **implication** or **conditional** is the statement "**If** P **then** Q" and is denoted by $P \to Q$. The statement $P \to Q$ is often read as "P **implies** Q,"

and we have seen in Section 1.1 that $P \to Q$ is false only when P is true and Q is false.

Some comments about the disjunction.
It is important to understand the use of the operator "or." In mathematics, we use the **"inclusive or"** unless stated otherwise. This means that $P \vee Q$ is true when both P and Q are true and also when only one of them is true. That is, $P \vee Q$ is true when at least one of P or Q is true, or $P \vee Q$ is false only when both P and Q are false.

A different use of the word "or" is the **"exclusive or."** For the exclusive or, the resulting statement is false when both statements are true. That is, "P exclusive or Q" is true only when exactly one of P or Q is true. In everyday life, we often use the exclusive or. When someone says, "At the intersection, turn left or go straight," this person is using the exclusive or.

Some comments about the negation. Although the statement, $\neg P$, can be read as "It is not the case that P," there are often better ways to say or write this in English. For example, we would usually say (or write):

- The negation of the statement, "391 is prime" is "391 is not prime."
- The negation of the statement, "$12 < 9$" is "$12 \geq 9$."

1. For the statements

$$P\colon \text{15 is odd} \qquad Q\colon \text{15 is prime}$$

write each of the following statements as English sentences and determine whether they are true or false. Notice that P is true and Q is false.

(**a**) $P \wedge Q$. (**b**) $P \vee Q$. (**c**) $P \wedge \neg Q$. (**d**) $\neg P \vee \neg Q$.

2. For the statements

$$P\colon \text{15 is odd} \qquad R\colon 15 < 17$$

write each of the following statements in symbolic form using the operators $\wedge$, $\vee$, and $\neg$.

(**a**) $15 \geq 17$.

(**b**) 15 is odd or $15 \geq 17$.

(**c**) 15 is even or $15 < 17$.

(**d**) 15 is odd and $15 \geq 17$.

Preview Activity 2 (**Truth Values of Statements**)
We will use the following two statements for all of this activity:

- P is the statement "It is raining."
- Q is the statement "Daisy is playing golf."

In each of the following four parts, a truth value will be assigned to statements P and Q. For example, in Question (1), we will assume that each statement is true. In Question (2), we will assume that P is true and Q is false. In each part, determine the truth value of each of the following statements:

(**a**) $(P \wedge Q)$ It is raining and Daisy is playing golf.

(**b**) $(P \vee Q)$ It is raining or Daisy is playing golf.

(**c**) $(P \to Q)$ If it is raining, then Daisy is playing golf.

(**d**) $(\neg P)$ It is not raining.

Which of the four statements [(a) through (d)] are true and which are false in each of the following four situations?

1. When P is true (it is raining) and Q is true (Daisy is playing golf).
2. When P is true (it is raining) and Q is false (Daisy is not playing golf).
3. When P is false (it is not raining) and Q is true (Daisy is playing golf).
4. When P is false (it is not raining) and Q is false (Daisy is not playing golf).

In the preview activities for this section, we learned about compound statements and their truth values. This information can be summarized with the following truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Rather than memorizing the truth tables, for many people it is easier to remember the rules summarized in Table 2.1.

Operator	**Symbolic Form**	**Summary of Truth Values**
Conjunction	$P \wedge Q$	True only when both P and Q are true
Disjunction	$P \vee Q$	False only when both P and Q are false
Negation	$\neg P$	Opposite truth value of P
Conditional	$P \to Q$	False only when P is true and Q is false

Table 2.1: Truth Values for Common Connectives

Other Forms of Conditional Statements

Conditional statements are extremely important in mathematics because almost all mathematical theorems are (or can be) stated as a conditional statement in the following form:

If "certain conditions are met," then "something happens."

It is imperative that all students studying mathematics thoroughly understand the meaning of a conditional statement and the truth table for a conditional statement.

