Probabilistic Relations Programming Examples - Machine Learning

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Abstract

This document lists some examples that use our probabilistic relations, based on Hehner's predicative probabilistic programming [1], for reasoning.

Contents

1 Example of probabilistic relation programming: cancer diagnosis

This example is developed based on the machine learning exercise that Dr. Thomas Gabel delivered and could be found at https://ml.informatik.uni-freiburg.de/former/_media/teaching/ss11/ml_ex07_solution.pdf. We also refer to Jason Brownlee's "A Gentle Introduction to Bayes Theorem for Machine Learning" at https://machinelearningmastery.com/bayes-theorem-for-machine-learning/ for some used terminologies.

If a randomly selected patient has a laboratory test for cancer, such as breast cancer, and the result is positive. Then what's the probability that the patient has cancer?

If the patient has the second laboratory test, would it be helpful to determine if the patient has cancer or not? How much could it contribute? This example aims to answer these questions.

theory *utp-prob-rel-cancer-diagnosis*
 imports
  UTP-prob-relations.utp-prob-rel-lattice-laws
begin

unbundle *UTP-Syntax*

declare [[*show-types*]]

datatype $LabTest = Pos \mid Neg$

c: true for cancer and false for no cancer.

alphabet $state =$

$c :: bool$
$lt :: LabTest$

The probability of a randomly selected patient has a cancer. It is the base rate or the prior.

abbreviation $p_1 \equiv 0.002$

The sensitivity of the laboratory test or the true positive rate.

abbreviation $p_2 \equiv 0.89$

The false negative rate. The specificity of the laboratory test or the true negative rate: $1 - p_3$.

abbreviation $p_3 \equiv 0.05$

definition *TestAction* :: *state prhfun* **where**
$TestAction = (if_c\ (c^<)\ then$
$\quad (if_p\ p_2\ then\ (lt := Pos)\ else\ (lt := Neg))$
$\ else$
$\quad (if_p\ p_3\ then\ (lt := Pos)\ else\ (lt := Neg))$
$\ )$

New knowledge or data learned: the test result is positive.

definition *TestResultPos* **where**
$TestResultPos = [\![lt^> = Pos]\!]_{\mathcal{I}e}$

definition *TestAction-altdef* :: *state rvhfun* **where**
$TestAction\text{-}altdef = ($
$\quad ([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![c^<]\!]_{\mathcal{I}e} * [\![c^> = c^<]\!]_{\mathcal{I}e} * p_2) +$
$\quad ([\![lt^> = Neg]\!]_{\mathcal{I}e} * [\![c^<]\!]_{\mathcal{I}e} * [\![c^> = c^<]\!]_{\mathcal{I}e} *(1-p_2)) +$
$\quad ([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![\neg c^<]\!]_{\mathcal{I}e} * [\![c^> = c^<]\!]_{\mathcal{I}e} *p_3) +$
$\quad ([\![lt^> = Neg]\!]_{\mathcal{I}e} * [\![\neg c^<]\!]_{\mathcal{I}e} * [\![c^> = c^<]\!]_{\mathcal{I}e} *(1-p_3))$
$)_e$

Initial knowledge, or prior.

definition *FirstTest* :: *state prhfun* **where**
$FirstTest = (if_p\ p_1\ then\ (c := True)\ else\ (c := False))\ ;\ \ TestAction$

definition *FirstTest-altdef* :: *state rvhfun* **where**
$FirstTest\text{-}altdef = ($
$\quad ([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![c^>]\!]_{\mathcal{I}e} * p_1 * p_2) +$
$\quad ([\![lt^> = Neg]\!]_{\mathcal{I}e} * [\![c^>]\!]_{\mathcal{I}e} * p_1 * (1 - p_2)) +$
$\quad ([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![\neg c^>]\!]_{\mathcal{I}e} * (1-p_1) * p_3) +$
$\quad ([\![lt^> = Neg]\!]_{\mathcal{I}e} * [\![\neg c^>]\!]_{\mathcal{I}e} * (1-p_1) * (1 - p_3))$
$)_e$

The result of the first laboratory test is positive.

definition *FirstTestPos* :: *state prhfun* **where**
$FirstTestPos = (FirstTest \parallel TestResultPos)$

definition *FirstTestPos-altdef* :: *state rvhfun* **where**
$FirstTestPos\text{-}altdef = ($
$\quad (([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![c^>]\!]_{\mathcal{I}e} * p_1 * p_2) + ([\![lt^> = Pos]\!]_{\mathcal{I}e} * [\![\neg c^>]\!]_{\mathcal{I}e} * (1 - p_1) * p_3)) /$
$\quad (p_1 * p_2 + (1 - p_1) * p_3)$
$)_e$

The result of the second laboratory test (which is independent to the first one) is also positive.

definition *SecondTest* :: *state prhfun* **where**
SecondTest = (*FirstTestPos* ; *TestAction*)

definition *SecondTest-altdef* :: *state rvhfun* **where**
SecondTest-altdef = ((
$([\![lt^{>} = Pos]\!]_{\mathcal{I}e} * [\![c^{>}]\!]_{\mathcal{I}e} * p_1 * p_2 * p_2)$ +
$([\![lt^{>} = Neg]\!]_{\mathcal{I}e} * [\![c^{>}]\!]_{\mathcal{I}e} * p_1 * p_2 * (1 - p_2))$ +
$([\![lt^{>} = Pos]\!]_{\mathcal{I}e} * [\![\neg c^{>}]\!]_{\mathcal{I}e} * (1 - p_1) * p_3 * p_3)$ +
$([\![lt^{>} = Neg]\!]_{\mathcal{I}e} * [\![\neg c^{>}]\!]_{\mathcal{I}e} * (1 - p_1) * p_3 * (1 - p_3))$
) / $(p_1 * p_2 + (1 - p_1) * p_3)$
$)_e$

