

# 104030 - Introduction to Partial Differential Equations

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## Abstract

## 1 Introduction

**PDE** In PDE, the solution is a function of a couple of variables  $u(x_1, x_2, \dots, x_m)$  such that:

$$F(x_1, x_2, \dots, x_m, u_{x_1}, u_{x_2}, \dots, u_{x_m}, u_{x_1 x_1}, \dots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually  $m = 2$ . For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_1 x_2}) = 0$$

Is PDE of two variables of order 2.

**Linear PDE** PDE is linear if  $F$  is linear in  $u$  and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$\begin{aligned} F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) = \\ = A(x_1, x_2)u_{x_1 x_1} + B(x_1, x_2)u_{x_1 x_2} + C(x_1, x_2)u_{x_2 x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0 \end{aligned}$$

**Quasilinear PDE** Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$\begin{aligned} F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) = \\ = A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2 x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0 \end{aligned}$$

For homogeneous linear PDE solution always exist. In addition, if  $u_1, u_2$ , then any linear combination of those  $\lambda_1 u_1 + \lambda_2 u_2$  will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

**Autonomous PDE** If  $F$  is independent on  $x_i$ , then if  $u(x_1, \dots, x_i, \dots, x_m)$  is solution then  $u(x_1, \dots, x_i + \lambda, \dots, x_m)$  is solution too.

In particular if  $u$  is independent on all  $x_i$ , then  $u(x_1 + \lambda_1, \dots, x_i + \lambda_i, \dots, x_m + \lambda_m)$ .

## 1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.

Lets start ODE describing harmonic oscillator, would be

$$m \frac{d^2 x}{dt^2} = k(x - x_0)$$

Now suppose that we have  $N$  such masses and position of mass is  $\bar{x}_i = x_i + u(x_i, t)$ , where  $u$  is displacement of mass and  $x_i - x_{i-1} = \Delta$ . Then the position of mass is described as

$$\frac{\partial^2 \bar{x}_i}{\partial t^2} = m \frac{\partial^2}{\partial t^2} u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m \frac{d^2}{dt^2} u(x, t) = k[u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)]$$

In limit  $\Delta \rightarrow 0$ :

$$\frac{d^2}{dt^2} u(x, t) = c^2 \frac{d^2}{dx^2} u(x, t)$$

Where

$$c^2 = \lim_{\Delta \rightarrow 0} \frac{\Delta^2 k \Delta}{m \Delta}$$

**Possible solutions** For each function  $f$  in  $\mathcal{C}^2$ ,  $u = f(x - ct)$  is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along  $x$  axis with constant velocity  $c$ . Since  $c$  can be negative too, we have solution

$$u(x, t) = f(x + ct) + g(x - ct)$$

## 1.2 Heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Here,  $u$  means amount of heat in point  $x$  at time  $t$ .

Amount of heat in interval  $[a, b]$  is

$$Q(t) = \int_a^b u(x, t) dx$$

And heat flux in point  $x$  at time  $t$  is  $k \frac{\partial u}{\partial x}$

Then flux out of interval is

$$\frac{dQ}{dt} = k \frac{\partial u}{\partial x} - k \frac{\partial u}{\partial x}$$

Thus

$$\int_a^b \frac{\partial}{\partial t} u(x, t) dx = k \frac{\partial u}{\partial x} - k \frac{\partial u}{\partial x}$$

In limit  $b \rightarrow a$  we get

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

**Example solution**

$$u(x, t) = e^{-kst} \sin(\sqrt{s}x)$$

for some parameter  $s$ . Here we also can add some constant to  $x$  and acquire additional solution:

$$U(x, t) = e^{-kst} \sin(\sqrt{s}(x + \lambda)) = \cos(\sqrt{s}\lambda) e^{-kst} \sin(\sqrt{s}x) + \sin(\sqrt{s}\lambda) e^{-kst} \cos(\sqrt{s}x)$$

Thus

$$w(x, t) = e^{-kst} \cos(\sqrt{s}x)$$

is solution too.

### 1.3 Diffusion equation

Suppose  $u(x_1, x_2, x_3, t)$  describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \nabla \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field  $v$  independent on  $u$ .

### 1.4 Elliptic PDEs

#### Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

#### Poisson equation

$$\nabla^2 u = f(x_1, x_2)$$

## 2 First-order PDE

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation:  $u(x, t) = f(x - ct)$  for some differentiable  $f$ .

Suppose we have initial conditions  $u(x, 0) = u_0(x)$ . Is it determines uniquely a solution of equation? Obviously,  $u(x, t) = u_0(x - ct)$  is a solution.

Lets show it's unique. Take a look at parametrization  $x(t) = s_1 + ct$ .

$$\frac{d}{dt}u(x(t), t) = c \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thus  $u$  is constant on every line of form  $x(t) = s + ct$ . Such lines, along which the PDE becomes an ordinary differential equation, are called characteristic curves or just characteristics. Thus if we know a value of  $u$  in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve  $x(t)$  for  $t \in [a, b]$ . So we want to find a solution such that the surface of solution comprises a given curve in 3D.

The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

### 2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where  $a, b, c$  are continuously differentiable in some neighborhood of point  $(x_0, y_0, z_0)$ .

Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left( -\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1 \right)$$

and

$$\nabla f \cdot (a, b, c) = -a \frac{\partial u}{\partial x} - b \frac{\partial u}{\partial y} + c = 0$$

Thus vector  $(a, b, c)$  is tangent to solution surface.

Now define curve such that

$$\begin{cases} \frac{dx}{dt} = a(x(t), y(t), z(t)) \\ \frac{dy}{dt} = b(x(t), y(t), z(t)) \\ \frac{dz}{dt} = c(x(t), y(t), z(t)) \end{cases}$$

The curve  $(x(t), y(t), z(t))$  is characteristic curve of PDE.

If there is no dependence on  $z$  (i.e. equation is linear) we can take a look on 2-dimensional curve in  $xy$ -plane.

**Theorem** If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

**Proof** Let  $(x(t), y(t), z(t))$  characteristic curve of PDE and suppose for some  $t_0$

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that  $w(t_0) = 0$ . Now

$$\begin{aligned} G(x(t), y(t), w(t)) &= \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}(x(t), y(t)) \frac{dx}{dt} - \frac{\partial u}{\partial y}(x(t), y(t)) \frac{dy}{dt} = c \left( x(t), y(t), w(t) + u(x(t), y(t)) \right) - \\ &\quad - \frac{\partial u}{\partial x}(x(t), y(t)) a \left( x(t), y(t), w(t) + u(x(t), y(t)) \right) - \frac{\partial u}{\partial y}(x(t), y(t)) b \left( x(t), y(t), w(t) + u(x(t), y(t)) \right) \end{aligned}$$

If we substitute  $w = 0$ , we get

$$G(x(t), y(t), 0) = c(x(t), y(t), u(x(t), y(t))) - \frac{\partial u}{\partial x} a(x(t), y(t), u(x(t), y(t))) - \frac{\partial u}{\partial y} b(x(t), y(t), u(x(t), y(t))) = 0$$

That means that  $w = 0$  is a solution of ODE, and since  $a, b, c \in C^1$ , the solution is unique, i.e.  $w = 0$  is the only solution, and thus characteristic curve is contained in the solution surface.

## 2.2 Existence and uniqueness theorem for first-order quasilinear PDE

**Existence and uniqueness theorem for first-order quasilinear PDE** Suppose we have initial curve  $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$  which around some point  $s_0$  is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0) \dot{\bar{y}}(s_0) - b(x_0, y_0, z_0) \dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).

Then in neighborhood of  $s_0$  exists unique solution of PDE.

**Proof** Define functions  $x(s, t), y(s, t), z(s, t)$  around  $(s_0, 0)$  such that

$$\begin{cases} x(s, 0) = \bar{x}(s) \\ y(s, 0) = \bar{y}(s) \\ z(s, 0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a(x(s, t), y(s, t), z(s, t)) \\ \frac{\partial y}{\partial t} = b(x(s, t), y(s, t), z(s, t)) \\ \frac{\partial z}{\partial t} = c(x(s, t), y(s, t), z(s, t)) \end{cases}$$

From uniqueness and existence of ODE, exists unique solution  $(x, y, z)$  in neighbourhood of  $s_0$ .

Lets try to find  $s, t$ , as a function of  $x, y$ . It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in  $(s_0, 0)$ .

Now define  $u(x, y) = z(s(x, y), t(x, y))$ .

$$\begin{aligned} a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} &= a \left[ \frac{\partial z}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial x} \right] + b \left[ \frac{\partial z}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial y} \right] = \frac{\partial z}{\partial t} \left[ a \frac{\partial t}{\partial x} + b \frac{\partial t}{\partial y} \right] + \frac{\partial z}{\partial s} \left[ a \frac{\partial s}{\partial x} + b \frac{\partial s}{\partial y} \right] = \\ &= \frac{\partial z}{\partial t} \underbrace{\left[ \frac{\partial x}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial y} \right]}_{\frac{dt}{dt}} + \frac{\partial z}{\partial s} \underbrace{\left[ \frac{\partial x}{\partial t} \frac{\partial s}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial s}{\partial y} \right]}_{\frac{ds}{dt}} = \frac{\partial z}{\partial t} = c \end{aligned}$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on  $xy$ -plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on  $xy$ -plane are different, there are different possibilities.

### Example

$$yu_x - xu_y = 0$$

with initial curve  $(s, 0, H(s))$  and  $0 < \alpha \leq s \leq \beta$

### Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s) \sin(t) + B(s) \cos(t) \\ y = A(s) \cos(t) - B(s) \sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s \cos(t) \\ y = -s \sin(t) \\ z = h(s) \end{cases}$$

Now we want to find  $s, t$  as a function of  $x, y$ :

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

### Example

$$yu_x - xu_y = u$$

with initial curve  $(s, 0, H(s))$  and  $0 < \alpha \leq s \leq \beta$

### Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s) \sin(t) + B(s) \cos(t) \\ y = A(s) \cos(t) - B(s) \sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s \cos(t) \\ y = -s \sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find  $s, t$  as a function of  $x, y$ :

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

## 2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y} F(u) = 0$$

for  $F = \frac{1}{2}u^2$ )

Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{dx}{dy} = -u \Rightarrow u = \frac{dx}{dy}$$

Here  $y$  denotes time.

To solve it, we take integral:

$$\begin{aligned} \int_a^b \left[ \frac{\partial u(x, y)}{\partial y} + \frac{\partial}{\partial x} F(u(x, y)) \right] dx &= 0 \\ \frac{\partial}{\partial y} \underbrace{\int_a^b u dx}_{Q(y)} + F(u(b, y)) - F(u(a, y)) &= 0 \\ \frac{dQ}{dy} = F(u(a, y)) - F(u(b, y)) & \end{aligned}$$

Now as for any quasilinear PDE:

$$\begin{cases} \dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = c_2 t + c_3 \\ y = t + c_1 \\ z = c_2 \end{cases}$$

For initial conditions  $(s, 0, h(s))$ :

$$\begin{cases} x = h(s)t + s \\ y = t \\ z = h(s) \end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{d\bar{x}}{ds} \cdot 1 - \frac{d\bar{y}}{ds} \cdot h(s) = 1 \neq 0$$

Since

$$\begin{aligned} \frac{\partial u}{\partial x} &= h'(x - yu) \cdot \left( 1 - y \frac{\partial u}{\partial x} \right) \\ \frac{\partial u}{\partial x} &= \frac{h'(x - yu)}{1 + h'(x - yu) \cdot y} \end{aligned}$$

even if we start from  $C^\infty$  function we can get  $1 + h'(x - yu) \cdot y = 0$  and thus undefined derivative.

Geometrically, the slope of projections of characteristic curves is equal to  $h(s)$  thus they can cross in some point.

**Weak solutions** We define a weak solution of equation, function  $u$  fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y} u(x, y) dx + F(u(b, y)) - F(u(a, y)) = 0$$

Intuitively,  $F$  is flux, and  $u$  is density, thus change in number of particles (integral) is difference between particles going in and out.

Suppose for solution  $u(x, y)$  exists curve of non-continuousness  $\gamma$ , i.e.  $u$  is not continuous in each point of curve:

$$u(y) = \begin{cases} u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y) \end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y) dx = \int_a^{\gamma(y)} u^+(x,y) dx + \int_{\gamma(y)}^b u^-(x,y) dx$$

$$\begin{aligned} \frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y} dx + u^+(x, \gamma(y)) \cdot \gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y} dx - u^-(x, \gamma(y)) \cdot \gamma'(y) = \\ &= - \int_a^{\gamma(y)} \frac{dF(u^+)}{dx} dx - \int_{\gamma(y)}^b \frac{dF(u^-)}{dx} dx + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = \\ &= - [F(u^+(\gamma(y), y)) - F(u^+(a, y))] - [F(u^-(b, y)) - F(u^-(\gamma(y), y))] + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] \end{aligned}$$

Meaning

$$- [F(u^+(\gamma(y), y)) - F(u^+(a, y))] - [F(u^-(b, y)) - F(u^-(\gamma(y), y))] + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = F(u^-(a, y)) - F(u^+(b, y))$$

$$\gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = F(u^+(\gamma(y), y)) - F(u^-(\gamma(y), y))$$

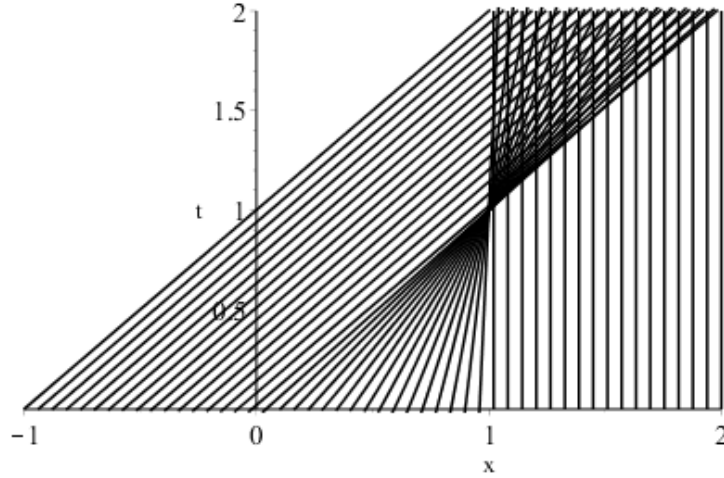
$$\gamma' = \frac{F(u^+(\gamma(y), y)) - F(u^-(\gamma(y), y))}{u^+(x, \gamma(y)) - u^-(x, \gamma(y))}$$

This equation is called Rankine–Hugoniot conditions. If  $F(u) = \frac{1}{2}u^2$ , we get  $\gamma'(y) = \frac{1}{2}(u^+ + u^-)$

**Example** Suppose we have initial conditions  $u(x, 0) = h(x)$  for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \leq x \leq \alpha \end{cases}$$

For  $0 < y < 1$  we have a triangle  $\Delta$  ( $0 < x < \alpha$  and  $y < \frac{x}{\alpha}$ ) for which there is intersection of two solution:



In point  $x, y$  we have slope  $u(x, y)$  thus the characteristic curve crosses  $x$ -axis at  $x_0 = x - uy$  and from initial conditions,  $u = 1 - \frac{x_0}{\alpha}$ . Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

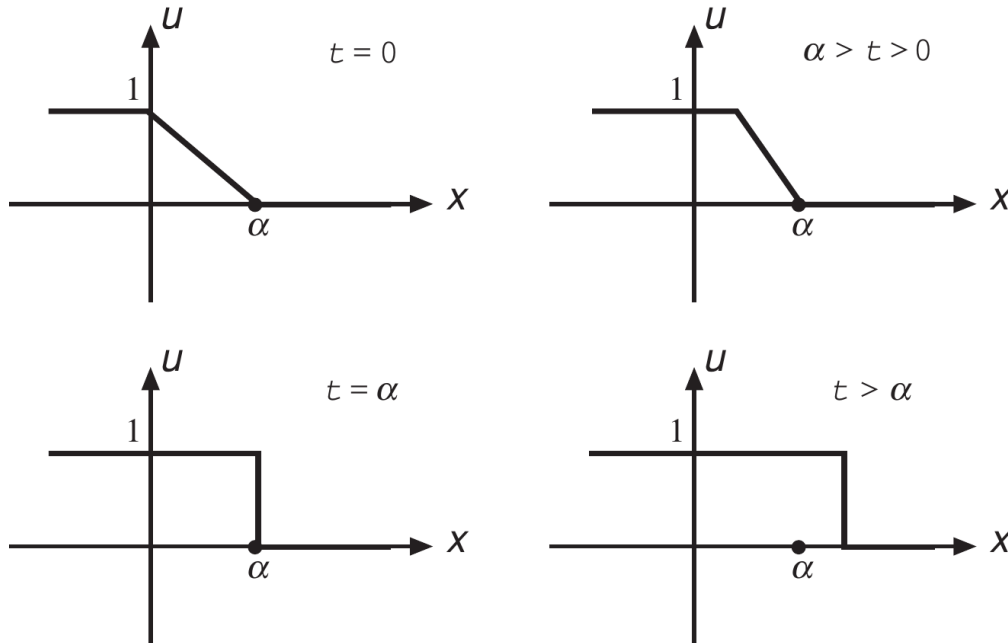
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for  $y > 1$  from Rankine–Hugoniot conditions

$$u(x, y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y - \alpha) \\ 0 & x > \alpha + \frac{1}{2}(y - \alpha) \end{cases}$$

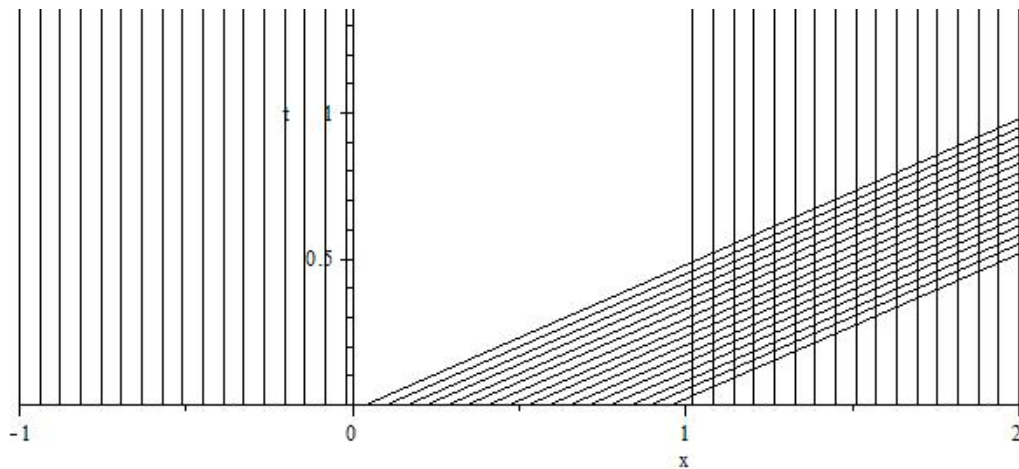
Such a solution is called a shock wave.



**Example** For

$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \leq x \leq \alpha \end{cases}$$

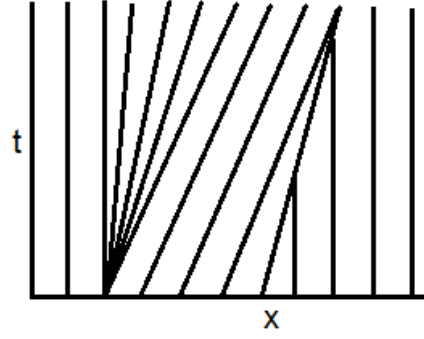
Now there is no place where characteristic curves meet



In the region without characteristic curves ( $0 \leq x \leq y$ ) we get the following: the solution starts from some point  $x_0 = x - uy$ , and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$





**What happens if  $\alpha \rightarrow 0$ ?** We get  $u = \frac{x}{y}$  for  $0 \leq x \leq y$ . We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.