We also need to be aware that in the English language, there are other ways for expressing the conditional statement $P \to Q$ other than "If P, then Q." Following are some common ways to express the conditional statement $P \to Q$ in the English language:

- If P, then Q.
- P implies Q.
- P only if Q.
- Q if P.
- Whenever P is true, Q is true.
- Q is true whenever P is true.
- Q is necessary for P. (This means that if P is true, then Q is necessarily true.)
- P is sufficient for Q. (This means that if you want Q to be true, it is sufficient to show that P is true.)

In all of these cases, P is the **hypothesis** of the conditional statement and Q is the **conclusion** of the conditional statement.

Progress Check 2.1 (The "Only If" Statement)
Recall that a quadrilateral is a four-sided polygon. Let S represent the following true conditional statement:

If a quadrilateral is a square, then it is a rectangle.

Write this conditional statement in English using

1. the word "whenever"
2. the phrase "only if"
3. the phrase "is necessary for"
4. the phrase "is sufficient for"

Constructing Truth Tables

Truth tables for compound statements can be constructed by using the truth tables for the basic connectives. To illustrate this, we will construct a truth table for $(P \wedge \neg Q) \to R$. The first step is to determine the number of rows needed.

- For a truth table with two different simple statements, four rows are needed since there are four different combinations of truth values for the two statements. We should be consistent with how we set up the rows. The way we will do it in this text is to label the rows for the first statement with (T, T, F, F) and the rows for the second statement with (T, F, T, F). All truth tables in the text have this scheme.

- For a truth table with three different simple statements, eight rows are needed since there are eight different combinations of truth values for the three statements. Our standard scheme for this type of truth table is shown in Table 2.2.

The next step is to determine the columns to be used. One way to do this is to work backward from the form of the given statement. For $(P \wedge \neg Q) \to R$, the last step is to deal with the conditional operator ($\to$). To do this, we need to know the truth values of $(P \wedge \neg Q)$ and R. To determine the truth values for $(P \wedge \neg Q)$, we need to apply the rules for the conjunction operator ($\wedge$) and we need to know the truth values for P and $\neg Q$.

Table 2.2 is a completed truth table for $(P \wedge \neg Q) \to R$ with the step numbers indicated at the bottom of each column. The step numbers correspond to the order in which the columns were completed.

P	Q	R	$\neg Q$	$P \wedge \neg Q$	$(P \wedge \neg Q) \to R$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T
1	1	1	2	3	4

Table 2.2: Truth Table for $(P \wedge \neg Q) \to R$

- When completing the column for $P \wedge \neg Q$, remember that the only time the conjunction is true is when both P and $\neg Q$ are true.
- When completing the column for $(P \wedge \neg Q) \to R$, remember that the only time the conditional statement is false is when the hypothesis $(P \wedge \neg Q)$ is true and the conclusion, R, is false.

The last column entered is the truth table for the statement $(P \wedge \neg Q) \to R$ using the setup in the first three columns.

Progress Check 2.2 (Constructing Truth Tables)
Construct a truth table for each of the following statements:

1. $P \wedge \neg Q$
2. $\neg (P \wedge Q)$
3. $\neg P \wedge \neg Q$
4. $\neg P \vee \neg Q$

Do any of these statements have the same truth table?

The Biconditional Statement

Some mathematical results are stated in the form "P if and only if Q" or "P is necessary and sufficient for Q." An example would be, "A triangle is equilateral if and only if its three interior angles are congruent." The symbolic form for the biconditional statement "P if and only if Q" is $P \leftrightarrow Q$. In order to determine a truth table for a biconditional statement, it is instructive to look carefully at the form of the phrase "P if and only if Q." The word "and" suggests that this statement is a conjunction. Actually it is a conjunction of the statements "P if Q" and "P only if Q." The symbolic form of this conjunction is $[(Q \to P) \wedge (P \to Q)]$.

Progress Check 2.3 (The Truth Table for the Biconditional Statement)
Complete a truth table for $[(Q \to P) \wedge (P \to Q)]$. Use the following columns: P, Q, $Q \to P$, $P \to Q$, and $[(Q \to P) \wedge (P \to Q)]$. The last column of this table will be the truth table for $P \leftrightarrow Q$.

Other Forms of the Biconditional Statement

As with the conditional statement, there are some common ways to express the biconditional statement, $P \leftrightarrow Q$, in the English language. For example,

- P if and only if Q.
- P implies Q and Q implies P.
- P is necessary and sufficient for Q.