definition *SecondTestPos* :: *state prhfun* **where**
SecondTestPos = (*SecondTest* ∥ *TestResultPos*)

definition *SecondTestPos-altdef* :: *state rvhfun* **where**
SecondTestPos-altdef = (
$(([\![lt^{>} = Pos]\!]_{\mathcal{I}e} * [\![c^{>}]\!]_{\mathcal{I}e} * p_1 * p_2 * p_2) + ([\![lt^{>} = Pos]\!]_{\mathcal{I}e} * [\![\neg c^{>}]\!]_{\mathcal{I}e} * (1 - p_1) * p_3 * p_3))$ /
$(p_1 * p_2 * p_2 + (1 - p_1) * p_3 * p_3)$
$)_e$

lemma *TestAction*: *TestAction* = *prfun-of-rvfun TestAction-altdef*
apply (*simp only*: *TestAction-def TestAction-altdef-def*)
apply (*simp add*: *prfun-seqcomp-right-unit*)
apply (*simp add*: *prfun-pcond-altdef*)
apply (*simp only*: *pchoice-def passigns-def*)
apply (*simp only*: *rvfun-assignment-inverse*)
apply (*simp only*: *rvfun-of-prfun-const*)
apply (*subst rvfun-pchoice-inverse-c'''*)
apply (*simp add*: *rvfun-assignment-is-prob*)
apply (*simp add*: *rvfun-assignment-is-prob*)
apply (*simp*)
apply (*subst rvfun-pchoice-inverse-c'''*)
apply (*simp add*: *rvfun-assignment-is-prob*)
apply (*simp add*: *rvfun-assignment-is-prob*)
apply (*simp*)
apply (*expr-simp-1 add*: *rel*)
apply (*rule HOL.arg-cong*[**where** *f*=*prfun-of-rvfun*])
apply (*subst fun-eq-iff*)
by (*pred-simp*)

lemma *pos-false*: $\{s::state.\ lt_v\ s = Pos \land \neg\ c_v\ s\} = \{(\!|c_v = False, lt_v = Pos|\!)\}$
apply (*simp add*: *set-eq-iff*)
apply (*rule allI*)
apply (*rule iffI*)
by *simp+*
lemma *neg-false*: $\{s::state.\ lt_v\ s = Neg \land \neg\ c_v\ s\} = \{(\!|c_v = False, lt_v = Neg|\!)\}$
apply (*simp add*: *set-eq-iff*)
apply (*rule allI*)
apply (*rule iffI*)
by *simp+*
lemma *summable-pos-false*: $(\lambda x::state.\ if\ lt_v\ x = Pos \land \neg\ c_v\ x\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))$ *summable-on UNIV*
apply (*rule infsum-constant-finite-states-summable*)
by (*simp add*: *pos-false*)

lemma *summable-neg-false*: $(\lambda x::state.\ if\ lt_v\ x = Neg \wedge \neg\ c_v\ x\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))$ *summable-on UNIV*
apply (*rule infsum-constant-finite-states-summable*)
by (*simp add*: *neg-false*)
lemma *pos-true*: $\{s::state.\ lt_v\ s = Pos \wedge c_v\ s\} = \{(\!|c_v = True, lt_v = Pos|\!)\}$
apply (*simp add*: *set-eq-iff*)
apply (*rule allI*)
apply (*rule iffI*)
by *simp+*
lemma *neg-true*: $\{s::state.\ lt_v\ s = Neg \wedge c_v\ s\} = \{(\!|c_v = True, lt_v = Neg|\!)\}$
apply (*simp add*: *set-eq-iff*)
apply (*rule allI*)
apply (*rule iffI*)
by *simp+*
lemma *summable-pos-true*: $(\lambda x::state.\ if\ lt_v\ x = Pos \wedge c_v\ x\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))$ *summable-on UNIV*
apply (*rule infsum-constant-finite-states-summable*)
by (*simp add*: *pos-true*)
lemma *summable-neg-true*: $(\lambda x::state.\ if\ lt_v\ x = Neg \wedge c_v\ x\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))$ *summable-on UNIV*
apply (*rule infsum-constant-finite-states-summable*)
by (*simp add*: *neg-true*)
lemma *TestAction-altdef-final*: *is-final-distribution TestAction-altdef*
apply (*simp add*: *dist-defs expr-defs TestAction-altdef-def*)
apply (*pred-auto*)
proof −
fix *c*
have $(\sum_\infty s::state.$
$(if\ lt_v\ s = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))\ /\ (20::\mathbb{R})\ +$
$(if\ lt_v\ s = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (19::\mathbb{R})\ /\ (20::\mathbb{R})) =$
$(\sum_\infty s::state.$
$(if\ lt_v\ s = Pos \wedge \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))\ /\ (20::\mathbb{R})\ +$
$(if\ lt_v\ s = Neg \wedge \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (19::\mathbb{R})\ /\ (20::\mathbb{R}))$
by (*smt* (*verit*, *ccfv-SIG*) *infsum-cong mult-cancel-right1 mult-eq-0-iff*)
also have ... $= (\sum_\infty s::state.\ (if\ lt_v\ s = Pos \wedge \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))\ /\ (20::\mathbb{R})) +$
$(\sum_\infty s::state.\ (if\ lt_v\ s = Neg \wedge \neg\ c_v\ s\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (19::\mathbb{R})\ /\ (20::\mathbb{R}))$
apply (*subst infsum-add*)
apply (*rule summable-on-cdiv-left*)
using *summable-pos-false* **apply** *blast*
apply (*rule summable-on-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-neg-false* **apply** *blast*
by *simp*
also have ... = *1*
apply (*subst infsum-cdiv-left*)
using *summable-pos-false* **apply** *blast*
apply (*subst infsum-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-neg-false* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
apply (*simp add*: *pos-false*)
apply (*subst infsum-cmult-left*)
using *summable-neg-false* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
apply (*simp add*: *neg-false*)
by (*simp add*: *pos-false neg-false*)
then show $(\sum_\infty s::state.$