However, also shock wave along  $y = x$  is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we can't reproduce the solution for some  $y < y_0$  even if I know the values for  $y = y_0$ .

**Entropy principle** Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

## 2.4 Fully non-linear equations

### Hamilton-Jacoby equation

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial p} \\ \frac{dy}{dt} = \frac{\partial F}{\partial q} \\ \frac{dz}{dt} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q} \\ \frac{dp}{dt} = -\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} \\ \frac{dq}{dt} = -\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve  $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$

We need to find  $\bar{p}$  and  $\bar{q}$ . We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by  $s$

$$\begin{aligned} \frac{\partial u}{\partial x} \frac{d\bar{x}}{ds} + \frac{\partial u}{\partial y} \frac{d\bar{y}}{ds} &= \frac{d\bar{z}}{ds} \\ \bar{p}(s) \frac{d\bar{x}}{ds} + \bar{q}(s) \frac{d\bar{y}}{ds} &= \frac{d\bar{z}}{ds} \end{aligned}$$

Now we can find  $p$  and  $q$ .

Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with  $u = 0$ , then characteristic curves are perpendicular to initial curve. We get  $u(x, y)$  equal to distance from initial curve, since absolute value of gradient of  $u$  is 1 due to equation.

If we have  $u = \phi(s)$  on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

**Higher dimension** We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

### 3 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + du_x + eu_y + fu = g$$

**Definition** Equation is called hyperbolic if  $b^2 - ac > 0$ , parabolic if  $b^2 - ac = 0$  and elliptic if  $b^2 - ac < 0$ . Wave equation is hyperbolic in the whole space.

We want to simplify the equation: we are searching for  $\xi(x, y)$  and  $\eta(x, y)$  such that

$$\frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y} \frac{\partial \xi}{\partial x} \neq 0$$

and solution  $u(x, y) = w(\xi(x, y), \eta(x, y))$ .

Derivatives of  $u$  are

$$\begin{aligned} u_y &= w_\xi \xi_y + w_\eta \eta_y \\ u_{yy} &= w_{\xi\xi} \xi_y^2 + w_{\xi\eta} \xi_y \eta_y + w_{\eta\xi} \xi_y \eta_y + w_{\eta\eta} \eta_y^2 + w_\eta \eta_{yy} \\ u_{xy} &= \frac{\partial}{\partial x} \frac{\partial u}{\partial y} = w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta} \xi_x \eta_y + w_{\eta\xi} \xi_x \eta_y + w_{\eta\eta} \xi_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy} \end{aligned}$$

Now we can get equation of form

$$A(\xi, \eta)w_{\xi\xi} + 2B(\xi, \eta)w_{\xi\eta} + C(\xi, \eta)w_{\eta\xi} + D(\xi, \eta)w_{\eta\eta} + F(\xi, \eta) = 0$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{\xi\eta} = 0$$

i.e.,

$$w(\xi, \eta) = f(\xi) + g(\eta)$$

If we substitute derivatives back into general equation

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} &= a[w_{\xi\xi} \xi_x^2 + w_{\xi\eta} \xi_x \eta_x + w_\xi \xi_{xx} + w_{\eta\xi} \eta_x \xi_x + w_{\eta\eta} \eta_x^2 + w_\eta \eta_{xx}] + \\ &\quad + 2b[w_{\xi\xi} \xi_x \xi_y + w_{\xi\eta} \xi_y \eta_x + w_{\eta\xi} \xi_x \eta_y + w_{\eta\eta} \eta_x \eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}] + \\ &\quad + c[w_{\xi\xi} \xi_y^2 + w_{\xi\eta} \xi_y \eta_y + w_\xi \xi_{yy} + w_{\eta\xi} \eta_y \xi_y + w_{\eta\eta} \eta_y^2 + w_\eta \eta_{yy}] = \\ &= \left( a\xi_x^2 + 3b\xi_x \xi_y + c\xi_y^2 \right) w_{\xi\xi} + 2 \left( a\xi_x \eta_x + c\eta_y \xi_y + b(\xi_x \eta_y + \xi_y \eta_x) \right) w_{\xi\eta} + \left( a\eta_x^2 + 3b\eta_x \eta_y + c\eta_y^2 \right) w_{\eta\eta} + \dots \end{aligned}$$

We can rewrite it in matrix form as

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \cdot \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}^2$$

Since the determinant is exactly  $ac - b^2$ , under the variable substitution the sign of  $b^2 - ac$  is conserved.

**Canonical form** The form  $w_{\xi\eta} + \ell_1[w] = G(\xi, \eta)$ , where  $\ell_1$  is first-order differential operator is called canonical form of hyperbolic equation.

**Theorem** Each hyperbolic equation can be written in canonical form

**Proof** We want to show that

$$\begin{cases} A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

i.e., that equation  $a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0$  has two independent solutions.

Dividing by  $\psi_y^2$ :

$$a\left(\frac{\psi_x}{\psi_y}\right)^2 + 2b\frac{\psi_x}{\psi_y} + c = 0$$

This is algebraic equation, with solutions

$$\frac{\psi_x}{\psi_y} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \lambda_{\pm}$$

We acquired a pair of equations

$$\psi_x - \lambda_{\pm}\psi_y = 0$$

And those are two independent solutions which result in  $A = 0$  and  $C = 0$ .

### Wave equation canonical form

$$u_{tt} - c^2 u_{xx} = 0$$

The canonical change of coordinates is

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u_t = -cw_{\xi} + cw_{\eta}$$

$$u_x = w_{\xi} + w_{\eta}$$

$$u_{tt} = c^2 w_{\xi\xi} - 2c^2 w_{\xi\eta} + c^2 w_{\eta\eta}$$

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

Then

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$$

The solution of canonical equation  $w_{\xi\eta} = 0$  is  $w(\xi, \eta) = F(\xi) + G(\eta)$ , thus solution of wave equation:

$$u(x, t) = F(x - ct) + G(x + ct)$$

An example for physical object fulfilling wave equation is infinite string. To find a solution we need initial conditions, for example, velocity and location at time  $t = 0$ :

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

, where  $f \in \mathcal{C}^2$ ,  $g \in \mathcal{C}^1$ .

**Theorem** Exists unique solution of wave equation with those initial conditions.

**Proof** Substituting initial conditions into general solutions:

$$\begin{cases} u(x, 0) = F(x) + G(x) = f(x) \\ u_t(x, 0) = c[F'(x) - G'(x)] = g(x) \end{cases}$$

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{g(x)}{c} \end{cases} \Rightarrow \begin{cases} F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c} \int_0^x g(s) ds + D_1 \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c} \int_0^x g(s) ds + D_2 \end{cases}$$

Now, since  $F(x) + G(x) = f(x)$ , thus  $D_1 + D_2 = 0$ .

Substituting into solution, we acquire what is called d'Alembert's formula:

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$$

From construction, the solution is unique.

### Example

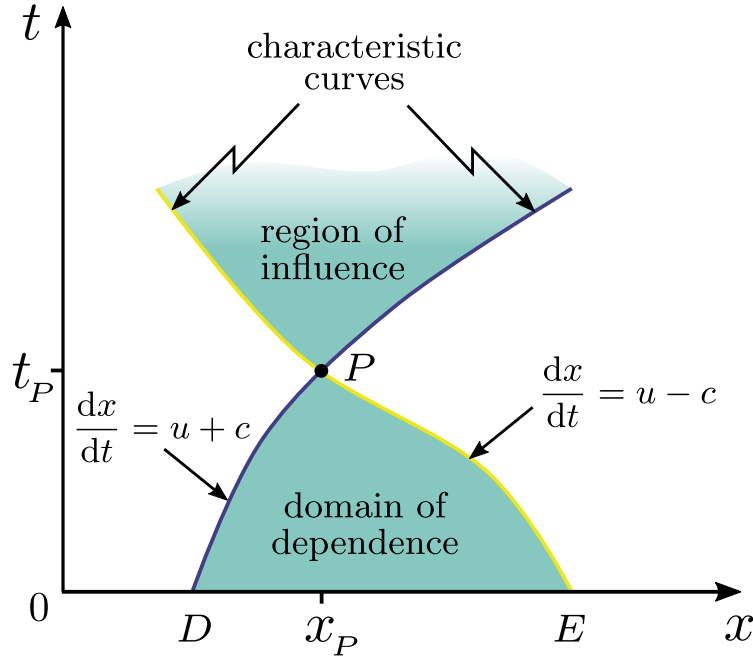
$$\begin{cases} g(x) = 0 \\ f(x) = e^{-x^2} \end{cases}$$

$$u(x, t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x-ct)^2}$$

**Standing wave** To get standing wave we want  $G = 0$ , i.e.,

$$\begin{cases} f(x) = F(x) \\ g(x) = cF'(x) \end{cases} \Rightarrow g(x) = cf'(x)$$

**Domain of dependence and region of influence** Domain of dependence of  $u$  in point  $(x_0, t_0)$  is a characteristic triangle with vertices  $(x_0 - ct_0, 0)$ ,  $(x_0 + ct_0, 0)$ ,  $(x_0, t_0)$ . Any point outside of triangle doesn't affect the value of  $u$  in point. Region of influence of point  $x_0$  is cone bounded by condition  $x_0 - ct < x < x_0 + ct$ .



### Weak solution

$$\begin{cases} g(x) = 0 \\ f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}$$

The weak solution

$$u(x, t) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct)$$

is not differentiable, but it solves the equation in some sense.

## 3.1 Generalization of d'Alembert's formula for non-homogeneous equations

Consider non-homogeneous equation

$$u_{tt} - c^2 u_{xx} = \varphi(x, t)$$

Remember Green's theorem, for differentiable  $P$  and  $Q$  defined in  $\Omega$ :

$$\iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t} \right) dx dt = \oint_{\partial\Omega} P(x, t) dx + Q(x, t) dt$$

Lets define  $Q = c^2 u_x$  and  $P = u_t$ , and choose  $\Omega(x_0, t_0)$  to be characteristic triangle.

$$\iint_{\Omega(x_0, t_0)} \varphi(x, t) dx dt = \iint_{\Omega(x_0, t_0)} u_{tt} - c^2 u_{xx} dx dt = \oint P(x, t) dx - Q(x, t) dt = - \left[ \oint_{\partial\Omega(x_0, t_0)} u_t dx + c^2 u_x dt \right]$$

Lets divide the curve integral into three integrals along each of lines. For first line  $dt = 0$ , for second  $dx + c dt = 0$  and for third  $dx - c dt = 0$ .

$$\begin{aligned} & \oint_{\partial\Omega(x_0, t_0)} u_t dx + c^2 u_x dt = \\ &= \int_{x_0 - ct_0}^{x_0 + ct_0} \underbrace{u_t}_{g(x) \text{ in } t=0} dx - \int_{(x_0 + ct_0, 0)}^{(x_0, t_0)} cu_t dt + u_x dx + \int_{(x_0, t_0)}^{(x_0 - ct_0, 0)} cu_t dt + u_x dx = \\ &= \int_{x_0 - ct_0}^{x_0 + ct_0} g(s) ds - c \int_{(x_0 + ct_0, 0)}^{(x_0, t_0)} du + c \int_{(x_0, t_0)}^{(x_0 - ct_0, 0)} du \end{aligned}$$

Since

$$\begin{aligned} \int_{(x_0 + ct_0, 0)}^{(x_0, t_0)} du &= u(x_0, t_0) - f(x_0 + ct_0) \\ \int_{(x_0, t_0)}^{(x_0 - ct_0, 0)} du &= f(x_0 - ct_0) - u(x_0, t_0) \end{aligned}$$

we get

$$\iint_{\Omega(x_0, t_0)} \varphi(x, t) dx dt = - \int_{x_0 - ct_0}^{x_0 + ct_0} g(s) ds + 2cu(x_0, t_0) - cf(x_0 + ct_0) - cf(x_0 - ct_0)$$

from which we get the solution

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds + \frac{1}{2c} \iint_{\Omega(x_0, t_0)} \varphi(\xi, \eta) d\xi d\eta$$

We've got a cadidate for the solution. Let's check that  $u$  is actually solving PDE. Define  $v$ ,  $w$ , such that  $w$  is a solution of homogeneous PDE and  $v = u - w$ , i.e.,

$$v(x, t) = \frac{1}{2c} \iint_{\Omega(x_0, t_0)} \varphi(\xi, \eta) d\xi d\eta$$

Let's show that  $v$  solves PDE. Rewrite  $v$  as double integral:

$$v(x, t) = \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi, \tau) d\xi d\tau$$

Define

$$H(x, t, \tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi, \tau) d\xi$$

and then

$$v(x, t) = \frac{1}{2c} \int_0^t H(x, t, \tau) d\tau$$

$$\frac{\partial v}{\partial t} = \frac{1}{2c} \underbrace{H(x, t, t)}_0 + \frac{1}{2c} \int_0^t \frac{\partial H}{\partial t} d\tau$$

$$\frac{\partial H}{\partial t} = c[\varphi(x + c(t - \tau), \tau) + \varphi(x - c(t - \tau), \tau)]$$

$$\frac{\partial^2 H}{\partial t^2} = c^2[\varphi_x(x + c(t - \tau), \tau) - \varphi_x(x - c(t - \tau), \tau)]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2c} \int_0^t \frac{\partial^2 H}{\partial t^2} d\tau = \varphi(x, t) + \frac{c}{2} \int_0^t \varphi_x(x + c(t - \tau), \tau) - \varphi_x(x - c(t - \tau), \tau) d\tau$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= \frac{1}{2c} \int_0^t \frac{\partial H}{\partial x} d\tau \\ \frac{\partial H}{\partial x} &= \varphi(x + c(t - \tau), \tau) - \varphi(x - c(t - \tau), \tau) \\ \frac{\partial^2 H}{\partial x^2} &= \varphi_x(x + c(t - \tau), \tau) - \varphi_x(x - c(t - \tau), \tau) \\ \frac{\partial^2 v}{\partial x^2} &= \frac{1}{2c} \int_0^t \varphi_x(x + c(t - \tau), \tau) - \varphi_x(x - c(t - \tau), \tau) d\tau\end{aligned}$$

Thus we got

$$\frac{\partial^2 v}{\partial t^2} - c^2 \frac{\partial^2 v}{\partial x^2} = \varphi(x, t)$$

Suppose we have two solutions  $u_1$  and  $u_2$  then  $u = u_1 - u_2$  is solution of homogeneous equation with 0 initial conditions, and thus  $u = 0$ . That means the solution is unique.

The presented initial condition problem has 3 properties:

1. Solution exist
2. It's unique
3. It's stable

**Stability of wave equation** For all  $\tau > 0$ ,  $\epsilon > 0$ , exists  $\delta > 0$  such that if

$$\begin{cases} |f(x) - \tilde{f}(x)| < \delta \\ |g(x) - \tilde{g}(x)| < \delta \\ |\varphi(x) - \tilde{\varphi}(x)| < \delta \end{cases}$$

For all  $-\infty < x < \infty$  and  $0 \leq t \leq \tau$  and if  $u, \tilde{u}$  are solutions of corresponding wave equations, then

$$|u(x, t) - \tilde{u}(x, t)| < \epsilon$$

**Proof** From the general solution:

$$\begin{aligned}u(x, t) - \tilde{u}(x, t) &= \\ &= \left| \frac{f(x + ct) + f(x - ct)}{2} - \frac{\tilde{f}(x + ct) + \tilde{f}(x - ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) - \tilde{g}(s) ds + \frac{1}{2c} \iint_{\Omega(x_0, t_0)} \varphi(\xi, \eta) - \tilde{\varphi}(\xi, \eta) d\xi d\eta \right| \leq \\ &\leq \left| \frac{f(x + ct) - \tilde{f}(x + ct)}{2} \right| + \left| \frac{f(x - ct) - \tilde{f}(x - ct)}{2} \right| + \frac{1}{2c} \int_{x-ct}^{x+ct} |g(s) - \tilde{g}(s)| ds + \frac{1}{2c} \iint_{\Omega(x_0, t_0)} |\varphi(\xi, \eta) - \tilde{\varphi}(\xi, \eta)| d\xi d\eta \leq \\ &\leq \frac{\delta}{2} + \frac{\delta}{2} \frac{1}{2c} \cdot 2c \cdot \delta + \frac{1}{2c} \frac{ct}{2} = 2\delta + \frac{\delta t}{4} \leq 2\delta + \frac{\delta \tau}{4} \leq \epsilon\end{aligned}$$

Thus we choose  $\delta < \frac{\epsilon}{2 + \frac{\tau}{4}}$ .

## 3.2 Wave equation with bound conditions

**Half-infinite string** Suppose string is fixed in one of its ends, at  $x = 0$ :  $u(0, t) = 0$ . This is called Dirichlet boundary condition. We want to solve the PDE for  $x > 0$ .

**Property of wave equation** If  $u(x, t)$  is solution, then  $u(-x, t)$  is also solution:

$$u(x, t) = F(x + ct) + G(x - ct)$$

$$u(-x, t) = F(-x + ct) + G(-x - ct) = \bar{F}(x + ct) + \bar{G}(x - ct)$$

Where  $\bar{F}(s) = G(-s)$  and  $\bar{G}(s) = F(-s)$ .

Lets extend  $f$  and  $g$  on the whole plane in odd way:

$$\bar{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(x) & x < 0 \end{cases}$$

and same for  $g$ .

Note that initial conditions have to be consistent, i.e.,  $f(0) = 0$ ,  $g(0) = 0$ , else the solution is discontinuous in 0. Lets use D'Lambert solution:

$$\bar{u}(x, t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \bar{g}(s) ds$$

Then the solution of half-infinite string is

$$u(x, t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds & x > ct \\ \frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds & x < ct \end{cases}$$

**Neumann boundary condition** In this case, instead of giving boundary condition on  $u$ , we give boundary condition of  $u_x$ :  $u_x(0, t) = 0$ . Physical meaning is that there is no force in this point. In this case we will extend function in even way (derivative of even function in 0 is 0). Then we get

$$u(x, t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds & x > ct \\ \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{c} \int_0^{ct-x} g(s) ds + \frac{1}{2c} \int_{ct-x}^{x+ct} g(s) ds & x < ct \end{cases}$$

**Uniqueness** Suppose we have two solutions, by subtracting them, we get

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x, 0) = u_x(x, 0) = 0 \\ u(0, t) = 0 \end{cases}$$

We acquire  $u(x, t) = 0$ . Since  $u(x, t)$  is of form  $F(x+ct) + G(x-ct)$ , we get that twosolutions are identical.

**Wave equation with finite string** Suppose we have string from  $a$  to  $b$ :

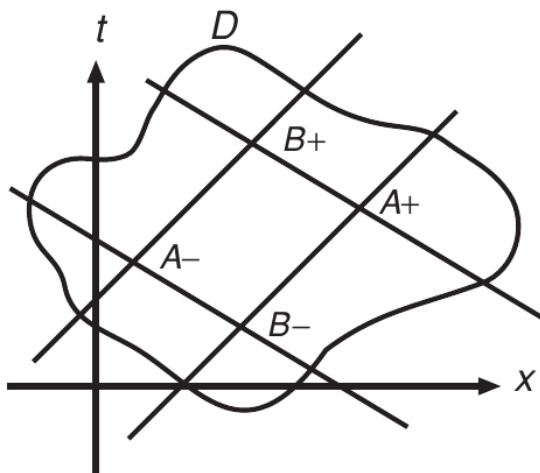
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & a \leq x \leq b \\ u(x, 0) = f(x) \\ u_t(a, t) = h(t) \\ u_t(b, t) = q(t) \end{cases}$$

Here the consistency conditions are

$$\begin{cases} h(0) = f(a) & q(0) = f(b) \\ h'(0) = g(a) & q'(0) = g(b) \end{cases}$$

Here we could have conditions derivatives instead of values as well.

