# 104030 - Introduction to Partial Differential Equations

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**Abstract**

## 1 Introduction

**PDE** In PDE, the solution is a function of a couple of variables $u(x_1, x_2, \dots x_m)$ such that:

$$F(x_1, x_2, \dots x_m, u_{x_1}, u_{x_2}, \dots, , u_{x_m}, u_{x_1x_1}, \dots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually $m = 2$. For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_2x_2}, u_{x_1x_0}) = 0$$

Is PDE of two variables of order 2.

**Linear PDE** PDE is linear if $F$ is linear in $u$ and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}) =$$
$$= A(x_1, x_2)u_{x_1x_1} + B(x_1, x_2)u_{x_1x_2} + C(x_1, x_2)u_{x_2x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

**Quasilinear PDE** Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}) =$$
$$= A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0$$

For homogeneous linear PDE solution always exist. In addition, if $u_1$, $u_2$, then any linear combination of those $\lambda_1 u_1 + \lambda_2 u_2$ will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

**Autonomous PDE** If $F$ is independent on $x_i$, then if $u(x_1, \dots, x_i, \dots, x_m)$ is solution then $u(x_1, \dots, x_i + \lambda, \dots x_m)$ is solution too.
In particular if $u$ is independent on all $x_i$, then $u(x_1 + \lambda_1, \dots, x_i + \lambda_i, \dots, x_m + \lambda_m)$.

## 1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.
Lets start ODE describing harmonic oscillator, would be

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = k(x - x_0)$$

Now suppose that we have $N$ such masses and position of mass is $\bar{x}_i = x_i + u(x_i, t)$, where $u$ is displacement of mass and $x_i - x_{i-1} = \Delta$. Then the position of mass is described as

$$\frac{\partial^2 \bar{x}_i}{\partial t^2} = m\frac{\partial^2}{\partial t^2} u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(x,t) = k\left[u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)\right]$$

In limit $\Delta \to 0$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u(x,t) = c^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x,t)$$

Where

$$c^2 = \lim_{\Delta \to 0} \frac{\Delta^2 k_\Delta}{m_\Delta}$$

**Possible solutions** For each function $f$ in $\mathcal{C}^2$, $u = f(x - ct)$ is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along $x$ axis with constant velocity $c$. Since $c$ can be negative too, we have solution

$$u(x,t) = f(x + ct) + g(x - ct)$$

## 1.2 Heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Here, $u$ means amount of heat in point $x$ at time $t$.
Amount of heat in interval $[a, b]$ is

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

And heat flux in point $x$ at time $t$ is $k\frac{\partial u}{\partial x}$
Then flux out of interval is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

Thus

$$\int_a^b \frac{\partial}{\partial t} u(x,t)\,\mathrm{d}x = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

In limit $b \to a$ we get

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

**Example solution**

$$u(x,t) = e^{-kst}\sin\left(\sqrt{s}x\right)$$

for some parameter $s$. Here we also can add some constant to $x$ and acquire additional solution:

$$U(x,t) = e^{-kst}\sin\left(\sqrt{s}(x + \lambda)\right) = \cos\left(\sqrt{s}\lambda\right)e^{-kst}\sin\left(\sqrt{s}x\right) + \sin\left(\sqrt{s}\lambda\right)e^{-kst}\cos\left(\sqrt{s}x\right)$$

Thus

$$w(x,t) = e^{-kst}\cos\left(\sqrt{s}x\right)$$

is solution too.

### 1.3 Diffusion equation

Suppose $u(x_1, x_2, x_3, t)$ describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field $v$ independent on $u$.

### 1.4 Elliptic PDEs

#### Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

#### Poisson equation

$$\nabla^2 u = f(x_1, x_2)$$

## 2 First-order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation: $u(x,t) = f(x - ct)$ for some differentiable $f$.
Suppose we have initial conditions $u(x, 0) = u_0(x)$. Is it determines uniquely a solution of equation? Obviously, $u(x,t) = u_0(x-ct)$ is a solution.
Lets show it's unique. Take a look at parametrization $x(t) = s_1 + ct$.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t), t) = c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thus $u$ is constant on every line of form $x(t) = s + ct$. Such lines, along which the PDE becomes an ordinary differential equation, are called characteristic curves or just characteristics. Thus if we know a value of $u$ in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve $x(t)$ for $t \in [a, b]$. So we want to find a solution such that the surface of solution comprises a given curve in 3D.
The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

### 2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where $a, b, c$ are continuously differentiable in some neighborhood of point $(x_0, y_0, z_0)$.
Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$$

and

$$\nabla f \cdot (a, b, c) = -a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + c = 0$$

Thus vector $(a, b, c)$ is tangent to solution surface.
Now define curve such that

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), z(t)) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), z(t)) \\ \frac{\mathrm{d}z}{\mathrm{d}t} = c(x(t), y(t), z(t)) \end{cases}$$

The curve $(x(t), y(t), z(t))$ is characteristic curve of PDE.
If there is no dependence on $z$ (i.e. equation is linear) we can take a look on 2-dimensional curve in xy-plane.

**Theorem** If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

**Proof** Let $(x(t), y(t), z(t))$ characteristic curve of PDE and suppose for some $t_0$

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that $w(t_0) = 0$. Now

$$G\big(x(t), y(t), w(t)\big) = \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}\big(x(t), y(t)\big)\frac{dx}{dt} - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)\frac{dy}{dt} = c\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) -$$
$$-\frac{\partial u}{\partial x}\big(x(t), y(t)\big)a\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)b\Big(x(t), y(t), w(t) + u\big(x(t), y(t))\big)\Big)$$

If we substitute $w = 0$, we get

$$G\big(x(t), y(t), 0\big) = c\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial x}a\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial y}b\big(x(t), y(t), u\big(x(t), y(t)\big)\big)\big) = 0$$

That means that $w = 0$ is a solution of ODE, and since $a, b, c \in \mathcal{C}^1$, te solution is unique, i.e. $w = 0$ is the only solution, and thus characteristic curve is contained in the solution surface.

## 2.2 Existence and uniquness theorem for first-order quasilinear PDE

**Existence and uniquness theorem for first-order quasilinear PDE** Suppose we have initial curve $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$ which around some point $s_0$ is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0)\dot{\bar{y}}(s_0) - b(x_0, y_0, z_0)\dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).
Then in neighborhood of $s_0$ exists unique solution of PDE.

**Proof** Define functions $x(s,t)$, $y(s,t)$, $z(s,t)$ around $(s_0, 0)$ such that

$$\begin{cases} x(s,0) = \bar{x}(s) \\ y(s,0) = \bar{y}(s) \\ z(s,0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial y}{\partial t} = b\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial z}{\partial t} = c\big(x(s,t), y(s,t), z(s,t)\big) \end{cases}$$

From uniquness and existance of ODE, exists unique solution $(x, y, z)$ in neighbourhood of $s_0$.
Lets try to find $s$, $t$, as a function of $x$,$y$. It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in $(s_0, 0)$.
Now define $u(x,y) = z\big(s(x,y), t(x,y)\big)$.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right] + b\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right] = \frac{\partial z}{\partial t}\left[a\frac{\partial t}{\partial x} + b\frac{\partial t}{\partial y}\right] + \frac{\partial z}{\partial s}\left[a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y}\right] =$$
$$= \frac{\partial z}{\partial t}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial y}\right]}_{\frac{dt}{dt}} + \frac{\partial z}{\partial s}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial s}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial s}{\partial y}\right]}_{\frac{ds}{dt}} = \frac{\partial s}{\partial t} = c$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on $xy$-plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on $xy$-plane are different, there are different possibilities.

**Example**

$$yu_x - xu_y = 0$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

**Solution**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s) \end{cases}$$

Now we want to find $s$, $t$ as a function of $x$,$y$:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

**Example**

$$yu_x - xu_y = u$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

**Solution**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ z = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find $s$, $t$ as a function of $x$,$y$:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

## 2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}F(u) = 0$$

for $F = \frac{1}{2}u^2$)
Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = -u \Rightarrow u = \frac{\mathrm{d}x}{\mathrm{d}y}$$

Here $y$ denotes time.
To solve it, we take integral:

$$\int_a^b \left[\frac{\partial u(x,y)}{\partial y} + \frac{\partial}{\partial x}F(u(x,y))\right]\mathrm{d}x = 0$$

$$\frac{\partial}{\partial y}\underbrace{\int_a^b u\,\mathrm{d}x}_{Q(y)} + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big)$$

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = F\Big(u(a,y)\Big) - F\Big(u(b,y)\Big)$$

Now as for any quasilinear PDE:

$$\begin{cases}\dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0\end{cases} \Rightarrow \begin{cases}x = c_2 t + c_3 \\ y = t + c_1 \\ \dot{z} = c_2\end{cases}$$

For initial conditions $\Big(s, 0, h(s)\Big)$:

$$\begin{cases}x = h(s)t + s \\ y = t \\ \dot{z} = h(s)\end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\cdot 1 - \frac{\mathrm{d}\bar{y}}{\mathrm{d}s}\cdot h(s) = 1 \neq 0$$

Since

$$\frac{\partial u}{\partial x} = h'(x - yu)\cdot\left(1 - y\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{h'(x - yu)}{1 + h'(x - yu)\cdot y}$$

even if we start from $\mathcal{C}^\infty$ function we can get $1 + h'(x - yu)\cdot y = 0$ and thus undefined derivative.
Geometrically, the slope of projections of characteristic curves is equal to $h(s)$ thus they can cross in some point.

**Weak solutions** We define a weak solution of equation, function $u$ fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y}u(x,y)\,\mathrm{d}x + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big) = 0$$

Intuitively, $F$ is flux, and $u$ is density, thus change in number of particles (integral) is difference between particles going in and out.
Suppose for solution $u(x,y)$ exists curve of non-continuousness $\gamma$, i.e, $u$ is not continuous in each point of curve:

$$u(y) = \begin{cases}u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y)\end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y)\,\mathrm{d}x = \int_a^{\gamma(y)} u^+(x,y)\,\mathrm{d}x + \int_{\gamma(y)}^b u^-(x,y)\,\mathrm{d}x$$

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y}\,\mathrm{d}x + u^+(x,\gamma(y))\cdot\gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y}\,\mathrm{d}x - u^-(x,\gamma(y))\cdot\gamma'(y) = \\ &= -\int_a^{\gamma(y)} \frac{\mathrm{d}F(u^+)}{\mathrm{d}x}\,\mathrm{d}x - \int_{\gamma(y)}^b \frac{\mathrm{d}F(u^-)}{\mathrm{d}x}\,\mathrm{d}x + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = \\ = -\left[F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^+(a,y)\Big)\right] &- \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right]\end{aligned}$$

Meaning

$$-\left[F\Big(u^+(\gamma(y),y)\Big) - F(u^+(a,y))\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+(x,\gamma(y)) - u^-(x,\gamma(y))\right] = F\Big(u^-(a,y)\Big) - F\Big(u^+(b,y)\Big)$$

$$\gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)$$

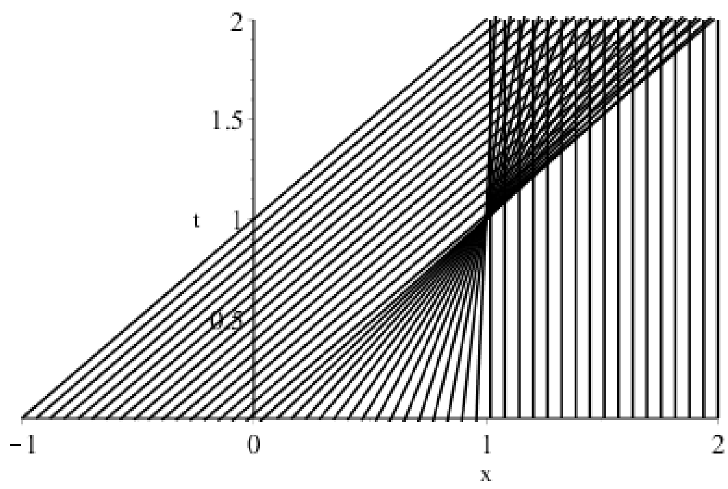
$$\gamma' = \frac{F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)}{u^+(x,\gamma(y)) - u^-(x,\gamma(y))}$$

