

# Test I Math 287 Spring 2022 (for review, Spring 2023)

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There are eight questions on this exam. Questions 1 and 2 form a pair, questions 3 and 4 form a pair and questions 5 and 6 form a pair. Your grade on a pair of questions will be seventy percent the grade on the one you do better on and thirty percent the grade on the one you do worse on. Questions 7 and 8 are free standing.

NOTE for 2023: questions will be free-standing, not paired as I describe, on your exam. I believe that on a takehome this device isn't needed.

You are allowed your test paper and your writing instrument. There is no use for a calculator on this exam.

A reference sheet with axioms, propositions and definitions you need is the last page of the exam. You may tear it off for reference.

NOTE for 2023: I will supply the reference sheet on your exam, though it is less needed, so that you know exactly what assumptions you are allowed to use in proofs.

NOT APPLICABLE in 2023: You will be handed a second copy of this test when it is distributed in class. You may complete as much of it at home as you wish, consulting no human being other than myself, but with free use of other resources, and submit what you do electronically by 11:55 pm on the 19th. No submission of the take-home copy is required. If you do submit a take-home, I may use it for partial credit decisions.

NOTE for 2023: Coverage on your exam will be as on this sample exam [with different logic content as noted on question 7]. You should expect not to see questions which are simply memorization (I won't ask you simply to state the Binomial Theorem, for example). You should also be aware that questions appearing on your test will not necessarily have exact models on

this test. One obvious source of model questions is your homework.

You should not seek assistance from any human being other than the instructor. You are allowed to use your book, your homework papers, and other sources, but words written on your paper should be your own words. I expect very good performance on this exam, as the conditions for you to work on it are optimal: what I expect and hope not to see is identical work on different student papers. If I do, Ill have to go for more gruelling proctoring options on subsequent exams. The actual exam may be somewhat longer than this one (a question or two) but not hugely longer.

1. (paired with 1) The FOIL identity you learned in school is

$$(a + b) \cdot (c + d) = (a \cdot c + a \cdot d) + (b \cdot c + a \cdot d)$$

(First, Outer, Inner, Last). We supply the parentheses for precision.

Use the axioms (parts of Axiom 1.1, listed in the attachments to the paper, which you may tear off for reference) to give a detailed step by step proof of FOIL.

Each step should be justified by a single axiom.

You may use references to parts of the axiom using the exact phrases I give, and be aware that the phrase distributive law refers to exactly the form in the axioms: you need to change things to apply it on the other side.

2. (paired with 2) Prove  $a \cdot 0 = 0$  using Proposition 1.9 and the axioms from chapter 1 in the reference sheet.

Each step should use one axiom or the proposition.

3. (paired with 4) Prove using the definition of divisibility (on the reference sheet) and algebra (you may be more informal about the algebra) that if  $d|a$  and  $d|b$ , it follows that  $d|(a + b)$ .

Your proof will start: Let  $a, b, d$  be integers and assume that  $d|a$  and  $d|b$ ...because  $d|a$ , there is an integer  $x$  such that  $a = d \cdot x$ ...carry on from there.

4. (paired with 3) Prove, using the axioms for  $\mathbf{N}$  (the set of positive integers: axiom 2.1 on the reference sheet) and the definition of  $<$  given on the reference sheet and algebra of equations with addition, subtraction and multiplication (about which you may be informal but be quite formal about applying the axioms for the positive integers (referencing the correct part of axiom 2.1) and the definition of  $<$ ) that if  $x < y$  and  $0 < z$ ,  $x \cdot z < y \cdot z$ .

Your proof will begin “Let  $x, y, z$  be integers. Suppose that  $x < y$  and  $0 < z$ . It follows that  $y - x \in \mathbf{N}$ , by the definition of  $<$  and...(carry on from there).

5. (paired with 6)

Prove by induction that the sum of the first  $n$  integers is  $\frac{n(n+1)}{2}$ : in symbols  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ .

6. (paired with 5)

A sequence  $a_1$  is defined recursively:  $a_1 = 1$ ,  $a_{k+1} = 2a_k + 1$ .

Compute the first six terms of this sequence.

Prove by induction that for each natural number  $k$ ,  $a_k = 2^k - 1$ .



7. (unpaired)

- (a) NOTE for 2023: consider this question as replaced with something like the proofs in the first homework from the logical style manual. Write the negation of the sentence “I like coffee and I don’t like tea” in natural English (the negation moved all the way in and applied to the verb).
- (b) Write the negation of the sentence in logical notation

$$(\exists x \in \mathbf{N} : (\forall y \in \mathbf{N} : x \geq y)),$$

in a form which doesn’t involve negation at all (move the negation all the way to the right and replace the order relation with its negation).

- (c) Say in English what the sentences

$$(\exists x \in \mathbf{N} : (\forall y \in \mathbf{N} : x > y))$$

and

$$(\forall y \in \mathbf{N} : (\exists x \in \mathbf{N} : x > y))$$

mean. Which one is true?

8. (unpaired)

State the Binomial Theorem using notation for binomial coefficients and summation notation.

State it for the exponent 4 and write the sum out in full, eliminating the summation notation and evaluating all the binomial coefficients (in other words, expand out  $(x + y)^4$  using the theorem).

# 1 Reference sheet

**Axiom 1.1.** If  $m$ ,  $n$ , and  $p$  are integers, then

- (a)  $m + n = n + m$  . (commutativity of addition)
- (b)  $(m + n) + p = m + (n + p)$  . (associativity of addition)
- (c)  $m \cdot (n + p) = m \cdot n + m \cdot p$  . (distributivity)
- (d)  $m \cdot n = n \cdot m$  . (commutativity of multiplication)
- (e)  $(m \cdot n) \cdot p = m \cdot (n \cdot p)$  . (associativity of multiplication)

**Axiom 1.2.** There exists an integer  $0$  such that whenever  $m \in \mathbb{Z}$ ,  $m + 0 = m$ . (identity element for addition)

**Axiom 1.3.** There exists an integer  $1$  such that  $1 \neq 0$  and whenever  $m \in \mathbb{Z}$ ,  $m \cdot 1 = m$  . (identity element for multiplication)

**Axiom 1.4.** For each  $m \in \mathbb{Z}$ , there exists an integer, denoted by  $-m$ , such that  $m + (-m) = 0$ . (additive inverse)

**Axiom 1.5.** Let  $m$ ,  $n$ , and  $p$  be integers. If  $m \cdot n = m \cdot p$  and  $m \neq 0$ , then  $n = p$ . (cancellation).

**Proposition 1.9.** Let  $m$ ,  $n$ , and  $p$  be integers. If  $m + n = m + p$ , then  $n = p$

**Axiom 2.1.** There exists a subset  $N \subseteq \mathbb{Z}$  with the following properties:

- (a) If  $m, n \in N$  then  $m + n \in N$ .
- (b) If  $m, n \in N$  then  $mn \in N$ .
- (c)  $0 \notin N$ .
- (d) For every  $m \in \mathbb{Z}$ , we have  $m \in N$  or  $m = 0$  or  $-m \in N$ .

**Definition:** The statements  $m < n$  ( $m$  is less than  $n$ ) and  $n > m$  ( $n$  is greater than  $m$ ) both mean that  $n - m \in N$  .

**Definition:** When  $m$  and  $n$  are integers, we say  $m$  is divisible by  $n$  (or alternatively,  $n$  divides  $m$ ) Do not confuse this with the notations  $n/m$  and  $n/m$  for fractions. if there exists  $j \in \mathbb{Z}$  such that  $m = jn$ . We use the notation  $n \mid m$ .