

# Math 189 Fall 2022: Draft Final Examination

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The final exam is officially given Wednesday, Dec 14 2:30–4:30 pm. Unless someone raises an objection, I will actually collect papers at 4:45 pm.

Solutions to this sample exam will be distributed before final exam week starts. This test was constructed largely by adapting problems from the hour exams, with two new questions on recent material. You should probably be sure you are ready for any question on Test I and Test II, in case I make some change in my design decisions.

1. Fill in each sentence with  $\in$  or  $\subseteq$  in such a way as to make it true. If both work, say both, if neither work, say neither.

(a)  $\{a, b\}$  \_\_\_  $\{b, \{a, b\}, a\}$  both

(b)  $\emptyset$   $\subseteq$   $\{d, e, f\}$

(c)  $-3$   $\in$   $\mathbb{Z}$

(d)  $x$  \_\_\_  $\{\{x, y\}\}$  neither

2. In a sophomore class of 22 students at a small school, every student takes at least one of English, Math, French. 16 take English, 10 take Math and 14 take French. 5 take English and Math, 7 take Math and French, and 9 take English and French. How many brave students are taking all three subjects?

$$|E \cup M \cup F| = |E| + |M| + |F| - |E \cap M| - |E \cap F| - |F \cap M| + |E \cap F \cap M|$$

so  $22 = 16 + 10 + 14 - 5 - 9 - 7 + x$

$$x = 22 - 16 - 10 - 14 + 5 + 9 + 7 = \boxed{3}$$

3 students took all three classes

3. Each of these four questions about  $k$  choices from  $n$  alternatives is answered in a different way, because of different combinations of conditions: in some we are allowed to repeat choices, and in some we are not; in some the order in which we make choice matters and in others it does not. Briefly answer each question, and **include calculations and brief explanation of the conditions which apply.**

- (a) A committee with 8 members wants to choose a three member executive committee. In how many ways can this be done?

$$\binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

order of choices does not matter  
repeated choices are not allowed

- (b) 8 scrabble tiles with different letters on them are on the table in front of you. You idly make a 5 letter "word" using these tiles (no requirement that it be in the dictionary or even possible to pronounce). How many ways can you do this?

~~$$26 \cdot 25 \cdot 24 \cdot 23 \cdot 22$$~~

~~$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$$~~

$$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 6720$$

order matters

repetition are not allowed

is correct, didn't need to erase anything

- (c) How many four letter "words" (they don't need to be in the dictionary or even pronounceable) are possible for you to make, assuming that that you have at least 4 of each letter in your bag of letter tiles?

~~26~~  $26^4$  456976  
~~repet~~ order matters,  
 repetition allowed

- (d) You go to the florist and order a bunch of a dozen roses. There are pink, white, red and exotic genetically engineered blue roses. How many bunches of a dozen are possible?

$$\binom{12+4-1}{12} = \binom{12+4-1}{3} = 455$$

order <sup>does</sup> not matter  
 repetition are allowed

4. Euclidean algorithm; find a modular reciprocal and solve a modular equation.

The three tasks are all connected!

- (a) Find integers  $x$  and  $y$  such that  $211x + 121y = \gcd(211, 121)$ . Show all work. This should include the usual table and should also make it clear that you know what  $x$  is, what  $y$  is and what  $\gcd(211, 121)$  is.

	$x$	$y$	$q$
211	1	0	
121	0	1	
90	1	-1	1
31	-1	2	1
28	3	-5	2
3	-4	7	1
1	39	-68	9

$$(39)(211) + (-68)(121) = \gcd(211, 121) = 1$$

- (b) Find the reciprocal of 121 in mod 211 arithmetic.

The reciprocal is  $211 - 68 = \boxed{143}$

- (c) Solve the equation  $121z \equiv_{211} 5$  for  $z$ . Your answer should be a remainder mod 211.

$$121z \equiv_{211} 5$$

$$(121)(143)z \equiv_{211} (5)(143)$$

became reciprocal  $\rightarrow 1z \equiv_{211} 715$

$$z \equiv_{211} 715 - 3(211) = \boxed{82} \leftarrow \text{the solution}$$

check:  $(121)(82) = 9922 - (47)(211) = 5$   
as desired

5. Number theory 2 Chinese remainder theorem

Solve the system of equations

$$x \equiv_{211} 179$$

$$x \equiv_{121} 4$$

Give the smallest positive solution and the general solution.