This equation is called Rankine–Hugoniot conditions. If $F(u) = \frac{1}{2}u^2$, we get $\gamma'(y) = \frac{1}{2}(u^+ + u^-)$

**Example** Suppose we have initial conditions $u(x,0) = h(x)$ for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

For $0 < y < 1$ we have a triangle $\Delta$ ($0 < x < \alpha$ and $y < \frac{x}{\alpha}$) for which there is intersection of two solution:




In point $x, y$ we have slope $u(x,y)$ thus the charecteristic curve crosses $x$-axis at $x_0 = x - uy$ and from initial conditions, $u = 1 - \frac{x_0}{\alpha}$. Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

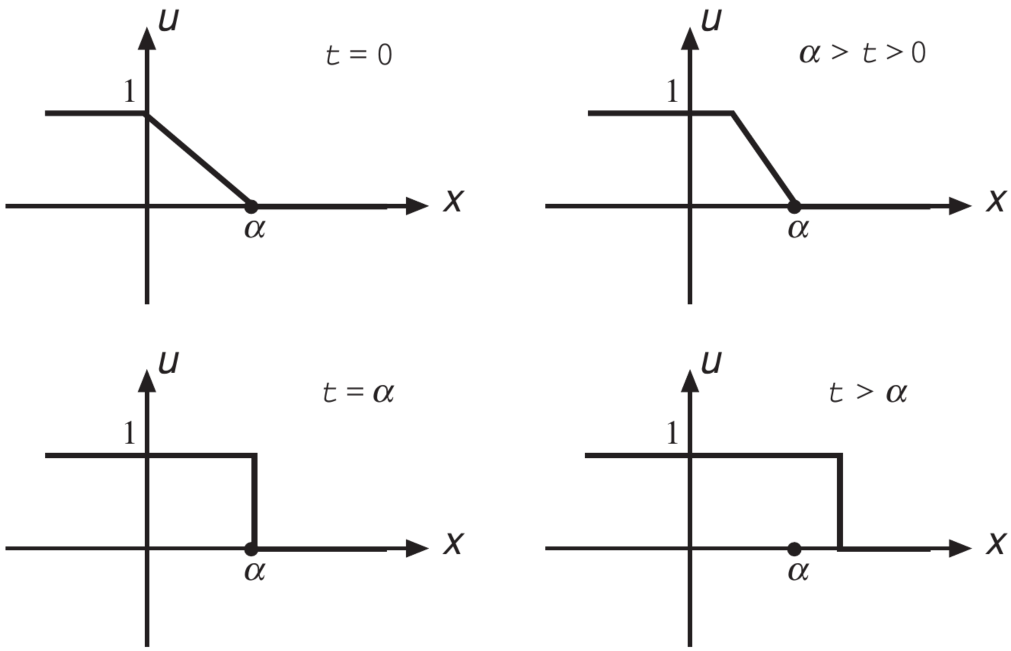
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for $y > 1$ from Rankine–Hugoniot conditions

$$u(x,y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y-\alpha) \\ 0 & x > \alpha + \frac{1}{2}(y-\alpha) \end{cases}$$

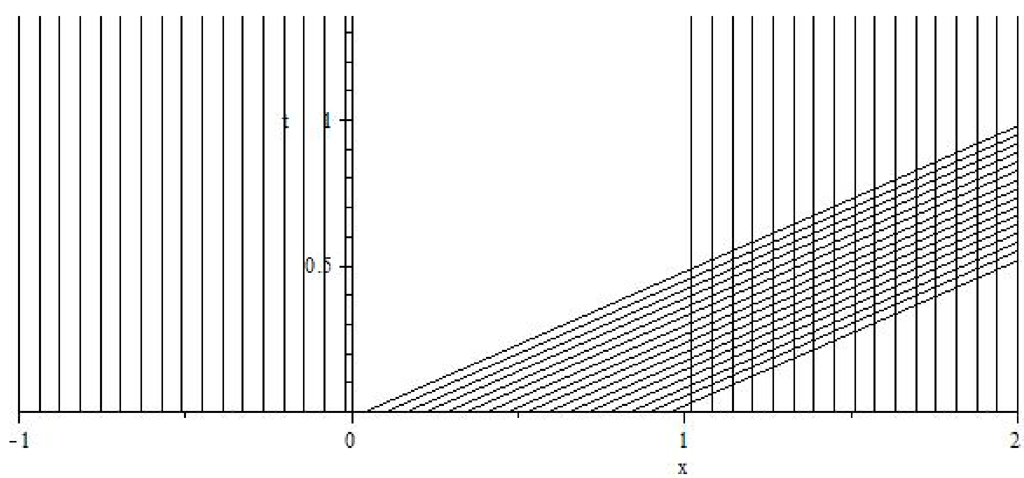
Such a solution is called a shock wave.




**Example** For

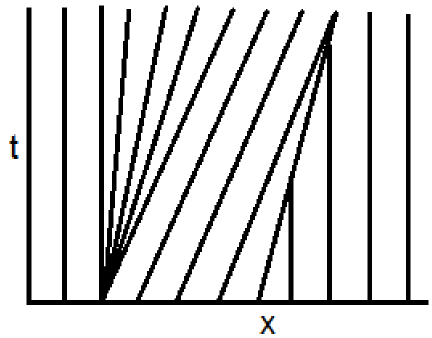
$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

Now there is no place where characteristic curves meet




In the region without characteristic curves ($0 \le x \le y$) we get the following: the solution starts from some point $x_0 = x - uy$, and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$




**What happens if $\alpha \to 0$?** We get $u = \frac{x}{y}$ for $0 \leq x \leq y$. We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.
However, also shock wave along $y = x$ is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we cant reproduce the solution for some $y < y_0$ even if I know the values for $y = y_0$.

**Entory principle** Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

## 2.4 Fully non-linear equations

**Hamilton-Jacoby equation**

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial F}{\partial p} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial F}{\partial q} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \\ \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} \\ \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$
We need to find $\bar{p}$ and $\bar{q}$. We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by $s$

$$\frac{\partial u}{\partial x}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

$$\bar{p}(s)\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \bar{q}(s)\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

Now we can find $p$ and $q$.
Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with $u = 0$, then characteristic curves are perpendicular to initial curve. We get $u(x, y)$ equal to distance from initial curve, since absolute value of gradient of $u$ is 1 due to equation.
If we have $u = \phi(s)$ on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

**Higher dimension** We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

# 3 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + du_x + eu_y + fu = g$$

**Definition** Equation is called hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$ and elliptic if $b^2 - ac < 0$. Wave equation is hyperbolic in the whole space.
We want to simplify the equation: we are searching for $\xi(x,y)$ and $\eta(x,y)$ such that

$$\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} \neq 0$$

and solution $u(x,y) = w\big(\xi(x,y), \eta(x,y)\big)$.
Derivatives of $u$ are

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta\eta_{yy}$$

$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}$$

Now we can get equation of form

$$A(\xi,\eta)w_{\xi\xi} + 2B(\xi,\eta)w_{\xi\eta} + C(\xi,\eta)w_{\eta\xi} + D(\xi,\eta)w_{\eta\eta} + F(\xi,\eta) = 0$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{\xi\eta} = 0$$

i.e.,

$$w(\xi,\eta) = f(\xi) + g(\eta)$$

If we substitute derivatives back into general equation

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} = a\big[w_{\xi\xi}\xi_x^2 + w_{\xi\eta}\xi_x\eta_x + w_\xi\xi_{xx} + w_{\eta\xi}\eta_x\xi_x + w_{\eta\eta}\eta_x^2 + w_\eta\eta_{xx}\big] + \\ +2b[w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}] + \\ +c\big[w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta\eta_{yy}\big] = \\ = \Big(a\xi_x^2 + 3b\xi_x\xi_y + c\xi_y^2\Big)w_{\xi\xi} + 2\Big(a\xi_x\eta_x + c\eta_y\xi_y + b(\xi_x\eta_y + \xi_y\eta_x)\Big)w_{\xi\eta} + \Big(a\eta_x^2 + 3b\eta_x\eta_y + c\eta_y^2\Big)w_{\eta\eta} + \dots \end{aligned}$$

We can rewrite it in matrix form as

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \cdot \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}^2$$

Since the determinant is exactly $ac - b^2$, under the variable substitution the sign of $b^2 - ac$ is conserved.

**Canonical form** The form $w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$, where $\ell_1$ is first-order differential operator is called canonical form of hyperbolic equation.

**Theorem** Each hyperbolic equation can be written in canonical form

**Proof** We want to show that

$$\begin{cases} A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

i.e., that equation $a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0$ has two independent solutions.
Dividing by $\psi_y^2$:

$$a\left(\frac{\psi_x}{\psi_y}\right)^2 + 2b\frac{\psi_x}{\psi_y} + c = 0$$

This is algebric equation, with solutions

$$\frac{\psi_x}{\psi_y} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \lambda_\pm$$

We acquired a pair of equations

$$\psi_x - \lambda_\pm \psi_y = 0$$

And those are two independent solutions which result in $A = 0$ and $C = 0$.

### Wave equation canonical form

$$u_{tt} - c^2 u_{xx} = 0$$

The canonical change of coordinates is

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u_t = -cw_\xi + cw_\eta$$

$$u_x = w_\xi + w_\eta$$

$$u_{tt} = c^2 w_{\xi\xi} - 2c^2 w_{\xi\eta} + c^2 w_{\eta\eta}$$

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

Then

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$$

The solution of canonical equation $w_{\xi\eta} = 0$ is $w(\xi, \eta) = F(\xi) + G(\eta)$, thus solution of wave equation:

$$u(x, t) = F(x + ct) + G(x - ct)$$

An example for physical object fulfilling wave equation is infinite string. To find a solution we need initial conditions, for example, velocity and location at time $t = 0$:

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

, where $f \in \mathcal{C}^2$, $g \in \mathcal{C}^1$.

**Theorem** Exists unique solution of wave equation with those initial conditions.

**Proof** Substituting initial conditions into general solutions:

$$\begin{cases} u(x, 0) = F(x) + G(x) = f(x) \\ u_t(x, 0) = c[F'(x) - G'(x)] = g(x) \end{cases}$$

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{g(x)}{c} \end{cases} \Rightarrow \begin{cases} F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_1 \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_2 \end{cases}$$

Now, since $F(x) + G(x) = f(x)$, thus $D_1 + D_2 = 0$.
Substituting into solution, we acquire what is called d'Alembert's formula:

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

From construction, the solution is unique.

