Notes on section 2E in Axler

Randall Holmes

November 8, 2023

These are my notes on section 2E in Axler. Since I had trouble with these myself, I am going to write out proofs of the main results in my own words.

Much of this section involves the interplay between two notions of convergence of a sequence of functions.

Definition (pointwise convergence): Let X be a set, let $\{f_k\}$ be a sequence of functions from X to $\mathbb{R}$. We say that $\{f_k\}$ converges pointwise to f just in case $\texttt{lim}_{k\to\infty} f_k(x) = f(x)$ for each $x \in X$.

We expand this definition. For every $x \in X$, for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that for $k \geq n$, $|f_k(x) - f(x)| < \epsilon$. Notice that the point n in the sequence where things fall within ϵ depends on x as well as ϵ.

Definition (uniform convergence): Let X be a set, let $\{f_k\}$ be a sequence of functions from X to $\mathbb{R}$. We say that $\{f_k\}$ converges uniformly to f just in case for every $\epsilon > 0$, there is $n \in \mathbb{N}$ such that for all $k \geq n$ and $x \in X$ we have $|f_k(x) - f(x)| < \epsilon$.

Clearly uniform convergence implies pointwise convergence. It makes the stronger statement that in effect the f_k's converge at each point in X at the same rate (speaking a bit loosely: we are not talking about derivatives here!): the choice of n to get within ϵ depends only on ϵ, not on x.

You might want to take a look at the example of a sequence of functions converging pointwise but not uniformly on p. 62 of Axler. One has a sequence of continuous functions with a steadily narrowing spike at a fixed x value converging pointwise to a function with a jump discontuity at the fixed x value.

Theorem: If a sequence $\{f_k\}$ of continuous functions from a subset B of the reals to the reals converges uniformly to f, then f is continous.

Proof: We want to show that for each $x \in B$ and $\epsilon > 0$, there is $\delta > 0$ such that for any $y \in B$, if $|x - y| < \delta$ then $f(x) - f(y) < \epsilon$.

Choose an n such that for all $k \geq n$ and all $y \in B$, $|f_k(y) - f(y)| < \frac{\epsilon}{3}$ (by uniform convergence).

Choose a δ such that for all $y \in B$, if $|x-y| < \delta$ then $|f_n(x)-f_n(y)| < \frac{\epsilon}{3}$.

It then follows that for any $y \in B$, if $|x - y| < \delta$, we have $f(y)$ within $\frac{\epsilon}{3}$ of $f_n(y)$, $f_n(y)$ within $\frac{\epsilon}{3}$ of $f_n(x)$, and $f_n(x)$ within $\frac{\epsilon}{3}$ of $f(x)$, from which it follows by the triangle inequality that $f(y)$ is within ϵ of $f(x)$, so we have shown that f is continuous.

We now get the first exciting theorem. Egoroff's theorem asserts that on any set X supporting a measure which gives all of X a finite measure, if we have a pointwise convergent sequence of measurable functions $\{f_k\}$ converging to a function f, and choose an $\epsilon > 0$, we can find an $E \subseteq X$ such that $X \setminus E$ has measure less than ϵ and $\{f_k\}$ converges uniformly to f on E.