Tautologies and Contradictions

Definition. A **tautology** is a compound statement S that is true for all possible combinations of truth values of the component statements that are part of S. A **contradiction** is a compound statement that is false for all possible combinations of truth values of the component statements that are part of S.

That is, a tautology is necessarily true in all circumstances, and a contradiction is necessarily false in all circumstances.

Progress Check 2.4 (Tautologies and Contradictions) For statements P and Q:

1. Use a truth table to show that $(P \vee \neg P)$ is a tautology.
2. Use a truth table to show that $(P \wedge \neg P)$ is a contradiction.
3. Use a truth table to determine if $P \to (P \vee Q)$ is a tautology, a contradiction, or neither.

Exercises for Section 2.1

⋆ **1.** Suppose that Daisy says, "If it does not rain, then I will play golf." Later in the day you come to know that it did rain but Daisy still played golf. Was Daisy's statement true or false? Support your conclusion.

⋆ **2.** Suppose that P and Q are statements for which $P \to Q$ is true and for which $\neg Q$ is true. What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) P (**b**) $P \wedge Q$ (**c**) $P \vee Q$

3. Suppose that P and Q are statements for which $P \to Q$ is false. What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) $\neg P \to Q$ (**b**) $Q \to P$ (**c**) $P \vee Q$

4. Suppose that P and Q are statements for which Q is false and $\neg P \to Q$ is true (and it is not known if R is true or false). What conclusion (if any) can be made about the truth value of each of the following statements?

(**a**) $\neg Q \to P$

(**b**) P

⋆ (**c**) $P \wedge R$

(**d**) $R \to \neg P$

⋆ **5.** Construct a truth table for each of the following statements:

(**a**) $P \to Q$

(**b**) $Q \to P$

(**c**) $\neg P \to \neg Q$

(**d**) $\neg Q \to \neg P$

Do any of these statements have the same truth table?

6. Construct a truth table for each of the following statements:

(**a**) $P \vee \neg Q$

(**b**) $\neg (P \vee Q)$

(**c**) $\neg P \vee \neg Q$

(**d**) $\neg P \wedge \neg Q$

Do any of these statements have the same truth table?

⋆ **7.** Construct truth tables for $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$. What do you observe?

8. Suppose each of the following statements is true.

- Laura is in the seventh grade.
- Laura got an A on the mathematics test or Sarah got an A on the mathematics test.
- If Sarah got an A on the mathematics test, then Laura is not in the seventh grade.

If possible, determine the truth value of each of the following statements. Carefully explain your reasoning.

(**a**) Laura got an A on the mathematics test.

(**b**) Sarah got an A on the mathematics test.

(**c**) Either Laura or Sarah did not get an A on the mathematics test.

9. Let P stand for "the integer x is even," and let Q stand for "x^2 is even." Express the conditional statement $P \to Q$ in English using

(**a**) The "if then" form of the conditional statement

(**b**) The word "implies"

- ⋆ **(c)** The "only if" form of the conditional statement
- ⋆ **(d)** The phrase "is necessary for"
- **(e)** The phrase "is sufficient for"

10. Repeat Exercise (9) for the conditional statement $Q \to P$.

⋆ **11.** For statements P and Q, use truth tables to determine if each of the following statements is a tautology, a contradiction, or neither.

(**a**) $\neg Q \vee (P \to Q)$.

(**b**) $Q \wedge (P \wedge \neg Q)$.

(**c**) $(Q \wedge P) \wedge (P \to \neg Q)$.

(**d**) $\neg Q \to (P \wedge \neg P)$.

12. For statements P, Q, and R:

(**a**) Show that $[(P \to Q) \wedge P] \to Q$ is a tautology. **Note**: In symbolic logic, this is an important logical argument form called **modus ponens**.

(**b**) Show that $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ is a tautology. **Note**: In symbolic logic, this is an important logical argument form called **syllogism**.