**Example**

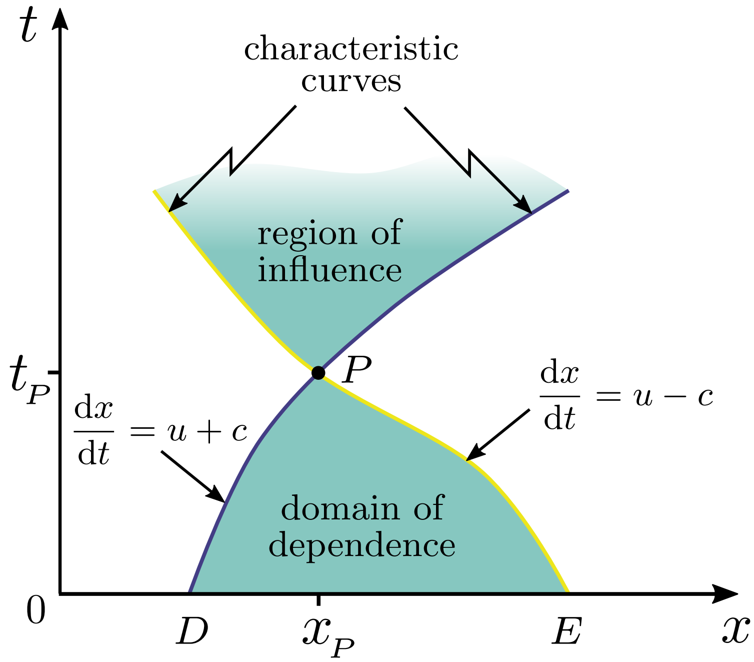
$$\begin{cases} g(x) = 0 \\ f(x) = e^{-x^2} \end{cases}$$

$$u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x+ct)^2}$$

**Standing wave** To get standing wave we want $G = 0$, i.e.,

$$\begin{cases} f(x) = F(x) \\ g(x) = cF'(x) \end{cases} \quad \Rightarrow g(x) = cf'(x)$$

**Domain of dependence and region of influence** Domain of dependence of $u$ in point $(x_0, t_0)$ is a characteristic triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$, $(x_0, t_0)$. Any point outside of triangle doesn't affect the value of $u$ in point.
Region of influence of point $x_0$ is cone bounded by condition $x_0 - ct < x < x_0 + ct$.




**Weak solution**

$$\begin{cases} g(x) = 0 \\ f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}$$

The weak solution

$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)$$

is not differentiable, but is solves the equation in some sense.

## 3.1 Generalization of d'Alembert's formula for non-homogeneous equations

Consider non-homogeneous equation

$$u_{tt} - c^2 u_{xx} = \varphi(x,t)$$

Remember Green's theorem, for differentiable $P$ and $Q$ defined in $\Omega$:

$$\iint\limits_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t} \,\mathrm{d}x\,\mathrm{d}t = \oint\limits_{\partial\Omega} P(x,t)\,\mathrm{d}x + Q(x,t)\,\mathrm{d}t$$

Lets define $Q = c^2 u_x$ and $P = u_t$, and choose $\Omega(x_0, t_0)$ to be characteristic triangle.

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\Omega(x_0,t_0)} u_{tt} - c^2 u_{xx}\,\mathrm{d}x\,\mathrm{d}t = \oint P(x,t)\,\mathrm{d}x - Q(x,t)\,\mathrm{d}t = -\left[\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t\right]$$

Lets divide the curve integral into three integrals along each of lines. For first line $\mathrm{d}t = 0$, for second $\mathrm{d}x + c\,\mathrm{d}t = 0$ and for third $\mathrm{d}x - c\,\mathrm{d}t = 0$.

$$\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} \underbrace{u_t}_{g(x)\text{ in }t=0}\,\mathrm{d}x - \int_{(x_0+ct_0,0)}^{(x_0,t_0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x + \int_{(x_0,t_0)}^{(x_0-ct_0,0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s - c\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u + c\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u$$

Since

$$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u = u(x_0,t_0) - f(x_0+ct_0)$$

$$\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u = f(x_0-ct_0) - u(x_0,t_0)$$

we get

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s + 2cu(x_0,t_0) - cf(x_0+ct_0) - cf(x_0-ct_0)$$

from which we get the solution

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

We've got a cadidate for the solution. Let's check that $u$ is actually solving PDE. Define $v$, $w$, such that $w$ is a solution of homogeneous PDE and $v = u - w$, i.e.,

$$v(x,t) = \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Let's show that $v$ solves PDE. Rewrite $v$ as double integral:

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Define

$$H(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi$$

and then

$$v(x,t) = \frac{1}{2c}\int_0^t H(x,t,\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial t} = \frac{1}{2c}\underbrace{H(x,t,t)}_{0} + \frac{1}{2c}\int_0^t \frac{\partial H}{\partial t}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial t} = c[\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 H}{\partial t^2} = c^2[\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2c}\int_0^t \frac{\partial^2 H}{\partial t^2}\,\mathrm{d}\tau = \varphi(x,t) + \frac{c}{2}\int_0^t \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial x} = \frac{1}{2c}\int_0^t \frac{\partial H}{\partial x}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial x} = \varphi(x + c(t-\tau), \tau) - \varphi(x - c(t-\tau), \tau)$$

$$\frac{\partial^2 H}{\partial x^2} = \varphi_x(x + c(t-\tau), \tau) - \varphi_x(x - c(t-\tau), \tau)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2c}\int_0^t \varphi_x(x + c(t-\tau), \tau) - \varphi_x(x - c(t-\tau), \tau)\,\mathrm{d}\tau$$

Thus we got

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = \varphi(x,t)$$

Suppose we have two solutions $u_1$ and $u_2$ then $u = u_1 - u_2$ is solution of homogeneous equation with 0 initial conditions, and thus $u = 0$. That means the solution is unique.
The presented initial condition problem has 3 properties:

1. Solution exist
2. It's unique
3. It's stable

**Stability of wave equation** For all $\tau > 0$, $\epsilon > 0$, exists $\delta > 0$ such that if

$$\begin{cases} |f(x) - \tilde{f}(x)| < \delta \\ |g(x) - \tilde{g}(x)| < \delta \\ |\varphi(x) - \tilde{\varphi}(x)| < \delta \end{cases}$$

For all $-\infty < x < \infty$ and $0 \le t \le \tau$ and if $u$, $\tilde{u}$ are solutions of corresponding wave equations, then

$$|u(x,t) - \tilde{u}(x,t)| < \epsilon$$

**Proof** From the general solution:

$$u(x,t) - \tilde{u}(x,t) =$$

$$= \left|\frac{f(x+ct) + f(x-ct)}{2} - \frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) - \tilde{g}(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right| \le$$

$$\le \left|\frac{f(x+ct) - \tilde{f}(x+ct)}{2}\right| + \left|\frac{f(x-ct) - \tilde{f}(x-ct)}{2}\right| + \frac{1}{2c}\int_{x-ct}^{x+ct} |g(s) - \tilde{g}(s)|\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} |\varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \le$$

$$\le \frac{\delta}{2} + \frac{\delta}{2}\frac{1}{2c}\cdot 2c \cdot \delta + \frac{1}{2c}\frac{ct}{2} = 2\delta + \frac{\delta t}{4} \le 2\delta + \frac{\delta\tau}{4} \le \epsilon$$

Thus we choose $\delta < \frac{\epsilon}{2 + \frac{\tau}{4}}$.

## 3.2 Wave equation with bound conditions

**Half-infinite string** Suppose string is fixed in one of its ends, at $x = 0$: $u(0,t) = 0$. This is called Dirichlet boundary condition. We want to solve the PDE for $x > 0$.

**Property of wave equation** If $u(x,t)$ is solution, then $u(-x,t)$ is also solution:

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$u(-x,t) = F(-x+ct) + G(-x-xt) = \bar{F}(x+ct) + \bar{G}(x-ct)$$

Where $\bar{F}(s) = G(-s)$ and $\bar{G}(s) = F(-s)$.

Lets extend $f$ and $g$ on the whole plane in odd way:

$$\bar{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(x) & x < 0 \end{cases}$$

and same for $g$.
Note that initial conditions have to be consistent, i.e., $f(0) = 0$, $g(0) = 0$, else the solution is discontinuous in 0. Lets use D'Lambert solution:

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(s)\,\mathrm{d}s$$

Then the solution of half-infinite string is

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

**Neumann boundary condition** In this case, instead of giving boundary condition on $u$, we give boundary condition of $u_x$: $u_x(0,t) = 0$. Physical meaning is that there is no force in this point. In this case we will extend function in even way (derivative of even function in 0 is 0). Then we get

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{c}\int_{0}^{ct-x} g(s)\,\mathrm{d}s + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

**Uniquness** Suppose we have two solutions, by subtracting them, we get

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = u_x(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

We acquire $u(x,t) = 0$. Since $u(x,t)$ is of form $F(x+ct) + G(x-ct)$, we get that twobsolutions are identical.

**Wave equation with finite string** Suppose we have string from $a$ to $b$:

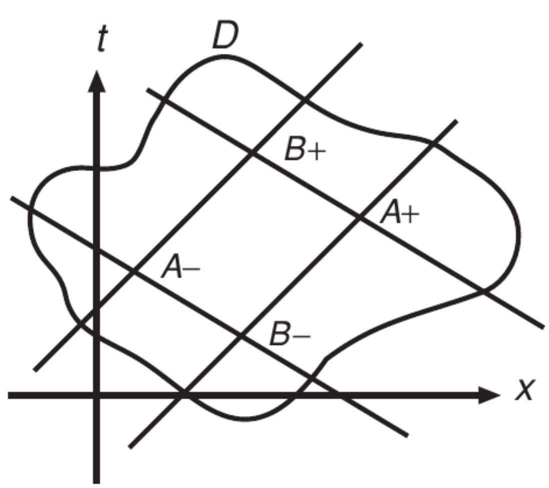
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & a \le x \le b \\ u(x,0) = f(x) \\ u_t(a,t) = h(t) \\ u_t(b,t)b = q(t) \end{cases}$$

Here the consistency conditions are

$$\begin{cases} h(0) = f(a) & q(0) = f(b) \\ h'(0) = g(a) & q'(0) = g(b) \end{cases}$$

Here we could have conditions derivatives instead of values as well.

