Last homework assignments (solutions)

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December 9, 2024

1. Prove that $\lim_{x\to 3} 2x + 3 = 9$

   Let $\epsilon > 0$ be chosen arbitrarily. [The original in the notes has a typo $\epsilon < 0$!]

   Let δ be ...

   Let x be chosen arbitarily.

   Assume $0 < |x - 3| < \delta$

   Goal: $|(2x + 3) - 9| < \epsilon$

   There is a gap in this development. We have to say what fills in the dots (what δ actually is). Notice that the choice of δ can depend on ϵ, but not on x.

   What we generally do is first do some "scratch work" to figure out a δ that might work. We do this by figuring backward from the desired outcome $|f(x) - L| < \epsilon$ to the $0 < |x - a| < \delta$ which will work. Notice that this is a dangerous activity, because the proof has to go in the other direction.

   The scratch work:

   We want $|(2x + 3) - 9| < \epsilon$

   This is equivalent to $|2x - 6| < \epsilon$

   and so to $2|x - 3| < \epsilon$

   and so to $|x - 3| < \frac{\epsilon}{2}$

   So we take away from the scratch work the idea that $\delta = \frac{\epsilon}{2}$

   and write **the official proof**:

Let $\epsilon > 0$ be chosen arbitrarily.

Let $\delta = \frac{\epsilon}{2}$

Let x be chosen arbitarily.

Assume $0 < |x - 3| < \delta = \frac{\epsilon}{2}$

Goal: $|(2x + 3) - 6| < \epsilon$

Since $0 < |x - 3| < \delta = \frac{\epsilon}{2}$, we have $|x - 3| < \frac{\epsilon}{2}$

Multiply both sides by 2 and we get $2|x - 3| < \epsilon$

From this we have $|(2x + 3) - 9| = |2x - 6| = 2|x - 3| < \epsilon$

so we have $|(2x + 3) - 9| < \epsilon$, our goal.

2. Prove that $\lim_{x\to 2} 12 - 2x = 8$, from the definition of limit.

   Let $\epsilon > 0$ be chosen arbitrarily.

   Let δ be ...

   Let x be chosen arbitarily.

   Assume $0 < |x - 2| < \delta$

   Goal: $|(12 - 2x) - 8| < \epsilon$

   There is a gap in this development. We have to say what fills in the dots (what δ actually is). Notice that the choice of δ can depend on ϵ, but not on x.

   What we generally do is first do some "scratch work" to figure out a δ that might work. We do this by figuring backward from the desired outcome $|f(x) - L| < \epsilon$ to the $0 < |x - a| < \delta$ which will work. Notice that this is a dangerous activity, because the proof has to go in the other direction.

   The scratch work:

   We want $|(12 - 2x) - 8| < \epsilon$

   This is equivalent to $|4 - 2x| < \epsilon$

   and so to $2|2 - x| = 2|x - 2| < \epsilon$ [notice what happens here...we are not going to divide by -2!]

   and so to $|x - 2| < \frac{\epsilon}{2}$

   So we take away from the scratch work the idea that $\delta = \frac{\epsilon}{2}$

and write **the official proof**:

Let $\epsilon > 0$ be chosen arbitrarily.

Let $\delta = \frac{\epsilon}{2}$

Let x be chosen arbitarily.

Assume $0 < |x-2| < \delta = \frac{\epsilon}{2}$

Goal: $|(12-2x)-8| < \epsilon$

Since $0 < |x-2| < \delta = \frac{\epsilon}{2}$, we have $|2-x| = |x-2| < \frac{\epsilon}{2}$

Multiply both sides by 2 and we get $2|2-x| < \epsilon$

From this we have $|(12-2x)-8| = |4-2x| = 2|2-x| < \epsilon$

so we have $|(12-2x)-8| < \epsilon$, our goal.

3. Prove that $\lim_{x\to 4} x^2 = 16$

   Choose an $\epsilon_0 > 0$ arbitrarily.

   Let $\delta_0 = \ldots$

   Choose x arbitrarily.

   Assume that $0 < |x-4| < \delta_0$

   Goal: $|x^2 - 16| < \epsilon_0$

   The next phase is scratch work to figure out what δ_0 should be. r We aim to make $|x^2-16| < \epsilon_0$.

   $|x^2-16| = |x+4||x-4| < \epsilon_0$ will be true if $|x-4| < \frac{\epsilon_0}{|x+4|}$

   We cannot set $\delta_0 = \frac{\epsilon_0}{|x+4|}$, because this expression depends on x. What we need is $|x-3| < \frac{\epsilon_0}{???} < \frac{\epsilon_0}{|x+4|}$ where ???, whatever it is, does not depend on x. We need ??? to be *greater than* $|x+4|$ (so that its reciprocal will be smaller). To get an upper bound on $|x+4|$, we impose an upper bound on x: the only way we have to do this is to make stipulations about δ_0. If we impose $\delta_0 \leq 1$, then we get $|x-4| < 1$, which is equivalent to $3 < x < 5$. We then get $7 < x+4 < 9$, and since $x+4 > 7 > 0$ we have $|x+4| = x+4 < 9$. 9 is the desired upper bound. So we get $|x-3| < \frac{\epsilon_0}{9} < \frac{\epsilon_0}{|x+4|}$ implies $|x^2-9| < \epsilon_0$ as long as we also stipulated $|x-4| < 1$. So a workable value of δ_0 is $\min(1, \frac{\epsilon_0}{9})$.

   Now we continue the proof, setting $\delta_0 = \min(1, \frac{\epsilon_0}{9})$.