$(\textit{if } lt_v\ s = Pos \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } \neg\ c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) / (20{::}\mathbb{R}) +$
$(\textit{if } lt_v\ s = Neg \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } \neg\ c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (19{::}\mathbb{R}) / (20{::}\mathbb{R})) =$
$(1{::}\mathbb{R})$
using *calculation* **by** *presburger*
next
fix *c*
have $(\sum_\infty s{::}state.$
$(\textit{if } lt_v\ s = Pos \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (89{::}\mathbb{R}) / (100{::}\mathbb{R}) +$
$(\textit{if } lt_v\ s = Neg \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R})) =$
$(\sum_\infty s{::}state.$
$(\textit{if } lt_v\ s = Pos \wedge c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R}))* (89{::}\mathbb{R}) / (100{::}\mathbb{R}) +$
$(\textit{if } lt_v\ s = Neg \wedge c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R}))$
by (*smt* (*verit*, *ccfv-SIG*) *infsum-cong mult-cancel-right1 mult-eq-0-iff*)
also have ... $= (\sum_\infty s{::}state.\ (\textit{if } lt_v\ s = Pos \wedge c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (89{::}\mathbb{R}) / (100{::}\mathbb{R})) +$
$(\sum_\infty s{::}state.\ (\textit{if } lt_v\ s = Neg \wedge c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R}))$
apply (*subst infsum-add*)
apply (*rule summable-on-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-pos-true* **apply** *blast*
apply (*rule summable-on-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-neg-true* **apply** *blast*
by *simp*
also have ... = *1*
apply (*subst infsum-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-pos-true* **apply** *blast*
apply (*subst infsum-cdiv-left*)
apply (*rule summable-on-cmult-left*)
using *summable-neg-true* **apply** *blast*
apply (*subst infsum-cmult-left*)
using *summable-pos-true* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
apply (*simp add*: *pos-true*)
apply (*subst infsum-cmult-left*)
using *summable-neg-true* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
apply (*simp add*: *neg-true*)
by (*simp add*: *pos-true neg-true*)

then show $(\sum_\infty s{::}state.$
$(\textit{if } lt_v\ s = Pos \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (89{::}\mathbb{R}) / (100{::}\mathbb{R}) +$
$(\textit{if } lt_v\ s = Neg \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (\textit{if } c_v\ s \textit{ then } 1{::}\mathbb{R} \textit{ else } (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R})) =$
$(1{::}\mathbb{R})$
using *calculation* **by** *presburger*
qed

lemma *FirstTest-simp*:
shows *FirstTest* = *prfun-of-rvfun FirstTest-altdef*
apply (*simp only*: *FirstTest-def FirstTest-altdef-def*)
apply (*simp add*: *TestAction*)
apply (*simp only*: *pseqcomp-def*)
apply (*subst rvfun-inverse*)
using *TestAction-altdef-final rvfun-prob-sum1-summable*′(*1*) **apply** *blast*
apply (*subst prfun-pchoice-assigns-inverse-c*′)

apply (*simp add*: *TestAction-altdef-def*)
apply (*expr-simp-1*)
apply (*simp add*: *real2eureal-inverse*)
apply (*rule HOL.arg-cong*[**where** *f*=*prfun-of-rvfun*])
apply (*subst fun-eq-iff*)
apply (*pred-auto*)
proof −
fix *lt c*
let $?f = (\sum_\infty v_0{::}state.$
$((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R})\ +$
$(499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (100{::}\mathbb{R})\ +$
$(if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R})))$
have $?f = (\sum_\infty v_0{::}state.$
$((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R})\ +$
$(499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R})))$
by (*smt* (*verit*) *divide-eq-0-iff infsum-cong mult-eq-0-iff*)
also have ... $= (\sum_\infty v_0{::}state.$
$((499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R})))$
by (*simp add*: *infsum-cong*)
also have ... $= (\sum_\infty v_0 \in \{\llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\}.\ ((499{::}\mathbb{R})\ /\ (10000{::}\mathbb{R})))$
apply (*subst infsum-cong-neutral*[**where** $S{=}UNIV$ **and** $T{=}\{\llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\}$ **and**
$f = \lambda v_0.\ ((499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R}))$ **and**
$g = \lambda v_0.\ ((499{::}\mathbb{R})\ /\ (10000{::}\mathbb{R}))$])
apply *blast*
by *simp+*
also have ... $= ((499{::}\mathbb{R})\ /\ (10000{::}\mathbb{R}))$
by *simp*
then show $(\sum_\infty v_0{::}state.$
$((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R})\ +$
$(499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (100{::}\mathbb{R})\ +$
$(if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R}))) *$
$(10000{::}\mathbb{R}) = (499{::}\mathbb{R})$
using *calculation* **by** *linarith*
next
fix *lt c*
have $(\sum_\infty v_0{::}state.$
$((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R})\ +$
$(499{::}\mathbb{R}) * (if\ v_0 = \llparenthesis c_v = False,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (500{::}\mathbb{R}) *$
$((if\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (100{::}\mathbb{R})\ +$
$(if\ \neg\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (if\ c_v\ v_0\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R}))\ /\ (20{::}\mathbb{R})))$
$= (\sum_\infty v_0{::}state.\ ((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (50000{::}\mathbb{R})))$
apply (*subst infsum-cong*[**where** $g = \lambda v_0{::}state.\ ((if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else$
$(0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (50000{::}\mathbb{R}))$])
by *auto*
also have ... $= (\sum_\infty v_0{::}state {\in} \{\llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\}.\ ((89{::}\mathbb{R})\ /\ (50000{::}\mathbb{R})))$
apply (*subst infsum-cong-neutral*[**where** $S{=}UNIV$ **and** $T{=}\{\llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\}$ **and**
$f = \lambda v_0.\ (if\ v_0 = \llparenthesis c_v = True,\ lt_v = lt \rrparenthesis\ then\ 1{::}\mathbb{R}\ else\ (0{::}\mathbb{R})) * (89{::}\mathbb{R})\ /\ (50000{::}\mathbb{R})$ **and**
$g = \lambda v_0.\ ((89{::}\mathbb{R})\ /\ (50000{::}\mathbb{R}))$])
by *simp+*
then show $(\sum_\infty v_0{::}state.$