**Parallelogram identity**



$u \in \mathcal{C}^2$ is the solution of wave equation iff for any parallelogram with sides parallel to characteristic lines with vertices $A_-$, $A_+$, $B_-$, $B_+$

$$u(A_-) + u(A_+) = u(B_-) + u(B_+)$$

**Proof** One direction is simple, since value of solution is constant along characteristic curves.
For second direction lets switch to canonical coordinates

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

So

$$w(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

$w$ is solution of wave equation iff $w_{\xi\eta} = 0$.
Note that parallelogram turned into rectangular in new coordinates, i.e.,

$$w(\xi_0, \eta_1) + w(\xi_1, \eta_0) = w(\xi_1, \eta_1) + w(xi_0, \eta_0)$$

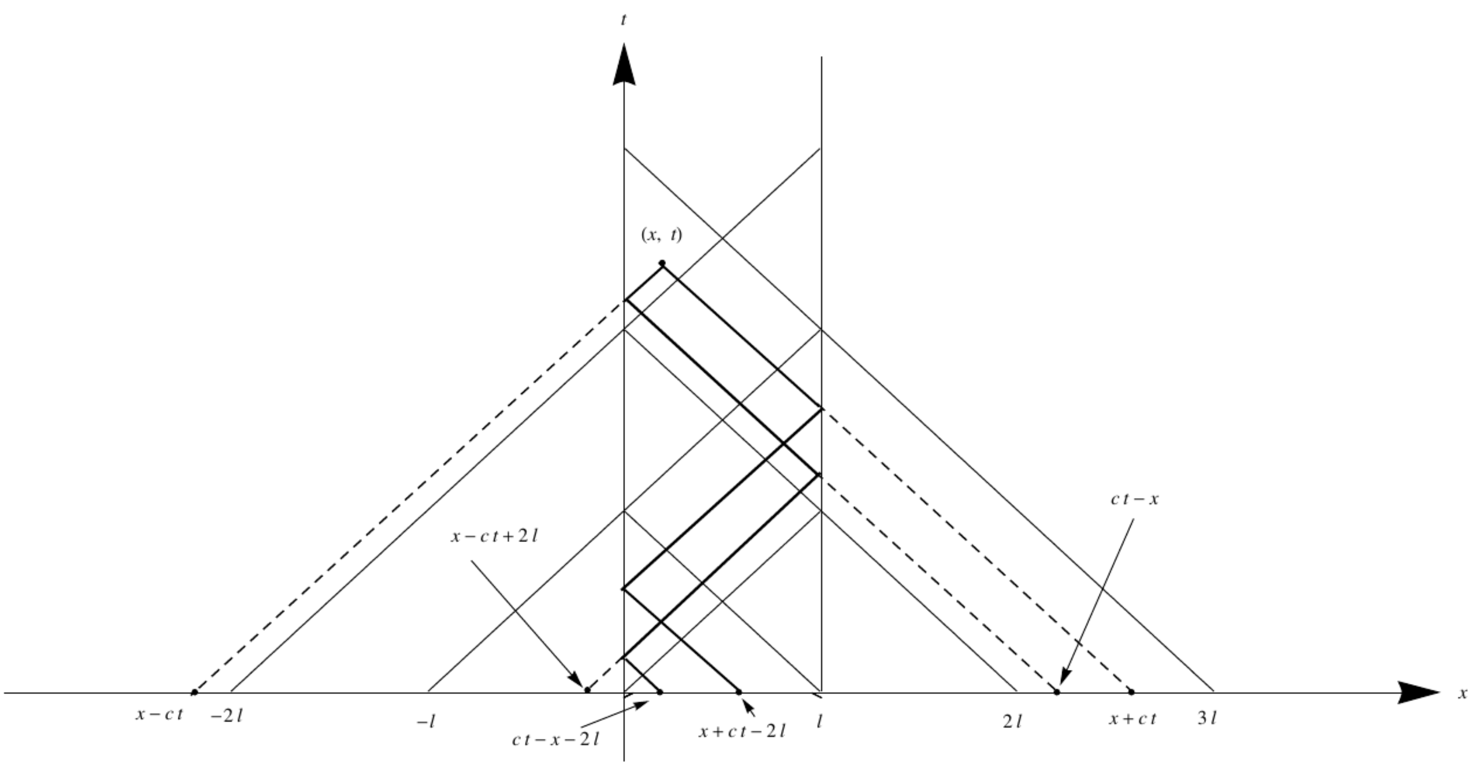
Dividing by $(\xi_1 - \xi_0)(\eta_1 - \eta_0)$:

$$\frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} - \frac{w(\xi_1, \eta_1) + w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = 0$$

Taking limit:

$$\lim_{\xi_1 \to \xi_0} \lim_{\eta_1 \to \eta_0} \frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0) - w(\xi_1, \eta_1) - w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = w_{\xi\eta} = 0$$

In this way we can solve wave equation on finite range:




Using this method we can get solution for homogeneous wave equation for finite string.

### Non-homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = \varphi(x, t)$$

In this case parallelogram identity doesn't work.
Lets extend $\varphi$ to the half plane $x > 0$ to some function $\tilde{\varphi} \in \mathcal{C}^1$.
Lets solve non-homogeneous equation with 0 initial conditions: $w = \frac{1}{2c} \int\limits_{\triangle} \tilde{\varphi}$.

with D'Lambert formula. Lets solve homogeneous equation in the interval:

$$\begin{cases} v(a,t) = h(t) + w(a,t) \\ v(b,t) = q(t) + w(b,t) \\ v(x,0) = f(x) \\ v_t(x,0) = g(x) \end{cases}$$

Then the solution is

$$u(x,t) = w(x,t) + v(x,t)$$

Checking the solution:

$$u_{tt} - c^2 u_{xx} = w_{tt} - c^2 w_{xx} + v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

which is $\varphi(x,t)$ in our interval.

**Energy method** Define

$$E(t) = \int_a^b \left[u_t^2(x,t) + c^2 u_x^2(x,t)\right] \mathrm{d}x$$

If $u \in \mathcal{C}^2$ we can differentiate it:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \left[2u_t u_{tt} + 2c^2 u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b \left[u_t u_{xx} + u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b (u_x u_t)_x \,\mathrm{d}x = 2c^2 [u_x(b,t)u_t(b,t) - u_x(a,t)u_t(a,t)]$$

If any combination of Dirichlet and Neumann conditions is fulfilled, the integral is 0, i.e., energy is conserved. Thus

$$E(t) = E(0) = \int_a^b g^2(x) + c^2 (f'(x))^2 \,\mathrm{d}x$$

From that we can conclude the solution is unique. As usual, suppose there are two solutions, $u$ and $v$. Subtracting we get a solution for homogeneous equation with homogeneous initial conditions $w = u - v$. Then $E_w(t) = E_w(0) = 0$, thus $w_x = w_t = 0$ and $w(x,t) = 0$.
Also, from energy difference, we can conclude the solutions are stable.

## 3.3 Variable separation

Lets guess solution of form

$$U(x,t) = A(x)B(t)$$

Substituting into wave equation:

$$u_{tt} - c^2 u_{xx} = A(x)B''(t) - c^2 A''(x)B(t) = 0$$

Dividing by $A(x)B(t)$ (assume they are not zero):

$$\frac{B''(t)}{B(t)} = c^2 \frac{A''(x)}{A(x)} = \mu$$

That means

$$\begin{cases} A'' = \frac{\mu}{c^2} A = -\lambda A \\ B'' = \mu B \end{cases}$$

Back to initial conditions $u(0,t) = u(1,t) = 0$, that means $A(0) = A(1) = 0$. The question is when

$$A'' + \lambda A = 0$$

If $\lambda < 0$, the solution is

$$A = \alpha e^{-\sqrt{-\lambda} x} + \beta e^{\sqrt{\lambda} x}$$

Substituting initial conditions we get

$$\begin{cases} \alpha + \beta = 0 \\ \alpha e^{-\sqrt{-\lambda}} + \beta e^{\sqrt{-\lambda}} = 0 \end{cases}$$

Since $\lambda \neq 0$, we conclude $\alpha = \beta = 0$ which is trivial solution. If $\lambda = 0$ we get $A = \alpha x + \beta$, which is also trivially $A = 0$. If $\lambda > 0$,

$$A = \alpha \sin\left(\sqrt{\lambda} x\right) + \beta \cos\left(\sqrt{\lambda} x\right)$$

Since $A(0) = 0$, $\beta = 0$. Since $A(1) = 0$, $\sqrt{\lambda} = k\pi$ for some $k \in \mathbb{N}$, i.e., $\lambda_k = k^2\pi^2$. The solution is

$$A_k = \sin(k\pi x)$$

Back to $B$:

$$\frac{B''}{B} = -c^2k^2\pi^2$$

i.e.,

$$B_k(t) = a_k \sin(ck\pi t) + b_k \cos(ck\pi t)$$

Thus the solution of wave equation

$$u_k(x,t) = a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

(note, that by using trigonometric identities we can get it to canonical form).
Define

$$u(x,t) \sim \sum_{k=1}^{\infty} a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

Substituting $t = 0$:

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f(x)$$

$$u_t(x,0) = c\pi \sum_{k=1}^{\infty} k a_k \sin(k\pi x) \ \ = g(x)$$

If we'll find $a_k$, $b_k$ fulfilling those conditions, then we have a "solution". How we find them? Look at following integral:

$$\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x = \int_0^1 \left[\sum_{k=1}^{\infty} b_k \sin(k\pi x)\right]\sin(n\pi x)\,\mathrm{d}x = \sum_{k=1}^{\infty} b_k \underbrace{\int_0^1 \sin(k\pi x)\sin(n\pi x)\,\mathrm{d}x}_{\frac{\delta_{nk}}{2}} = \frac{b_n}{2}$$

Thus,

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x$$

Exactly in the same way we can get

$$a_n = \frac{2}{c\pi n}\int_0^1 g(x)\sin(n\pi x)\,\mathrm{d}x$$

**Convergence** If $\sum |ka_k| < \infty$ and $\sum |b_k| < \infty$, our series converge uniformly. If also $\sum |k^2 a_k| < \infty$ and $\sum |kb_k| < \infty$, then $u(x,t) \in \mathcal{C}^1$. Analogously, if $\sum |k^3 a_k| < \infty$ and $\sum |k^2 b_k| < \infty$, $u \in \mathcal{C}^2$.
Suppose $\max\limits_{(0,1)} |f| < M_0$, then

$$|b_n| \leq 2\int_0^1 |f(x)\sin(n\pi x)|\,\mathrm{d}x \leq 2M_0$$

. Suppose also that $f \in \mathcal{C}^1$ and $\max\limits_{(0,1)} |f'| < M_1$ then

$$b_n = -\frac{2}{n\pi}\int_0^1 f(x)(\cos(n\pi x))'\,\mathrm{d}x$$

Integrating by parts and using the fact $f(0) = f(1) = 0$:

$$b_n = \frac{2}{n\pi}\int_0^1 f'(x)\cos(n\pi x)\,\mathrm{d}x \leq \frac{2}{n\pi}M_1$$

To show that $b_n < \frac{B_2}{n^2}$, we need $f \in \mathcal{C}^2$, $\max\limits_{(0,1)} |f''| < M_2$ and $f'(0) + f'(1) = 0$.
In general, if $f \in \mathcal{C}^l$ and sum $f^{(l-1)}(0) + f^{(l-1)}(1) = 0$, we can bound $|b_n| < \frac{B_l}{n^l}$.

**Generalization**

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(0) \\ u(0,t) = h(t) \\ u(1,t) = q(t) \end{cases}$$

Define $w(x,t)$ such that $w(0,t) = h(t)$ and $w(1,t) = q(t)$ and $v = u - w$. Then

$$v_{tt} - c^2 v_{xx} = u_{tt} - c^2 u_{xx} - w_{tt} + c^2 w_{xx} = \varphi(x,t) - w_{tt} + c^2 w_{xx} = \tilde{\varphi}(x,t)$$

Thus we can assume bound conditions are 0, as soon as we can solve non-homogeneous equation.

$$v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

Guess solution

$$v(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\pi x)$$

Substituting:

$$\sum_{k=1}^{\infty} \left(q_k''(t) + k^2 c^2 \pi^2 q_k(t)\right) \sin(k\pi x) = \tilde{\varphi}(x,t)$$

Suppose we can expand

$$\tilde{\varphi}(x,t) = \sum_{k=1}^{\infty} p_k(t) \sin(k\pi x)$$

.

$$\int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x = \sum_{k=1}^{\infty} p_k(t) \int_0^1 \sin(k\pi x) \sin(n\pi x) \, \mathrm{d}x = \frac{p_n(t)}{2}$$

Thus

$$p_n(t) = 2 \int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x$$

By coefficient comparison:

$$q_k''(t) + k^2 c^2 \pi^2 q_k(t) = 2 \int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x$$

Since we know that

$$\begin{cases} v(x,0) = \sum_{k=1}^{\infty} q_k(0) \sin(k\pi x) = f(x) \\ v_t(x,0) = \sum_{k=1}^{\infty} q_k'(0) \sin(k\pi x) = g(x) \end{cases}$$

i.e.,

$$q_k(0) = 2 \int_0^1 f(x) \sin(k\pi x) \, \mathrm{d}x$$

$$q_k'(0) = 2 \int_0^1 g(x) \sin(k\pi x) \, \mathrm{d}x$$

Meaning we can solve ODE, and get solution for wave equation.

