70092 ExerciseTypes.CW2

Maths Coursework

Submitters

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Math Coursework 2

Q1.

According to the definition of least squares fit of a linear function, the minimized squared errors occurred when $m = \frac{\sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}}{\sum_{k=1}^{n} x_k{}^2 - n\bar{x}^2}$ and $c = \bar{y} - m\bar{x}$. And more often, we use the formula for m as: $m = \frac{\sum_{k=1}^{n}(x_k-\bar{x})(y_k-\bar{y})}{\sum_{k=1}^{n}(x_k-\bar{x})^2}$.

From the table given, we can calculate that $\bar{x} = \frac{55+60+65+70+75}{5} = 65$, $\bar{y} = \frac{107+109+114+118+123}{5} = 114.2$. After applying these values to the formulas, we can get m=0.82, c = 60.9.

The codes are shown in the figure:

```
int main(){
    double x[] = {55,60,65,70,75};
    double y[] = {107,109,114,118,123};

    double up = 0.0;
    double down = 0.0;
    double avg_x = 0.0;
    double avg_y = 0.0;
    for(int i=0;i<5;i++){
        avg_x += x[i];
        avg_y += y[i];
    }
    avg_x /= 5;
    avg_y /= 5;
    for(int i=1;i<=5;i++){
        // up += x[i-1]*y[i-1] - i*avg_x*avg_y;
        // down += x[i-1]*x[i-1] - i*avg_x*avg_x;
        up += (x[i-1]-avg_x)*(y[i-1]-avg_y);
        down += (x[i-1]-avg_x)*(x[i-1]-avg_x);
    }
    double m = up/down;
    double c = avg_y - m*avg_x;
    cout << "m=" << m << " , c = "<< c << endl;
```

would be better if this was dynamic

why this shift?

ok? Next time just calculate it...

6/6

Q2.

Q2 1. We know the definition of a 3×3 square matrix's ($B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$) determinant is $|B| = b_{11}\begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix} - b_{12}\begin{vmatrix} b_{21} & b_{23} \\ b_{31} & b_{33} \end{vmatrix} + b_{13}\begin{vmatrix} b_{21} & b_{22} \\ b_{31} & b_{32} \end{vmatrix}$.

$C=\begin{vmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{vmatrix}$

So, in our case $|C| = 5\begin{vmatrix} 6 & 2 \\ 2 & 7 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 2 \\ 0 & 7 \end{vmatrix} + 0\begin{vmatrix} -2 & 6 \\ 0 & 2 \end{vmatrix} = 5\times 38 - (-2)\times(-14) + 0 = 162$ ✓

2. The adjoint matrix is defined as the transpose of the matrix of cofactors. eg, for matrix A, $adjA = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$

for our matrix $C = \begin{vmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{vmatrix}$

$C_{11} = (-1)^{1+1}\begin{vmatrix} 6 & 2 \\ 2 & 7 \end{vmatrix} = 38$ $\quad C_{12} = (-1)^{1+2}\begin{vmatrix} -2 & 2 \\ 0 & 7 \end{vmatrix} = 14$ $\quad C_{13} = (-1)^{1+3}\begin{vmatrix} -2 & 6 \\ 0 & 2 \end{vmatrix} = -4$

$C_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 0 \\ 2 & 7 \end{vmatrix} = 14$ $\quad C_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 0 \\ 0 & 7 \end{vmatrix} = 35$ $\quad C_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -2 \\ 0 & 2 \end{vmatrix} = -10$

$C_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 0 \\ 6 & 2 \end{vmatrix} = -4$ $\quad C_{32} = (-1)^{3+2}\begin{vmatrix} 5 & 0 \\ -2 & 2 \end{vmatrix} = -10$ $\quad C_{33} = (-1)^{3+3}\begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix} = 26$

$adjC = \begin{bmatrix} 38 & 14 & -4 \\ 14 & 35 & -10 \\ -4 & -10 & 26 \end{bmatrix}^T = \begin{bmatrix} 38 & 14 & -4 \\ 14 & 35 & -10 \\ -4 & -10 & 26 \end{bmatrix}$ ✓