((*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ) +
(*499*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ)) ∗
((*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*89*::ℝ) / (*100*::ℝ) +
(*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) / (*20*::ℝ))) ∗
(*50000*::ℝ) = (*89*::ℝ)
using *calculation* **by** *fastforce*
next
fix *lt c*
have ($\sum_\infty$ v_0::*state*.
((*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ) +
(*499*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ)) ∗
((*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*100*::ℝ) +
(*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*19*::ℝ) / (*20*::ℝ))) =
($\sum_\infty$ v_0::*state*.
((*9481*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*10000*::ℝ)))
apply (*subst infsum-cong*[**where** *g* = λv_0::*state*. ((*9481*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*10000*::ℝ))])
by *auto*
also have ... = ($\sum_\infty$ v_0::*state*∈{⦇c_v = *False*, lt_v = *lt*⦈}. ((*9481*::ℝ) / (*10000*::ℝ)))
apply (*subst infsum-cong-neutral*[**where** *S*=*UNIV* **and** *T*={⦇c_v = *False*, lt_v = *lt*⦈} **and**
f = λv_0. ((*9481*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*10000*::ℝ)) **and**
g = λv_0. ((*9481*::ℝ) / (*10000*::ℝ))])
by *simp+*
then show ($\sum_\infty$ v_0::*state*.
((*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ) +
(*499*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ)) ∗
((*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*100*::ℝ) +
(*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*19*::ℝ) / (*20*::ℝ))) ∗
(*10000*::ℝ) = (*9481*::ℝ)
using *calculation* **by** *fastforce*
next
fix *lt c*
have ($\sum_\infty$ v_0::*state*.
((*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ) +
(*499*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ)) ∗
((*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*100*::ℝ) +
(*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*19*::ℝ) / (*20*::ℝ)))
= ($\sum_\infty$ v_0::*state*. (*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) ∗(*11*::ℝ) / (*50000*::ℝ))
apply (*subst infsum-cong*[**where** *g* = λv_0::*state*. (*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) ∗(*11*::ℝ) / (*50000*::ℝ)])
by *auto*
also have ... = ($\sum_\infty$ v_0::*state*∈{⦇c_v = *True*, lt_v = *lt*⦈}. ((*11*::ℝ) / (*50000*::ℝ)))
apply (*subst infsum-cong-neutral*[**where** *S*=*UNIV* **and** *T*={⦇c_v = *True*, lt_v = *lt*⦈} **and**
f = λv_0. (*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) ∗(*11*::ℝ) / (*50000*::ℝ) **and**
g = λv_0. ((*11*::ℝ) / (*50000*::ℝ))])
by *simp+*
then show ($\sum_\infty$ v_0::*state*.
((*if* v_0 = ⦇c_v = *True*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ) +
(*499*::ℝ) ∗ (*if* v_0 = ⦇c_v = *False*, lt_v = *lt*⦈ *then* *1*::ℝ *else* (*0*::ℝ)) / (*500*::ℝ)) ∗
((*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*100*::ℝ) +
(*if* ¬ c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then* *1*::ℝ *else* (*0*::ℝ)) ∗ (*19*::ℝ) / (*20*::ℝ))) ∗
(*50000*::ℝ) = (*11*::ℝ)
using *calculation* **by** *force*
qed

lemma *FirstTestPos*: *FirstTestPos* = *prfun-of-rvfun FirstTestPos-altdef*
  apply (*simp add*: *FirstTestPos-def FirstTestPos-altdef-def*)
  apply (*simp add*: *FirstTest-simp TestResultPos-def*)
  apply (*simp add*: *pfun-defs*)
  apply (*subst rvfun-inverse*)
  apply (*simp add*: *FirstTest-altdef-def*)
  apply (*expr-simp-1 add*: *dist-defs*)
  apply (*rule HOL.arg-cong*[**where** *f*=*prfun-of-rvfun*])
  apply (*subst fun-eq-iff*)
  apply (*simp add*: *FirstTest-altdef-def dist-defs*)
  apply (*pred-auto*)
proof −
  fix *c*
  have *f1*: ($\sum_\infty v_0$::*state*.
    ((*if* lt_v v_0 = *Pos then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*89*::ℝ) / (*50000*::ℝ) +
    (*if* lt_v v_0 = *Neg then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*50000*::ℝ) +
    (*if* lt_v v_0 = *Pos then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*499*::ℝ) / (*10000*::ℝ) +
    (*9481*::ℝ) ∗ ((*if* lt_v v_0 = *Neg then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ))) / (*10000*::ℝ)) ∗
    (*if* lt_v v_0 = *Pos then 1*::ℝ *else* (*0*::ℝ))) =
    ($\sum_\infty v_0$::*state*.
    ((*if* lt_v v_0 = *Pos* ∧ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*89*::ℝ) / (*50000*::ℝ) +
    (*if* lt_v v_0 = *Pos* ∧ ¬ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*499*::ℝ) / (*10000*::ℝ)))
    apply (*rule infsum-cong*)
    by *simp*
  also have *f2*: ... = (*89*::ℝ) / (*50000*::ℝ) + (*499*::ℝ) / (*10000*::ℝ)
    apply (*subst infsum-add*)
    apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-true*)
    apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-false*)
    apply (*subst infsum-cdiv-left*)
    using *summable-on-cmult-left summable-pos-true* **apply** *blast*
    apply (*subst infsum-cmult-left*)
    using *summable-pos-true* **apply** *blast*
    apply (*subst infsum-cdiv-left*)
    using *summable-on-cmult-left summable-pos-false* **apply** *blast*
    apply (*subst infsum-cmult-left*)
    using *summable-pos-false* **apply** *blast*
    apply (*subst infsum-constant-finite-states*)
    using *pos-true* **apply** *force*
    apply (*subst infsum-constant-finite-states*)
    using *pos-false* **apply** *force*
    using *pos-false pos-true* **by** *force*
  show (*161177*::ℝ) /
    ((*1250*::ℝ) ∗
    ($\sum_\infty v_0$::*state*.
    ((*if* lt_v v_0 = *Pos then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*89*::ℝ) / (*50000*::ℝ) +
    (*if* lt_v v_0 = *Neg then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*11*::ℝ) / (*50000*::ℝ) +
    (*if* lt_v v_0 = *Pos then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ)) ∗ (*499*::ℝ) / (*10000*::ℝ) +
    (*9481*::ℝ) ∗ ((*if* lt_v v_0 = *Neg then 1*::ℝ *else* (*0*::ℝ)) ∗ (*if* ¬ c_v v_0 *then 1*::ℝ *else* (*0*::ℝ))) /