**Parallelogram identity**



$u \in \mathcal{C}^2$  is the solution of wave equation iff for any parallelogram with sides parallel to characteristic lines with vertices  $A_-$ ,  $A_+$ ,  $B_-$ ,  $B_+$

$$u(A_-) + u(A_+) = u(B_-) + u(B_+)$$

**Proof** One direction is simple, since value of solution is constant along characteristic curves.  
For second direction lets switch to canonical coordinates

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

So

$$w(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

$w$  is solution of wave equation iff  $w_{\xi\eta} = 0$ .

Note that parallelogram turned into rectangular in new coordinates, i.e.,

$$w(\xi_0, \eta_1) + w(\xi_1, \eta_0) = w(\xi_1, \eta_1) + w(\xi_0, \eta_0)$$

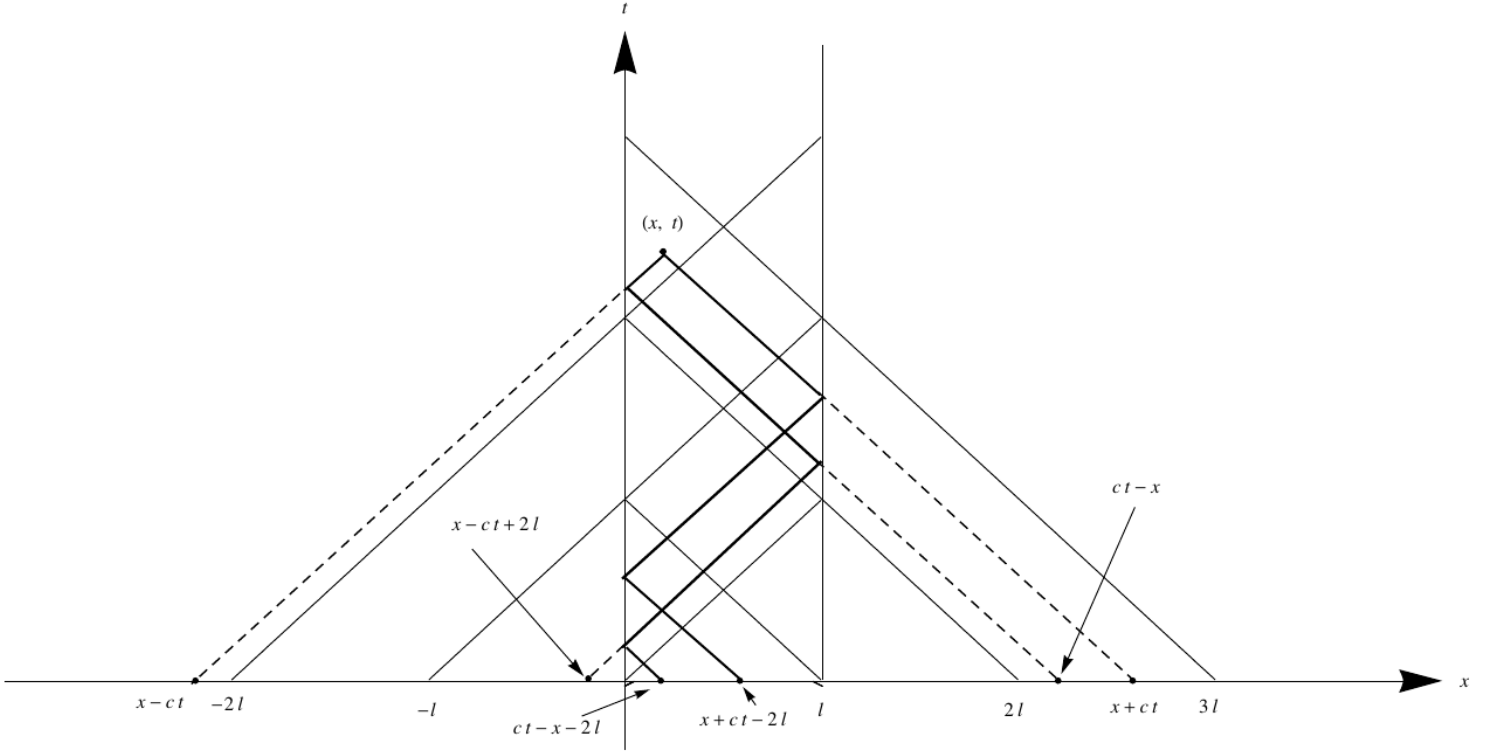
Dividing by  $(\xi_1 - \xi_0)(\eta_1 - \eta_0)$ :

$$\frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} - \frac{w(\xi_1, \eta_1) + w(\xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = 0$$

Taking limit:

$$\lim_{\xi_1 \rightarrow \xi_0} \lim_{\eta_1 \rightarrow \eta_0} \frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0) - w(\xi_1, \eta_1) - w(\xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = w_{\xi\eta} = 0$$

In this way we can solve wave equation on finite range:



Using this method we can get solution for homogeneous wave equation for finite string.

### Non-homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = \varphi(x, t)$$

In this case parallelogram identity doesn't work.

Lets extend  $\varphi$  to the half plane  $x > 0$  to some function  $\tilde{\varphi} \in \mathcal{C}^1$ .

Lets solve non-homogeneous equation with 0 initial conditions:  $w = \frac{1}{2c} \int_{\Delta} \tilde{\varphi}$ .



with D'Lambert formula. Lets solve homogeneous equation in the interval:

$$\begin{cases} v(a, t) = h(t) + w(a, t) \\ v(b, t) = q(t) + w(b, t) \\ v(x, 0) = f(x) \\ v_t(x, 0) = g(x) \end{cases}$$

Then the solution is

$$u(x, t) = w(x, t) + v(x, t)$$

Checking the solution:

$$u_{tt} - c^2 u_{xx} = w_{tt} - c^2 w_{xx} + v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x, t)$$

which is  $\varphi(x, t)$  in our interval.

**Energy method** Define

$$E(t) = \int_a^b [u_t^2(x, t) + c^2 u_x^2(x, t)] dx$$

If  $u \in \mathcal{C}^2$  we can differentiate it:

$$\frac{dE}{dt} = \int_a^b [2u_t u_{tt} + 2c^2 u_x u_{xt}] dx = 2c^2 \int_a^b [u_t u_{xx} + u_x u_{xt}] dx = 2c^2 \int_a^b (u_x u_t)_x dx = 2c^2 [u_x(b, t)u_t(b, t) - u_x(a, t)u_t(a, t)]$$

If any combination of Dirichlet and Neumann conditions is fulfilled, the integral is 0, i.e., energy is conserved. Thus

$$E(t) = E(0) = \int_a^b g^2(x) + c^2 (f'(x))^2 dx$$

From that we can conclude the solution is unique. As usual, suppose there are two solutions,  $u$  and  $v$ . Subtracting we get a solution for homogeneous equation with homogeneous initial conditions  $w = u - v$ . Then  $E_w(t) = E_w(0) = 0$ , thus  $w_x = w_t = 0$  and  $w(x, t) = 0$ .

Also, from energy difference, we can conclude the solutions are stable.

### 3.3 Variable separation

Lets guess solution of form

$$U(x, t) = A(x)B(t)$$

Substituting into wave equation:

$$u_{tt} - c^2 u_{xx} = A(x)B''(t) - c^2 A''(x)B(t) = 0$$

Dividing by  $A(x)B(t)$  (assume they are not zero):

$$\frac{B''(t)}{B(t)} = c^2 \frac{A''(x)}{A(x)} = \mu$$

That means

$$\begin{cases} A'' = \frac{\mu}{c^2} A = -\lambda A \\ B'' = \mu B \end{cases}$$

Back to initial conditions  $u(0, t) = u(1, t) = 0$ , that means  $A(0) = A(1) = 0$ . The question is when

$$A'' + \lambda A = 0$$

If  $\lambda < 0$ , the solution is

$$A = \alpha e^{-\sqrt{-\lambda}x} + \beta e^{\sqrt{-\lambda}x}$$

Substituting initial conditions we get

$$\begin{cases} \alpha + \beta = 0 \\ \alpha e^{-\sqrt{-\lambda}} + \beta e^{\sqrt{-\lambda}} = 0 \end{cases}$$

Since  $\lambda \neq 0$ , we conclude  $\alpha = \beta = 0$  which is trivial solution. If  $\lambda = 0$  we get  $A = \alpha x + \beta$ , which is also trivially  $A = 0$ . If  $\lambda > 0$ ,

$$A = \alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)$$

Since  $A(0) = 0$ ,  $\beta = 0$ . Since  $A(1) = 0$ ,  $\sqrt{\lambda} = k\pi$  for some  $k \in \mathbb{N}$ , i.e.,  $\lambda_k = k^2\pi^2$ . The solution is

$$A_k = \sin(k\pi x)$$

Back to  $B$ :

$$\frac{B''}{B} = -c^2 k^2 \pi^2$$

i.e.,

$$B_k(t) = a_k \sin(ck\pi t) + b_k \cos(ck\pi t)$$

Thus the solution of wave equation

$$u_k(x, t) = a_k \sin(k\pi x) \sin(ck\pi t) + b_k \sin(k\pi x) \cos(ck\pi t)$$

(note, that by using trigonometric identities we can get it to canonical form).

Define

$$u(x, t) \sim \sum_{k=1}^{\infty} a_k \sin(k\pi x) \sin(ck\pi t) + b_k \sin(k\pi x) \cos(ck\pi t)$$

Substituting  $t = 0$ :

$$u(x, 0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f(x)$$

$$u_t(x, 0) = c\pi \sum_{k=1}^{\infty} k a_k \sin(k\pi x) = g(x)$$

If we'll find  $a_k$ ,  $b_k$  fulfilling those conditions, then we have a "solution". How we find them? Look at following integral:

$$\int_0^1 f(x) \sin(n\pi x) dx = \int_0^1 \left[ \sum_{k=1}^{\infty} b_k \sin(k\pi x) \right] \sin(n\pi x) dx = \sum_{k=1}^{\infty} b_k \underbrace{\int_0^1 \sin(k\pi x) \sin(n\pi x) dx}_{\frac{\delta_{nk}}{2}} = \frac{b_n}{2}$$

Thus,

$$b_n = 2 \int_0^1 f(x) \sin(n\pi x) dx$$

Exactly in the same way we can get

$$a_n = \frac{2}{c\pi n} \int_0^1 g(x) \sin(n\pi x) dx$$

**Convergence** If  $\sum |ka_k| < \infty$  and  $\sum |b_k| < \infty$ , our series converge uniformly. If also  $\sum |k^2 a_k| < \infty$  and  $\sum |kb_k| < \infty$ , then  $u(x, t) \in \mathcal{C}^1$ . Analogously, if  $\sum |k^3 a_k| < \infty$  and  $\sum |k^2 b_k| < \infty$ ,  $u \in \mathcal{C}^2$ .

Suppose  $\max_{(0,1)} |f| < M_0$ , then

$$|b_n| \leq 2 \int_0^1 |f(x) \sin(n\pi x)| dx \leq 2M_0$$

. Suppose also that  $f \in \mathcal{C}^1$  and  $\max_{(0,1)} |f'| < M_1$  then

$$b_n = -\frac{2}{n\pi} \int_0^1 f(x) (\cos(n\pi x))' dx$$

Integrating by parts and using the fact  $f(0) = f(1) = 0$ :

$$b_n = \frac{2}{n\pi} \int_0^1 f'(x) \cos(n\pi x) dx \leq \frac{2}{n\pi} M_1$$

To show that  $b_n < \frac{B_2}{n^2}$ , we need  $f \in \mathcal{C}^2$ ,  $\max_{(0,1)} |f''| < M_2$  and  $f'(0) + f'(1) = 0$ .

In general, if  $f \in \mathcal{C}^l$  and  $\sum f^{(l-1)}(0) + f^{(l-1)}(1) = 0$ , we can bound  $|b_n| < \frac{B_l}{n^l}$ .

## Generalization

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x, t) \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \\ u(0, t) = h(t) \\ u(1, t) = q(t) \end{cases}$$

Define  $w(x, t)$  such that  $w(0, t) = h(t)$  and  $w(1, t) = q(t)$  and  $v = u - w$ . Then

$$v_{tt} - c^2 v_{xx} = u_{tt} - c^2 u_{xx} - w_{tt} + c^2 w_{xx} = \varphi(x, t) - w_{tt} + c^2 w_{xx} = \tilde{\varphi}(x, t)$$

Thus we can assume bound conditions are 0, as soon as we can solve non-homogeneous equation.

$$v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x, t)$$

Guess solution

$$v(x, t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\pi x)$$

Substituting:

$$\sum_{k=1}^{\infty} (q_k''(t) + k^2 c^2 \pi^2 q_k(t)) \sin(k\pi x) = \tilde{\varphi}(x, t)$$

Suppose we can expand

$$\tilde{\varphi}(x, t) = \sum_{k=1}^{\infty} p_k(t) \sin(k\pi x)$$

.

$$\int_0^1 \tilde{\varphi} \sin(n\pi x) dx = \sum_{k=1}^{\infty} p_k(t) \int_0^1 \sin(k\pi x) \sin(n\pi x) dx = \frac{p_n(t)}{2}$$

Thus

$$p_n(t) = 2 \int_0^1 \tilde{\varphi} \sin(n\pi x) dx$$

By coefficient comparison:

$$q_k''(t) + k^2 c^2 \pi^2 q_k(t) = 2 \int_0^1 \tilde{\varphi} \sin(k\pi x) dx$$

Since we know that

$$\begin{cases} v(x, 0) = \sum_{k=1}^{\infty} q_k(0) \sin(k\pi x) = f(x) \\ v_t(x, 0) = \sum_{k=1}^{\infty} q_k'(0) \sin(k\pi x) = g(x) \end{cases}$$

i.e.,

$$q_k(0) = 2 \int_0^1 f(x) \sin(k\pi x) dx$$

$$q_k'(0) = 2 \int_0^1 g(x) \sin(k\pi x) dx$$

Meaning we can solve ODE, and get solution for wave equation.

**Neumann bound conditions** Once again guessing the solution

$$\begin{cases} u(x, t) = A(x)B(t) \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \\ A'(0) = A'(1) = 0 \end{cases}$$

We get once again

$$A(x) = a \sin(\sqrt{\lambda}x) + b \cos(\sqrt{\lambda}x)$$

$$A'(x) = \sqrt{\lambda}a \cos(\sqrt{\lambda}x) - \sqrt{\lambda}b \sin(\sqrt{\lambda}x)$$

Substituting initial conditions:

$$A'(0) = \sqrt{\lambda}a \Rightarrow a = 0$$

$$A'(1) = \sqrt{\lambda}b \sin(\sqrt{\lambda}x)$$

Thus we get the same  $\lambda_k = k\pi$ , however the series contains cosines instead of sines:

$$A_k(x) = \cos(k\pi x)$$

and we can solve in a similar way.

**Operator of hyperbolic equation** We can define linear operator

$$L(u) = \left( a \frac{\partial^2}{\partial x^2} + 2b \frac{\partial^2}{\partial x \partial y} + c \frac{\partial^2}{\partial y^2} + d \frac{\partial}{\partial x} + f \frac{\partial}{\partial y} + g \right)$$

Then the equation is

$$L(u) = h$$

We can turn it into canonical form:

$$L'(u) = \left( \frac{\partial^2}{\partial \xi \partial \eta} + d' \frac{\partial}{\partial x} + f' \frac{\partial}{\partial y} + g' \right)$$

**Cauchy problem for hyperbolic equation** Given a curve in space  $\vec{r}(s) = (\bar{x}(s), \bar{y}(s))$ , we define initial conditions

$$\begin{cases} u(x(s), y(s)) = h(s) \\ u_x(x(s), y(s)) = \varphi(s) \\ u_y(x(s), y(s)) = \psi(s) \end{cases}$$

However, since we need two conditions, there is consistency requirement on those functions:

$$\frac{dh}{ds} = u_x(x(s), y(s)) \frac{\partial \bar{x}}{\partial s} + u_y(x(s), y(s)) \frac{\partial \bar{y}}{\partial s} = \varphi(s) \frac{\partial \bar{x}}{\partial s} + \psi(s) \frac{\partial \bar{y}}{\partial s}$$

Suppose we have equation of form

$$\begin{cases} au_{xx} + 2bu_{xy} + cu_{yy} = d \\ u_{xx} \frac{d\bar{x}}{ds} + u_{xy} \frac{d\bar{y}}{ds} = \frac{d\varphi}{ds} \\ u_{xy} \frac{d\bar{x}}{ds} + u_{yy} \frac{d\bar{y}}{ds} = \frac{d\psi}{ds} \end{cases}$$

To have an opportunity to evaluate second derivatives, we need to find solution of this linear system, i.e., we need that

$$\begin{vmatrix} a & 2b & c \\ \frac{d\bar{x}}{ds} & \frac{d\bar{y}}{ds} & 0 \\ 0 & \frac{d\bar{x}}{ds} & \frac{d\bar{y}}{ds} \end{vmatrix} \neq 0$$

or

$$a \left( \frac{d\bar{y}}{ds} \right)^2 - 2b \frac{d\bar{x}}{ds} \frac{d\bar{y}}{ds} + c \left( \frac{d\bar{x}}{ds} \right)^2 \neq 0$$

meaning the direction of tangent line is not in direction of characteristic lines.

We can derive the system once again and thus find third-order derivatives, doing it up to infinity, we get all the partial derivative.

**Cauchy–Kowalevski theorem** If the coefficients and initial curve are analytic functions, then exists unique analytic solution.

## 4 Heat equation

For some positive  $k$

$$u_t - ku_{xx} = 0$$

Temperature  $u(x, t)$  fulfills heat equation.

**Dirichlet bound conditions**

$$u(a, 0) = u(b, 0) = 0$$

**Neumann bound conditions** The meaning of Neumann bound condition is that there is heat isolation in interval bounds:

$$\begin{aligned} u_x(a, t) &= u_x(b, t) = 0 \\ Q(t) &= \int_a^b u(x, t) \, dx \\ \frac{dQ}{dt} &= \int_a^b \frac{\partial Q}{\partial t} \, dx = k \int_a^b \frac{\partial^2 u}{\partial x^2} \, dx = k \left[ \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right] = 0 \end{aligned}$$

Thus  $Q(t)$  is constant.

$$Q(t) = Q(0) = \int_a^b f(x) \, dx$$

**Solution of heat equation** Suppose, for Dirichlet bound condition, that solution is series of sines.

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} a_n(t) \sin(n\pi x) \\ \frac{\partial u}{\partial t} &= \sum_{n=1}^{\infty} a'_n(t) \sin(n\pi x) \\ \frac{\partial^2 u}{\partial x^2} &= \sum_{n=1}^{\infty} -(n\pi)^2 a_n(t) \sin(n\pi x) \end{aligned}$$

Then we get

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} [a'_n(t) + k(n\pi)^2 a_n(t)] \sin(n\pi x) = 0$$

Thus, by coefficient comparison

$$\begin{aligned} a'_n(t) + k(n\pi)^2 a_n(t) &= 0 \\ a_n(t) &= a_n(0) e^{-k(n\pi)^2 t} \\ u(x, t) &= \sum_{n=1}^{\infty} a_n(0) e^{-k(n\pi)^2 t} \sin(n\pi x) \end{aligned}$$

From initial conditions we can find  $a_n(0)$ :

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} a_n(0) \sin(n\pi x)$$

Our series is infinite differentiable if  $t > 0$ .