Since $|x-4| < \delta_0$, we also have $|x-4| < 1$ and $|x-4| < \frac{\epsilon_0}{9}$. From this we deduce $7 < |x+4| = x+4 < 9$ just as we did above in the scratch work. Now $|x^2-16| = |x+3||x-3| < 9|x-3| < 9\frac{\epsilon_0}{9} = \epsilon_0$.

[I don't actually recall why I used ϵ_0 and δ_0 in the original from which this is edited, but I didn't change it: variable names can be arbitrary!]

4. Read example 1 on page 75, then prove $\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$

   Start the proof as usual.

   Let $\epsilon > 0$ be chosen arbitrarily.

   We do some scratch work to find δ: we want to make $|\frac{1}{x} - \frac{1}{3}| < \epsilon$ by making $|x-3|$ small enough.

   $|\frac{1}{x} - \frac{1}{3}| = |\frac{3-x}{3x}| = \frac{|x-3|}{|3x|}$ and this will be less than ϵ just in case $|x-3| < |3x|\epsilon$.

   δ cannot depend on x: but we can get a condition which works if we can place a lower bound on x.

   Assume $|x-3| < 1$, or equivalently $2 < x < 4$, so we now want $|x-3| < 3 \cdot 2 \cdot \epsilon < |3x|\epsilon$

   so we set $\delta = \mathtt{min}(1, 6\epsilon)$.

   Continue the proof. Let $\delta = \mathtt{min}(1, 6\epsilon)$.

   Choose x arbitrarily.

   Suppose that $0 < |x-3| < \delta = \mathtt{min}(1, 6\epsilon)$.

   It follows that $|x-3| < 1$ so $2 < x < 4$.

   Also $|x-2| < 6\epsilon$.

   Now $|\frac{1}{x} - \frac{1}{3}| = \frac{|x-3|}{|3x|} = \frac{|x-3|}{3x} < \frac{|x-2|}{6}$ (because $x > 2$) $< \frac{6\epsilon}{6} = \epsilon$.

5. Prove the subtraction rule for limits from the definition directly (not from the addition rule and the constant multiple rule): if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}(f(x) - g(x)) = L - M$. This should look very much like the proof of the addition rule with slightly different manipulations of absolute values.

   Goal: if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then

   $$\lim_{x\to a} f(x) - g(x) = L - M.$$

The proof starts.

Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Choose $\epsilon_0 > 0$.

Let $\delta_0 = \ldots$

Choose x arbitrarily.

Assume $0 < |x - a| < \delta_0$.

Goal: $|(f(x) + g(x)) - (L - M)| < \epsilon_0$

We pause for scratch work.

$|(f(x) - g(x)) - (L - M)| = |(f(x) - L) + (M - g(x))| \leq |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M|$. This will be less than ϵ_0 if we make $|f(x)-L| < \frac{\epsilon_0}{2}$ and $|g(x)-M| < \frac{\epsilon_0}{2}$ By limit assumptions, we can choose δ_1 so that if $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon_0}{2}$ and choose δ_2 so that if $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon_0}{2}$. The point is that the limit statements about f and g allow us to find δ's corresponding to any value of ϵ, in this case a value half as large as the value being considered for the sum function. Let $\delta_0 = \min(\delta_1, \delta_2)$. Continue the proof.

Since we have assumed $0 < |x - a| < \delta_0$, we also have $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$, so we have $|f(x) - L| < \frac{\epsilon_0}{2}$ and $|g(x) - M| < \frac{\epsilon_0}{2}$.
Thus $|(f(x) - g(x)) - (L - M)| = |(f(x) - L) + (M - g(x))|$
$\leq |f(x) - L| + |M - g(x)| = |f(x) - L| + |g(x) - M| < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0$.

6. Prove that $|x||y| = |xy|$. $|x|$ is defined as x if $x \geq 0$ and $-x$ otherwise. This is a straightforward argument by cases: make sure you write out everything you need to say.

   Note that $x \leq 0$ implies $|x| = -x$. If $x < 0$ this is true by the definition directly, and if $x = 0$ we have $-x = 0 = |x|$.

   By trichotomy, either $x \geq 0$ or $x < 0$, and either $y \geq 0$ or $y < 0$. This gives four cases.

   If $x \geq 0$ and $y > 0$ then $|x||y| = xy$, and further $xy \geq 0$, so $xy = |xy|$ and we are done.

   If $x \geq 0$ and $y < 0$, then $|x||y| = x(-y)$ and further, $xy \leq 0$, so $x(-y) = -xy = |xy|$.

If $x < 0$ and $y \geq 0$, then $|x||y| = (-x)y$ and further, $xy \leq 0$, so $(-x)y = -xy = |xy|$.

If $x < 0$ and $y < 0$ then $|x||y| = (-x)(-y) = xy$ and $xy > 0$ so $xy = |xy|$.

7. (depends on Thursday's lecture) Write out the proof that any nonempty set of real numbers which is bounded below has a greatest lower bound, using the Completeness Axiom, which asserts that each nonempty set of real numbers which is bounded above has a least upper bound.

   I'm not going to write out the full answer to this one. It will appear as a bonus question on the exam.

   Hint: if A is a nonempty set of real numbers which is bounded below, what can you say about the set $-A = \{-x : x \in A\}$? Don't just say it, prove it. The point is to write out all the details of the straightforward manipulations of set notation and order which are involved. I believe I did something very similar in arguing in class and in the notes that it follows from the Well-Ordering Principle that any set of integers bounded above has a maximum: I believe this appears in the construction of the gcd. You can surely find this written out in a book or on the web...