$(\mathit{10000}::\mathbb{R})) *$
$(\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})))) = (\mathit{2495}::\mathbb{R})$
by (*simp add*: *f1 f2*)
next
fix *c*
have *f1*: $(\sum_\infty v_0::state.$
$((\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{89}::\mathbb{R}) / (\mathit{50000}::\mathbb{R})$
$+$
$(\mathit{if}\ lt_v\ v_0 = Neg\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{11}::\mathbb{R}) / (\mathit{50000}::\mathbb{R})$
$+$
$(\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{499}::\mathbb{R}) /$
$(\mathit{10000}::\mathbb{R}) +$
$(\mathit{9481}::\mathbb{R}) * ((\mathit{if}\ lt_v\ v_0 = Neg\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R}))) /$
$(\mathit{10000}::\mathbb{R})) *$
$(\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R}))) =$
$(\sum_\infty v_0::state.$
$((\mathit{if}\ lt_v\ v_0 = Pos \wedge c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{89}::\mathbb{R}) / (\mathit{50000}::\mathbb{R}) +$
$(\mathit{if}\ lt_v\ v_0 = Pos \wedge \neg\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{499}::\mathbb{R}) / (\mathit{10000}::\mathbb{R})))$
apply (*rule infsum-cong*)
by *simp*
have *f2*: ... $= (\mathit{89}::\mathbb{R}) / (\mathit{50000}::\mathbb{R}) + (\mathit{499}::\mathbb{R}) / (\mathit{10000}::\mathbb{R})$
apply (*subst infsum-add*)
apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-true*)
apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-false*)
apply (*subst infsum-cdiv-left*)
using *summable-on-cmult-left summable-pos-true* **apply** *blast*
apply (*subst infsum-cmult-left*)
using *summable-pos-true* **apply** *blast*
apply (*subst infsum-cdiv-left*)
using *summable-on-cmult-left summable-pos-false* **apply** *blast*
apply (*subst infsum-cmult-left*)
using *summable-pos-false* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
using *pos-true* **apply** *force*
apply (*subst infsum-constant-finite-states*)
using *pos-false* **apply** *force*
using *pos-false pos-true* **by** *force*
show $(\mathit{28747}::\mathbb{R}) /$
$((\mathit{6250}::\mathbb{R}) *$
$(\sum_\infty v_0::state.$
$((\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{89}::\mathbb{R}) / (\mathit{50000}::\mathbb{R})$
$+$
$(\mathit{if}\ lt_v\ v_0 = Neg\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{11}::\mathbb{R}) / (\mathit{50000}::\mathbb{R})$
$+$
$(\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{499}::\mathbb{R}) /$
$(\mathit{10000}::\mathbb{R}) +$
$(\mathit{9481}::\mathbb{R}) * ((\mathit{if}\ lt_v\ v_0 = Neg\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})) * (\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R}))) /$
$(\mathit{10000}::\mathbb{R})) *$
$(\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ \mathit{1}::\mathbb{R}\ \mathit{else}\ (\mathit{0}::\mathbb{R})))) = (\mathit{89}::\mathbb{R})$
by (*simp add*: *f1 f2*)
qed

What's the probability that the patient has cancer, given a positive test? $P(Cancer \mid Test{=}Pos)$

lemma *FirstTestPos-Cancer*:
rvfun-of-prfun FirstTestPos ; $[\![c^<]\!]_{\mathcal{I}e} = ((p_1 * p_2) / (p_1 * p_2 + (1 - p_1) * p_3))_e$

apply (*simp add*: *FirstTestPos-altdef-def FirstTestPos*)
apply (*subst rvfun-inverse*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*pred-auto*)
proof −
have *f1*: $(\sum_\infty v_0{::}state.$
$((89{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}) +$
$(2495{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}))$
$*$
$(\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) / (323{::}\mathbb{R})) =$
$(\sum_\infty v_0{::}state.\ (((\mathit{if}\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) * ((89{::}\mathbb{R}) / (8{::}\mathbb{R}) / (323{::}\mathbb{R}))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... $= ((89{::}\mathbb{R}) / (8{::}\mathbb{R}) / (323{::}\mathbb{R}))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-true*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-true* **apply** *auto*[*1*]
by (*smt* (*verit*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1 pos-true*)
show $(\sum_\infty v_0{::}state.$
$((89{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}) +$
$(2495{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}))$
$*$
$(\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) / (323{::}\mathbb{R})) * (2584{::}\mathbb{R}) = (89{::}\mathbb{R})$
using *f1 f2* **by** *linarith*
qed