If  $t = 0$ , we need

$$\lim_{t \rightarrow 0^-} u(x, t) = f(x)$$

For  $t < 0$ , coefficients diverge, and thus we can't find solutions for  $t < 0$ . Physically we can see it from entropy grows.

**Example**

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x, 0) = 1 \\ u(0, t) = u(1, t) = 0 \quad t > 0 \end{cases}$$

We acquire

$$\begin{aligned} a_n &= \frac{1}{2} \int_0^1 1 \cdot \sin(n\pi x) \, dx \\ |a_n| &\leq \int_0^1 dx = \frac{1}{2} \end{aligned}$$

Thus

$$|a_n| \leq \frac{1}{2} e^{-k(n\pi)^2 t}$$

Thus the series absolutely converges for  $t > 0$ .

In limit  $t \rightarrow \infty$ ,  $a_n \rightarrow 0$ , and thus  $u(x, t) \rightarrow 0$ .

**Stability** For each  $\epsilon > 0$  exists  $\delta > 0$  such that if

$$\max_{a \leq x \leq b} |u(x, 0)| < \delta$$

then

$$\max_{a \leq x \leq b} |u(x, t)| < \epsilon$$

**Proof** We'll proof the weaker version of the theorem, with condition that  $\sum |a_n| < \delta$ . Then coefficients of  $u(x, t)$  are bounded by

$$a_n(t) \leq e^{-k(n\pi)^2 t} \frac{\delta}{2}$$

i.e.

$$|u(x, t)| \leq \sum_{n=1}^{\infty} |a_n| e^{-k(n\pi)^2 t} < \sum |a_n| < \delta$$

## 5 Potential equation

$$u_{xx} + u_{yy} = 0$$

Bound conditions are

$$\begin{cases} u(x, 0) = f(x) \\ u_x(x, 0) = g(x) \end{cases}$$

By variable separation we get

$$u_n = A_n(x) B_n(y)$$

we know that

$$A_n(x) = \sin(n\pi x)$$

Since

$$(n\pi)^2 = \frac{A_n''}{A_n} = -\frac{B_n''}{B_n}$$

$$B_n(y) = \alpha_n \sinh(n\pi y) + \beta_n \cosh(n\pi y)$$

**Stability** Is potential equation stable? Suppose  $\max |f(x)| < \delta$  and  $\max |g(x)| < \delta$ . No. For example

$$u(x, y) = \frac{1}{n^3} e^{(n\pi)^2 y} \sin(n\pi x)$$

Then

$$u(x, 0) = \frac{\sin(n\pi x)}{n^2}$$

$$u_y(x, 0) = \frac{\sin(n\pi x)}{n}$$

This doesn't fulfill stability condition

$$|u_n(x, y)| < \epsilon$$

for large enough  $n$ .

### Laplace equation

$$\nabla^2 u = 0$$

If  $u$  fulfills  $\nabla^2 u = 0$ , it is called harmonic function.

### Poisson equation

$$\nabla^2 u = f$$

Where  $f$  describes mass/charge distribution in space.

**Elliptic PDEs** For elliptic equation  $b^2 - 4ac < 0$ . In this case, we can get the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + L_1(\xi, \eta) = f$$

where  $L_1$  is first-order differential operator.

## 5.1 Laplace equation

$$\nabla \cdot \nabla u = \nabla^2 u = 0$$

**Gauss law** For vector field  $\vec{\mathbf{w}} \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega})$

$$\iint_{\Omega} \nabla \cdot \vec{\mathbf{w}} \, d^3x = \oint_{\partial\Omega} \vec{\mathbf{w}} \cdot \hat{\mathbf{n}} \, ds$$

Thus

$$\iiint_{\Omega} \nabla^2 u \, d^3x = \oint_{\partial\Omega} \frac{\partial u}{\partial n} \, ds$$

i.e., if function is harmonic,

$$\oint_{\partial\Omega} \frac{\partial u}{\partial n} \, ds = 0$$

**Conclusion** The equation  $\nabla^2 u = f$  in  $\Omega$  for  $u$  fulfilling  $\frac{\partial u}{\partial n} = g$  in each point of  $\partial\Omega$  there is no solution if

$$\iiint_{\Omega} f \, d^3x \neq \oint_{\partial\Omega} g \, ds$$

The necessary condition for solution of Neumann problem

$$\iiint_{\Omega} f \, d^3x = \oint_{\partial\Omega} g \, ds$$

For Dirichlet problem, there is no such constraint.

**Examples of harmonic functions** In  $n = 1$ , linear functions are harmonic.

In  $n = 2$ , any real or imaginary part of analytic function is harmonic, e.g.,  $e^x \sin y$ .

**The mean value property of harmonic function** If  $u$  is harmonic in  $\Omega$  which contains  $B_R(x) = \{y | |x - y| < R\}$  then

$$u(x) = \frac{1}{|\partial B_R(x)|} \oint_{\partial\Omega} u \, ds$$

and

$$u(x) = \frac{1}{|B_R(x)|} \int_{\Omega} u(y) \, d^n y$$

**Proof** Suppose  $x = 0$ . Rewrite  $y \in B_R(x)$  as  $y = \rho\alpha$  for  $\alpha = \frac{y}{\|y\|}$  and  $\rho = \|y\|$ .

For each  $\rho \in [0, R]$ :

$$\int_{B_\rho(0)} \frac{\partial u}{\partial n} \, ds_y = \int_{B_\rho(0)} \frac{\partial u}{\partial n} \, ds_y = \iint \nabla^2 u \, d^n x = 0$$

With variable substitution  $ds_y = \rho^{n-1} ds_\alpha$ :

$$\int_{B_\rho(0)} \frac{\partial u}{\partial n} \, ds_y = \int_{B_\rho(0)} \frac{\partial u}{\partial n} \rho^{n-1} \, ds_\alpha = \rho^{n-1} \int_{B_\rho(0)} \frac{\partial u}{\partial \rho} \, ds_\alpha = \rho^{n-1} \frac{\partial}{\partial \rho} \int_{B_\rho(0)} u(\rho\alpha) \, ds_\alpha$$

Thus

$$\frac{\partial}{\partial \rho} \int_{B_\rho(0)} u(\rho \alpha) \, ds_\alpha = 0$$

meaning

$$H(\rho) = \int_{B_\rho(0)} u(\rho \alpha) \, ds_\alpha = \text{const}$$

Denote volume of unit ball as  $\omega_n$ , then  $|B_n| = \omega_n R^n$  and  $|\partial B_n| = \omega_{n-1} R^{n-1}$

$$H(0) = \omega_n \cdot u(0)$$

And since  $H(1) = H(0)$ :

$$u(0) = \frac{1}{n\omega_n R^{n-1}} \int_{\partial B_R(0)} ds$$

as required.

$$\begin{aligned} \omega_n \rho^{n-1} u(0) &= \int_{\partial B_\rho(0)} u(y) \, ds_y \\ \int_0^R n\omega_n \rho^{n-1} u(0) \, d\rho &= \int_0^R d\rho \int_{\partial B_\rho(0)} ds_y u(y) \\ \omega_n R^n u(0) &= \iint_{B_r(0)} u(y) \, d^n y \end{aligned}$$

i.e.,

$$u(0) = \frac{1}{|B_r(0)|} \iint_{B_r(0)} u(y) \, d^n y$$

**Strong maximum principle** If  $u$  is harmonic and it acquires maximum or minimum, then it is constant.

**Subharmonic and superharmonic functions** Subharmonic function is function for which  $\nabla^2 u \leq 0$  and superharmonic is one for which  $\nabla^2 u \geq 0$ .

In this case, strong maximum principle applies only in one direction (maximum for subharmonic, minimum for superharmonic). For mean value theorem we get inequality instead of equality.

**Proof** let  $u$  subharmonic, and  $m = u(x) = \max_{\Omega} u$ . The set  $W = \{y : u(y) = m\}$  is closed relatively to  $\Omega$ . Let  $z \in W$ ,  $B_R(z) \in \Omega$ .

$$m = u(z) \leq \frac{1}{|B_R(z)|} \int_{B_R(z)} u(y) \, d^n y = m$$

Thus for all  $z \in W$

$$u(z) = \frac{1}{|B_R(z)|} \int_{B_R(z)} u(y) \, d^n y$$

That means

$$\int_{B_r(z)} u(x) - m \, d^n x = 0$$

Thus  $u(x) = m$  for all  $x \in B_R(z)$ , which turns  $W$  is open set. Thus  $W$  is both open and closed, i.e.  $W = \Omega$ .

**Conclusion** Poisson equation  $\nabla^2 u = f$  in  $\Omega$  with bound condition  $u_{\partial\Omega} = y$  has not more than one solution.

**Proof** Suppose there are two solutions,  $u_1 = u_2$ , define  $v = u_1 - u_2$  is harmonic function with bound condition  $v_{\partial\Omega} = 0$ . The function  $v$  is harmonic. If  $v \neq 0$ , it has either maximum or minimum, in contradiction with strong maximum principle.



**Weak maximum principle** For compact connected  $\Omega$  and  $u = \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$

If  $\nabla^2 u \leq 0$ ,

$$\max_{\Omega} u = \max_{\partial\Omega} u$$

If  $\nabla^2 u \geq 0$ ,

$$\min_{\Omega} u = \min_{\partial\Omega} u$$

**Harnack's inequality** If  $u$  harmonic and non-negative in interval  $\Omega$  and  $\bar{\Omega}' \subsetneq \Omega$ , then exists constant  $c(\Omega, \Omega')$  independent on  $u$  such that

$$\sup_{\Omega'} u \leq c \inf_{\Omega} u$$

i.e., for any two points  $x, y \in \Omega'$

$$u(x) \leq cu(y)$$

**Proof** Let  $\Omega = B_{4R}(y)$  and  $\Omega' = B_R(y)$ . Choose  $x_1, x_2 \in \Omega'$ .

Since  $B_R(x_1) \subset B_{4R}(y)$ , we can use the mean value property:

$$u(x_1) = \frac{1}{\omega_n R^n} \int_{B_R(x_1)} dx u \leq \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} dx u$$

Similarly, since  $B_{3R}(x_1) \subset B_{4R}(y)$ , we can use the mean value property:

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int_{B_{3R}(x_2)} dx u \geq \frac{1}{\omega_n (3R)^n} \int_{B_{2R}(y)} dx u = \frac{1}{3^n} \frac{1}{\omega_n R^n} \int_{B_{2R}(y)} dx u \geq u(x_1)$$

We got

$$3^n u(x_2) \leq u(x_1)$$

Since  $\bar{\Omega}' \subsetneq \Omega$ , there exists  $R > 0$  such that distance from any point of  $\Omega'$  to any point of  $\Omega^c$  is greater than  $4R$ .

For any pair of points  $x_1, x_2 \in \Omega$ , the path between them can be covered by  $m$  balls  $B_j$  of radius  $R$ , such that intersection of each pair of consecutive balls is non-empty.

So, let  $y_j \in B_j \cap B_{j+1}$  and  $y_1 = p, y_m = q$ , then  $u(y_j) \leq 3^n u(y_{j+1})$ , since  $y_j, y_{j+1} \in B_{j+1}$  and  $B_{4R}(y_{j+1}) \subset \Omega$ . This means that

$$u(q) \leq 3^{nm} u(p)$$

**Radial harmonic functions** Let's search for harmonic functions of form  $u(x) = f(r)$  for  $f$  defined on  $\mathbb{R}^+$ .

$$|x| = \sqrt{\sum x_i^2} = r$$

is harmonic on  $\mathbb{R}^n \setminus 0$  and radial.

$$\begin{aligned} \frac{\partial r}{\partial x_i} &= \frac{x_i}{r} \\ \frac{\partial^2 r}{\partial x_i^2} &= \frac{1}{r} - \frac{x_i^2}{r^2} \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial u}{\partial x_i} &= f'(r) \frac{\partial r}{\partial x_i} = f'(r) \frac{x_i}{r} \\ \frac{\partial^2 u}{\partial x_i^2} &= f''(r) \left( \frac{x_i}{r} \right)^2 - f'(r) \left( \frac{1}{r} - \frac{x_i^2}{r^2} \right) \\ \nabla^2 u &= f''(r) \sum \left( \frac{x_i}{r} \right)^2 - f'(r) \left( \frac{n}{r} - \frac{\sum x_i^2}{r^2} \right) = f''(r) + \frac{n-1}{n} f'(r) = 0 \end{aligned}$$

This is Euler equation with solution For  $n = 2$

$$f(r) = c_1 \ln r + c_2$$

For  $n > 2$ :

$$f(r) = \frac{c_1}{r^{n-2}} + c_2$$

**Fundamental solution** Define fundamental solution of Laplace equation:

$$\begin{cases} \Gamma(r) = \frac{1}{2\pi} \ln(r) & n = 2 \\ \Gamma(r) = \frac{1}{n(2-n)\omega_n} r^{2-n} & n > 2 \end{cases}$$

We conclude that  $\Gamma(|x-y|)$  is harmonic function in  $\mathbb{R}^n \setminus \{y\}$ .

For  $n = 2$

$$\lim_{r \rightarrow \infty} \Gamma(r) = \infty$$

and for  $n > 2$

$$\lim_{r \rightarrow \infty} \Gamma(r) = 0$$

Also, for any  $n \geq 2$

$$\lim_{r \rightarrow 0} \Gamma(r) = -\infty$$

**Homogeneity of  $\Gamma$**  For  $n > 2$

$$\begin{aligned} \frac{\partial \Gamma(x-y)}{\partial x_i} &= \frac{1}{n\omega_n} \frac{x_i - y_i}{|x-y|^n} \\ \frac{\partial^2 \Gamma}{\partial x_i \partial x_j} &= \frac{1}{n\omega_n} \left[ |x-y|^2 \delta_{ij} - n(x_i - y_i)(x_j - y_j) \right] |x-y|^{-n-2} \\ \left| \frac{\partial \Gamma(x-y)}{\partial x_i} \right| &\leq \frac{1}{n\omega_n} |x-y|^{1-n} \\ \left| \frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j} \right| &\leq \frac{1}{n\omega_n} |x-y|^{-n} \end{aligned}$$

Also

$$\begin{cases} \Gamma'(r) = \frac{1}{n\omega_n r} & n = 2 \\ \Gamma'(r) = \frac{1}{n\omega_n} r^{1-n} & n > 2 \end{cases}$$

**Green identities** If  $\Omega \subset \mathbb{R}^n$  bounded set with boundcin  $\mathcal{C}^2$  and  $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$\begin{aligned} \int_{\Omega} v \nabla^2 u + \nabla u \cdot \nabla v \, dx &= \int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds \\ \int_{\Omega} v \nabla^2 u - u \nabla^2 v \, dx &= \int_{\partial\Omega} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \, ds \end{aligned}$$

If  $\Omega \subset \mathbb{R}^n$  bounded set with bound in  $\mathcal{C}^1$  and  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$u(y) = \int_{\Omega} \Gamma(x-y) \nabla^2 u \, dx + \int_{\partial\Omega} \left[ u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] \, ds_x$$

If  $u$  harmonic

$$u(y) = \int_{\partial\Omega} \left[ u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] \, ds_x$$

**Proof**  $\Gamma(x-y)$  is harmonic in  $\Omega \setminus B_\rho(y) = \Omega_\delta$ . Choose  $v(x) = \Gamma(x-y)$ ,

$$\int_{\Omega_\delta} \Gamma \nabla^2 u \, dx = \int_{\partial\Omega} \Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \, ds_x - \int_{\partial B_\rho(y)} \Gamma \frac{\partial u}{\partial n} - u \frac{\partial \Gamma}{\partial n} \, ds$$

Now

$$\begin{aligned} \int_{\Omega} B_\rho(y) \Gamma(|x-y|) \nabla^2 u(x) \, dx &\leq \max_{B_\rho(y)} \cdot \int_{B_\rho(y)} B_\rho(y) \Gamma(|x-y|) \, dx = \frac{\omega_n}{n\omega_n} \int_0^\rho r^{2-n} r^{n-1} \, dr = \frac{1}{n} \int_0^\rho r \, dr = \frac{\rho^2}{2n} \xrightarrow{\rho \rightarrow 0} 0 \\ \left| \frac{\partial u}{\partial n} \right| &= |\nabla^2 u \cdot n| \leq C \end{aligned}$$

$$\left| \int_{\partial B_\rho(y)} \Gamma \frac{\partial u}{\partial n} \right| \leq C \int_{B_\rho(y)} |\Gamma| = \frac{c}{n\omega_n} \rho^{2-n} n\omega_n \rho^{n-1} = C\rho \rightarrow 0$$

$$\int_{B_\rho(y)} u \frac{\partial \Gamma}{\partial n} ds_x = \Gamma'(\rho) \int_{B_\rho(y)} u ds_x = \frac{1}{n\omega_n \rho^{n-1}} \int_{B_\rho(y)} u ds_x \xrightarrow{\rho \rightarrow 0} u(y)$$

Substituting it back into equation we get exactly what was needed:

$$u(y) = \int_{\Omega} \Gamma(x-y) \nabla^2 u dx + \int_{\partial\Omega} \left[ u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] ds_x$$

**Green function** Green function can be understood as inverse of Laplacian operator in a sense that if  $\nabla^2 u = f$ ,  $u(c) = \int G(x,y) f(y) dy$ . For all  $y \in \Omega$  define  $h^y(x)$  such that

- $h^y(x)$  is harmonic in  $\Omega$
- $h^y(x) = -\Gamma(|x-y|)$  for all  $x \in \partial\Omega$

(suppose there exists one).

Now define Green function  $G(x,y) = \Gamma(|x-y|) + h^y(x)$ . Properties of  $G$ :

1.  $G$  harmonic for  $x \neq y$
2.  $G(x,y) = 0$  for all  $x \in \partial\Omega$ ,  $y \in \Omega$

Using second Green identity

$$\int_{\Omega} h \nabla^2 u dx = \int_{\partial\Omega} h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n} ds$$

and summing it with  $u(y)$  we get:

$$u(y) = \int_{\Omega} G(x,y) \nabla^2 u dx + \int_{\partial\Omega} u(x) \frac{\partial}{\partial n_x} G(x,y) ds_x$$

**Conclusion** If  $\nabla^2 u = f$  on  $\Omega$  and  $u = g$  on  $\partial\Omega$  the solution

$$u(y) = \int_{\Omega} G(x,y) f(x) dx + \int_{\partial\Omega} g(x) \frac{\partial G}{\partial n} ds_x$$

**Lemma** For all  $x \neq y$ ,  $x, y \in \Omega$

$$G(x,y) = G(y,x)$$

In particular, for constant  $x$ ,  $G$  is harmonic in  $y$ .

