Math 287 Homework 7 rubric

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1. There are 8 functions. None of them are one-to-one. Six of them are onto (all but the two constant functions).

2. There are 9 functions. None of them are onto $\{1, 2, 3\}$. Six of them are one-to-one (all but the three constant functions).

3. There are six functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$ which are onto $\{1, 2, 3\}$. All of them are one-to-one.

4. Claim 1 is the assertion that if E is an equivalence relation on A, $P = \{[a]_E : a \in A\}$ is a partition of A.

   We verify this.

   P needs to be a collection of nonempty subsets of A. Obviously each element $[a]_E = \{x \in A : x \, E \, a\}$ of P is a subset of A. We need to show that each set $[a]_E = \{x \in A : x \, E \, a\}$ is nonempty. Because E is reflexive, $a \, E \, a$, so $a \in \{x \in A : x \, E \, a\} = [a]_E$, establishing that each element of P is nonempty.

   for any $A, B \in P$, we need either $A = B$ or $A \cap B = \emptyset$. We will have $A = [a]_E$ for some $a \in A$ and $B = [b]_E$ for some $b \in A$, by definition of P, and we showed that for any a, b, either $[a]_E = [b]_E$ or $[a]_E \cap [b]_E = \emptyset$, so this is verified.

   We need to show that any $a \in A$ belongs to some element of P, and we already showed this above when we showed $a \in [a]_E$ (when we showed elements of P are nonempty).

5. Suppose P is a partition of A. Claim 2 is the claim that the relation $x \equiv_P y$ defined as $(\exists B \in P : x \in B \wedge y \in B)$ is an equivalence relation.

We first show that $\equiv_P$ is reflexive. For any $x \in A$, there is $B_0 \in P$ such that $x \in B_0$, so we have $x \in B_0 \wedge x \in B_0$, so we have $(\exists B \in P : x \in B \wedge x \in B)$, that is, $x \equiv_P x$.

Now we show the $\equiv_P$ is symmetric. Suppose that $x \equiv_P y$. Our aim is to show $y \equiv_P x$. Because $x \equiv_P y$, we have $B_0 \in P$ such that $x \in B_0 \wedge y \in B_0$. So $y \in B_0 \wedge x \in B_0$. So $(\exists B \in P : y \in B \wedge x \in B)$, that is, $y \equiv_P x$.

Now we show that $\equiv_P$ is transitive. This is where those with some success on this problem dropped the ball, alas. Suppose that $x \equiv_P y$ and $y \equiv_P z$. Our goal is to show $x \equiv_P z$. Because $x \equiv_P y$ there is $B_1 \in P$ such that $x \in B_1$ and $y \in B_1$. Because $y \equiv_P z$ there is $B_2 \in P$ such that $y \in B_2$ and $z \in B_2$. You do *not* get to give these witnesses the same name: you have to show that they are the same! Now we know that either $B_1 = B_2$ or $B_1 \cap B_2 = \emptyset$. But y belongs to both B_1 and B_2, so $B_1 \cap B_2 \neq \emptyset$, so $B_1 = B_2$, so $x \in B_1$ and $z \in B_2 = B_1$, so $(\exists B \in P : x \in B \wedge z \in B)$, so $x \equiv_P z$.

6. Project 6.7. For each of the following relations defined on Z, determine whether it is an equivalence relation. If it is, determine the equivalence classes.

   (a) $<$ is not reflexive: 3 is not less than 3.

   (b) $\leq$ is not symmetric: $2 \leq 3$ but not $3 \leq 2$.

   (c) The relation $|x| = |y|$ is an equivalence relation.

   (d) $\neq$ is reflexive, but neither symmetric nor transitive.

   (e) The relation $xy > 0$ is not reflexive: $(0)(0) = 0$. It is symmetric and transitive: it would be an equivalence relation on nonzero numbers.

   (f) the relation "$x|y \vee y|x$" is reflexive and symmetric, but it is not transitive. For example, it relates 2 to 10 and 10 to 5, but not 2 to 5.

7. We define $(x, y)SD(z, w)$ if $x + w = y + z$. Show that this is an equivalence relation.

   reflexive: $(x, y)SD(x, y)$ means $x + y = x + y$, which is true.

symmetric: Assume that $(x, y)SD(z, w)$. Our goal is to show that $(z, w)SD(x, y)$. Because $(x, y)SD(z, w)$ we have $x + w = y + z$. It follows that $z + y = w + x$ (commutativity of addition and symmetry of equality) so $(z, w)SD(x, y)$.

transitive: Assume that $(x, y)SD(z, w)$ and $(z, w)SD(u, v)$. Show that $(x, y)SD(u, v)$. Because $(x, y)SD(z, w)$ we have $x+w = y+z$. Because $(z, w)SD(u, v)$ we have $z + v = w + u$. Thus we have $x + w + z + v = y + z + w + u$ from which (by subtracting w and z from both sides we get $x + v = y + u$, so $(x, y)SD(u, v)$.

What is the same about (x, y) and (z, w) if $(x, y)SD(z, w)$? What I was looking for was $x - y = z - w$: the differences of the numbers are the same. Nobody got that, but some noticed that (x, y) and (z, w) lie on the same line of slope 1 in the plane, which is a correct answer.