What's the probability that the patient has no cancer, given a positive test? $P(\neg Cancer \mid Test{=}Pos)$

lemma *FirstTestPos-NotCancer*:
rvfun-of-prfun FirstTestPos ; $[\![\neg c^<]\!]_{\mathcal{I}e} = ((1 - p_1) * p_3 / (p_1 * p_2 + (1 - p_1) * p_3))_e$
apply (*simp add*: *FirstTestPos-altdef-def FirstTestPos*)
apply (*subst rvfun-inverse*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*pred-auto*)
proof −
have *f1*: $(\sum_\infty v_0{::}state.$
$((89{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}) +$
$(2495{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}))$
$*$
$(\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) / (323{::}\mathbb{R})) =$
$(\sum_\infty v_0{::}state.\ (((\mathit{if}\ \neg\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) * ((2495{::}\mathbb{R}) / (8{::}\mathbb{R}) / (323{::}\mathbb{R}))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... $= ((2495{::}\mathbb{R}) / (8{::}\mathbb{R}) / (323{::}\mathbb{R}))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-false*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-false* **apply** *auto*[*1*]
by (*smt* (*verit*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1 pos-false*)
show $(\sum_\infty v_0{::}state.$

$((89::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (8::\mathbb{R}))$
$*$
$(if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (323::\mathbb{R})) * (2584::\mathbb{R}) = (2495::\mathbb{R})$
using *f1 f2* **by** *linarith*
qed

lemma *SecondTest*: *SecondTest = prfun-of-rvfun SecondTest-altdef*
apply (*simp add*: *SecondTest-def SecondTest-altdef-def*)
apply (*simp add*: *FirstTestPos TestAction*)
apply (*simp add*: *pseqcomp-def*)
apply (*subst rvfun-inverse*)
apply (*simp add*: *FirstTestPos-altdef-def*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*subst rvfun-inverse*)
apply (*simp add*: *TestAction-altdef-def*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*simp add*: *FirstTestPos-altdef-def TestAction-altdef-def*)
apply (*rule HOL.arg-cong*[**where** *f=prfun-of-rvfun*])
apply (*subst fun-eq-iff*)
apply (*simp add*: *FirstTest-altdef-def dist-defs*)
apply (*pred-auto*)
proof $-$
fix *c*
have *f1*: $(\sum_\infty v_0::state.$
$((89::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (89::\mathbb{R}) / (100::\mathbb{R}) +$
$(if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (20::\mathbb{R})) / (323::\mathbb{R}))$
$= (\sum_\infty v_0::state.$
$((if\ \neg\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * ((2495::\mathbb{R}) / ((8::\mathbb{R}) * (20::\mathbb{R})*(323::\mathbb{R})))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: $... = ((2495::\mathbb{R}) / ((8::\mathbb{R}) * (20::\mathbb{R})*(323::\mathbb{R})))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-false*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-false* **apply** *auto*[*1*]
by (*smt* (*verit*, *best*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1-eq-iff pos-false*)
show $(10336::\mathbb{R}) *$
$(\sum_\infty v_0::state.$
$((89::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (89::\mathbb{R}) / (100::\mathbb{R}) +$
$(if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (20::\mathbb{R})) /$
$(323::\mathbb{R})) = (499::\mathbb{R})$
using *f1 f2* **by** *linarith*
next
fix *c*
have *f1*: $(\sum_\infty v_0::state.$
$((89::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$

$(2495::\mathbb{R}) * ((\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (11::\mathbb{R}) / (100::\mathbb{R}) +$
$(\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (19::\mathbb{R}) / (20::\mathbb{R})) /$
$(323::\mathbb{R}))$
$= (\sum_\infty v_0::\mathit{state}.$
$((\mathit{if} \neg c_v\ v_0 \wedge lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * ((2495::\mathbb{R})*19 / ((8::\mathbb{R}) * (20::\mathbb{R})*(323::\mathbb{R})))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... $= ((2495::\mathbb{R})*19 / ((8::\mathbb{R}) * (20::\mathbb{R})*(323::\mathbb{R})))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-false*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-false* **apply** *auto*[*1*]
by (*smt* (*verit*, *best*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1-eq-iff pos-false*)
show $(544::\mathbb{R}) *$
$(\sum_\infty v_0::\mathit{state}.$
$((89::\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (11::\mathbb{R}) / (100::\mathbb{R}) +$
$(\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (19::\mathbb{R}) / (20::\mathbb{R})) /$
$(323::\mathbb{R})) = (499::\mathbb{R})$
using *f1 f2* **by** *linarith*
next
fix *c*
have *f1*: $(\sum_\infty v_0::\mathit{state}.$
$((89::\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (89::\mathbb{R}) / (100::\mathbb{R}) +$
$(\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) / (20::\mathbb{R})) / (323::\mathbb{R}))$
$= (\sum_\infty v_0::\mathit{state}.$
$(((\mathit{if}\ c_v\ v_0 \wedge lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) * (89 * (89::\mathbb{R}) / ((100::\mathbb{R}) * (323::\mathbb{R}) *$
$(8::\mathbb{R})))))$
apply (*rule infsum-cong*)
by *simp*
have *f2*: ... $= (89 * (89::\mathbb{R}) / ((100::\mathbb{R}) * (323::\mathbb{R}) * (8::\mathbb{R})))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-true*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-true* **apply** *auto*[*1*]
by (*smt* (*verit*, *best*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1-eq-iff pos-true*)
show $(258400::\mathbb{R}) *$
$(\sum_\infty v_0::\mathit{state}.$
$((89::\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) / (8::\mathbb{R}) +$
$(2495::\mathbb{R}) * ((\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = \mathit{Pos}\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R}))) /$
$(8::\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (89::\mathbb{R}) / (100::\mathbb{R}) +$
$(\mathit{if} \neg c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1::\mathbb{R}\ \mathit{else}\ (0::\mathbb{R})) / (20::\mathbb{R})) / (323::\mathbb{R})) =$
$(7921::\mathbb{R})$