Explorations and Activities

13. Working with Conditional Statements. Complete the following table:

English Form	**Hypothesis**	**Conclusion**	**Symbolic Form**
If P, then Q.	P	Q	$P \to Q$
Q only if P.	Q	P	$Q \to P$
P is necessary for Q.			
P is sufficient for Q.			
Q is necessary for P.			
P implies Q.			
P only if Q.			
P if Q.			
If Q then P.			
If $\neg Q$, then $\neg P$.			
If P, then $Q \wedge R$.			
If $P \vee Q$, then R.			

14. Working with Truth Values of Statements. Suppose that P and Q are true statements, that U and V are false statements, and that W is a statement and it is not known if W is true or false.

Which of the following statements are true, which are false, and for which statements is it not possible to determine if it is true or false? Justify your conclusions.

(a) $(P \vee Q) \vee (U \wedge W)$
(b) $P \wedge (Q \to W)$
(c) $P \wedge (W \to Q)$
(d) $W \to (P \wedge U)$
(e) $W \to (P \wedge \neg U)$
(f) $(\neg P \vee \neg U) \wedge (Q \vee \neg V)$
(g) $(P \wedge \neg V) \wedge (U \vee W)$
(h) $(P \vee \neg Q) \to (U \wedge W)$
(i) $(P \vee W) \to (U \wedge W)$
(j) $(U \wedge \neg V) \to (P \wedge W)$

2.2 Logically Equivalent Statements

Preview Activity 1 (Logically Equivalent Statements)
In Exercises (5) and (6) from Section 2.1, we observed situations where two different statements have the same truth tables. Basically, this means these statements are equivalent, and we make the following definition:

> **Definition**. Two expressions are **logically equivalent** provided that they have the same truth value for all possible combinations of truth values for all variables appearing in the two expressions. In this case, we write $X \equiv Y$ and say that X and Y are logically equivalent.

1. Complete truth tables for $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$.

2. Are the expressions $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ logically equivalent?

3. Suppose that the statement "I will play golf and I will mow the lawn" is false. Then its negation is true. Write the negation of this statement in the form of a disjunction. Does this make sense?

Sometimes we actually use logical reasoning in our everyday living! Perhaps you can imagine a parent making the following two statements.

Statement 1	If you do not clean your room, then you cannot watch TV.
Statement 2	You clean your room or you cannot watch TV.

4. Let P be "you do not clean your room," and let Q be "you cannot watch TV." Use these to translate Statement 1 and Statement 2 into symbolic forms.

5. Construct a truth table for each of the expressions you determined in Part (4). Are the expressions logically equivalent?

6. Assume that Statement 1 and Statement 2 are false. In this case, what is the truth value of P and what is the truth value of Q? Now, write a true statement in symbolic form that is a conjunction and involves P and Q.

7. Write a truth table for the (conjunction) statement in Part (6) and compare it to a truth table for $\neg (P \to Q)$. What do you observe?

Preview Activity 2 (Converse and Contrapositive)
We now define two important conditional statements that are associated with a given conditional statement.

Definition. If P and Q are statements, then

- The **converse** of the conditional statement $P \to Q$ is the conditional statement $Q \to P$.
- The **contrapositive** of the conditional statement $P \to Q$ is the conditional statement $\neg Q \to \neg P$.

1. For the following, the variable x represents a real number. Label each of the following statements as true or false.

   (**a**) If $x = 3$, then $x^2 = 9$.
   (**b**) If $x^2 = 9$, then $x = 3$.
   (**c**) If $x^2 \neq 9$, then $x \neq 3$.
   (**d**) If $x \neq 3$, then $x^2 \neq 9$.

2. Which statement in the list of conditional statements in Part (1) is the converse of Statement (1a)? Which is the contrapositive of Statement (1a)?

3. Complete appropriate truth tables to show that

   - $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$.

- $P \to Q$ is not logically equivalent to its converse $Q \to P$.

In Preview Activity 1, we introduced the concept of logically equivalent expressions and the notation $X \equiv Y$ to indicate that statements X and Y are logically equivalent. The following theorem gives two important logical equivalencies. They are sometimes referred to as **De Morgan's Laws**.