**Neumann bound conditions** Once again guessing the solution

$$\begin{cases} u(x,t) = A(x) B(t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ A'(0) = A'(1) = 0 \end{cases}$$

We get once again

$$A(x) = a \sin\left(\sqrt{\lambda} x\right) + b \cos\left(\sqrt{\lambda} x\right)$$

$$A'(x) = \sqrt{\lambda} a \cos\left(\sqrt{\lambda} x\right) - \sqrt{\lambda} b \sin\left(\sqrt{\lambda} x\right)$$

Substituting initial conditions:

$$A'(0) = \sqrt{\lambda}a \Rightarrow a = 0$$

$$A'(1) = \sqrt{\lambda}b\sin\left(\sqrt{\lambda}x\right)$$

Thus we get the same $\lambda_k = k\pi$, however the series contains cosines instead of sines:

$$A_k(x) = \cos(k\pi x)$$

and we can solve in a similar way.

**Operator of hyperbolic equation** We can define linear operator

$$L(u) = \left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\right)$$

Then the equation is

$$L(u) = h$$

We can turn it into canonical form:

$$L'(u) = \left(\frac{\partial^2}{\partial\xi\partial\eta} + d'\frac{\partial}{\partial x} + f'\frac{\partial}{\partial y} + g'\right)$$

**Cauchy problem for hyperbolic equation** Given a curve in space $\vec{\mathbf{r}}(s) = \big(\bar{x}(s), \bar{y}(s)\big)$, ge define initial conditions

$$\begin{cases} u\big(x(s), y(s)\big) = h(s) \\ u_x\big(x(s), y(s)\big) = \varphi(s) \\ u_y\big(x(s), y(s)\big) = \psi(s) \end{cases}$$

However, since we need two conditions, there is consistency requirement on those functions:

$$\frac{\mathrm{d}h}{\mathrm{d}s} = u_x\big(x(s), y(s)\big)\frac{\partial\bar{x}}{\partial s} + u_y\big(x(s), y(s)\big)\frac{\partial\bar{x}}{\partial s} == \varphi(s)\frac{\partial\bar{x}}{\partial s} + \psi(s)\frac{\partial\bar{x}}{\partial s}$$

Suppose we have equation of form

$$\begin{cases} au_{xx} + 2bu_{xy} + cu_{yy} = d \\ u_{xx}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{xy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s} \\ u_{xy}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{yy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}s} \end{cases}$$

To have an opportunity to evaluate second derivatives, we need to find solution of this linear system, i.e., we need that

$$\begin{vmatrix} a & 2b & c \\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} & 0 \\ 0 & \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \end{vmatrix} \neq 0$$

or

$$a\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}s}\right)^2 - 2b\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} + c\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\right)^2 \neq 0$$

meaning the direction of tangent line is not in direction of characteristic lines.
We can derive the system once again and thus find third-order derivatives, doing it up to infinity, we get all the partial derivative.

**Cauchy–Kowalevski theorem** If the coefficients and initial curve are analytic functions, then exists unique analytic solution.

# 4 Heat equation

For some positive $k$

$$u_t - ku_{xx} = 0$$

Temperature $u(x, t)$ fulfills heat equation.

**Dirichlet bound conditions**

$$u(a, 0) = u(b, 0) = 0$$

**Neumann bound conditions** The meaning of Neumann bound condition is that there is heat isolation in interval bounds:

$$u_x(a,t) = u_x(b,t) = 0$$

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_a^b \frac{\partial Q}{\partial t}\,\mathrm{d}x = k\int_a^b \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = k\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right] = 0$$

Thus $Q(t)$ is constant.

$$Q(t) = Q(0) = \int_a^b f(x)\,\mathrm{d}x$$

**Solution of heat equation** Suppose, for Dirichlet bound condition, that solution is series of sines.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t)\sin(n\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -(n\pi)^2 a_n(t)\sin(n\pi x)$$

Then we get

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\left[a_n'(t) + k(n\pi)^2 a_n(t)\right]\sin(n\pi x) = 0$$

Thus, by coefficient comparison

$$a_n'(t) + k(n\pi)^2 a_n(t) = 0$$

$$a_n(t) = a_n(0)e^{-k(n\pi)^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(0)e^{-k(n\pi)^2 t}\sin(n\pi x)$$

From initial conditions we can find $a_n(0)$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\sin(n\pi x)$$

Our series is infinite differentiable if $t > 0$.
If $t = 0$, we need

$$\lim_{t\to 0^-} u(x,t) = f(x)$$

For $t < 0$, coefficients diverge, and thus we can't find solutions for $t < 0$. Physically we can see it from entropy grows.

**Example**

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = 1 \\ u(0,t) = u(1,t) = 0 \quad t > 0 \end{cases}$$

We acquire

$$a_n = \frac{1}{2}\int_0^1 1\cdot\sin(n\pi x)\,\mathrm{d}x$$

$$|a_n| \le \int_0^1 \mathrm{d}x = \frac{1}{2}$$

Thus

$$|a_n| \le \frac{1}{2}e^{-k(n\pi)^2 t}$$

Thus the series absolutely converges for $t > 0$.
In limit $t \to \infty$, $a_n \to 0$, and thus $u(x,t) \to 0$.

**Stability** For each $\epsilon > 0$ exists $\delta > 0$ such that if

$$\max_{a \leq x \leq b} |u(x,0)| < \delta$$

then

$$\max_{a \leq x \leq b} |u(x,t)| < \epsilon$$

**Proof** We'll proof the weaker version of the theorem, with condition that $\sum |a_n| < \delta$. Then coefficients of $u(x,t)$ are bounded by

$$a_n(t) \leq e^{-k(n\pi)^2 t} \frac{\delta}{2}$$

i.e.

$$|u(x,t)| \leq \sum_{n=1}^{\infty} |a_n| e^{-k(n\pi)^2 t} < \sum |a_n| < \delta$$

# 5 Potential equation

$$u_{xx} + u_{yy} = 0$$

Bound conditions are

$$\begin{cases} u(x,0) = f(x) \\ u_x(x,0) = g(x) \end{cases}$$

By variable separation we get

$$u_n = A_n(x) B_n(y)$$

we know that

$$A_n(x) = \sin(n\pi x)$$

Since

$$(n\pi)^2 = \frac{A_n''}{A_n} = -\frac{B_n''}{B_n}$$

$$B_n(y) = \alpha_n \sinh(n\pi y) + \beta_n \cosh(n\pi y)$$

**Stability** Is potential equation stable? Suppose $\max |f(x)| < \delta$ and $\max |g(x)| < \delta$
No. For example

$$u(x,y) = \frac{1}{n^3} e^{(n\pi)^2 y} \sin(n\pi x)$$

Then

$$u(x,0) = \frac{\sin(n\pi x)}{n^2}$$

$$u_y(x,0) = \frac{\sin(n\pi x)}{n}$$

This doesn't fulfills stability condition

$$|u_n(x,y)| < \epsilon$$

for large enough $n$.

**Laplace equation**

$$\nabla^2 u = 0$$

If $u$ fulfills $\nabla^2 u = 0$, it is called harmonic function.

**Poisson equation**

$$\nabla^2 u = f$$

Where $f$ describes mass/charge distribution in space.

**Elliptic PDEs** For elliptic equation $b^2 - 4ac < 0$. In this case, we can get the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + L_1(\xi, \eta) = f$$

where $L_1$ is first-order differential operator.

## 5.1 Laplace equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \nabla^2 u = 0$$

**Gauss law** For vector field $\vec{\mathbf{w}} \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega}$

$$\iint\limits_{\Omega} \boldsymbol{\nabla} \cdot \vec{\mathbf{w}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \vec{\mathbf{w}} \cdot \hat{\mathbf{n}} \, \mathrm{d}s$$

Thus

$$\iiint\limits_{\Omega} \nabla^2 \vec{\mathbf{u}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s$$

i.e., if function is harmonic,

$$\oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s = 0$$

**Conclusion** The equation $\nabla^2 u = f$ in $\Omega$ for $u$ fulfilling $\frac{\partial u}{\partial n} = g$ in each point of $\partial\Omega$ there is no solution if

$$\iiint\limits_{\Omega} \neq \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

The necessary condition for solution of Neumann problem

$$\iiint\limits_{\Omega} = \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

For Dirichlet problem, there is no such constraint.

**Examples of harmonic functions** In $n = 1$, linear functions are harmonic.
In $n = 2$, any real or imaginary part of analytic function is harmonic, e.g., $e^x \sin y$.

**The mean value property of harmonic function** If $u$ is harmonic in $\Omega$ which contains $B_R(x) = \{y||x - y| < R\}$ then

$$u(x) = \frac{1}{|\partial B_R(x)|} \oint\limits_{\partial\Omega} u \, \mathrm{d}s$$

and

$$u(x) = \frac{1}{|B_R(x)|} \int\limits_{\Omega} u(y) \, \mathrm{d}^n y$$

**Proof** Suppose $x = 0$. Rewrite $y \in B_R(x)$ as $y = \rho\alpha$ for $\alpha = \frac{y}{\|y\|}$ and $\rho = \|y\|$.
For each $\rho \in [0, R]$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \iint \nabla^2 u \, \mathrm{d}^n x = 0$$

With variable substitution $\mathrm{d}s_y = \rho^{n-1} \, \mathrm{d}s_\alpha$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \rho^{n-1} \, \mathrm{d}s_\alpha = \rho^{n-1} \int\limits_{B_\rho(0)} \frac{\partial u}{\partial \rho} \, \mathrm{d}s_\alpha = \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha) \, \mathrm{d}s_\alpha$$

Thus

$$\frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha)\,\mathrm{d}s_\alpha = 0$$

meaning

$$H(\rho) = \int\limits_{B_\rho(0)} u(\rho\alpha)\,\mathrm{d}s_\alpha = \text{const}$$

Denote volume of unit ball as $\omega_n$, then $|B_n| = \omega_n R^n$ and $|\partial B_n| = \omega_{n-1} R^{n-1}$

$$H(0) = \omega_n \cdot u(0)$$

And since $H(1) = H(0)$:

$$u(0) = \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(0)} \mathrm{d}s$$

as required.

$$\omega_n \rho^{n-1} u(0) = \int\limits_{\partial B_\rho(0)} u(y)\,\mathrm{d}s_y$$

$$\int_0^R n\omega_n \rho^{n-1} u(0)\,\mathrm{d}\rho = \int_0^R \mathrm{d}\rho \int\limits_{\partial B_\rho(0)} \mathrm{d}s_y\, u(y)$$

$$\omega_n R^n u(0) = \iint\limits_{B_r(0)} u(y)\,\mathrm{d}^n y$$

i.e.,

$$u(0) = \frac{1}{|B_r(0)|} \iint\limits_{B_r(0)} u(y)\,\mathrm{d}^n y$$

**Strong maximum principle** If $u$ is harmonic and it acquires maximum or minimum, then it is constant.

**Subharmonic and superharmonic functions** Subharmonic function is function for which $\nabla^2 u \leq 0$ and superharmonic is one for which $\nabla^2 u \geq 0$.
In this case, strong maximum principle applies only in one direction (maximum for subharmonic, minimum for superharmonic). For mean value theorem we get inequality instead of equality.