3. The principle of using the adjoint matrix to find the inverse matrix is that: if matrix A is non-singular, then $|A| \neq 0$ and $A^{-1} = \frac{adjA}{|A|}$

if A is singular, then $|A| = 0$ and A^{-1} does not exist. In this case, since from (1) we get $|C| = 162 \neq 0$, so C^{-1} does exist

and $C^{-1} = \frac{adjC}{|C|} = \begin{bmatrix} \frac{38}{162} & \frac{14}{162} & \frac{-4}{162} \\ \frac{14}{162} & \frac{35}{162} & \frac{-10}{162} \\ \frac{-4}{162} & \frac{-10}{162} & \frac{26}{162} \end{bmatrix}$ ✓ ok, could have simplified

1/4

4. From above, we now have $C = \begin{vmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{vmatrix}$ and $C^{-1} = \frac{1}{162} \begin{vmatrix} 38 & 14 & -4 \\ 14 & 35 & -10 \\ -4 & -10 & 26 \end{vmatrix}$.

$\|C\|_\infty = \max_{1\le i\le 3} \sum_{j=1}^{3} |a_{ij}| = \max(7, 10, 9) = 10$ ✓ $\|C^{-1}\|_\infty = \max\left(\frac{56}{162}, \frac{59}{162}, \frac{40}{162}\right) = \frac{59}{162}$ ✓

$\|C\|_1 = \max_{1\le j\le 3} \sum_{i=1}^{3} |a_{ij}| = \max(7, 10, 9) = 10$ $\|C^{-1}\|_1 = \max\left(\frac{56}{162}, \frac{59}{162}, \frac{40}{162}\right) = \frac{59}{162}$

$K_\infty(C) = \|C\|_\infty \|C^{-1}\|_\infty = 10 \cdot \frac{59}{162} = \frac{590}{162}$ ✓ ≈ 3.64

$K_1(C) = \|C\|_1 \|C^{-1}\|_1 = 10 \cdot \frac{56}{162} = \frac{590}{162}$ ✓ ≈ 3.64

5. In order to calculate eigenvalues λ_i, we need to solve $|C - \lambda I| = 0$.

In our case, $|C-\lambda I| = \begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = (5-\lambda)\begin{vmatrix} 6-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} - (-2)\begin{vmatrix} -2 & 2 \\ 0 & 7-\lambda \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & 6-\lambda \\ 0 & 2 \end{vmatrix} = (5-\lambda)((6-\lambda)(7-\lambda)-4) + 2((-2)(7-\lambda)-0) + 0$

$= -\lambda^3 + 18\lambda^2 - 99\lambda + 162 = (\lambda-3)(\lambda-6)(\lambda-9)$

How? And no, it's missing a − sign

−0.5 marks

Hence, it gives $\lambda_1 = 3, \lambda_2 = 6, \lambda_3 = 9$ ✓ (in decreasing order by convention)

but your order is increasing?

Having obtained the eigenvalues λ_i, $i = 1,2,3$, the corresponding eigenvectors $x = e_i$ are obtained by solving the homogeneous equation $(C - \lambda I)x = 0$ in form of $(C - \lambda_i I)e_i = 0$ as follows:

when $i=1$, $\lambda_1 = 3$ and $(C - \lambda_1 I)e_1 = 0$ follows:

$$\begin{bmatrix} 5-3 & -2 & 0 \\ -2 & 6-3 & 2 \\ 0 & 2 & 7-3 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & 2 \\ 0 & 2 & 4 \end{bmatrix}\begin{bmatrix} e_{11} \\ e_{12} \\ e_{13} \end{bmatrix} = 0, \text{ that is } \begin{matrix} 2e_{11} - 2e_{12} + 0e_{13} = 0 \\ -2e_{11} + 3e_{12} + 2e_{13} = 0 \\ 0e_{11} + 2e_{12} + 4e_{13} = 0 \end{matrix}$$

$\hat{e}_1 = \left[\frac{-2}{3}, \frac{-2}{3}, \frac{1}{3}\right]^T$ ✓

Letting $e_{13} = \beta_1$, we have $e_{12} = -2\beta_1$, $e_{11} = -2\beta_1$. Thus, the eigenvector e_1 corresponding to $\lambda_1 = 3$ is: $e_1 = \beta_1[-2, -2, 1]^T$ when β_1 is an arbitrary non-zero scalar.