**Proof** Define

$$V_x(z) = G(z,x)$$

and

$$W_y(z) = G(z,y)$$

We want to show that  $V_x(y) = W_y(x)$  on  $\Omega \setminus (B_\epsilon(x) \cap B_\epsilon(y))$

$$\begin{aligned} 0 &= \int_{\Omega_\epsilon} V_x \nabla^2 W_y - V_y \nabla^2 W_x dz = \int_{\partial\Omega_\epsilon} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] ds = \\ &= \underbrace{\int_{\partial B_\epsilon(x)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] ds}_{I_1} + \underbrace{\int_{\partial B_\epsilon(y)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] ds}_{I_2} = I_1(\epsilon) + I_2(\epsilon) \end{aligned}$$

$$W_y(x) = \int_{B_\epsilon(x)} V_x \nabla^2 W_y \, dz - \int_{\partial B_\epsilon(x)} \left[ W_x \frac{\partial V_y}{\partial n} - V_y \frac{\partial W_x}{\partial n} \right] \, ds = I_1(\epsilon)$$

$$V_x(z) = \Gamma(|z - x|) + h^y(z)$$

Now

$$V_x(y) = \int_{B_\epsilon(y)} W_y \nabla^2 V_x \, dz - \int_{\partial B_\epsilon(y)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \, ds = -I_2(\epsilon)$$

Thus

$$W_y(x) - V_x(y) = I_1(\epsilon) + I_2(\epsilon) = 0 \Rightarrow W_y(x) = V_x(y)$$

**Green function in ball** Take a look at  $y \in B_R(0)$ , define reflection point  $y^* = \frac{R^2}{|y|^2}y$ . Note that

$$|y^*||y| = R^2$$

and that if  $y \rightarrow \partial B_R(0)$ ,  $y^* \rightarrow \partial B_R(0)$ . Define

$$h^y = -\Gamma(|x - y^*|) \left( \frac{|y|}{R} \right)^{2-n}$$

Then

$$G(x, y) = \Gamma(|x - y|) - \Gamma(|x - y^*|) \left( \frac{|y|}{R} \right)^{2-n}$$

From harmony of  $\Gamma$ ,  $G$  is harmonic in  $x$ . For  $x \in \partial B_r$ :

$$(x - y^*)^2 = \left( x - \frac{R^2}{|y|^2}y \right)^2 = |x|^2 - 2x \frac{R^2}{|y|^2}y + \frac{R^4}{|y|^2} = \frac{R^2}{|y|^2} [R^2 - 2xy + y^2] = \frac{R^2}{|y|^2} (x - y)^2$$

$$|x - y^*| = \frac{R}{|y|} |x - y|$$

$$h^y(x) = -\left( \frac{|y|}{R} \right)^{2-n} \frac{1}{n(2-n)\omega_n} (x - y^*)^{2-n} = -\frac{1}{n(2-n)\omega_n} (x - y)^{2-n} = -\Gamma(|x - y|)$$

i.e.,  $h^y(x)$  we defined fulfills conditions on  $h^y(x)$ .

For  $n = 2$

$$h^y(x) = \begin{cases} -\frac{1}{2\pi} \ln |x - y^*| + \frac{1}{2\pi} \ln \frac{R}{|y|} & y \neq 0 \\ -\frac{1}{2\pi} \ln R & y = 0 \end{cases}$$

For  $n > 2$

$$h^y(x) = \begin{cases} -\left( \frac{|y|}{R} \right)^{n-2} \frac{1}{n\omega_n} |x - y^*|^{2-n} & y \neq 0 \\ -\Gamma(R) & y = 0 \end{cases}$$

And thus for  $n = 2$

$$G(x, y) = \frac{1}{2\pi} \ln \left[ \left( \frac{R}{|y|} \right) \frac{|x - y|}{|x - y^*|} \right]$$

For  $n > 2$ :

$$G(x, y) = \frac{1}{n\omega_n} \left[ |x - y|^{2-n} - \left( \frac{|y|}{R} \right)^{n-2} |x - y^*|^{2-n} \right]$$

If  $u \in \bar{B}_R$  harmonic,

$$u(y) = \oint_{\partial B_R} u(x) \frac{\partial G}{\partial n_x} \, ds_x$$

**Definition**  $K(x, y) = \frac{\partial G}{\partial \hat{\mathbf{n}}_x}$  is called Poisson kernel:

$$\begin{aligned}\frac{\partial}{\partial x_i} G(x, y) &= \frac{\partial}{\partial x_i} \Gamma(x - y) - \frac{\partial}{\partial x_i} \Gamma\left(\frac{|y|}{R} |x - y^*|\right) \\ \frac{\partial}{\partial x_i} \Gamma(x - y) &= \frac{1}{n\omega_n} \frac{x_i - y_i}{|x - y|} |x - y|^{-n+1} = \frac{1}{n\omega_n} \frac{x_i - y_i}{|x - y|^n} \\ \frac{\partial}{\partial x_i} \Gamma\left(\frac{|y|}{R} |x - y^*|\right) &= \frac{1}{n\omega_n} \left(\frac{|y|}{R}\right)^2 (x_i - y_i^*) |x - y|^{-n}\end{aligned}$$

Since we are on ball,  $\hat{\mathbf{n}}_x = \sum \frac{x_i}{R} \hat{\mathbf{x}}_i$ :

$$\begin{aligned}\frac{\partial}{\partial n_x} G(x, y) &= \sum_{i=1}^n \frac{\partial}{\partial x_i} \Gamma(x - y) = \frac{1}{n\omega_n} |x - y|^n \sum_{i=1}^n \left[ \frac{x_i}{R} (x_i - y_i) - \frac{x_i}{R} \left( \frac{|y|^2}{R^2} (x_i - y_i) \right) \right] = \\ &= \frac{1}{n\omega_n} \frac{1}{R} |x - y|^n \sum_{i=1}^n \left[ x_i^2 - x_i y_i - \frac{x_i^2 |y|^2}{R^2} + x_i y_i \right] = \frac{1}{n\omega_n} \frac{1}{R} |x - y|^n (R^2 - |y|^2)\end{aligned}$$

**Conclusion** If  $u \in \bar{B}_R$  harmonic,

$$u(y) = \oint_{\partial B_R} u(x) K(x, y) \, ds_x$$

In case  $y = 0$  we get  $K(x, 0) = \frac{1}{n\omega_n R^{n-1}}$  and acquire the mean value theorem.

**Claim**  $K$  fulfills following conditions

1.  $K(x, y) > 0$ ,  $y \in B_R$ ,  $x \in \partial B_R$ .
2.  $\nabla_y^2 K(x, y) = 0$ ,  $y \in B_R$
3.  $\oint_{\partial B_R} K(x, y) \, ds_x = 1$  for all  $y \in B_R$

**Theorem** If  $g \in \mathcal{C}(\partial B_R)$  then

$$u(y) = \oint_{\partial B_R} K(x, y) g(x) \, ds_x$$

is harmonic function and  $u(x) = g(x)$  for all  $x \in \partial B_R$ .

**Proof** Note that

$$K(x, y) = \frac{1}{Rn\omega_n} |x - y|^n (R^2 - |y|^2)$$

is harmonic in  $y$  in  $B_R$  for  $x \in \partial B_R$

$$\nabla_y^2 u(y) = \oint_{\partial B_R} \nabla_y^2 K(x, y) g(x) \, ds_x = 0$$

Lets show that  $\lim_{y \rightarrow y_0 \in \partial B_R} u(y) = g(y_0)$ . Lets choose  $\epsilon > 0$ . We choose  $\delta_1$  such that  $|g(x) - g(y_0)| < \epsilon$  if  $x \in \partial B_R \cap B_{\delta_1}(y_0)$

$$\begin{aligned}& \oint_{\partial B_R} g(x) K(x, y) \, ds_x - g(y_0) = \oint_{\partial B_R} (g(x) - g(y_0)) K(x, y) \, ds_x = \\ &= \underbrace{\oint_{\partial B_R \setminus \partial B_R \cap B_{\delta_1}(y_0)} (g(x) - g(y_0)) K(x, y) \, ds_x}_{I_1} + \underbrace{\oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(x) - g(y_0)) K(x, y) \, ds_x}_{I_2} \\ & |I_2| \leq \oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(y) - g(y_0)) K(x, y) \, ds_y \leq \epsilon \oint_{\partial B_R} K(x, y) \, ds_x \leq \epsilon\end{aligned}$$

$|x - y_0| \geq \delta_1$  and  $|y - y_0| > \frac{\delta_1}{2}$ , thus

$$|x - y| \geq |x - y_0| - |y - y_0| > \delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2}$$

$$K(x, y) \leq \frac{1}{Rn\omega_n} \left( \frac{\delta_1}{2} \right)^{-n} (R^2 - |y|^2) \xrightarrow{y \rightarrow y_0} 0$$

$$|I_1(y)| \leq (R^2 - |y|^2) \max_{\partial B_R} |y| \frac{\delta_1^{-n}}{Rn\omega_n} \xrightarrow{y \rightarrow y_0} 0$$

**Conclusion**  $u$  is continuous in  $y_0 \in \partial B_R$  and  $\lim_{y \rightarrow y_0 \in \partial B_R} u(y) = g(y_0)$ , i.e. Dirichlet problem has solution if the bound is continuous.

**Theorem** Continuous function in  $\Omega$  is harmonic iff it fulfills mean value theorem.

**Proof** Suppose  $u$  fulfills mean value theorem, take a look at  $B_R(x_0) \subset \Omega$  for some  $x_0 \in \Omega$ .

Let  $v$  harmonic function that equals to  $u$  on  $\partial B_R$ . Since  $v$  fulfills mean value theorem, then  $u - v$  also fulfills it. However,  $u - v = 0$  on ball bound, and thus from weak extremum theorem we get that  $u - v = 0$  in the whole ball, i.e.  $u = v$ .

**Conclusion** If  $u_n$  a sequence of harmonic functions uniformly converging to  $u$ ,  $u$  is harmonic.

**Proof** If  $u_n \rightarrow u$  uniformly in  $\Omega$ , then for all  $x_0 \in \Omega$ , exists  $r > 0$  such that  $B_r(x_0) \subset \Omega$ , and since  $u_n \rightarrow u$  uniformly in  $\partial B_r(x_0)$  thus

$$\lim_{n \rightarrow \infty} \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u_n(x) \, ds = \int_{\partial B_r(x_0)} u(x) \, ds$$

and also

$$\lim_{n \rightarrow \infty} u_n(x_0) = u(x_0)$$

However, from harmony,

$$u_n(x_0) = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u_n(x) \, ds$$

i.e.

$$u(x_0) = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(x) \, ds$$

and thus  $u$  is harmonic.

**Conclusion** If  $u$  is harmonic in  $\Omega$ ,  $u \in C^\infty(\Omega)$ .

**Proof** If  $u$  harmonic,  $u(y) = \int_{\partial B_R} K(x, y) u(x) \, ds_x$  and  $K$  is infinitely differentiable by  $y \in B_R$ :

$$\frac{\partial^m}{\partial y_j^m} u(y) = \int_{\partial B_R} \frac{\partial^m}{\partial y_j^m} K(x, y) u(x) \, ds_x$$

**Conclusion** If  $u_n$  monotonic sequence of harmonic functions in  $\Omega$  and exists  $y \in \mathbb{R}$  such that  $\{u_n(y)\}$  is bounded then in every bounded interval  $\bar{\Omega}' \subsetneq \Omega$  sequence  $u_n$  converges uniformly to harmonic function.

**Proof**  $u_{n+1}(x) \geq u_n(x)$  for all  $x \in \Omega$ , lets show that for all  $x \in \Omega'$  sequence converges to finite limit. Denote  $\omega_{mn} = u_m - u_n$ . By Harnack's identity for  $m > n$

$$\sup_{\Omega'} \omega_{mn} \leq C \inf_{\Omega} \omega_{mn}$$

$$u_m(x) - u_n(x) = \omega_{mn}(x) \leq C[u_m(y_0) - u_n(y_0)] < K$$

$K$  is constant independent on  $n$  and  $m$ , in particular,  $u_n$  is bounded for all  $x \in \Omega'$ , thus exists  $u(x) = \lim_{n \rightarrow \infty} u_n(x)$  in  $\Omega'$ . It's left to proof sequence converges uniformly and thus  $u$  is harmonic.

**Definition** A sequence of function  $u_n$  is equicontinuous in  $x_0$  if for all  $\epsilon > 0$  exists  $\delta > 0$ ,  $N > 0$  such that if  $x - x_0 < \delta$  and  $n > N$ ,

$$|u_n(x) - u_n(x_0)| < \epsilon$$

The particular case is when derivatives are bounded, since if  $|\nabla u_n| < K$ ,

$$|u_n(x) - u_n(x_0)| < K|x - x_0|$$

**Arzelà–Ascoli theorem** If  $u_n$  sequence of functions equicontinuous in compact interval and bounded, then exists uniformly converging subsequence.

**Note** The only bounded harmonic function in  $\mathbb{R}^n$  is constant one.

**Theorem** If  $u$  harmonic in  $\Omega \subset \mathbb{R}^n$  then

$$\left| \frac{\partial u}{\partial x_j} \right| \leq \sup_{\Omega} |u| \cdot \frac{n}{d(x, \partial\Omega)}$$

**Proof**

$$u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} u(y) \, dy$$

Since  $u$  is harmonic and in particular  $\frac{\partial u}{\partial x_i}$  is harmonic

$$\nabla^2 u = 0 \Rightarrow \nabla^2 \frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} \nabla^2 u = 0$$

we get

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} \frac{\partial}{\partial x_j} u(y) \, dy$$

Consider function

$$v_i(y) = \begin{cases} 0 & i \neq j \\ u(y) & i = j \end{cases}$$

then

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int_{B_r(x)} \nabla \cdot v \, dy = \frac{1}{\omega_n r^n} \oint_{\partial B_r(x)} u(y) n_j(y) \, ds_y$$

where  $n_j(y) = \hat{\mathbf{n}} \cdot \hat{\mathbf{x}}_j$ , and thus  $|n_j| \leq 1$ :

$$\left| \frac{\partial}{\partial x_j} u(x) \right| \leq \frac{1}{\omega_n r^n} \left| \oint_{\partial B_r(x)} u(y) n_j(y) \, ds_y \right| \leq \frac{1}{\omega_n r^n} \oint_{\partial B_r(x)} |u(y)| \, ds_y < \frac{1}{\omega_n r^n} \sup |u| n \omega_n r^{n-1} = \sup |u| \frac{n}{r}$$

for all  $r < d(x, \partial\Omega)$ .

**Theorem** If  $\{u_n\}$  sequence of bounded harmonic functions in  $\Omega$  then on each compact set  $K \subset \Omega$  exists subsequence uniformly converging to  $u$  harmonic on  $\Omega$ .

**Proof** By previous theorem each compact set  $K \subset \Omega$  fulfills  $d(x, \partial\Omega) < C(K)$  for all  $x \in K$  and thus partial derivatives are bounded  $|\partial u_n x_i| < C(K)$  thus  $\{u_n\}$  is equicontinuous on  $K$  and thus it has subsequence uniformly converging to  $u$  harmonic in  $K$ .

Lets choose compact sets

$$K_j = \left\{ x \in \Omega, |x| \leq j, d(x, \Omega^C) \geq \frac{1}{j} \right\}$$

such that  $\bigcup_{n=1}^{\infty} K_n = \Omega$ . Suppose  $u_{n_j}$  is converging subsequence in  $K_n$  thus exists subsequence of  $u_{n_j}$  converging in  $K_{n+1}$ . Diagonal sequence (of mutual indices) converges uniformly for all  $K$  and its limit  $u$  is harmonic.

## 5.2 Dirichlet problem in arbitrary interval

Let  $\Omega$  bounded interval and  $g \in \mathcal{C}(\partial\Omega)$  We want to solve the following problem:

$$\begin{cases} \nabla^2 u = 0 & u \in \mathcal{C}(\bar{\Omega}) \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

**Definition**  $u$  is generalized subharmonic function iff  $u$  is continuous, for any ball  $\bar{B} \subsetneq \Omega$  and for all harmonic and continuous  $h$  such that  $u \leq h$  in  $B$ ,  $u \leq h$  on  $\partial B$ .

**Example**  $u(x) = |x - x_0|$  is generalized subharmonic function in  $\mathbb{R}^n$ .

**Conclusion** Generalized subharmonic function fulfills mean value inequality:

$$u(x_0) \leq \frac{1}{|\partial B_r(x_0)|} \oint_{\partial B_r(x_0)} u$$

Also, if function fulfills mean value inequality it is generalized subharmonic function.

**Theorem** If  $\nabla^2 u \geq 0$  on  $\Omega$ ,  $u$  is generalized subharmonic function on  $\Omega$ .

**Lemma** Let  $u$  generalized subharmonic function in  $\Omega$ ,  $v$  generalized superharmonic function in  $\Omega$  and  $u, v \in \mathcal{C}(\bar{\Omega})$  and  $u \geq v$  on  $\partial\Omega$ , then  $u \geq v$  in  $\Omega$

**Proof** Directly from weak extremum.

**Lemma** If  $u, v$  generalized subharmonic functions, then  $\max(u, v)(x) = \max(u(x), v(x))$  is generalized subharmonic function. This is right for any finite amount of functions.

**Proof** If  $h$  harmonic on ball and  $h \geq \max(u, v)(x)$  on  $\partial B$ , in particular  $h \geq u$ ,  $h \geq v$  on  $\partial B$  and thus  $h \geq u$ ,  $h \geq v$  on  $B$  and thus  $h \geq \max(u, v)(x)$  on  $B$ .

**Definition** Let  $\Omega \subset \mathbb{R}^n$  bounded. Let  $g$  bounded on  $\partial\Omega$ , define  $S_g$  as set of all generalized subharmonic functions continuous on  $\bar{\Omega}$  such that for all  $x \in \partial\Omega$   $u(x) \leq g(x)$ .

## 5.3 Perron method

**Lemma** Let  $u$  generalized subharmonic function in  $\Omega$ ,  $\bar{B} \subsetneq \Omega$ , define  $U(x) = u(x)$  for  $x \in \Omega \setminus B$  such that  $\nabla^2 U(x) = 0$  for  $x \in B$  and  $U(x) = u(x)$  on  $\partial B$ .  $U$  is generalized subharmonic function.  $U(x) > u(x)$  for all  $x$ .  $U(x)$  is called harmonic lifting of  $u$  and is unique from maximum principle.

**Proof** Let  $h$  harmonic on  $B$  such that  $h(x) \geq U(x) \geq u(x)$  on  $\partial B_1$  and thus  $h(x) \geq u(x) = U(x)$  in  $x \in B_1 \setminus B$ .

$$\partial(B \cap B_1) = (\partial B \cap \bar{B}_1) \cup (\partial B_1 \cap \bar{B})$$

For  $\partial B \cap \bar{B}_1$   $U = u < h$ . For  $\partial B_1 \cap \bar{B}$   $h \geq U$ . Thus  $H \geq U$  on  $\partial(B \cap B_1)$  however  $h - U$  is nonnegative and harmonic on  $\partial(B \cap B_1)$  and thus  $H \geq U$  in  $B \cap B_1$ .

**Theorem**  $u(x) = \sup_{w \in S_g} w(x)$  is harmonic.



**Proof** Constant  $M$  is harmonic (constant) and it fulfills  $M \geq y$  on  $\partial\Omega$  thus  $w \leq M$  for all  $w \in S_g$  in particular  $w(x) \leq M$  for all  $x \in \Omega$  and  $w \in S_g$ .

Let  $y \in \Omega$ , there exists a sequence of functions  $v_n \in S_g$  such that

$$\lim_{n \rightarrow \infty} v_n(y) = u(y)$$

Let  $\bar{B}_r(y) \subsetneq \Omega$  and define  $V_n$  that is harmonic lifting of  $v_n$ . Since  $V_n(x) \geq v_n(x)$  for all  $x \in \Omega$  and  $V_n \in S_g$  (it is generalized subharmonic and equals to  $v_n$  on  $\partial\Omega$ ),

$$v_n(y) \leq V_n(y) \leq u(y)$$

for all  $y \in \Omega$  thus  $V_n(y) \rightarrow u(y)$ . Thus there is subsequence of  $V_n$  uniformly converging to  $v$  in  $B_\rho(y) \subset B_r(y)$ . In  $B_\rho$   $v \leq u$ . Suppose exists  $z \in B_\rho(y)$  such that  $v(z) < u(z)$ . Exists  $\bar{u} \in S_g$  such that  $v(z) < \bar{u}(z) \leq u(z)$ . Then

$$\bar{u} < w_j = \bar{u} \vee V_n \in S_g$$

Let  $W_j$  harmonic lift of  $w_j$  on  $B_r(y)$ .  $W_j \in S_g$  thus once again exists subsequence uniformly converging to  $w$  in  $B_\rho$ . We know that

$$v \leq w \leq u$$

and  $v(y) = w(y) = u(y)$ . Thus, since both are harmonic,  $w - v$  is harmonic and non-negative, thus  $w - v = 0$ . We got the contradiction, thus  $v = u$ .