**Proof** let $u$ subharmonic, and $m = u(x) = \max\limits_{\Omega} u$. The set $W = \{y : u(y) = m\}$ is closed relatively to $\Omega$. Let $z \in W$, $B_R(z) \in \Omega$.

$$m = u(z) \leq \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\,\mathrm{d}^n y = m$$

Thus for all $z \in W$

$$u(z) = \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\,\mathrm{d}^n y$$

That means

$$\int\limits_{B_r(z)} u(x) - m\,\mathrm{d}^n x = 0$$

Thus $u(x) = m$ for all $x \in B_R(z)$, which turns $W$ is open set. Thus $W$ is both open and closed, i.e. $W = \Omega$.

**Conclusion** Poisson equation $\nabla^2 u = f$ in $\Omega$ with bound condition $u_{\partial\Omega} = y$ has not more than one solution.

**Proof** Suppose there are two solutions, $u_1 = u_2$, define $v = u_1 - u_2$ is harmonic function with ound condition $v_{\partial\Omega} = 0$. The function $v$ is harmonic. If $v \neq 0$, it has either maximum or minimum, in contradiction with strong maximum principle.

**Weak maximum principle** For compact connected $\Omega$ and $u = \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$
If $\nabla^2 u \leq 0$,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

If $\nabla^2 u \geq 0$,

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$$

**Harnack's inequality** If $u$ harmonic and non-negative in interval $\Omega$ and $\bar{\Omega}' \subsetneq \Omega$, then exists constant $c(\Omega, \Omega')$ independent on $u$ such that

$$\sup_{\Omega'} u \leq c \inf_{\Omega} u$$

i.e., for any two points $x, y \in \Omega'$

$$u(x) \leq cu(y)$$

**Proof** Let $\Omega = B_{4R}(y)$ and $\Omega' = B_R(y)$. Choose $x_1, x_2 \in \Omega'$.
Since $B_R(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_1) = \frac{1}{\omega_n R^n} \int\limits_{B_R(x_1)} \mathrm{d}x\, u \leq \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u$$

Similarly, since $B_{3R}(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int\limits_{B_{3R}(x_2)} \mathrm{d}x\, u \geq \frac{1}{\omega_n (3R)^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u = \frac{1}{3^n} \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u \geq u(x_1)$$

We got

$$3^n u(x_2) \leq u(x_1)$$

Since $\bar{\Omega}' \subsetneq \Omega$, there exists $R > 0$ such that distance from any point of $\Omega'$ to any point of $\Omega^c$ is greater than $4R$.
For any pair of points $x_1, x_2 \in \Omega$, the path between then can be covered by $m$ balls $B_j$ of radius $R$, such that intersection of each pair of consequentive balls is non-empty.
So, let $y_j \in B_j \cap B_{j+1}$ and $y_1 = p$, $y_m = q$, then $u(y_j) \leq 3^n u(y_{j+1})$, since $y_j, y_{j+1} \in_{j+1}$ and $B_{4R}(y_{j+1}) \subset \Omega$. Than means that

$$u(q) \leq 3^{nm} u(p)$$

**Radial harmonic functions** Lets search for harmonic functions of form $u(x) = f(r)$ for $f$ defined on $\mathbb{R}^+$.

$$|x| = \sqrt{\sum x_i} = r$$

is harmonic on $\mathbb{R}^n \setminus 0$ and radial.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} - \frac{x_i^2}{r^2}$$

Thus

$$\frac{\partial u}{\partial x_i} = f'(r) \frac{\partial r}{\partial x_i} = f'(r) \frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i^2} = f''(r) \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^2}\right)$$

$$\nabla^2 u = f''(r) \sum \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{n}{r} - \frac{\sum x_i^2}{r^2}\right) = f''(r) + \frac{n-1}{n} f'(r) = 0$$

This is Euler equation with solution For $n = 2$

$$f(r) = c_1 \ln r + c_2$$

For $n > 2$:

$$f(r) = \frac{c_1}{r^{n-2}} + c_2$$

**Fundamental solution** Define fundamental solution of Laplace equation:

$$\begin{cases} \Gamma(r) = \frac{1}{2\pi}\ln(r) & n=2 \\ \Gamma(r) = \frac{1}{n(2-n)\omega_n} r^{2-n} & n>2 \end{cases}$$

We conclude that $\Gamma(|x-y|)$ is harmonic function in $\mathbb{R}^n \setminus \{y\}$.
For $n = 2$

$$\lim_{r\to\infty} \Gamma(n) = \infty$$

and for $n > 2$

$$\lim_{r\to\infty} \Gamma(n) = 0$$

Also, for any $n \geq 2$

$$\lim_{r\to 0} \Gamma(n) = -\infty$$

**Homogenity of $\Gamma$** For $n > 2$

$$\frac{\partial \Gamma(x-y)}{\partial x_i} = \frac{1}{n\omega_n}\frac{x_i - y_i}{|x-y|^n}$$

$$\frac{\partial^2 \Gamma}{\partial x_i \partial x_j} = \frac{1}{n\omega_n}\Big[|x-y|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)\Big]|x-y|^{-n-2}$$

$$\left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| \leq \frac{1}{n\omega_n}|x-y|^{1-n}$$

$$\left|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}\right| \leq \frac{1}{n\omega_n}|x-y|^{-n}$$

Also

$$\begin{cases} \Gamma'(r) = \frac{1}{n\omega_n r} & n=2 \\ \Gamma'(r) = \frac{1}{n\omega_n} r^{1-n} \end{cases}$$

**Green identities** If $\Omega \subset \mathbb{R}^n$ bounded set with boundcin $\mathcal{C}^2$ and $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$\int\limits_{\Omega} v\nabla^2 u + \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \, \mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}s$$

$$\int\limits_{\Omega} v\nabla^2 u - u\nabla^2 v \, \mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\,\mathrm{d}s$$

If $\Omega \subset \mathbb{R}^n$ bounded set with bound in $\mathcal{C}^1$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$u(y) = \int\limits_{\Omega} \Gamma(x-y)\nabla^2 u\,\mathrm{d}x + \int\limits_{\partial\Omega}\left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

If $u$ harmonic

$$u(y) = \int\limits_{\partial\Omega}\left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

**Proof** $\Gamma(x-y)$ is harmonic in $\Omega \setminus B_\rho(y) = \Omega_\delta$. Choose $v(x) = \Gamma(x-y)$,

$$\int\limits_{\Omega_\delta} \Gamma\nabla^2 u\,\mathrm{d}x = \int\limits_{\partial\Omega}\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\,\mathrm{d}s_x - \int\limits_{\partial B_\rho(y)}\Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\,\mathrm{d}s$$

Now

$$\int B_\rho(y)\Gamma(|x-y|)\nabla^2 u(x)\,\mathrm{d}x \leq \max_{B_\rho(y)} \cdot \int B_\rho(y)\Gamma(|x-y|)\,\mathrm{d}x = \frac{\omega_n}{n\omega_n}\int_0^\rho r^{2-n}r^{n-1}\,\mathrm{d}r = \frac{1}{n}\int_0^\rho r\,\mathrm{d}r = \frac{\rho^2}{2n} \xrightarrow{\rho\to 0} 0$$

$$\left|\frac{\partial u}{\partial n}\right| = \left|\nabla^2 u \cdot n\right| \leq C$$

$$\left| \int\limits_{\partial B_\rho(y)} \Gamma \frac{\partial u}{\partial n} \right| \le C \int\limits_{B_\rho(y)} |\Gamma| = \frac{c}{n\omega_n} \rho^{2-n} n\omega_n \rho^{n-1} = C\rho \to 0$$

$$\int\limits_{B_\rho(y)} u \frac{\partial \Gamma}{\partial n} \,\mathrm{d}s_x = \Gamma'(\rho) \int\limits_{B_\rho(y)} u \,\mathrm{d}s_x = \frac{1}{n\omega_n \rho^{n-1}} \int\limits_{B_\rho(y)} u \,\mathrm{d}s_x \xrightarrow{\rho \to 0} u(y)$$

Substituting it back into equation we get exactly what was needed:

$$u(y) = \int\limits_{\Omega} \Gamma(x-y) \nabla^2 u \,\mathrm{d}x + \int\limits_{\partial\Omega} \left[ u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] \mathrm{d}s_x$$

**Green function** Green function can be understood as inverse of Laplacian operator in a sense that if $\nabla^2 u = f$, $u(c) = \int G(x,y) f(y) \,\mathrm{d}y$. For all $y \in \Omega$ define $h^y(x)$ such that

- $h^y(x)$ is harmonic in $\Omega$
- $h^y(x) = -\Gamma(|x-y|)$ for all $x \in \partial\Omega$

(suppose there exists one).
Now define Green function $G(x,y) = \Gamma(|x-y|) + h^y(x)$. Properties of $G$:

1. $G$ harmonic for $x \neq y$
2. $G(x,y) = 0$ for all $x \in \partial\Omega$, $y \in \Omega$

Using second Green identity

$$\int\limits_{\Omega} h \nabla^2 u \,\mathrm{d}x = \int\limits_{\partial\Omega} h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n} \,\mathrm{d}s$$

and summing it with $u(y)$ we get:

$$u(y) = \int\limits_{\Omega} G(x,y) \nabla^2 u \,\mathrm{d}x + \int\limits_{\partial\Omega} u(x) \frac{\partial}{\partial n_x} G(x,y) \,\mathrm{d}s_x$$

**Conclusion** If $\nabla^2 u = f$ on $\Omega$ and $u = y$ on $\partial\Omega$ the solution

$$u(y) = \int\limits_{\Omega} G(x,y) f(x) \,\mathrm{d}x + \int\limits_{\partial\Omega} g(x) \frac{\partial G}{\partial n} \,\mathrm{d}s_x$$

**Lemma** For all $x \neq y$, $x, y \in \Omega$

$$G(x,y) = G(y,x)$$

In particular, for constant $x$, $G$ is harmonic in $y$.

**Proof** Define

$$V_x(z) = G(z,x)$$

and

$$W_y(z) = G(z,y)$$

We want to show that $V_x(y) = W_y(x)$ on $\Omega \setminus (B_\epsilon(x) \cap B_\epsilon(y))$

$$0 = \int\limits_{\Omega_\epsilon} V_x \nabla^2 W_y - V_y \nabla^2 W_x \,\mathrm{d}z = \int\limits_{\partial\Omega_\epsilon} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s =$$

$$= \underbrace{\int\limits_{\partial B_\epsilon(x)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_1} + \underbrace{\int\limits_{\partial B_\epsilon(y)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_2} = I_1(\epsilon) + I_2(\epsilon)$$

$$W_y(x) = \int\limits_{B_\epsilon(x)} V_x \nabla^2 W_y \, \mathrm{d}z - \int\limits_{\partial B_\epsilon(x)} \left[ W_x \frac{\partial V_y}{\partial n} - V_y \frac{\partial W_x}{\partial n} \right] \mathrm{d}s = I_1(\epsilon)$$

$$V_x(z) = \Gamma(|z-x|) + h^y(z)$$

Now

$$V_x(y) = \int\limits_{B_\epsilon(y)} W_y \nabla^2 V_x \, \mathrm{d}z - \int\limits_{\partial B_\epsilon(y)} \left[ V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s = -I_2(\epsilon)$$

Thus

$$W_y(x) - V_x(y) = I_1(\epsilon) + I_2(\epsilon) = 0 \Rightarrow W_y(x) = V_x(y)$$

