This is the end, page down for the start, sorry...

Prop 2.4

for $m, n, p \in \mathbb{Z}$

if $m < n$ and $n < p$ then $m < p$

$<$ is transitive

Assume ① $m < n$ and ② $n < p$.

Goal: $m < p$

③ Since $m < n$, $n - m \in \mathbb{N}$

④ Since $n < p$ $p - n \in \mathbb{N}$

Notice that (again using the defn) what we want to show is that $p - m \in \mathbb{N}$.

$(n-m)+(p-n)$ with familiar manipulation

leaves $p-m$ and so $p-m \in \mathbb{N}$

by 2.1.i

$$p+-n+n+-m = p+-n$$

def -

$$= p-m$$

$(n-m)+(p-n) =$

$(n+(-m))+(p+(-n)) =$ ~~def~~ comm + 1

$(\underbrace{(-m)+n}_{x})+(\underbrace{-n}_{y}+\underbrace{p}_{z}) =$ assoc + ~~comm~~

$((-m+n)+-n)+p =$ assoc +

$(-m+(n+-n))+p =$ ~~def~~ inv +

$(-m+0)+p =$ id +

$(-m)+p =$ comm +

$p+-m =$ def -

$p-m$

so $m < p$ by def $<$.

Suppose $m < n$ and $p > 0$

Then $mp < np$.

Since ① $m < n$ ② $p < 0$ $0 < p$ $p - 0 \in \mathbb{N}$

This is gives us ③ $n - m \in \mathbb{N}$

④ $p \in \mathbb{N}$

then (assuming we've proved by chart subtraction)

$p(n-m) \in \mathbb{N}$

but $p(n-m) = pn - pm = np - mp$

so $np - mp \in \mathbb{N}$

so $mp < np$ def $<$

For each $m, n \in \mathbb{Z}$ there is exactly one x such that $m + x = n$.

prop 1.23

Definition (subtraction): For each $m, n \in \mathbb{Z}$, $n - m$ is defined as $n + (-m)$.

Note that $m + (n - m)$

$$= m + (n + (-m)) \quad \text{def}$$

$$= m + (-m + n) \quad \text{comm } +$$

$$= (m + -m) + n \quad \text{assoc } +$$

$$= 0 + n \quad \text{inv } + 1.2$$

$$= n \quad \text{thm.}$$

by prop 1.23

if $m + x = n$, x must be $n + (-m)$

We do not have anywhere near a complete specification of the integers at this point.

$\mathbb{Z} = \{0, 1, 2\}$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

mod 3 arithmetic satisfies axioms 1-5

$$1 + 1 + 1 = 0$$

Another example of this a relation $<$

$$x < y$$

Axiom

1. For each x, y exactly one of

$$x < y$$

$$x = y$$

$$y < x$$

is true.

2. If $x < y$, $y < z$ then $x < z$

3. If $x < y$, $x + z < y + z$

4. If $x < y$, and $0 < z$ then $xz < yz$.

Theorem: $0 < 1$

Theorem $0<1$:

(recall that an axiom tells us $0 \leq 1$.)

Suppose that $0<1$ is not true.

We know that we have

$$\cancel{0<1} \text{ or } \cancel{0=1} \text{ or } 1<0$$

so we must have $1<0$.

then we have by ax2 $1+(-1)<0+(-1)$

so $0<-1$

if $x<y$ and $0<z$ then $xz<yz$

Since so $\overset{x}{0}<\overset{y}{-1}$ and $0<\overset{z}{-1}$ then $(0\cdot -1)<(-1)(-1)$

$0<1$

but we assumed this is false

(X)

so $0<1$ must be true.

so $0+1 < 1+1$

so $0+1+1 < 1+1+1$

$1+1 < 1+1+1$

so $0 < 1+1+1$

so $0 \neq 1+1+1$.

Forget $<$ for the moment

We introduce a subset $\mathbb{N}$ of $\mathbb{Z}$

elements of $\mathbb{N}$ are called natural numbers

Axiom 2.1:

1. if $m, n \in \mathbb{N}$ then $m+n \in \mathbb{N}$
2. if $m, n \in \mathbb{N}$ then $mn \in \mathbb{N}$
3. $0 \notin \mathbb{N}$
4. ~~$m\in$~~ For each m, $m \in \mathbb{Z}$ or $m = 0$ or $-m \in \mathbb{Z}$

what this says is, that $\mathbb{N}$ is closed under $+$ and $\cdot$ and 0 is not a natural number

Proposition 2.2 for $m \in \mathbb{Z}$
exactly one of the following is true:

$m \in \mathbb{N}$, $m = 0$, $-m \in \mathbb{N}$

Proof:

either $m = 0$ or $m \neq 0$
since $m \in \mathbb{Z}$

Case 1: $m = 0$

By Prop 2.1, $m \notin \mathbb{N}$ $(0 \notin \mathbb{N})$

and $-m = -0 = 0$ so $-m \notin \mathbb{N}$

so just one of the ~~stat~~ statements is true.

Case 2 $m \neq 0$

Case 2a $m \in \mathbb{N}$

since $m \in \mathbb{N}$ $m \neq 0$ because $0 \notin \mathbb{N}$.

Suppose $-m \in \mathbb{N}$, then (ax 2.1)

$m + -m = 0 \in \mathbb{N}$ $\times$

Case 2b $m \notin \mathbb{N}$

we know that $m \neq 0$.

we know that $m \notin \mathbb{N}$

so we have by the fourth part of 2.1 that

$-m \in \mathbb{N}$.

Definition:

$m < n$ is defined as $n - m \in \mathbb{N}$

$\ldots \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \ldots$

we believe $1 < 2$

and $2 - 1 = 1 \in \mathbb{N}$

$-1 < 1$

$1 - (-1) = 2 \in \mathbb{N}$

define $m > n$ as $n < m$

define $m \geq n$ as $m > n$ or $m = n$

define $m \leq n$ as $m < n$ or $m = n$