The way we usually do it, with the larger modulus

$$x = 179 + 211k \text{ for some integer } k \text{ by the first equation}$$

So

$$179 + 211k \equiv_{121} 4$$

~~$$211k \equiv_{121} -175$$~~  

$$58 + 90k \equiv_{121} 4 \quad \left( \begin{array}{l} \text{subtract 121 from the} \\ \text{large when on left} \end{array} \right)$$

$$90k \equiv_{121} 121 - 58 + 4 = 67$$

reciprocal of 90 mod 121 is 39 and this can be read from the table in the previous problem!

the reciprocal of 211 mod 121 is 39, and

$211 \equiv_{121} 90$ ! You can set up a table to check of course

$$\begin{array}{l} (39)(90)k \equiv_{121} (67)(39) \\ \hline 1 \end{array} \quad \begin{array}{l} (67)(39) = 2613 \\ 2613 - (21)(121) = 72 \end{array}$$

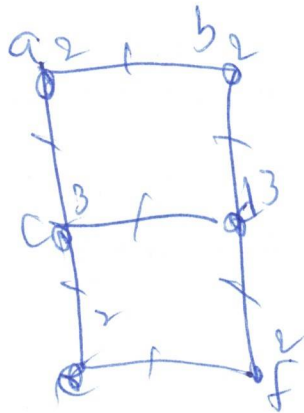
$$x \equiv 179 + (211)(72) = 15371 \text{ \# smallest solution}$$

General solution is  $15371 + 25531n$  for some integer  $n$   
 $25531$  being  $(211)(121)$

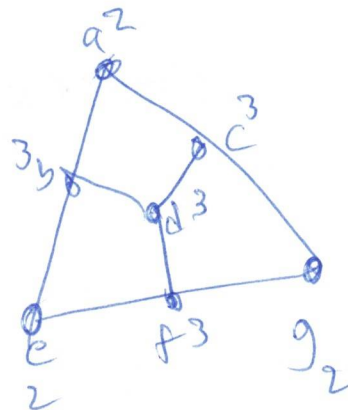
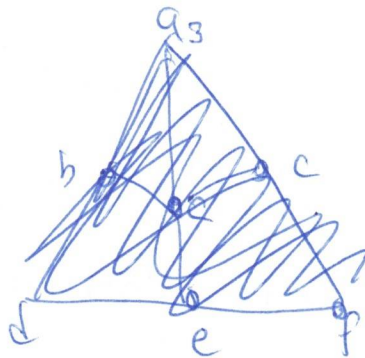


## 6. Eulerian walks and trails

Two graphs are pictured. In one there is an Eulerian walk (a walk which visits each edge in the graph exactly once); in the other there is not. Present the walk in the graph which has one as a sequence of vertices (vertices can be repeated, of course); explain briefly why the graph which does not have one cannot have one.



we not start at a vertex  
of odd degree and end  
at the other.  
c, a, b, d, c, e, f, d  
for example



This one  
has  
4 vertices  
of degree  
3.

A graph with  
an Eulerian  
walk can have  
no more than two  
vertices of odd  
degree.



7. For each statement given, write a formal proof using our rules of propositional logic. The one you do better on will count for 7 points out of 10 and the one you do worse on for 3. A summary of our formal rules is attached to the test (not the sample test, you can use the manual for that).

I'm giving a third one because it is a practice test.

(a) Prove  $((A \rightarrow B) \wedge (\neg C \rightarrow \neg B)) \rightarrow (A \rightarrow C)$

This one is lucky for a reason that won't happen on the actual test.

Pre  $((A \rightarrow B) \wedge (\neg C \rightarrow \neg B)) \rightarrow A \rightarrow C$

Assume ①  $(A \rightarrow B) \wedge (\neg C \rightarrow \neg B)$

Goal:  ~~$A \rightarrow C$~~

Assume ②  $A$

Goal  $C$

③  $A \rightarrow B$  simp 1

④  $B$  mp

⑤  $\neg \neg B$  dni. (double negation intro)

~~⑥  $\neg C$  m.p.~~

⑥  $\neg C \rightarrow \neg B$  simp 1

⑦  $C$

⑧  $A \rightarrow C$  deduction 1-7

9 The theorem ded 1-8

I ~~ought~~ to allow  
 ~~$A \rightarrow B$~~   
 ~~$\neg B$~~   
 ~~$A$~~   
~~as a form~~  
 $A \rightarrow \neg B$   
 $B$   
 $A$   
 as a form of  
 modus tollens  
 I'll also  
 let dni  
 as an  
 allowable  
 rule

(b) Prove  $((A \wedge B) \vee (B \wedge C)) \rightarrow B$  (hint: use proof by cases)

$\text{Pre}((A \wedge B) \vee (B \wedge C)) \rightarrow D$

Assume ①  $(A \wedge B) \vee (B \wedge C)$

Goal  $B$

Prove by cases on ①

Case ① Assume 1<sup>a</sup>  $A \wedge B$

Goal  $B$

②<sup>a</sup>  $B$  simp 1

Case 2 Assume 1<sup>b</sup>  $B \wedge C$

Goal:  $B$

2<sup>b</sup>  $B$  simp 1<sup>b</sup>

③  $B$  proof by cases 1, 1<sup>a</sup>-2<sup>a</sup>, 1<sup>b</sup>-2<sup>b</sup>

④

$((A \wedge B) \vee (B \wedge C)) \rightarrow B$  deduction 1-3

(c) Prove  $((A \vee \neg B) \wedge (\neg A \vee C)) \rightarrow (B \rightarrow C)$  (hint: use disjunctive syllogism)

Pre ~~Ass~~  $((A \vee \neg B) \wedge (\neg A \vee C)) \rightarrow (B \rightarrow C)$