**Conclusion** If Dirichlet problem

$$\begin{cases} \nabla^2 w(x) = 0 & x \in \Omega \\ w(y) = g(y) & x \in \partial\Omega \end{cases}$$

has solution, then  $u(x) = \sup_{v \in S_g} v(x)$  is solution.

**Solution**  $u(x)$  is harmonic and  $g \geq u$  on  $\partial\Omega$ , thus  $w \geq u$  on  $\Omega$ .

On the other hand,  $w$  is harmonic and in particular subharmonic, thus  $w \in S_g$  and  $w \leq u$ . We conclude  $w = u$ .

**Exercise**

$$\Omega = B_1 \setminus \{0\}$$

Then

$$\partial\Omega = \partial B_1 \cup \{0\}$$

For bound condition  $u(\partial B_1) = 1$  and  $u(0) = 0$  there is no solution.

**Definition**  $y \in \partial\Omega$  is called regular if exists generalized subharmonic function  $w$  on  $\bar{\Omega}$  such that  $w(y) = 0$  and  $w(\bar{\Omega} \setminus \{0\}) < 0$ .

**Exercise** If there exist a ball  $B_\epsilon(z)$  in  $\Omega^C$  tangent to  $\partial\Omega$  in  $y$ , then  $y$  is regular.

**Theorem** If  $\Omega$  is bounded interval,  $\partial\Omega$  is regular in every point,  $g \in \mathcal{C}(\partial\Omega)$  exists  $u \in \mathcal{C}(\bar{\Omega})$  such that  $u = g$  on  $\partial\Omega$  and  $\nabla u = 0$ .

**Proof** We need

$$\lim_{x \rightarrow y \in \partial\Omega} u(x) = g(y)$$

Define

$$\begin{cases} v_1(x) = g(y) - \epsilon + hw(x) \\ v_2(x) = g(y) + \epsilon - hw(x) \end{cases}$$

where  $h > 0$  is constant to be determined later and  $w$  is the function from definition of regular point.

$v_1$  is subharmonic and  $v_2$  is superharmonic. For  $h$  big enough we know that  $v_1(x) < g(x)$  and  $v_2(x) > g(x)$  for  $x \in \partial\Omega$ .

$0 > -m \geq w(x)$  if  $|x - y| > \delta$ . Thus

$$v_1(x) \leq g(x) - \epsilon - hm$$

Choose  $h$  such that

$$g(x) + \epsilon - hm < \min_{x \in \partial\Omega} g(x)$$

Since  $|g(x) - g(y)| < \epsilon$  if  $|x - y| < \delta$  thus  $v_1(x) < g(x)$  if  $|x - y| \leq \delta$  and thus  $v_1(x) \leq g$  in  $\partial\Omega$  and similarly  $v_2(x) \geq g$  in  $\partial\Omega$ . Thus

$$v_2 \geq u \geq v_1$$

for  $u = \sup_{w \in S_g} w(x)$ .

$$\begin{aligned} g(x) + \epsilon - hw(x) &\geq u(x) \geq g(x) - \epsilon + hw(x) \\ -\epsilon &\leq \liminf_{x \rightarrow y} u(x) - g(x) \leq \limsup_{x \rightarrow y} u(x) - g(x) < \epsilon \end{aligned}$$

**Theorem** Regularity is sufficient condition for solution existence.

**Proof** Suppose there exists solution of Dirichlet problem in interval with bound condition  $g(x) = -|x - y|$  for  $y \in \partial\Omega$ . Thus exists harmonic function  $h$  which equals to  $g$  on  $\partial\Omega$ , then  $h < 0$  in  $\Omega$  and  $h(y) = 0$ , i.e.,  $y$  is regular.

**Hilbert proof of Dirichlet problem** For all  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$  and  $u = g$  on  $\partial\Omega$ . Define Dirichlet integral- functional

$$I(u) = \int_{\Omega} |\nabla u|^2$$

Suppose there exists minimum of  $I$ ,  $u_0$  (this is not easy to show). Then

$$I(u_0) \leq \int_{\Omega} |\nabla u|^2$$

Let  $\phi \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$  such that  $\phi = 0$  on  $\partial\Omega$ . Then  $u_0 + \epsilon\phi$  fulfills the condition.

$$I(u_0 + \epsilon\phi) = \int_{\Omega} |\nabla(u_0 + \epsilon\phi)|^2 = \int_{\Omega} |\nabla u_0|^2 + 2\epsilon \int_{\Omega} \nabla u_0 \nabla \phi + \epsilon^2 \int_{\Omega} |\nabla \phi|^2 \geq I(u_0) = \int_{\Omega} |\nabla u_0|^2$$

Thus

$$\int_{\Omega} \nabla u_0 \nabla \phi \geq 0$$

$$\nabla(\nabla u_0 \phi) = \nabla u_0 \nabla \phi + \nabla^2 u_0 \phi$$

$$\int_{\Omega} \nabla u_0 \nabla \phi = - \int_{\Omega} \nabla^2 u_0 \phi + \int_{\Omega} \nabla(\nabla u_0 \phi) = - \int_{\Omega} \nabla^2 u_0 \phi + \oint_{\partial\Omega} \phi \nabla u_0 \geq 0$$

Thus

$$\int_{\Omega} \nabla^2 u_0 \phi = 0$$

for any  $\phi \in \mathcal{C}(\bar{\Omega})$  thus

$$\nabla^2 u_0 = 0$$

## 5.4 Neumann bound condition

Is there solution of Neumann bound condition

$$\begin{cases} \nabla^2 u(x) = 0 & x \in \Omega \\ \frac{\partial u}{\partial n} = g & y \in \partial\Omega \end{cases}$$

**Proof** Let  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$ . Define functional

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \oint_{\partial\Omega} u g$$

If there exists minimum  $u_0$ , take a look on  $u_0 + \epsilon\phi$  for  $\phi \in \mathcal{C}(\bar{\Omega})$ .

$$0 \leq I(u_0 + \epsilon\phi) - I(u_0) = \epsilon \iint_{\Omega} \left[ \nabla u_0 \nabla \phi + \frac{1}{2} \epsilon |\nabla \phi|^2 \right] + \epsilon \int_{\partial\Omega} \phi g$$

Thus

$$\iint_{\Omega} \nabla u_0 \nabla \phi + \oint_{\partial\Omega} \phi g = 0$$

For any  $\phi \in \mathcal{C}^2 \cap \mathcal{C}(\bar{\Omega})$ .

$$- \int \nabla^2 u_0 \phi + \int \phi \frac{\partial u}{\partial n} + \iint_{\Omega} \nabla u_0 \nabla \phi = 0$$

thus

$$\oint \left( \frac{\partial u}{\partial n} - g \right) \phi = 0$$

i.e.

$$\frac{\partial u}{\partial n} = g$$

If  $\oint_{\partial\Omega} g > 0$  obviously there is no minimum of  $I$ : for  $u = \lambda$

$$I(u) = -\lambda \oint_{\partial\Omega} g$$

Thus  $\oint g = 0$  is necessary condition for  $I$  to be bounded. By Gauss

$$0 = \int_{\Omega} \nabla^2 u = \oint_{\partial\Omega} \frac{\partial u}{\partial n} = \oint_{\partial\Omega} g$$

## 6 Heat equation in higher dimensions

$$\frac{\partial u}{\partial t} = \nabla_x^2 u$$

for  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}^+$ .

Note the equation is unchanged under transformation

$$\begin{cases} x \rightarrow ax = x' \\ t \rightarrow a^2 t = t' \end{cases}$$

Thus if  $u(x, t)$  is solution then  $u(ax, a^2 t)$  is solution. Thus, lets search for solutions dependent on  $\xi = \frac{|x|}{\sqrt{t}}$ :

$$u(x, t) = w\left(\frac{|x|}{\sqrt{t}}\right)$$

Define heat as

$$Q(t) = \int_{\mathbb{R}^n} u(x, t) dx = \int_{\mathbb{R}^n} w\left(\frac{|x|}{\sqrt{t}}\right) dx = \int w(\xi) t^{\frac{n}{2}} d\xi = t^{\frac{n}{2}} \int w(\xi) d\xi = \text{const}$$

So, to keep  $Q$  constant we want

$$u(x, t) = t^{-\frac{n}{2}} w\left(\frac{|x|}{\sqrt{t}}\right)$$

Substituting into equation:

$$\begin{aligned}(\xi^{n-1}w')' + \frac{1}{2}(\xi^n w)' &= 0 \\ \xi^{n-1}w' + \frac{1}{2}\xi^n w &= c\end{aligned}$$

Substituting  $\xi = 0$  we get  $c = 0$ :

$$\begin{aligned}\xi^{n-1}w' + \frac{1}{2}\xi^n w &= 0 \\ w' + \frac{1}{2}\xi w &= 0 \\ w(\xi) &= ce^{-\frac{|\xi|^2}{4}}\end{aligned}$$

If we want to normalize

$$\int u(x, t) \, dx = 1$$

We get

$$u(x, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Which is  $n$  dimensional Gaussian.

### Kernel of heat

$$K(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

Then

$$\frac{\partial K}{\partial t} = \nabla_x^2 K$$

Also note that  $K > 0$  and that, from normalization,

$$\int_{\mathbb{R}^4} K(x, y, t) \, d^n x = 1$$

Moreover, for all  $\delta > 0$  and for all  $y \in \mathbb{R}^n$

$$\lim_{t \rightarrow 0} \int_{|x-y| \geq \delta} K(x, y, t) \, d^n x = 0$$

i.e.,

$$\lim_{t \rightarrow 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|x-y| \geq \delta} e^{-\frac{|x-y|^2}{4t}} \, d^n x = 0$$

**Proof** By variable substitution  $\xi = \frac{x}{\sqrt{t}}$ :

$$\lim_{t \rightarrow 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|x-y| \geq \delta} e^{-\frac{|x-y|^2}{4t}} \, d^n x = \lim_{t \rightarrow 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{|x| \geq \delta} e^{-\frac{x^2}{4t}} \, d^n x = \lim_{t \rightarrow 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \int_{|\xi| \geq \frac{\delta}{\sqrt{t}}} e^{-\frac{\xi^2}{4}} \, d^n \xi = 0$$

Define

$$u(x, t) = \int_{\mathbb{R}^n} K(x, y, t) f(y) \, d^n y$$

**Conclusion** If  $u_t = \nabla^2 u$  then  $\min_{\Omega} u, \max_{\Omega} u$  are acquired on  $\partial_p \Omega$ . In particular there is unique solution of heat equation Dirichlet problem in  $\Omega = D \times [0, T]$ : if  $u_1, u_2$  are solutions of heat equation in  $\Omega$  and  $u_1 = u_2$  on  $\partial_p \Omega$  then  $u_1 = u_2$  on  $\Omega$ .

**Theorem** If  $|f|$  is bounded and continuous,  $u(x, t)$  is solution of the heat equation in  $\mathbb{R}^n \times (0, \infty)$  fulfilling  $\lim_{t \rightarrow 0} u(x, t) = f(x)$ .

### Proof

$$\frac{\partial u}{\partial t} = \iint_{\mathbb{R}^n} \frac{\partial K}{\partial t} f(x) \, dx$$

$$\nabla_x^2 u = \iint_{\mathbb{R}^n} \nabla_x^2 K f(x) \, dx$$

To be allowed to switch between integral and derivative we need to show that

$$u(x, t+h) - u(x, t) = \int_{\mathbb{R}^n} [K(x, y, t+h) - K(x, y, t)] f(y) \, dy$$

is bounded by integrable function. Since  $K$  is decaying exponent, it is true.

Lets show that  $\lim_{t \rightarrow 0} u(x, t) = f(x)$ .

$$u(x, t) = \int_{|x-y| \leq \delta} K(x, y, t) f(y) \, dy + \int_{|x-y| > \delta} K(x, y, t) f(y) \, dy$$

Since integral of  $K$  is 1

$$f(x) = \iint_{\mathbb{R}^n} K(x, y, t) f(y) \, dy$$

and thus

$$u(x, t) - f(x) = \int_{|x-y| \leq \delta} K(x, y, t) (f(y) - f(x)) \, dy + \int_{|x-y| > \delta} K(x, y, t) (f(y) - f(x)) \, dy$$

Since  $f$  is bounded:

$$\int_{|x-y| > \delta} K(x, y, t) (f(y) - f(x)) \, dy \leq 2c \int_{|x-y| > \delta} K(x, y, t) \, dy \rightarrow 0$$

Always exists  $\epsilon$  such that  $|f(x) - f(y)| < \epsilon$

$$\int_{|x-y| \leq \delta} K(x, y, t) (f(y) - f(x)) \, dy \leq \int_{|x-y| \leq \delta} K(x, y, t) |f(y) - f(x)| \, dy \leq \epsilon \int_{|x-y| \leq \delta} K(x, y, t) \, dy \leq \epsilon \int_{\mathbb{R}^n} K(x, y, t) \, dy \leq \epsilon$$

Thus

$$\limsup_{t \rightarrow 0} |u(x, t) - f(x)| = \epsilon$$

and

$$\lim_{t \rightarrow 0} u(x, t) = f(x)$$

Note that temperature has infinite velocity - even if we start with  $f$  with compact support, for any  $t > 0$  for any point we have  $u > 0$ .

**Heat equation in compact region** Let  $\Omega = D \times [0, T]$  for some compact  $D \subsetneq \mathbb{R}^n$ . We call  $\partial_p \Omega \{D \times \{0\}\} \cup \{\partial D \times (0, T]\}$  parabolic bound of  $\Omega$ .

**Theorem** If  $u \in \mathcal{C}(\bar{\Omega})$ ,  $u_t, u_{x_i x_j} \in \mathcal{C}(\Omega)$  and  $u_t - \nabla^2 u \leq 0$  on  $\Omega$  then

$$\max_{(x,t) \in \Omega} u(x, t) = \max_{(x,t) \in \partial_p \Omega} u(x, t)$$

**Proof** Suppose  $u_t - \nabla^2 u < 0$  for all  $(x, t) \in \Omega$ . Suppose  $(x_0, t_0) \in \Omega$  is maximum of  $u$ . Then

$$\nabla^2 u \leq 0$$

$$u_t = 0$$

i.e.,  $u_t - \nabla^2 u \geq 0$  in contradiction.

If  $(x_0, T)$  is maximum, then

$$\begin{aligned} \nabla^2 u &\leq 0 \\ \frac{\partial u}{\partial t} \Big|_{t=T} &= T \end{aligned}$$

thus still  $u_t - \nabla^2 u \geq 0$  in contradiction.

If we suppose that  $u_t - \nabla^2 u \leq 0$  define  $v(x, t) = u(x, t) - \delta t$ . Then  $v$  fulfills  $v_t - \nabla^2 v < 0$ .

Suppose  $\max u(x, t) = u(x_0, t_0) = m$ , then

$$\max_{\Omega} v \geq m - \delta T$$

is acquired in  $(x_\delta, t_\delta) \in \partial_p \Omega$ . Looking on sequence  $\delta_n \rightarrow 0$ , since  $\partial_p \Omega$  is compact,  $(x_{\delta_n}, t_{\delta_n}) \rightarrow (x_0, t_0) \in \partial_p \Omega$ , i.e.

$$\lim_{\delta \rightarrow 0} v(x_\delta, t_\delta) = u(x_0, t_0) = m$$

Thus maximum is acquired on parabolic boundary.

**Theorem** If  $f$  is bounded on  $\mathbb{R}^n$  and  $u(x, t)$  fulfills

$$u_t - \nabla^2 u \geq 0$$

and  $u \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^+) \cap \mathcal{C}(\mathbb{R}^n \times \bar{\mathbb{R}}^+)$ ,  $u(x, 0) = f$  and in addition

$$u(x, t) \leq M e^{a|x|^2}$$

for some  $M, a > 0$  then

$$u(x, t) < \sup f(x)$$

for all  $x, t$ .

Note that it's enough to show theorem for  $0 < t \leq T$  for some constant  $T > 0$ , since then we can use  $u(x, T)$  as new initial conditions to get to arbitrary  $t$ .

$$K(x, y, t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

is solution of heat equation. So is

$$K(x, y, t + T) = \frac{1}{(4\pi(t + T))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+T)}}$$

However  $K(x, y, -t)$  is not solution of heat equation. But

$$K'(x, y, -t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{+\frac{|x|^2}{-4t}}$$

is solution of heat equation. It can be seen by extending to complex plane, we can note that under transformation

$$\begin{cases} t \rightarrow -t \\ x \rightarrow ix \end{cases}$$

heat equation is preserved.

So lets look at

$$w_{y,T}(x, t) = K'(x, y, T - t) = \frac{1}{(4\pi(T - t))^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T-t)}}$$

with

$$w_{y,T}(x, 0) = \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{|x|^2}{4T}}$$

for  $0 < t \leq T$ . This function solves heat equation, however this function is increasing and goes to infinity in finite time. This happens due to value of  $w_{y,T}(x, 0)$  in infinity - it is infinite, i.e. there is infinite amount of heat in  $x \rightarrow \infty$ .

**Proof** Define

$$v_\mu(x, t) = u(x, t) = \mu w_{T, y}(x, t)$$

Then

$$\frac{\partial}{\partial t} v_\mu(x, t) - \nabla^2 v_\mu \leq 0$$

Define

$$\Omega_{\rho, T} = \{(x, t) : |x - y| < \rho, 0 \leq t < T\}$$

We know that

$$\max_{\Omega_{\rho, T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho, T}} v_\mu(t)$$

In particular, for  $x = y$

$$v_\mu(y, t) \leq \max_{\Omega_{\rho, T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho, T}} v_\mu(t)$$

In ball  $B_\rho(y)$

$$v_\mu(x, 0) \leq u(x, 0)$$

For  $0 \leq t \leq \frac{T}{2}$

$$v_\mu(x, t) = u(x, t) - \mu w(x, t) \leq M e^{a|x|^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} \leq M e^{a(|y|+\rho)^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} = e^{a\rho^2} \left[ M e^{2|y|\rho+a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2(\frac{1}{4T}-a)} \right]$$

Since we are free to choose  $T$  and  $\rho$ , we can choose them such that  $\frac{1}{4T} - a > 0$  and then for  $\rho$  big enough

$$v_\mu(x, t) \leq e^{a\rho^2} \left[ M e^{2|y|\rho+a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2(\frac{1}{4T}-a)} \right] < C$$

for any  $C$ . Thus,  $v_\mu(x, t)$  can acquire maximum only in  $t = 0$ :

$$\max_{\substack{|x-y| \leq \rho \\ 0 \leq t \leq \frac{T}{2}}} v_\mu(x, t) = \max_{|x-y| \leq \rho} v_\mu(x, 0)$$

Thus, taking  $\mu$  to 0:

$$u(x, t) \leq \max_{\mathbb{R}^n} u(x, 0)$$

for  $y \in \mathbb{R}^n$  and  $0 \leq t \leq \frac{2}{a}$ .

**Conclusion** If  $u_t - \nabla^2 u = 0$  then

$$\sup_{x \in \mathbb{R}^n} |u(x, t)| \leq \sup_{x \in \mathbb{R}^n} |u(x, 0)|$$

**Conclusion** If  $u_1, u_2$  fulfill heat equation such that  $u_1(x, 0) = u_2(x, 0)$  and in addition

$$\begin{cases} |u_1(x, t)| < M e^{a|x|^2} \\ |u_2(x, t)| < M e^{a|x|^2} \end{cases}$$

then  $u_1(x, t) = u_2(x, t)$  for all  $x, t \in \mathbb{R}^n \times \mathbb{R}^+$ .

