Charged Black Holes: Why We May Ignore Them

Randy Kayser
General Relativity final

December 11, 2017

1 Determining the Metric of a Charged Black Hole

We seek a static, spherically symmetric solution to the EFEs in the presence of an electromagnetic field $A_\mu = (-f(r), 0, 0, 0)$. Knowing that if $A_\mu = 0$, our result should reproduce the Schwartzschild metric, guess a metric of the following form:

$$\mathrm{d}s^2 = -(1 + h(r))\mathrm{d}t^2 + (1 + h(r))^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

The maxwell tensor is given by:

$$F_{\mu\nu} = 2\nabla_{[\mu}A_{\nu]} = 2\partial_{[\mu}A_{\nu]}$$

Whose only nonvanishing components are:

$$F_{tr} = -F_{rt} = -F^{tr} = F^{rt} = \partial_r f(r)$$

The Stress-Energy tensor of the Electromagnetic Field is given by:

$$T_{\mu\nu} = F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

The non-zero $(1,1)$ components of this tensor are given by:

$$\begin{aligned}
T^t{}_t &= -\frac{1}{2}\partial_r f(r) \\
T^r{}_r &= -\frac{1}{2}\partial_r f(r) \\
T^\theta{}_\theta &= \frac{1}{2}\partial_r f(r) \\
T^\phi{}_\phi &= \frac{1}{2}\partial_r f(r) \\
\implies T^\mu{}_\mu &= 0 \\
\implies G^\mu{}_\mu &= 0
\end{aligned}$$

Where the last implication is satisfied due to the EFEs. Computing the non-zero Christoffel symbols yields:

$$\begin{aligned}
\Gamma^t_{rt} &= \frac{1}{2}\frac{h'(r)}{1+h(r)} \\
\Gamma^r_{tt} &= \frac{1}{2}(1+h(r))h'(r) \\
\Gamma^r_{rr} &= -\frac{1}{2}\frac{h'(r)}{1+h(r)} \\
\Gamma^r_{\theta\theta} &= -r(1+h(r)) \\
\Gamma^r_{\phi\phi} &= -r(1+h(r))\sin^2\theta \\
\Gamma^\theta_{\theta r} &= \frac{1}{r} \\
\Gamma^\theta_{\phi\phi} &= -\cos\theta\sin\theta \\
\Gamma^\phi_{\phi\theta} &= \cot\theta
\end{aligned}$$

Thus, the (non-zero) components of the Ricci Tensor and the Ricci scalar are given by:

$$\begin{aligned}
R_{tt} &= \frac{(1+h(r))(2h'(r)+rh''(r))}{2r} \\
R_{rr} &= \frac{h'(r)+rh''(r)}{2r(1+h(r))} \\
R_{\theta\theta} &= -h'(r)-rh''(r) \\
R_{\phi\phi} &= -\sin^2\theta(h(r)+rh'(r)) \\
R &= -\frac{2h(r)+4rh'(r)+r^2h''(r)}{r^2}
\end{aligned}$$

Computing the trace of the resulting einstein tensor yields:

$$\frac{2h(r)+4rh'(r)+r^2h''(r)}{r^2} = 0$$

Guess a solution of the form $h(r) = r^\alpha$. Then, obtain:

$$\begin{aligned}
2r^{\alpha-2} + r\alpha r^{\alpha-2} + \alpha(\alpha-1)r^{\alpha-2} &= 0 \\
\implies \alpha^2 + 3\alpha + 2 &= 0 \\
\implies \alpha &= -1, -2
\end{aligned}$$

Thus, $h(r) = \frac{A}{r} + \frac{B}{r^2}$. Now, use the 3,3 component of the EFEs to determine $f(r)$.

$$\begin{aligned}
\frac{1}{2}r\big(2h'(r)+rh''(r)\big) &= 4\pi r^2\big(f'(r)\big)^2 \\
\implies f(r) &= \sqrt{\frac{B}{4\pi}\frac{1}{r^2}}
\end{aligned}$$

Identifying the resulting stress energy tensor with that of a point charge Q located at the origin fixes $B = Q^2$ in gaussian units.

2 Numerically Integrating the Resulting Geodesic Equation

See github link below:

`http://github.com/RandallKayser/GR_Final`