Theorem 2.5 (De Morgan's Laws)
For statements P and Q,

- *The statement $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$. This can be written as $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$.*

- *The statement $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$. This can be written as $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.*

The first equivalency in Theorem 2.5 was established in Preview Activity 1. Table 2.3 establishes the second equivalency.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Table 2.3: Truth Table for One of De Morgan's Laws

It is possible to develop and state several different logical equivalencies at this time. However, we will restrict ourselves to what are considered to be some of the most important ones. Since many mathematical statements are written in the form of conditional statements, logical equivalencies related to conditional statements are quite important.

Logical Equivalencies Related to Conditional Statements

The first two logical equivalencies in the following theorem were established in Preview Activity 1, and the third logical equivalency was established in Preview Activity 2.

Theorem 2.6. *For statements P and Q,*

1. *The conditional statement $P \to Q$ is logically equivalent to $\neg P \vee Q$.*
2. *The statement $\neg (P \to Q)$ is logically equivalent to $P \wedge \neg Q$.*
3. *The conditional statement $P \to Q$ is logically equivalent to its contrapositive $\neg Q \to \neg P$.*

The Negation of a Conditional Statement

The logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$ is interesting because it shows us that **the negation of a conditional statement is not another conditional statement.** The negation of a conditional statement can be written in the form of a conjunction. So what does it mean to say that the conditional statement

If you do not clean your room, then you cannot watch TV,

is false? To answer this, we can use the logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$. The idea is that if $P \to Q$ is false, then its negation must be true. So the negation of this can be written as

You do not clean your room and you can watch TV.

For another example, consider the following conditional statement:

$$\text{If } -5 < -3, \text{ then } (-5)^2 < (-3)^2.$$

This conditional statement is false since its hypothesis is true and its conclusion is false. Consequently, its negation must be true. Its negation is not a conditional statement. The negation can be written in the form of a conjunction by using the logical equivalency $\neg (P \to Q) \equiv P \wedge \neg Q$. So, the negation can be written as follows:

$$-5 < -3 \text{ and } \neg \left((-5)^2 < (-3)^2 \right).$$

However, the second part of this conjunction can be written in a simpler manner by noting that "not less than" means the same thing as "greater than or equal to." So we use this to write the negation of the original conditional statement as follows:

$$-5 < -3 \text{ and } (-5)^2 \geq (-3)^2.$$

This conjunction is true since each of the individual statements in the conjunction is true.

Another Method of Establishing Logical Equivalencies

We have seen that it is often possible to use a truth table to establish a logical equivalency. However, it is also possible to prove a logical equivalency using a sequence of previously established logical equivalencies. For example,

- $P \to Q$ is logically equivalent to $\neg P \vee Q$. So
- $\neg (P \to Q)$ is logically equivalent to $\neg (\neg P \vee Q)$.
- Hence, by one of De Morgan's Laws (Theorem 2.5), $\neg (P \to Q)$ is logically equivalent to $\neg (\neg P) \wedge \neg Q$.
- This means that $\neg (P \to Q)$ is logically equivalent to $P \wedge \neg Q$.

The last step used the fact that $\neg (\neg P)$ is logically equivalent to P.

When proving theorems in mathematics, it is often important to be able to decide if two expressions are logically equivalent. Sometimes when we are attempting to prove a theorem, we may be unsuccessful in developing a proof for the original statement of the theorem. However, in some cases, it is possible to prove an equivalent statement. Knowing that the statements are equivalent tells us that if we prove one, then we have also proven the other. In fact, once we know the truth value of a statement, then we know the truth value of any other logically equivalent statement. This is illustrated in Progress Check 2.7.

Progress Check 2.7 (Working with a Logical Equivalency)
In Section 2.1, we constructed a truth table for $(P \wedge \neg Q) \to R$. See page 38.

1. Although it is possible to use truth tables to show that $P \to (Q \vee R)$ is logically equivalent to $(P \wedge \neg Q) \to R$, we instead use previously proven logical equivalencies to prove this logical equivalency. In this case, it may be easier to start working with $(P \wedge \neg Q) \to R$. Start with

$$(P \wedge \neg Q) \to R \equiv \neg (P \wedge \neg Q) \vee R,$$

which is justified by the logical equivalency established in Part (5) of Preview Activity 1. Continue by using one of De Morgan's Laws on $\neg (P \wedge \neg Q)$.