using *f1 f2* **by** *linarith*
next
fix *c*
have *f1*: $(\sum_\infty v_0{::}state.$
$((89{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}) +$
$(2495{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) /$
$(8{::}\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R}) +$
$(\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (19{::}\mathbb{R}) / (20{::}\mathbb{R})) /$
$(323{::}\mathbb{R}))$
$= (\sum_\infty v_0{::}state.$
$(((\mathit{if}\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) * (89 * (11{::}\mathbb{R}) / ((100{::}\mathbb{R}) * (323{::}\mathbb{R}) *$
$(8{::}\mathbb{R})))))$
apply (*rule infsum-cong*)
by *simp*
have *f2*: ... $= (89 * (11{::}\mathbb{R}) / ((100{::}\mathbb{R}) * (323{::}\mathbb{R}) * (8{::}\mathbb{R})))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-true*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-true* **apply** *auto*[*1*]
by (*smt* (*verit*, *best*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1-eq-iff pos-true*)
show $(258400{::}\mathbb{R}) *$
$(\sum_\infty v_0{::}state.$
$((89{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (8{::}\mathbb{R}) +$
$(2495{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) /$
$(8{::}\mathbb{R})) *$
$((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (11{::}\mathbb{R}) / (100{::}\mathbb{R}) +$
$(\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (19{::}\mathbb{R}) / (20{::}\mathbb{R})) /$
$(323{::}\mathbb{R})) =$
$(979{::}\mathbb{R})$
using *f1 f2* **by** *linarith*
qed

lemma *SecondTestPos*: *SecondTestPos* = *prfun-of-rvfun SecondTestPos-altdef*
apply (*simp add*: *SecondTestPos-def SecondTestPos-altdef-def*)
apply (*simp add*: *SecondTest*)
apply (*simp add*: *pfun-defs*)
apply (*subst rvfun-inverse*)
apply (*simp add*: *SecondTest-altdef-def*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*rule HOL.arg-cong*[**where** *f*=*prfun-of-rvfun*])
apply (*subst fun-eq-iff*)
apply (*simp add*: *SecondTest-altdef-def TestResultPos-def dist-defs*)
apply (*pred-auto*)
proof −
fix *c*
have *f1*: $(\sum_\infty v_0{::}state.$
$((7921{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) /$
$(800{::}\mathbb{R}) +$
$(979{::}\mathbb{R}) * ((\mathit{if}\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Neg\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) / (800{::}\mathbb{R})$
$+$
$(499{::}\mathbb{R}) * ((\mathit{if}\ \neg\ c_v\ v_0\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R})) * (\mathit{if}\ lt_v\ v_0 = Pos\ \mathit{then}\ 1{::}\mathbb{R}\ \mathit{else}\ (0{::}\mathbb{R}))) /$
$(32{::}\mathbb{R}) +$

$(9481::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (32::\mathbb{R})) *$
$(if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (323::\mathbb{R})) =$
$(\sum_\infty v_0::state.$
$(((if\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) * ((7921::\mathbb{R}) / ((800::\mathbb{R}) * 323)) +$
$((if\ \neg\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) * ((499::\mathbb{R}) / ((32::\mathbb{R}) * (323::\mathbb{R})))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... $= ((7921::\mathbb{R}) / ((800::\mathbb{R}) * 323)) + ((499::\mathbb{R}) / ((32::\mathbb{R}) * (323::\mathbb{R})))$
apply (*subst infsum-add*)
apply (*subst summable-on-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-true*)
apply (*simp*)
apply (*subst summable-on-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-false*)
apply (*simp*)
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*, *ccfv-SIG*) *summable-on-cong summable-pos-true*)
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*, *ccfv-SIG*) *summable-on-cong summable-pos-false*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-true* **apply** *auto*[*1*]
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-false* **apply** *auto*[*1*]
by (*metis* (*no-types*, *lifting*) *Collect-cong One-nat-def card.empty card.insert equals0D finite.emptyI mult-cancel-right2 of-nat-1 pos-false pos-true*)
show $(2544401::\mathbb{R}) / ((2584::\mathbb{R}) *$
$(\sum_\infty v_0::state.$
$((7921::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (800::\mathbb{R}) +$
$(979::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (800::\mathbb{R}) +$
$(499::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (32::\mathbb{R}) +$
$(9481::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (32::\mathbb{R})) *$
$(if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (323::\mathbb{R}))) = (12475::\mathbb{R})$
apply (*simp only*: *f1 f2*)
by *auto*
next
fix *c*
have *f1*: $(\sum_\infty v_0::state.$
$((7921::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (800::\mathbb{R}) +$
$(979::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (800::\mathbb{R}) +$
$(499::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (32::\mathbb{R}) +$
$(9481::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) / (32::\mathbb{R})) *$
$(if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) / (323::\mathbb{R})) =$
$(\sum_\infty v_0::state.$
$((if\ lt_v\ v_0 = Pos \wedge c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (7921::\mathbb{R}) / ((800::\mathbb{R})*(323::\mathbb{R})) +$
$(if\ lt_v\ v_0 = Pos \wedge \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (499::\mathbb{R}) / ((32::\mathbb{R})*(323::\mathbb{R}))))$
apply (*rule infsum-cong*)