Assume  $① (A \vee \neg B) \wedge (\neg A \vee C)$

Goal:  $B \rightarrow C$

Assume  $② B$

Goal:  $C$

③  $A \vee \neg B$  simp 1

④  $A$  d.s. 1,3

⑤  $\neg A \vee C$  simp 1

⑥  $C$  d.s. 4,5

⑦  $A \rightarrow C$  det. 2-6

⑧ the theorem deduction 1-7

Definition:

An integer  $n$  is odd

an integer  $n$  is even

iff there is an integer  $k$  s.t.  $n = 2k$

8. Give a proof in the style discussed in class of the statement "The sum of two odd numbers is even". First write the sentence out with appropriate use of variables and of quantifiers (English "for all" or "for some") is fine), making it clear that there is an implication in the statement, then prove it.

iff  
there is an integer  
 $k$  s.t.  $n = 2k+1$

Proof: The sum of two odd numbers is even

Restate: For any integers  $x, y$ , if  $x$  is odd and  $y$  is odd, then  $x+y$  is even

Let  $x, y \in \mathbb{Z}$  be chosen arbitrarily

Assume ①  $x$  is odd

②  $y$  is odd

Goal:  $x+y$  is even  
by ① we can choose  $k \in \mathbb{Z}$  s.t. ③  $x = 2k+1$

by ② we can choose  $l \in \mathbb{Z}$  s.t. ④  $y = 2l+1$

Now our goal is " $x+y$  is even"

which we can restate as: Find an integer  $m$  s.t.  $2m = x+y$

$$x+y \stackrel{\text{by ③, ④}}{=} (2k+1) + (2l+1) \stackrel{\text{algebra}}{=} 2k+2l+2 \stackrel{\text{algebra}}{=} 2(k+l+1)$$

Let  $m$  be  $k+l+1$ , which is an integer (by closure properties of the integers)

then we have  $2m = x+y$  and  $m$  is an integer,

so  $x+y$  is even.

This completes the proof.

9. Do both parts. Proofs by mathematical induction are expected. The part on which you do better will count 70 percent and the part you do worse on 30 percent.

In both parts, be sure to clearly identify the basis step, the induction hypothesis, the induction goal, and show where the induction hypothesis is used in the proof of the induction goal.

I give an extra part because it is a practice exam.

- (a) Prove that the sum of the first  $n$  square integers is  $\frac{n(n+1)(2n+1)}{6}$ . State the theorem using summation notation, then prove it by mathematical induction.

Re-write in summation form:  
 Prove that  ~~$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$~~   $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Basis:  $\sum_{i=1}^1 i^2 = 1^2 = 1 = \frac{1(1+1)(2 \cdot 1 + 1)}{6} \checkmark$

Induction Step:

Let  $n \geq 1$  be chosen arbitrarily

Assume  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Goal:  $\sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$

$$\begin{aligned} \sum_{i=1}^{n+1} i^2 &= \left( \sum_{i=1}^n i^2 \right) + (n+1)^2 \stackrel{\text{ind hyp!}}{=} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \stackrel{\text{algebra steps omitted (which you would be expected to show)}}{=} \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$



(b) Prove using mathematical induction that  $n^3 + 5n$  is divisible by 3 for each natural number  $n$ .

(c) The Fibonacci numbers are defined by  $F_1 = 1, F_2 = 1, F_i + F_{i+1} = F_{i+2}$ .

Prove by mathematical induction that the sum of the first  $n$  Fibonacci numbers is  $F_{n+2} - 1$ .

Prve  $3 | n^3 + 5n$  for any  $n \geq 0$

Basis:  $3 | 0^3 + 5 \cdot 0 = 0$  is true.

Ind Step. Let  $k \geq 0$  be chosen arbitrarily.

Assume  $3 | (k^3 + 5k)$ .

Goal:  $3 | (k+1)^3 + 5(k+1)$

$$\begin{aligned} (k+1)^3 + 5(k+1) &= \underbrace{(k^3 + 5k)}_{\substack{\text{divisible by } 3, \\ \text{ind hyp}}} + \underbrace{(3k^2 + 3k + 6)}_{\substack{\text{divisible by } 3 \\ \text{by inspection}}} \\ &= (k^3 + 5k) + (3k^2 + 3k + 6) \text{ which is divisible by } 3. \end{aligned}$$

Prve by math induction that for all  $n \geq 1$ ,  $\sum_{i=1}^n F_i = F_{n+2} - 1$ .

Basis ~~Induction~~:  $\sum_{i=1}^1 F_i = F_1 = 1 = F_3 - 1 = 2 - 1 \checkmark$

Induction step: Let  $k \geq 1$  be chosen arbitrarily.

Assume  $\sum_{i=1}^k F_i = F_{k+2} - 1$

Goal:  $\sum_{i=1}^{k+1} F_i = F_{k+3} - 1$

$$\begin{aligned} \sum_{i=1}^{k+1} F_i &= \sum_{i=1}^k F_i + F_{k+1} \stackrel{\text{ind hyp}}{=} (F_{k+2} - 1) + F_{k+1} = (F_{k+1} + F_{k+2}) - 1 \\ &= F_{k+3} - 1 \\ &\quad \text{def of } F_n! \end{aligned}$$