Continued. when $i=2$, $\lambda_2 = 6$ and $(C - \lambda_2 I)e_2 = 0$ follows:

$$\begin{bmatrix} 5-6 & -2 & 0 \\ -2 & 6-6 & 2 \\ 0 & 2 & 7-6 \end{bmatrix}\begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} e_{21} \\ e_{22} \\ e_{23} \end{bmatrix} = 0, \text{ that is } \begin{cases} -1e_{21} - 2e_{22} + 0e_{23} = 0 \\ -2e_{21} + 0e_{22} + 2e_{23} = 0 \\ 0e_{21} + 2e_{22} + e_{23} = 0 \end{cases}$$

Letting $e_{22} = \beta_2$, we have $e_{23} = -2\beta_2$, $e_{21} = -2\beta_2$. when β_2 is an arbitrary non-zero scalar.

Thus, the eigenvector e_2 corresponding to $\lambda_2 = 6$ is $e_2 = \beta_2[-2, 1, -2]^T$

Its normalised form is $\hat{e}_2 = \left[\frac{-2}{3}, \frac{1}{3}, -\frac{2}{3}\right]^T$ ✓

when $i=3$, $\lambda_3 = 9$ and $(C - \lambda_3 I)e_3 = 0$ follows:

$$\begin{bmatrix} 5-9 & -2 & 0 \\ -2 & 6-9 & 2 \\ 0 & 2 & 7-9 \end{bmatrix}\begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = \begin{bmatrix} -4 & -2 & 0 \\ -2 & -3 & 2 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} e_{31} \\ e_{32} \\ e_{33} \end{bmatrix} = 0, \text{ that is } \begin{cases} -4e_{31} - 2e_{32} + 0e_{33} = 0 \\ -2e_{31} + -3e_{32} + 2e_{33} = 0 \\ 0e_{31} + 2e_{32} + -2e_{33} = 0 \end{cases}$$

Letting $e_{33} = \beta_3$, we have $e_{32} = \beta_3$, $e_{31} = -\frac{1}{2}\beta_3$ when β_3 is an arbitrary non-zero scalar.

Thus, the eigenvector e_3 corresponding to $\lambda_3 = 9$ is $e_3 = \beta_3\left[-\frac{1}{2}, 1, 1\right]^T = \frac{1}{2}\beta_3[-1, 2, 2]^T$

Its normalised form is $\left[-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]^T$ ✓

To sum up, $\hat{e}_1 = \left[-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right]^T$, $\hat{e}_2 = \left[-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right]^T$, $\hat{e}_3 = \left[-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]^T$

6. $U = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$ ✓ $U^T = \begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

$$D = U^TCU = \begin{bmatrix} \frac{-10+4}{3} & \frac{4-12+2}{3} & \frac{-4+7}{3} \\ \frac{-10-2}{3} & \frac{4+6-4}{3} & \frac{2-14}{3} \\ \frac{-5+4+0}{3} & \frac{2+12+4}{3} & \frac{4+14}{3} \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$C = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & 1 \\ -4 & 2 & -4 \\ -3 & 6 & 6 \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Continued. $= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{bmatrix} = D$ ✓

don't use this line

We can tell that $D^T = D$ So it is a diagonal matrix.