**Green function in ball** Take a look at $y \in B_R(0)$, define reflection point $y^* = \frac{R^2}{y^2} y$. Note that

$$|y^*||y| = R^2$$

and that if $y \to \partial B_R(0)$, $y^* \to \partial B_R(0)$. Define

$$h^y = -\Gamma(|x-y^*|) \left( \frac{|y|}{R} \right)^{2-n}$$

Then

$$G(x,y) = \Gamma(|x-y|) - \Gamma(|x-y^*|) \left( \frac{|y|}{R} \right)^{2-n}$$

From harmony of $\Gamma$, $G$ is harmonic in $x$. For $x \in \partial B_r$:

$$(x-y^*)^2 = \left( x - \frac{R^2}{y^2} y \right)^2 = |x|^2 - 2x \frac{R^2}{y^2} y + \frac{R^4}{|y|^2} = \frac{R^2}{y^2} \left[ R^2 - 2xy + |x|^2 \right] = \frac{R^2}{y^2} (x-y)^2$$

$$|x-y^*| = \frac{R}{|y|} |x-y|$$

$$h^y(x) = -\left( \frac{|y|}{R} \right)^{2-n} \frac{1}{n(2-n)\omega_n} (x-y^*)^{2-n} = -\frac{1}{n(2-n)\omega_n} (x-y)^{2-n} = -\Gamma(|x-y|)$$

i.e., $h^y(x)$ we defined fulfills conditions on $h^y(x)$.
For $n = 2$

$$h^y(x) = \begin{cases} -\frac{1}{2\pi} \ln |x-y^*| + \frac{1}{2\pi} \ln \frac{R}{|y|} & y \neq 0 \\ -\frac{1}{2\pi} \ln R & y = 0 \end{cases}$$

For $n > 2$

$$h^y(x) = \begin{cases} -\left( \frac{|y|}{R} \right)^{n-2} \frac{1}{n\omega_n} |x-y^*|^{2-n} & y \neq 0 \\ -\Gamma(R) & y = 0 \end{cases}$$

And thus for $n = 2$

$$G(x,y) = \frac{1}{2\pi} \ln \left[ \left( \frac{R}{|y|} \right) \frac{|x-y|}{|x-y^*|} \right]$$

For $n > 2$:

$$G(x,y) = \frac{1}{n\omega_n} \left[ |x-y|^{2-n} - \left( \frac{|y|}{R} \right)^{n-2} |x-y^*|^{2-n} \right]$$

If $u \in \bar{B}_R$ harmonic,

$$u(y) = \oint\limits_{\partial B_R} u(x) \frac{\partial G}{\partial n_x} \mathrm{d}s_x$$

**Definition** $K(x,y) = \frac{\partial G}{\partial \hat{\mathbf{n}}_x}$ is called Poisson kernel:

$$\frac{\partial}{\partial x_i} G(x,y) = \frac{\partial}{\partial x_i}\Gamma(x-y) - \frac{\partial}{\partial x_i}\Gamma\left(\frac{|y|}{R}|x-y^*|\right)$$

$$\frac{\partial}{\partial x_i}\Gamma(x-y) = \frac{1}{n\omega_n}\frac{x_i-y_i}{|x-y|}|x-y|^{-n+1} = \frac{1}{n\omega_n}\frac{x_i-y_i}{|x-y|^n}$$

$$\frac{\partial}{\partial x_i}\Gamma\left(\frac{|y|}{R}|x-y^*|\right) = \frac{1}{n\omega_n}\left(\frac{|y|}{R}\right)^2 (x_i - y_i^*)|x-y|^{-n}$$

Since we are on ball, $\hat{\mathbf{n}}_x = \sum \frac{x_i}{R}\hat{\mathbf{x}}_\mathbf{i}$:

$$\frac{\partial}{\partial n_x} G(x,y) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\Gamma(x-y) = \frac{1}{n\omega_n}|x-y|^n \sum_{i=1}^n \left[\frac{x_i}{R}(x_i-y_i) - \frac{x_i}{R}\left(\frac{|y|^2}{R^2}(x_i-y_i)\right)\right] =$$

$$= \frac{1}{n\omega_n}\frac{1}{R}|x-y|^n \sum_{i=1}^n \left[x_i^2 - x_i y_i - \frac{x_i^2|y|^2}{R^2} + x_i y_i\right] = \frac{1}{n\omega_n}\frac{1}{R}|x-y|^n (R^2 - |y|^2)$$

**Conclusion** If $u \in \bar{B}_R$ harmonic,

$$u(y) = \oint_{\partial B_R} u(x) K(x,y)\,\mathrm{d}s_x$$

In case $y=0$ we get $K(x,0) = \frac{1}{n\omega_n R^{n-1}}$ and acquire the mean value theorem.

**Claim** K fulfills following conditions

1. $K(x,y) > 0$, $y \in B_R$, $x \in \partial B_R$.
2. $\nabla_y^2 K(x,y) = 0$, $y \in B_R$
3. $\oint_{\partial B_R} K(x,y)\,\mathrm{d}s_x = 1$ for all $y \in B_R$

**Theorem** If $g \in \mathcal{C}(\partial B_R)$ then

$$u(y) = \oint_{\partial B_R} K(x,y) g(x)\,\mathrm{d}s_x$$

is harmonic function and $u(x) = g(x)$ for all $x \in \partial B_R$.

**Proof** Note that

$$K(x,y) = \frac{1}{Rn\omega_n}|x-y|^n (R^2 - |y|^2)$$

is harmonic in $y$ in $B_R$ for $x \in \partial B_R$

$$\nabla_y^2 u(y) = \oint_{\partial B_R} \nabla_y^2 K(x,y) g(x)\,\mathrm{d}s_x = 0$$

Lets show that $\lim_{y\to y_0 \in \partial B_R} u(y) = g(y_0)$. Lets choose $\epsilon > 0$. We choose $\delta_1$ such that $|g(x) - g(y_0)| < \epsilon$ if $x \in \partial B_R \cap B_{\delta_1}(y_0)$

$$\oint_{\partial B_R} g(x)K(x,y)\,\mathrm{d}s_x - g(y_0) = \oint_{\partial B_R} (g(x) - g(y_0))K(x,y)\,\mathrm{d}s_x =$$

$$= \underbrace{\oint_{\partial B_R \setminus \partial B_R \cap B_{\delta_1}(y_0)} (g(x)-g(y_0))K(x,y)\,\mathrm{d}s_x}_{I_1} + \underbrace{\oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(x)-g(y_0))K(x,y)\,\mathrm{d}s_x}_{I_2}$$

$$|I_2| \le \oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(y) - g(y_0))K(x,y)\,\mathrm{d}s_y \le \epsilon \oint_{\partial B_R} K(x,y) \le \epsilon$$

$|x-y_0| \geq \delta_1$ and $|y-y_0| > \frac{\delta_1}{2}$, thus

$$|x-y| \geq |x-y_0| - |y-y_0| > \delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2}$$

$$K(x,y) \leq \frac{1}{Rn\omega_n}\left(\frac{\delta_1}{2}\right)^{-n}(R^2-|y|^2) \overset{y\to y_0}{\longrightarrow} 0$$

$$|I_1(y)| \leq (R^2-|y|^2)\max_{\partial B_R}|y|\frac{\delta_1^{-n}}{Rn\omega_n} \overset{y\to y_0}{\longrightarrow} 0$$

**Conclusion** $u$ is continuous in $y_0 \in \partial B_R$ and $\lim_{y\to y_0 \in \partial B_R} u(y) = g(y_0)$, i.e. Dirichlet problem has solution if the bound is continuous.

**Theorem** Continuous function in $\Omega$ is harmonic iff it fulfills mean value theorem.

**Proof** Suppose $u$ fulfills mean value theorem, take a look at $B_R(x_0) \subset \Omega$ for some $x_0 \in \Omega$.
Let $v$ harmonic function that equals to $u$ on $\partial B_R$. Since $v$ fulfills mean value theorem, then $u-v$ also fulfills it. However, $u-v=0$ on ball bound, and thus from weak extremum theorem we get that $u-v=0$ in the whole ball, i.e. $u=v$.

**Conclusion** If $u_n$ a sequence of harmonic functions uniformly converging to $u$, $u$ is harmonic.

**Proof** If $u_n \to u$ uniformly in $\Omega$, then for all $x_0 \in \Omega$, exists $r>0$ such that $B_r(x_0) \subset \Omega$, and since $u_n \to u$ uniformly in $\partial B_r(x_0)$ thus

$$\lim_{n\to\infty} \frac{1}{|\partial B_r(x_0)|} \int\limits_{\partial B_r(x_0)} u_n(x)\,\mathrm{d}s = \int\limits_{\partial B_r(x_0)} u(x)\,\mathrm{d}s$$

and also

$$\lim_{n\to\infty} u_n(x_0) = u(x_0)$$

However, from harmony,

$$u_n(x_0) = \frac{1}{|\partial B_r(x_0)|}\int\limits_{\partial B_r(x_0)} u_n(x)\,\mathrm{d}s$$

i.e.

$$u(x_0) = \frac{1}{|\partial B_r(x_0)|}\int\limits_{\partial B_r(x_0)} u(x)\,\mathrm{d}s$$

and thus $u$ is harmonic.

**Conclusion** If $u$ is harmonic in $\Omega$, $u \in C^\infty(\Omega)$.

**Proof** If $u$ harmonic, $u(y) = \int\limits_{\partial B_R} K(x,y)u(x)\,\mathrm{d}s_x$ and $K$ is infinitely differentiable by $y \in B_R$:

$$\frac{\partial^m}{\partial y_j^m}u(y) = \int\limits_{\partial B_R} \frac{\partial^m}{\partial y_j^m}K(x,y)u(x)\,\mathrm{d}s_x$$

**Conclusion** If $u_n$ monotonic sequence of harmonic functions in $\Omega$ and exists $y \in \mathbb{R}$ such that $\{u_n(y)\}$ is bounded then in every bounded interval $\bar{\Omega}' \subsetneq \Omega$ sequence $u_n$ converges uniformly to harmonic function.

**Proof** $u_{n+1}(x) \geq u_n(x)$ for all $x \in \Omega$, lets show that for all $x \in \Omega'$ sequence converges to finite limit. Denote $\omega_{mn} = u_m - u_n$.
By Harnack's identity for $m > n$

$$\sup_{\Omega'} \omega_{mn} \leq C \inf_{\Omega} \omega_{mn}$$

$$u_m(x) - u_n(x) = \omega_{mn}(x) \leq C[u_m(y_0) - u_n(y_0)] < K$$

$K$ is constant independent on $n$ and $m$, in particular, $u_n$ is bounded for all $x \in \Omega'$, thus exists $u(x) = \lim_{n\to\infty} u_n(x)$ in $\Omega'$.
It's left to proof sequence converges uniformly and thus $u$ is harmonic.

**Definition** A sequence of function $u_n$ is equicontinuous in $x_0$ if for all $\epsilon > 0$ exists $\delta > 0$, $N > 0$ such that if $x - x_0 < \delta$ and $n > N$,

$$|u_n(x) - u_n(x_0)| < \epsilon$$

The particular case is when derivatives are bounded, since if $|\boldsymbol{\nabla} u_n| < K$,

$$|u_n(x) - u_n(x_0)| < K|x - x_0|$$

**Arzelá–Ascoli theorem** If $u_n$ sequence of functions equicontinuous in compact interval and bounded, then exists uniformly converging subsequence.