Take a look on time integral of heat kernel,

$$G(x, y) = \int_0^\infty K(x, y, t) dt$$

If  $n \geq 3$  the integral converges both in infinity and in 0, for  $x \neq y$ .

Also

$$u(x, t) = \int_{\mathbb{R}^n} dy K(x, y, t) f(y)$$

Assume  $f$  has compact support (we don't have to, but it makes things easier). We see that

$$u = \mathcal{O}\left(\frac{1}{t^{\frac{n}{2}}}\right)$$

$$\int_0^\infty dt u(x, t) = \int_0^\infty \int_{\mathbb{R}^n} dy K(x, y, t) f(y) = \int_{\mathbb{R}^n} dy f(y) \int_0^\infty K(x, y, t) dt = \int_{\mathbb{R}^n} dy f(y) G(x, y)$$

Integrating heat equation

$$\int_0^\infty \frac{\partial u}{\partial t} dt = u(x, \infty) - u(x, 0) = -f(x)$$

since  $u(x, 0) = f(x)$  and  $u(x, \infty) = 0$ . on the other hand

$$\int_0^\infty \nabla_x^2 u(x, t) dt = \nabla_x^2 \int_0^\infty u(x, t) dt$$

Denote

$$V = \int_0^\infty u(x, t) dt$$

then

$$\nabla^2 V = -f$$

This is Laplace equation, with  $n \geq 3$ , i.e.

$$V(x) = - \int \Gamma(|x - y|) f(y) dy$$

where  $\Gamma(r) = \frac{1}{(2-n)\omega_n} r^{2-n}$ .

## 7 Wave equation in $\mathbb{R}^n \times \mathbb{R}$

We look at

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \nabla_x^2 u \\ u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

Lets assume radial solution:  $u(x, t) = w(|x|, t)$  for some  $w = w(r, t)$  Then we get

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r} \frac{\partial w}{\partial r}$$

Define  $v(r) = rw$ , then

$$\begin{aligned} v_r &= w + w_r r \\ v_{rr} &= 2w_r + w_{rr} r \\ \frac{1}{r} v_{rr} &= w_{rr} + \frac{2}{r} w_r \end{aligned}$$

which exactly Laplacian for  $n = 3$ .

$$\frac{\partial^2 v}{\partial t^2} = r \frac{\partial^2 w}{\partial t^2} = c^2 \left( r \frac{\partial^2 w}{\partial r^2} + (n-1) \frac{\partial w}{\partial r} \right)$$

For  $n = 3$ :

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial r^2}$$

which is one-dimensional wave equation for  $r > 0$ . Since

$$\frac{\partial}{\partial x_i} w(|x|, t) = w_r(0, t) \frac{\partial |x|}{\partial x_i}$$

and  $\frac{\partial |x|}{\partial x_i}$  is not continuous in 0, we require

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial r} w_t(0) = 0$$

Assume initial condition of form  $w(r, 0) = f(r)$ ,  $w_t(r, 0) = g(r)$  and  $f'(0) = g'(0) = 0$ . Define extension of  $f$  and  $g$  on  $\mathbb{R}$ :  $f(|r|)$  and  $g(|r|)$ .

Then, for  $v(r, 0) = rf(|r|)$  and  $v_t(r, 0) = rg(|r|)$ . The D'Alembert solution is

$$rw(r, t) = v(r, t) = \frac{(r+ct)f(|r+ct|) + (r-ct)f(|r-ct|)}{2} + \frac{1}{2ct} \int_{r-ct}^{r+ct} sg(|s|) ds$$



i.e.,

$$w(r, t) = \frac{(r + ct)f(|r + ct|) + (r - ct)f(|r - ct|)}{2r} + \frac{1}{2ctr} \int_{r-ct}^{r+ct} sg(|s|) ds$$

Note that if  $g$  and  $f$  has compact support, exist time  $t_0$  such that  $w(r, t > t_0) = 0$   
 In 1D wave equation we required  $f \in \mathcal{C}^2$  and  $g \in \mathcal{C}^1$ , however in 3D it's not enough.  
 For example, assuming  $t = 0$

$$\lim_{r \rightarrow 0} w(r, t) = f(ct) + ct f'(ct) = w(0, t)$$

**Spherical average** For  $h(x)$ ,  $x \in \mathbb{R}^n$ , spherical average of  $h$  is

$$M_h(x, r) = \frac{1}{|\partial B_r(x)|} \int_{|\partial B_r(x)|} h(y) ds_y$$

Note that

$$M_h(x, 0) = 0$$

With variable substitution  $x + r\xi = y$  we get

$$M_h(x, r) = \frac{1}{n\omega_x r^{n-1}} \int_{\partial B_1(0)} r^{n-1} h(x + r\xi) ds_\xi = \frac{1}{n\omega_x} \int_{\partial B_1(0)} h(x + r\xi) ds_\xi$$

Now  $M_h(x, r)$  is even in  $r$ .

$$\begin{aligned} \frac{\partial}{\partial r} M_h(x, r) &= \frac{1}{n\omega_x} \int_{\partial B_1(0)} \sum_{x_i} \xi_i \frac{\partial h}{\partial x_i}(x + r\xi) ds_\xi = \frac{1}{n\omega_x r} \int_{\partial B_1(0)} \vec{\xi} \cdot \nabla_\xi h(x + r\xi) ds_\xi \stackrel{\text{Guass}}{=} \frac{1}{n\omega_x r} \int_{B_1(0)} \nabla_\xi^2 h(x + r\xi) d\xi = \\ &= \frac{r}{n\omega_x} \int_{B_1(0)} \nabla_x^2 h(x + r\xi) d^n \xi = \frac{1}{n\omega_n r^{n-1}} \int_{B_r(x)} \nabla_x^2 h(y) d^n y = \frac{1}{n\omega_n r^{n-1}} \nabla^2 \int_0^r d\rho \int_{\partial B_\rho(x)} h(y) ds_y = \\ &= \frac{1}{r^{n-1}} \nabla^2 \int_0^r d\rho \rho^{n-1} \underbrace{\frac{1}{n\omega_n \rho^{n-1}} \int_{\partial B_\rho(x)} h(y) ds_y}_{M_h(x, \rho)} = r^{1-n} \nabla_x^2 \int_0^r d\rho \rho^{n-1} M_h(x, \rho) \end{aligned}$$

Since radial Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} = r^{1-n} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right)$$

$$\frac{\partial}{\partial r} \left[ r^{n-1} \frac{\partial}{\partial r} M_h(x, r) \right] = \frac{\partial}{\partial r} \nabla_x^2 \int_0^r d\rho \rho^{n-1} M_h(x, \rho) = \nabla_x^2 r^{n-1} M_h(x, r) = r^{n-1} \nabla_x^2 M_h(x, r) = r^{n-1} M_{\nabla^2 h}(x, r)$$

Thus

$$r^{1-n} \frac{\partial}{\partial r} \left[ r^{n-1} \frac{\partial}{\partial r} M_h(x, r) \right] = \frac{\partial^2}{\partial r^2} M_h(x, r) + \frac{n-1}{r} M_h(x, r) = \nabla_x^2 M_h(x, r) = M_{\nabla^2 h}(x, r)$$

Let  $u$  solution of wave equation, then

$$M_u(x, r, t) = \frac{1}{n\omega_n} \int_{|\xi|=1} u(x + r\xi, t) ds_\xi$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla_x^2 u$$

$$\frac{\partial^2}{\partial t^2} M_u(x, r, t) = c^2 M_{\nabla^2 u}(x, r, t) = c^2 \left[ \frac{\partial^2 M_u}{\partial r^2} + \frac{n-2}{r} \frac{\partial}{\partial r} M_u \right]$$

**Conclusion** Spherical average of wave equation in  $x$  is acquired as solution of wave equation for  $M_u(x, r, t)$  as a function of  $r$  and  $t$ .

For  $n = 3$ :

$$M_u(x, r, t) = \frac{1}{2r}[(ct + r)M_f(x, ct + r) - (ct - r)M_f(x, ct - r)] + \frac{1}{c} \frac{1}{2r} \int_{ct-r}^{ct+r} \rho M_g(x, \rho) d\rho$$

Then

$$\begin{aligned} u(x, t) &= \lim_{r \rightarrow 0} M_u(x, r, t) = \frac{\partial}{\partial \rho} \rho M_f(x, \rho) \Big|_{\rho=ct} + \frac{1}{c} \rho M_g(x, \rho) \Big|_{\rho=ct} = M_f(x, ct) + ct \frac{\partial}{\partial \rho} M_f(x, \rho) \Big|_{\rho=ct} + t M_g(x, ct) = \\ &= \frac{\partial}{\partial t} (t M_f(x, ct)) + t M_g(x, ct) = \left( \frac{t}{n\omega_n} \int_{|\xi|=1} f(x + ct\xi) d\xi \right) + \frac{t}{n\omega_n} \int_{|\xi|=1} g(x + ct\xi) d\xi = \\ &= \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} f(x) d s_y \right) + \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} g(x) d s_y = \\ &= \frac{1}{4\pi} \int_{\xi=1} f(x + ct\xi) d s_\xi + \frac{ct}{4\pi} \int_{\xi=1} (\nabla_{x+ct\xi} f) \cdot \xi d s_\xi + \frac{t}{4\pi} \int_{\xi=1} g(x + ct\xi) d s_\xi \end{aligned}$$

**2D wave equation** We can reduce dimensionality by with initial conditions. If

$$\begin{cases} u(x_1, x_2, x_3, 0) = f(x_1, x_2) \\ u_t(x_1, x_2, x_3, 0) = g(x_1, x_2) \end{cases}$$

Thus  $u = u(x_1, x_2, t)$  is solution of 2D wave equation. Take a look at sphere around  $(x_1, x_2, 0)$ :

$$|y - x| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2} = ct$$

$$y_3 = \sqrt{c^2 t^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2}$$

And thus

$$d s_y = \sqrt{1 + \left( \frac{\partial y_3}{\partial y_1} \right)^2 + \left( \frac{\partial y_3}{\partial y_2} \right)^2} d y_1 d y_2 = \left| \frac{ct}{y_3} \right| d y_1 d y_2$$

Define

$$r = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

and

$$\begin{aligned} u(x_1, x_2, t) &= \frac{\partial}{\partial t} \left( \frac{2}{4\pi c^2 t} \iint_{r < ct} f(y_1, y_2) \left| \frac{ct}{y_3} \right| d y_1 d y_2 \right) + \frac{2}{4\pi c^2 t} \iint_{r < ct} g(y_1, y_2) \left| \frac{ct}{y_3} \right| d y_1 d y_2 = \\ &= \frac{\partial}{\partial t} \left[ \frac{1}{2\pi c} \iint_{r < ct} \frac{f(y_1, y_2)}{\sqrt{(ct)^2 - r^2}} d y_1 d y_2 \right] + \frac{1}{2\pi c} \iint_{r < ct} \frac{g(y_1, y_2)}{\sqrt{(ct)^2 - r^2}} d y_1 d y_2 \end{aligned}$$

## 8 Algebraic theory of PDEs

Take a look at heat equation

$$\frac{\partial u}{\partial t} = \nabla_x^2 u$$

Define  $L(u) = \nabla^2 u$ .  $L$  is linear operator on functions in  $\mathcal{C}^2$ . However since for  $f \in \mathcal{C}^k$ ,  $L(f) \in \mathcal{C}^{k-2}$ , we can say that  $L : \mathcal{C}^\infty \rightarrow \mathcal{C}^\infty$  and then

$$\frac{\partial u}{\partial t} = L(u)$$

In case of wave equation Define  $v = \frac{\partial u}{\partial t}$  and then

$$\frac{d}{dt} \begin{pmatrix} u \\ v \end{pmatrix} = L \begin{pmatrix} u \\ v \end{pmatrix}$$

Then wave equation is

$$L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \nabla^2 u \end{pmatrix}$$

In case of Poisson equation

$$Lu = f$$

i.e.  $u = L^{-1}f$ .

To solve such equations we want to diagonalize  $L$ , for that we'd like  $L$  to be symmetric.

## 8.1 Sturm-Liouville problem

$$\frac{\partial u}{\partial t} = s(x)u_{xx} + b(x)u_x c(x)u$$

Given interval  $[a, b]$  with initial conditions we want to rewrite the equation as

$$\frac{\partial u}{\partial t} = (pu_x)_x + qu$$

for some  $p(x)$   $q(x)$ . It's always possible to find  $r(x)$  such that multiplication of first equation by  $r(x)$  will give the second one for  $v = ru$ .

Define operator

$$L(v) = -\frac{1}{r(x)}[(p(x)v_x)_x + qv]$$

Then we write equation as

$$L(v) = \frac{\partial v}{\partial t}$$

And we want to diagonalize the operator, i.e. find  $v$  and  $\lambda$  such that

$$Lv - \lambda v = 0$$

$L$  is defined over vector space  $\mathcal{C}^2[a, b]$  and fulfills a pair of bound conditions:

$$\begin{cases} B_a(u) = \alpha u(a) + \beta u'(a) = 0 & |\alpha| + |\beta| > 0 \\ B_b(u) = \gamma u(b) + \delta u'(b) = 0 & |\gamma| + |\delta| > 0 \end{cases}$$

Denote

$$X = \{f \in \mathcal{C}^2[a, b], B_a(f) = B_b(f) = 0\}$$

Alternatively,

$$X = \{f \in \mathcal{C}^2[a, b], u(a) = u(b), u'(a) = u'(b)\}$$

Note that for  $f \in X$  it's possible that  $L(f) \notin X$ .

$$\langle u, v \rangle = \int_a^b \int_a^b u(x)v(x)r(x) dx$$

This is inner product, since it's linear, symmetric and positive.

**Definition**  $L$  is symmetric operator relatively to inner product iff

$$\langle u, Lv \rangle = \langle Lu, v \rangle$$

Let's check if  $L$  we defined is symmetric:

$$uLv - vLu = \frac{1}{r} [u[(pv_x)_x + qv] - v[(pu_x)_x + qu]] = \frac{1}{r} [pu_xv - pv_xu]_x$$

$$\langle u, Lv \rangle - \langle Lu, v \rangle = \int_a^b [pu_xv - pv_xu]_x dx = \left[ pu_xv - pv_xu \right]_a^b = 0$$

**Theorem** Sturm-Liouville has real eigenvalues and orthogonal eigenfunctions. In case of divided bound conditions  $X$ , each eigenvalue is simple. In any case, multiplicity of each eigenvalue is not greater than 2.

**Proof** For each there are two independent solutions  $u_1^\lambda, u_2^\lambda$ . If  $\lambda$  is eigenvalue, exist  $A, B \in \mathbb{R}$  such that

$$\begin{cases} B_a(Au_1^\lambda + Bu_2^\lambda) = 0 \\ B_b(Au_1^\lambda + Bu_2^\lambda) = 0 \end{cases}$$

$$\begin{cases} A(\alpha u_1^\lambda(a) + \beta u_1^{\lambda'}(a)) + B(\alpha u_2^\lambda(a) + \beta u_2^{\lambda'}(a)) = 0 \\ A(\gamma u_1^\lambda(b) + \delta u_1^{\lambda'}(b)) + B(\gamma u_2^\lambda(b) + \delta u_2^{\lambda'}(b)) = 0 \end{cases}$$

If this linear system has solution,

$$(\alpha u_1^\lambda(a) + \beta u_1^{\lambda'}(a))(\gamma u_2^\lambda(b) + \delta u_2^{\lambda'}(b)) - (\alpha u_2^\lambda(a) + \beta u_2^{\lambda'}(a))(\gamma u_1^\lambda(b) + \delta u_1^{\lambda'}(b)) = 0$$

This is actually equation in  $\lambda$ , that has infinite number of solution.

For  $\lambda_1 \neq \lambda_2$ ,

$$Lu_1 = \lambda_1 u_1$$

$$Lu_2 = \lambda_2 u_2$$

$$\langle u_2, Lu_1 \rangle = \lambda_1 \langle u_1, u_2 \rangle$$

$$\langle u_2, Lu_1 \rangle = \langle u_1, Lu_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle$$

$$(\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0$$

Thus if  $\lambda_1 \neq \lambda_2$ ,  $\langle u_1, u_2 \rangle = 0$ .

Suppose  $\lambda \in \mathbb{C}$ :

$$Lu = \lambda u$$

$$\overline{Lu} = \overline{\lambda u}$$

$$\bar{L}\bar{u} = L\bar{u} = \bar{\lambda}\bar{u}$$

Thus both  $u$  and  $\bar{u}$  are eigenfunctions, with different eigenvalues, i.e.,  $\langle u, \bar{u} \rangle = 0$ , meaning

$$\int_a^b |u|^2 dx = 0$$

$$u = 0$$

$$Lu = Lu_1 + iLu_2$$

$$\overline{Lu} = Lu_1 - iLu_2 = L(u_1 - iu_2) = L\bar{u}$$

Now

$$Lu = \lambda u$$

$$Lv = \lambda v$$

$$0 = vLu - uLv = \frac{1}{2}[(pu')' + qu]v - [(pv')' + qv]u = \frac{1}{2}[p \cdot (u'v - v'u)]' = 0$$

Thus

$$p \cdot (u'v - v'u) = c = \text{const}$$

$$u'v - v'u = \frac{c}{p}$$

Since  $p > 0$  and  $u'v - v'u$  is 0 on bound,  $c = 0$ , thus

$$u'v = v'u$$

i.e.,  $(\frac{u}{v})'$  or  $(\frac{v}{u})'$  is 0, i.e.,  $u$  and  $v$  are linearly dependent.

**Example**

$$Lu = u''$$

with periodic bound conditions on  $[-\pi, \pi]$ . Solutions are  $\sin(\theta) \cos(\theta)$ .

Suppose  $u_i$  are normalized eigenfunctions of  $L$ , and

$$V_n = \text{span}(u_1, \dots, u_n)$$

If  $f \in X$ , what is closest function to  $f$  in  $V_n$ :

$$\min_{w \in V_N} \|w - f\|$$

We call  $w$  a projection of  $f$  on  $V_N$ .

**Claim**

$$w = \sum_i^N \langle f, u_i \rangle u_i \in V_N$$

**Proof**

$$w \in V_N \Rightarrow w = \sum_1^N \alpha_i u_i$$

$$\min_{\alpha_1, \dots, \alpha_N} \left\| f - \sum_1^N \alpha_i u_i \right\|^2$$

$$\left\| f - \sum_1^N \alpha_i u_i \right\|^2 = \left\langle f - \sum_1^N \alpha_i u_i, f - \sum_1^N \alpha_i u_i \right\rangle = \langle f, f \rangle - 2 \sum \alpha_i \langle f, u_i \rangle + \sum_{i,j} \alpha_i \alpha_j \langle u_i, u_j \rangle = \|f\|^2 - 2 \sum \alpha_i \langle f, u_i \rangle + \sum \alpha_i^2 = H(\alpha_1, \dots, \alpha_N)$$

$$\frac{\partial H}{\partial \alpha_i} = 2[\alpha_i - \langle f, u_i \rangle] = 0$$

Thus

$$\alpha_i = \langle f, u_i \rangle$$

**Theorem** Let  $\{u_i\}$  is infinite sequence of orthonormal functions in  $X$ , for all  $f \in X$

$$\sum_i |\langle f, u_i \rangle|^2 \leq \|f\|^2$$

this is Bessel equation.

$$0 \leq \left\| f - \sum_1^N \langle f, u_i \rangle u_i \right\|^2 = \|f\|^2 - \sum_{i=1}^N |\langle f, u_i \rangle|^2$$

If

$$\sum_{i=1}^{\infty} |\langle f, u_i \rangle|^2 = \|f\|^2$$

Then

$$\lim_{N \rightarrow \infty} \left\| f - \sum_1^N \langle f, u_i \rangle u_i \right\| = 0$$

and we say that series

$$\sum_i \langle f, u_i \rangle u_i = f$$

Note this is not pointwise convergence:

$$\sum_i \langle f, u_i \rangle u_i(x) \neq f(x)$$

but rather norm convergence:

$$\lim_{N \rightarrow \infty} \int_a^b \left| f(x) - \sum_i \langle f, u_i \rangle u_i(x) \right|^2 r(x) dx$$

**Definition** Orthonormal sequence is called total if for all  $f \in X$

$$\sum_{i=1}^{\infty} |\langle f, u_i \rangle|^2 = \|f\|^2$$

(Parseval's identity).