8.5/9

Q3

Q3. $f(x) = 2e^{-x}(x-1)^3$

1. According to the definition, ~~the~~ a solution to $f'(x)=0$ is ~~the~~ a stationary point.

Since $f(x) = 2e^{-x}\cdot(x-1)^3$, $f'(x) = 2\cdot e^{-x}\cdot(-1)\cdot(x-1)^3 + 2e^{-x}\cdot 3(x-1)^2 = 2e^{-x}\cdot[3(x-1)^2 - (x-1)^3] = 2e^{-x}\cdot(x-1)^2\cdot(3-x+1)$

$= 2e^{-x}(x-1)^2(4-x)$ ✓ $x_1 = 1, x_2 = 4$ ✓

$f(1) = 2\cdot e^{-1}\cdot 0 = 0$ ✓ $f(4) = 2\cdot e^{-4}\cdot 27 = \frac{54}{e^4}$ ✓

Hence, the stationary points are $x_1=1, x_2=4$ and the value of the function at these points are $f(1)=0, f(4)=\frac{54}{e^4}$ respectively ✓

Approach 1: Use the sign of $f'(x)$ to determine

$(f''(x) = (2e^{-x}(x-1)^2(4-x))' = (2e^{-x}\cdot(-x^3+6x^2-9x+4))' = 2e^{-x}\cdot(-1)(-x^3+6x^2-9x+4) + 2e^{-x}\cdot(-3x^2+12x-9) = 2e^{-x}(x^3-9x^2+21x-13)$ ✓

2.

x	$(-\infty, 1)$	1	$(1,4)$	4	$(4,+\infty)$
$f''(x)$	$-$	0		$-$	
$f'(x)$	$+$	0	$+$	0	$-$
$f(x)$	↗		↗		↘

✓ ✓ ✓

e^{-x} is always greater than or equals to 0

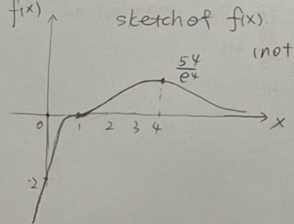
$(x-1)^2$ is always greater than or equals to 0.

$g(x) = 4-x \begin{cases} <0 & \text{when } x>4 \\ =0 & x=4 \\ >0 & x<4 \end{cases}$

So, in order to determine the value of $f'(x)$, we only focus on $g(x) = 4-x$

sketch of $f(x)$

(not in scale) ✓



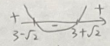
Although at $x_1 = 1$, $f'(x)=0$, but the value of $f'(x)$ besides the points are all greater than 0. so x_1 is a point of inflection ✓

At $x_2 = 4$, we have $f'(x)>0$ at its left side and $f'(x)<0$ at its right side, so we call $(4, \frac{54}{e^4})$ as a local maximum ✓

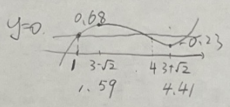
Approach 2:
analyze of $f''(x)$ for prove

Notes: $f''(x) = 2e^{-x} \cdot (x^3-9x^2+21x-13)$ we have $\forall x \in R$, $e^{-x} \geq 0$

so let $g(x) = x^3-9x^2+21x-13$. $g'(x) = 3x^2-18x+21 = 3(x^2-6x+7) = 3(x^2-6x+9-2) = 3[(x-3)^2-2]$

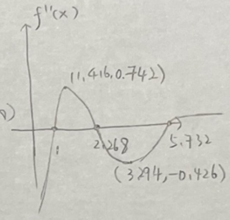


so $g(x)$:



$f''(3-\sqrt{2}) \approx 0.68$

$f''(3+\sqrt{2}) \approx -0.23$



x	$(-\infty, 1)$	1	$(1, 2.268)$	2.268	$(2.268, 4)$	4	$(4, 5.732)$	5.732	$(5.732, +\infty)$
$f''(x)$	−	0	+	0	−	−	−	0	+
$f'(x)$	+	0	+	+	+	0	−	−	−

At $x_1=1$, $f'(x)=0$ and $f''(x)=0$, cannot determine. Use approach 1 to see the sign of $f'(x)$

At $x_2=4$, $f'(x)=0$ and $f''(x)<0$, then it is a local maximum

In summary, $(1, 0)$ is a point of inflection

$(4, \frac{54}{e^4})$ is a local maximum.

✓

5/5

19.5/20