by *simp*
have *f2*: ... = $(7921::\mathbb{R})\ /\ ((800::\mathbb{R})*(323::\mathbb{R})) + (499::\mathbb{R})\ /\ ((32::\mathbb{R})*(323::\mathbb{R}))$
apply (*subst infsum-add*)
apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-true*)
apply (*simp add*: *summable-on-cdiv-left summable-on-cmult-left summable-pos-false*)
apply (*subst infsum-cdiv-left*)
using *summable-on-cmult-left summable-pos-true* **apply** *blast*
apply (*subst infsum-cmult-left*)
using *summable-pos-true* **apply** *blast*
apply (*subst infsum-cdiv-left*)
using *summable-on-cmult-left summable-pos-false* **apply** *blast*
apply (*subst infsum-cmult-left*)
using *summable-pos-false* **apply** *blast*
apply (*subst infsum-constant-finite-states*)
using *pos-true* **apply** *force*
apply (*subst infsum-constant-finite-states*)
using *pos-false* **apply** *force*
using *pos-false pos-true* **by** *force*
show $(40389179::\mathbb{R})\ /\ ((64600::\mathbb{R})\ *$
$(\sum_\infty v_0::state.$
$((7921::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /$
$(800::\mathbb{R}) +$
$(979::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /\ (800::\mathbb{R})$
$+$
$(499::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /$
$(32::\mathbb{R}) +$
$(9481::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Neg\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /$
$(32::\mathbb{R})) *$
$(if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))\ /\ (323::\mathbb{R}))) = (7921::\mathbb{R})$
apply (*simp only*: *f1 f2*)
by *auto*
qed

What's the probability that the patient has cancer, given a positive test? $P(Cancer\ |\ Test{=}Pos)$

lemma *SecondTestPos-Cancer*:
rvfun-of-prfun SecondTestPos ; $[\![c^<]\!]_{\mathcal{I}e} = ((p_1 * p_2 * p_2)\ /\ (p_1 * p_2 * p_2 + (1 - p_1) * p_3 * p_3))_e$
apply (*simp add*: *SecondTestPos-altdef-def SecondTestPos*)
apply (*subst rvfun-inverse*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*pred-auto*)
proof −
have *f1*: $(\sum_\infty v_0::state.$
$((7921::\mathbb{R}) * ((if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /\ (4::\mathbb{R}) +$
$(12475::\mathbb{R}) * ((if\ \neg\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})) * (if\ lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R})))\ /$
$(4::\mathbb{R})) *$
$(if\ c_v\ v_0\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))\ /\ (5099::\mathbb{R})) =$
$(\sum_\infty v_0::state.\ (((if\ c_v\ v_0 \wedge lt_v\ v_0 = Pos\ then\ 1::\mathbb{R}\ else\ (0::\mathbb{R}))) * ((7921::\mathbb{R})\ /\ (4::\mathbb{R})\ /\ (5099::\mathbb{R}))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... = $((7921::\mathbb{R})\ /\ (4::\mathbb{R})\ /\ (5099::\mathbb{R}))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-true*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-true* **apply** *auto*[*1*]

by (*smt* (*verit*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1 pos-true*)
show ($\sum_\infty v_0$::*state*.
$((7921::\mathbb{R}) * ((\textit{if } c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R}) + (12475::\mathbb{R}) * ((\textit{if } \neg\ c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R})) *$
$(\textit{if } c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) / (5099::\mathbb{R})) * (20396::\mathbb{R}) = (7921::\mathbb{R})$
using *f1 f2* **by** *linarith*
qed

What's the probability that the patient has no cancer, given a positive test? $P(\neg Cancer \mid Test{=}Pos)$

lemma *SecondTestPos-NotCancer*:
rvfun-of-prfun SecondTestPos ; $[\![\neg c^<]\!]_{\mathcal{I}e} = ((1 - p_1) * p_3 * p_3 / (p_1 * p_2 * p_2 + (1 - p_1) * p_3 * p_3))_e$
apply (*simp add*: *SecondTestPos-altdef-def SecondTestPos*)
apply (*subst rvfun-inverse*)
apply (*expr-simp-1 add*: *dist-defs*)
apply (*pred-auto*)
proof −
have *f1*: ($\sum_\infty v_0$::*state*.
$((7921::\mathbb{R}) * ((\textit{if } c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R}) + (12475::\mathbb{R}) * ((\textit{if } \neg\ c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R})) *$
$(\textit{if } \neg\ c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) / (5099::\mathbb{R})) =$
$(\sum_\infty v_0::state.\ (((\textit{if } \neg\ c_v\ v_0 \wedge lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) * ((12475::\mathbb{R}) / (4::\mathbb{R}) / (5099::\mathbb{R}))))$
apply (*rule infsum-cong*)
by *simp*
also have *f2*: ... = $((12475::\mathbb{R}) / (4::\mathbb{R}) / (5099::\mathbb{R}))$
apply (*subst infsum-cmult-left*)
apply (*smt* (*verit*) *summable-on-cong summable-pos-false*)
apply (*simp*)
apply (*subst infsum-constant-finite-states*)
using *finite.simps pos-false* **apply** *auto*[*1*]
by (*smt* (*verit*) *Collect-cong One-nat-def card.empty card.insert empty-iff finite.emptyI of-nat-1 pos-false*)
show ($\sum_\infty v_0$::*state*.
$((7921::\mathbb{R}) * ((\textit{if } c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R}) + (12475::\mathbb{R}) * ((\textit{if } \neg\ c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) * (\textit{if } lt_v\ v_0 = Pos \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R}))) / (4::\mathbb{R})) *$
$(\textit{if } \neg\ c_v\ v_0 \textit{ then } 1::\mathbb{R} \textit{ else } (0::\mathbb{R})) / (5099::\mathbb{R})) * (20396::\mathbb{R}) = (12475::\mathbb{R})$
using *f1 f2* **by** *linarith*
qed

end

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References

[1] E. C. R. Hehner, “A probability perspective,” vol. 23, no. 4, pp. 391–419. [Online]. Available: https://doi.org/10.1007/s00165-010-0157-0