### Theorem

1. Sequence of eigenfunctions of Sturm–Liouville is total sequence.
2. Eigenvalues are  $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$  and  $\lim_{n \rightarrow \infty} \lambda_n = \infty$ .

**Conclusion** If  $f \in X$  then series

$$\sum_{n=1}^{\infty} \langle f, u_i \rangle u_i = \sum_{n=1}^{\infty} \frac{2}{\pi} \left[ \int_0^{\pi} f(y) \sin(ny) dy \right] \sin(nx)$$

$$\langle f, u_i \rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin(nx) dx$$

$$a_n(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$f(x) = \sum a_n \sin(nx)$$

### Theorem

$$l = \int_a^b \sqrt{\frac{r(x)}{p(x)}} dx$$

$$\lambda_n \sim \left( \frac{n\pi}{l} \right)^2$$

which means

$$\lim_{n \rightarrow \infty} \lambda_n \sim \left( \frac{l}{n\pi} \right)^2 = 1$$

In addition, exist constants

$$|u_n| < c$$

$$|u'_n| < cn$$

$$|u''_n| < cn^2$$

### Generalized heat equation

$$\begin{cases} r(x)u_t = (pu_x)_x + qu \\ B_a(u) = B_b(u) = 0 \\ u(x, 0) = f \end{cases}$$

$$(pu_x^{(i)})_x + qu^{(i)} + r\lambda_i^{(i)} = 0$$

Lets search the solution of form

$$u(x, t) = \sum_{j=1}^{\infty} a_j(t) u^{(j)}(x)$$

$$r(x) \sum_{j=1}^{\infty} \dot{a}_j(t) u^{(j)}(x) = r(x) \sum_{j=1}^{\infty} a_j(t) \lambda_j u^{(j)}(x)$$

i.e.,

$$\dot{a}_j(t) = \lambda_j a_j(t)$$

$$a_j(t) = a_j(0) e^{-\lambda_j t}$$

Thus

$$u(x, t) = \sum_{j=1}^{\infty} a_j(0) e^{-\lambda_j t} u^{(j)}(x)$$

$a_j(0)$  are generalized Fourier coefficients of  $f$ :

$$a_j(0) = \int_a^b r f u^{(j)} dx$$

$$u(x, t) = \sum_{j=1}^{\infty} \left( \int_a^b r(y) f(y) u^{(j)}(y) dy \right) e^{-\lambda_j t} u^{(j)}(x) = \int_a^b \left[ \sum_{j=1}^{\infty} r(y) u^{(j)}(y) e^{-\lambda_j t} u^{(j)}(x) \right] f(y) dy$$

Define kernel of heat

$$K(x, y, t) = r(y) \sum_{j=1}^{\infty} u^{(j)}(x) u^{(j)}(y) e^{-\lambda_j t}$$

And then

$$u(x, t) = \int_a^b K(x, y, t) f(y) dy$$

Note that  $K$  is symmetric:

$$K(x, y, t) = K(y, x, t)$$

We can say that

$$K(x, y, t) \cong e^{Lt}$$

For operator  $L$  with eigenvalues  $\lambda_n$  and eigenfunctions  $u_n$ , define

$$\mathcal{L}(x, y) = \sum_{n=1}^{\infty} \lambda_n u_n(x) u_n(y)$$

Then

$$Lu = \int \mathcal{L}(x, y) u(y) dy = \sum_{n=1}^{\infty} \lambda_n \langle u, u_n \rangle u_n(x)$$

i.e.,

$$Lu_n = \lambda_n u_n$$

Thus

$$e^{\mathcal{L}t} = \sum e^{-\lambda_n t} u_n(x) u_n(y) = K$$

The last question is whether the series converge:

$$\frac{\partial u}{\partial t} = - \sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t} a_j(f) u_j(x)$$

$$\frac{\partial^2 u}{\partial x^2} = - \sum_{j=1}^{\infty} e^{-\lambda_j t} a_j(f) u_j''(x)$$

$$a_j(f) = \int_a^b f(x) u_j(x) r(x) dx$$

$$|a_j(f)| < c$$

and

$$\sum_{j=1}^{\infty} a_j^2(f) - \|f\|^2 < \infty$$

Thus

$$\lim_{j \rightarrow \infty} a_j(f) = 0$$

(this is Riemann–Lebesgue lemma).

Since

$$|\lambda_j e^{-\lambda_j t} a_j(f)| < j^2 e^{-j^2 t} c$$

If  $t > 0$  coefficient series absolutely converges and thus the solution is in  $\mathcal{C}^\infty$ .

## Non-homogeneous equations

$$u_t = (pu_x)_x + qu + f(x, t)$$

First of all, let's solve the following ODE:

$$\dot{x} = ax + f(t)$$

The homogeneous solution is

$$x(t) = e^{at}$$

thus

$$x(t) = a_0 e^{at} + e^{at} \int_0^t e^{-as} f(s) ds$$

We want to solve

$$\dot{a}_j = -\lambda_j a_j + f_j(t)$$

where

$$f_j(t) = \int_a^b f(t) u_j(x) dx$$

thus

$$a_j(t) = \underbrace{a_j(0)}_0 + e^{-\lambda_j t} \int_0^t e^{\lambda_j s} f_j(s) ds$$
$$|\dot{a}_j(t)| |f_j(t)| + |\lambda_j| |a_j|$$

Since  $|a_j| \sim e^{-j^2}$ , we want only

$$\sum_{j=1}^{\infty} |f_j(t)| < \infty$$

and then

$$\frac{\partial u}{\partial t} = \sum \dot{a}_j u_j(x) < c$$

If  $f$  is differentiable,  $\sum_{j=1}^{\infty} |f_j(t)| < \infty$ :

## Note

**Definition**  $E(a, b)$  are all the piecewise continuously differentiable functions in  $[a, b]$ , which means function is continuously differentiable in all but countable number of points and has one-side derivative everywhere.

**Theorem** For each  $u \in E(a, b)$  generalized Fourier series

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \langle u, \phi_i \rangle \phi_i(x) = \frac{u(x^+) + u(x^-)}{2}$$

In particular, if  $u$  continuous in  $x$ , then series converges to  $u(x)$ . If  $u \in E(a, b) \cap \mathcal{C}(a, b)$  and fulfills bound condition then series uniformly converges to  $u$ .

**Definition** For all  $u \in X$  define

$$B(u) = -\langle \mathcal{L}u, u \rangle_r = -\int_a^b dx \left[ (pu')' + qu \right] u = \int_a^b [p(u')^2 - qu^2] dx - p(b)u'(b)u(b) + p(a)u'(a)u(a)$$

We know  $\mathcal{L}$  has infinite increasing sequence of eigenvalues.

## Theorem

$$\lambda_0 = \inf_{\substack{u \in X \\ \|u\|_r = 1}} B(u) = B(\phi_0) = \inf_{\substack{v \in X \\ \|v\|_r \neq 0}} \frac{B(v)}{\|v\|^2} = \inf_{\substack{B(w) \in \mathbb{R} \\ \|w\|_r = 1}} B(w) = B(\phi_0)$$

where  $\|\phi_0\| = 1$  is eigenfunction of  $\mathcal{L}$  of  $\lambda_0$ .



**(Partial) proof** Take a look at

$$X_N = \text{span}\{\phi_0, \phi_1, \dots, \phi_N\}$$

Then  $u \in X_N \Rightarrow u = \sum_1^N \alpha_i \phi_i$ . Obviously  $\|u\| = 1$ .

$$B(u) = -\langle \mathcal{L}u, u \rangle_r$$

$$\mathcal{L}u = \sum_i \alpha_i \mathcal{L}\phi_i = -\sum_i \alpha_i \lambda_i \phi_i$$

Then

$$B(u) = -\langle \mathcal{L}u, u \rangle_r = -\left\langle \sum_i \alpha_i \lambda_i \phi_i, \sum_i \alpha_i \phi_i \right\rangle_r = \sum_i \lambda_i \alpha_i^2 \geq \lambda_0$$

Thus

$$B(u) \geq \lambda_0 \|u\|^2$$

with obvious equality if  $u = \phi_0$ .

**Conclusion**  $\phi_0 > 0$  and  $\lambda_0$  is simple eigenvalue.

**Proof** Suppose  $\phi_0$  has zero in  $(a, b)$ .  $|\phi_0| \in E(a, b) \cap \mathcal{C}(a, b)$ . Also,  $B(\phi_0) = B(|\phi_0|)$ , i.e.,  $(|u'|)^2 = (u')^2$  in all points  $u$  is differentiable and  $u \neq 0$ .

Thus  $|\phi_0|$  is also minimum of  $B(u)$ , i.e.,  $|\phi_0|$  is eigenfunction. Thus  $|\phi_0| \in \mathcal{C}^2$ . If there is point where  $\phi_0(x) = 0$ ,  $|\phi_0|$  if differentiable in same point iff  $\phi'_0 = 0$ , however then  $\phi_0 = 0$  since it solves second-order ODE. In addition, if there were two eigenfunctions corresponding to  $\lambda_0$ , we could find two orthogonal eigenfunctions of  $\lambda_0$ , but both are positive and integral of their multiplication is positive, in contradiction to orthogonality.

**Exercise** Proof that  $\{u : B(u) \in \mathbb{R}\}$  is inner product space, if  $B(u_1), B(u_2) < \infty$  then  $B(u_1 + u_2) < \infty$  and

$$B(u, v) = -\langle \mathcal{L}u, v \rangle_r$$

is inner product.

**Example** For both Neumann and Dirichlet bound condition:

$$B(u) = \int_a^b p \cdot (u')^2 - qu^2 dx$$

However, in case of Dirichlet we need  $u(a) = u(b) = 0$ , and in case of Neumann we don't have special requirements. This is due to small change in  $u$  is enough to force  $u'(a) = u'(b) = 0$ , however to force  $u(a) = u(b) = 0$  we need bigger change in  $u$ , and, as a result  $u'$ , resulting in divergence of  $B$ .

**Heat equation**

$$ru_t = (pu_x)_x + qu$$

$u(x, 0) = f(x)$  and  $B_a[u(x, t)] = B_b[u(x, t)] = 0$  for all  $t > 0$ .

$$u(x, t) = \sum_{n=1}^{\infty} \langle f, \phi_i \rangle e^{-\lambda_n t} \phi_i(x)$$

If  $\lambda_0 < 0$ ,

$$\lim_{t \rightarrow 0} u(x, t) = \infty$$

And

$$Q(t) = \langle u(x, t), \phi_0 \rangle = \langle f, \phi_0 \rangle e^{-\lambda_0 t}$$

This if  $\lambda_0 < 0$ ,  $Q(t) \rightarrow \infty$  exponentially.

**Example** Given two bound conditions:

$$\begin{cases} u(0, t) = 0 \\ u_x(1, t) = \beta u(1, t) \end{cases}$$

and

$$\begin{cases} u_x(0, t) = 0 \\ u_x(1, t) = \beta u(1, t) \end{cases}$$

In both cases,

$$B(u) = \int_0^1 (u')^2 - \beta u^2(x)$$

however, in first case we want  $u(a) = 0$ .

To show that  $\lambda_0$  it's enough to find  $\phi$  such that  $B(\phi) < 0$ .

Choose  $\phi(x) = \sin(\alpha x)$ :

$$B(\phi) = \alpha^2 \int_0^1 \cos^2(\alpha x) dx - \beta \sin^2(\alpha x) \leq \alpha^2 - \beta \sin^2(\alpha) = \alpha^2 \left[ 1 - \beta \left( \frac{\sin(\alpha)}{\alpha} \right)^2 \right] \rightarrow \alpha^2 [1 - \beta] < 0$$

if  $\beta > 1$ , we get  $\lambda_0 < 0$ .

In second case, we can choose  $\phi(x) = \cos(\alpha x)$ , since we don't need  $\phi(0) = 0$ :

$$B(\phi) = \alpha^2 \int_0^1 \sin^2(\alpha x) dx - \beta \cos^2(\alpha x) \leq \alpha^2 - \beta \cos^2(\alpha) \rightarrow -\beta$$

Thus the condition is weaker – it's enough  $\beta > 0$  to have negative eigenvalue.

**Lemma** Define

$$L_1(u) = (pu')' + q_1 u = 0$$

$$L_2(v) = (pv')' + q_2 v = 0$$

where  $u(x_1) = u(x_2) = 0$ ,  $> 0$  on  $(x_1, x_2)$  and  $q_2 > q_1$ . Then  $v$  has at least one zero in  $(x_1, x_2)$ .

**Proof** Take a look at

$$vL_1u - uL_1v = p(u'v - v'u)'$$

$$L_1v + (q_2 - q_1)v = L_2v$$

$$L_1v = L_2v - (q_2 - q_1)v$$

$$vL_1u - uL_2v + (q_2 - q_1)uv = p(u'v - v'u)'$$

Since  $L_2v = L_1u = 0$ , if  $v > 0$ :

$$0 < \int (q_2 - q_1)uv = p(x_2)u'(x_2)v(x_2) - p(x_1)u'(x_1)v(x_1) < 0$$

in contradiction, thus  $v$  has 0.

**Theorem** For Sturm–Liouville problem with Dirichlet bound conditions, eigenfunction  $\phi_n$  has exactly  $n$  zeros in  $(a, b)$ .

**Proof** If  $\phi_n(x_j)$  the sequence  $\{x_j\} \subset (a, b)$  then there is subsequence

$$x_j \rightarrow x_0 \in [a, b]$$

We get that

$$\phi(x_j) - \phi(x_0) = 0 \Rightarrow \begin{cases} \phi(x_0) = 0 \\ \phi'(x_0) = 0 \end{cases} \Rightarrow \phi(x_0) = \phi'(x_0) = 0 \Rightarrow \phi = 0$$

$$L(u) = (pu')' + qu + \lambda_n u = 0$$

Then  $L_1 = q_1 = q + \lambda_1$ , choose  $x_1, x_2$  to be subsequent zeros of  $u$ . Then  $q_2 = q + \lambda_{n+1} > q_1$  and from previous theorem  $v$  has zero between  $x_1$  and  $x_2$ .

## 8.2 Sturm–Liouville problem in $\mathbb{R}^n$

Let  $\Omega \in \mathbb{R}^n$  bounded interval and  $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$ . Take a look at family of operators

$$Lu = \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i \frac{\partial u}{\partial x_i} + cu$$

where  $a_{ij}, b_i, c \in \mathcal{C}(\Omega)$ . If

$$Lu = \frac{1}{r} \left[ \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \right) + cu \right]$$

we say operator in divergence form. If  $n = 1$ ,

$$Lu = au'' + bu' + c$$

which is equivalent to  $\frac{1}{r}[(pu')' + qu]$ . In higher dimensions it's not always possible.

Operator  $L$  is called elliptical if  $A = \{a_{ij}\}$  is positive defined symmetric matrix.

Alternatively

$$\sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) \xi_i \xi_j \geq \mu \sum_{i=1}^n \xi_i^2$$

for all  $\xi \in \mathbb{R}^n$  and  $\mu > 0$ .

Take a look at  $Lu = f$  in  $\Omega$  and bound condition:

$$\sigma_2(A \cdot \nabla u) \cdot \hat{\mathbf{n}} + \sigma_1 u = 0$$

for  $x \in \partial\Omega$ , where  $\hat{\mathbf{n}}$  is normal vector to  $\partial\Omega$  and  $\sigma_1, \sigma_2 \in \mathcal{C}(\partial\Omega)$  such thatn  $|\sigma_1| + |\sigma_2| > 0$  for  $x \in \partial\Omega$ .

Define inner product

$$\langle u, v \rangle = \iint_{\Omega} ruv \, d^n r$$

$$\langle Lu, v \rangle = \iint_{\Omega} v \cdot \left[ \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \sum_{j=1}^n a_{ij} \frac{\partial u}{\partial x_j} \right) \right] d^n r + \iint_{\Omega} cuv \, d^n r = - \iint_{\Omega} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d^n r + \int_{\partial\Omega} (A \cdot \nabla u) \cdot \hat{\mathbf{n}} \, ds + \iint_{\Omega} cuv \, d^n r$$

And

$$\int_{\partial\Omega} (A \cdot \nabla u) \cdot \hat{\mathbf{n}} \, ds = \int_{\partial\Omega} \frac{\sigma_1}{\sigma_2} uv \, ds$$

We define quadratic form

$$B(u) = -\langle Lu, u \rangle = \iint_{\Omega} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} d^n r - \int_{\partial\Omega} \frac{\sigma_1}{\sigma_2} u^2 \, ds - \iint_{\Omega} cu^2 \, d^n r$$

Also, since

$$\langle Lu, v \rangle = \langle u, Lv \rangle$$

$L$  is self-adjoint, and thus it has orthogonal eigenfunctions and real eigenvalues.

**Theorem** Under this conditions there us infinite sequence of eigenvalues:

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

and orthonormal eigenfunctions under defined inner product.

### Example

$$\nabla^2 u + \lambda u = 0$$

in  $B = \{|x| \leq 1\} \subset \mathbb{R}^2$  with Dirichlet bound conditions.

Rewrite in polar coordinates:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0$$

and  $u(1, \theta) = 0$ . Guess solution of form

$$u(r, \theta) = R(r)\Theta(\theta)$$

$$\frac{r^2}{R\Theta} \left[ \Theta \left( R''(r)\Theta(\theta) + \frac{R'}{r} \right) + \frac{R}{r^2} \Theta' + \lambda R\Theta \right] = 0$$

i.e.

$$\frac{r^2(R'' + \frac{1}{r}R')}{R} + \frac{\Theta''}{\Theta} + \lambda r^2 = 0$$

Thus for  $\Theta$

$$\frac{\Theta''}{\Theta} = -n^2$$

i.e.

$$\Theta = a \sin(n\theta) + b \cos(n\theta)$$

For  $R$

$$R'' + \frac{1}{r}R' - \frac{Rn^2}{r^2} + \lambda R = 0$$

with  $R(1) = 0$  and  $R(0), R'(0) < \infty$ . To simplify, define  $x = r$ , we got Sturm–Liouville:

$$(xR')' - \frac{n^2}{x}R + \lambda xR = 0$$

with  $p(x) = x$ ,  $q(x) = -\frac{n^2}{x}$  and  $r(x) = x$ .

However, since we don't have bound condition in one of bounds ( $x_1 = 0$ ), there is degeneracy in problem, due to  $p(x_1) = 0$ . The operator is

$$Lu = \frac{1}{x} \left[ (xu')' - \frac{n^2}{x}u \right]$$

and

$$\langle u, v \rangle = \int_0^1 xuv \, dx$$

$$\langle Lu, v \rangle - \langle u, Lv \rangle = u(1)v'(1) - v(1)u'(1)$$