**Note** The only bounded harmonic function in $\mathbb{R}^n$ is constant one.

**Theorem** If $u$ harmonic in $\Omega \subset \mathbb{R}^n$ then

$$\left|\frac{\partial u}{\partial x_j}\right| \leq \sup_{\Omega} |u| \cdot \frac{n}{d(x, \partial\Omega)}$$

**Proof**

$$u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} u(y) \, \mathrm{d}y$$

Since $u$ is harmonic and in particular $\frac{\partial u}{\partial x_i}$ is harmonic

$$\nabla^2 u = 0 \Rightarrow \nabla^2 \frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} \nabla^2 u = 0$$

we get

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} \frac{\partial}{\partial x_j} u(y) \, \mathrm{d}y$$

Consider function

$$v_i(y) = \begin{cases} 0 & i \neq j \\ u(y) & i = j \end{cases}$$

then

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} \boldsymbol{\nabla} \cdot v \, \mathrm{d}y = \frac{1}{\omega_n r^n} \oint\limits_{\partial B_r(x)} u(y) n_j(y) \, \mathrm{d}s_y$$

where $n_j(y) = \hat{\mathbf{n}} \cdot \hat{\mathbf{x}}_j$, and thus $|n_j| \leq 1$:

$$\left|\frac{\partial}{\partial x_j} u(x)\right| \leq \frac{1}{\omega_n r^n} \left| \oint\limits_{\partial B_r(x)} u(y) n_j(y) \, \mathrm{d}s_y \right| \leq \frac{1}{\omega_n r^n} \oint\limits_{\partial B_r(x)} |u(y)| \, \mathrm{d}s_y < \frac{1}{\omega_n r^n} \sup |u| n \omega_n r^{n-1} = \sup |u| \frac{n}{r}$$

for all $r < d(x, \partial\Omega)$.

**Theorem** If $\{u_n\}$ sequence of bounded harmonic functions in $\Omega$ then on each compact set $K \subset \Omega$ exists subsequence uniformly converging to $u$ harmonic on $\Omega$.

**Proof** By previous theorem each compact set $K \subset \Omega$ fulfills $d(x, \partial\Omega) < C(K)$ for all $x \in K$ and thus partial derivatives are bounded $|\partial u_n x_i| < C(K)$ thus $\{u_n\}$ is equicontinuous on $K$ and thus it has subsequence uniformly converging to $u$ harmonic in $K$.
Lets choose compact sets

$$K_j = \left\{ x \in \Omega, \ |x| \leq j, \ d(x, \Omega^C) \geq \frac{1}{j} \right\}$$

such that $\bigcup_{n=1}^{\infty} K_n = \Omega$. Suppose $u_{nj}$ is converging subsequence in $K_n$ thus exists subsequence of $u_{nj}$ converging in $K_{n+1}$. Diagonal sequence (of mutual indices) converges uniformly for all $K$ and its limit $u$ is harmonic.

## 5.2 Dirichlet problem in arbitrary interval

Let $\Omega$ bounded interval and $g \in \mathcal{C}(\partial\Omega)$ We want to solve the following problem:

$$\begin{cases} \nabla^2 u = 0 & u \in \mathcal{C}(\bar{\Omega}) \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

**Definition** $u$ is generalized subharmonic function iff $u$ is continuous, for any ball $\bar{B} \subsetneq \Omega$ and for all harmonic and continuous $h$ such that $u \leq h$ in $B$, $u \leq h$ on $\partial B$.

**Example** $u(x) = |x - x_0|$ is generalized subharmonic function in $\mathbb{R}^n$.

**Conclusion** Generalized subharmonic function fulfills mean value inequality:

$$u(x_0) \leq \frac{1}{|\partial B_r(x_0)|} \oint\limits_{\partial B_r(x_0)} u$$

Also, if function fulfills mean value inequality it is generalized subharmonic function.

**Theorem** If $\nabla^2 u \geq 0$ on $\Omega$, $u$ isgeneralized subharmonic function on $\Omega$.

**Lemma** Let $u$ generalized subharmonic function in $\Omega$, $v$ generalized superharmonic function in $\Omega$ and $u, v \in \mathcal{C}(\bar{\Omega})$ and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in $\Omega$

**Proof** Directly from weak extermum.

**Lemma** If $u$,$v$ generalized subharmonic functions, then $\max(u, v)(x) = \max(u(x), v(x))$ is generalized subharmonic function. This is right for any finite amount of functions.

**Proof** If $h$ harmonic on ball and $h \geq \max(u, v)(x)$ on $\partial B$, in particular $h \geq u$, $h \geq v$ on $\partial B$ and thus $h \geq u$, $h \geq v$ on $B$ and thus $h \geq \max(u, v)(x)$ on $B$.

**Definition** Let $\Omega \subset \mathbb{R}^n$ bounded. Let $g$ bounded on $\partial\Omega$, define $S_g$ as set of all generalized subharmonic functions continuous on $\bar{\Omega}$ such that for all $x \in \partial\Omega$ $u(x) \leq g(x)$.

## 5.3 Perron method

**Lemma** Let $u$ generalized subharmonic function in $\Omega$, $\bar{B} \subsetneq \Omega$, define $U(x) = u(x)$ for $x \in \Omega \setminus B$ such that $\nabla^2 U(x) = 0$ for $x \in B$ and $U(x) = u(x)$ on $\partial B$. $U$ is generalized subharmonic function. $U(x) > u(x)$ for all $x$. $U(x)$ is called harmonic lifting of $u$ and is unique from maximum principle.

**Proof** Let $h$ harmonic on $B$ such that $h(x) \geq U(x) \geq u(x)$ on $\partial B_1$ and thus $h(x) \geq u(x) = U(x)$ in $x \in B_1 \setminus B$.

$$\partial(B \cap B_1) = (\partial B \cap \bar{B}_1) \cup (\partial B_1 \cap \bar{B})$$

For $\partial B \cap \bar{B}_1$ $U = u < h$. For $\partial B_1 \cap \bar{B}$ $h \geq U$. Thus $H \geq U$ on $\partial(B \cap B_1)$ however $h - U$ is nonnegative and harmonic on $\partial(B \cap B_1)$ and thus $H \geq U$ in $B \cap B_1$.

**Theorem** $u(x) = \sup\limits_{w \in Sg} w(x)$ is harmonic.

**Proof** Constant $M$ is harmonic (constant) and it fulfills $M \geq y$ on $\partial\Omega$ thus $w \leq M$ for all $w \in S_g$ in particular $w(x) \leq M$ for all $x \in \Omega$ and $w \in S_g$.
Let $y \in \Omega$, there exists a sequence of functions $v_n \in S_g$ such that

$$\lim_{n\to\infty} v_n(y) = u(y)$$

Let $\bar{B}_r(y) \subsetneq \Omega$ and define $V_n$ that is harmonic lifting of $v_n$. Since $V_n(x) \geq v_n(x)$ for all $x \in \Omega$ and $V_n \in S_g$ (it is generalized subharmonic and equals to $v_n$ on $\partial\Omega$),

$$v_n(y) \leq V_n(y) \leq u(y)$$

for all $y \in \Omega$ thus $V_n(y) \to u(y)$. Thus there is subsequence of $V_n$ uniformly converging to $v$ in $B_\rho(y) \subset B_r(y)$. In $B_\rho$ $v \leq u$. Suppose exists $z \in B_\rho(y)$ such that $v(z) < u(z)$. Exists $\bar{u} \in S_g$ such that $v(z) < \bar{u}(z) \leq u(z)$. Then

$$\bar{u} < w_j = \bar{u} \vee V_n \in S_g$$

Let $W_j$ harmonic lift of $w_j$ on $B_r(y)$. $W_j \in S_g$ thus once again exists subsequence uniformly converging to $w$ in $B_\rho$. We know that

$$v \leq w \leq u$$

and $v(y) = w(y) = u(y)$. Thus, since both are harmonic, $w - v$ is harmonic and non-negative, thus $w - v = 0$. We got the contradiction, thus $v = u$.

**Conclusion** If Direchelet problem

$$\begin{cases} \nabla^2 w(x) = 0 & x \in \Omega \\ w(y) = g(y) & x \in \partial\Omega \end{cases}$$

has solution, then $u(x) = \sup_{v \in Sg} v(x)$ is solution.

**Solution** $u(x)$ is harmonic and $g \geq u$ on $\partial\Omega$, thus $w \geq u$ on $\Omega$.
On the other hand, $w$ is harmonic and in particular subharmonic, thus $w \in S_g$ and $w \leq u$. We conclude $w = u$.

**Exercise**

$$\Omega = B_1 \setminus \{0\}$$

Then

$$\partial\Omega = \partial B_1 \cup \{0\}$$

For bound condition $u(\partial B_1) = 1$ and $u(0) = 0$ there is no solution.

**Definition** $y \in \partial\Omega$ is called regular if exists generalized subharmonic function $w$ on $\bar{\Omega}$ such that $w(y) = 0$ and $w(\bar{\Omega} \setminus \{0\}) < 0$.

**Exercise** If there exist a ball $B_\epsilon(z)$ in $\Omega^C$ tangent to $\partial\Omega$ in $y$, then $y$ is regular.

**Theorem** If $\Omega$ is bounded interval, $\partial\Omega$ is regular in every point, $g \in \mathcal{C}(\partial\Omega)$ exists $u \in \mathcal{C}(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$ and $\boldsymbol{\nabla} u = 0$.

**Proof** We need

$$\lim_{x\to y\in\partial\Omega} u(x) = g(y)$$

Define

$$\begin{cases} v_1(x) = g(y) - \epsilon + hw(x) \\ v_2(x) = g(y) + \epsilon - hw(x) \end{cases}$$

where $h > 0$ is constant to be determined later and $w$ is the function from definition of regular point.
$v_1$ is subharmonic and $v_2$ is superharmonic. For $h$ big enough we know that $v_1(x) < g(x)$ and $v_2(x) > g(x)$ for $x \in \partial\Omega$.
$0 > -m \geq w(x)$ if $|x - y| > \delta$. Thus

$$v_1(x) \leq g(x) - \epsilon - hm$$

Choose $h$ such that

$$g(x) + \epsilon - hm < \min_{x\in\partial\Omega} g(x)$$

Since $|g(x)-g(y)|<\epsilon$ if $|x-y|<\delta$ thus $v_1(x)<g(x)$ if $|x-y|\leq\delta$ and thus $v_1(x)\leq g$ in $\partial\Omega$ and similarly $v_2(x)\geq g$ in $\partial\Omega$. Thus

$$v_2 \geq u \geq v_1$$

for $u=\sup_{w\in S_g} w(x)$.

$$g(x)+\epsilon-hw(x)\geq u(x)\geq g(x)-\epsilon+hw(x)$$

$$-\epsilon\leq \liminf_{x\to y} u(x)-g(x)\leq \limsup_{x\to y} u(x)-g(x)<\epsilon$$

**Theorem** Regularity is sufficient condition for solution existence.

**Proof** Suppose there exists solution of Dirichlet problem in interval with bound condition $g(x)=-|x-y|$ for $y\in\partial\Omega$. Thus exists harmonic function $h$ which equals to $g$ on $\partial\Omega$, then $h<0$ in $\Omega$ and $h(y)=0$, i.e., $y$ is regular.

**Hilbert proof of Direchlet problem** For all $u\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\partial\Omega)$ and $u=g$ on $\partial\Omega$.
Define Direchlet integral- functional

$$I(u)=\int\limits_{\Omega}|\boldsymbol{\nabla} u|^2$$

Suppose there exists minimum of $I$ , $u_0$ (this is not easy to show). Then

$$I(u_0)\leq\int\limits_{\Omega}|\boldsymbol{\nabla} u|^2$$

Let $\phi\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\partial\Omega)$ such that $\phi=0$ on $\partial\Omega$. Then $u_0+\epsilon\phi$ fulfills the condition.

$$I(u_0+\epsilon\phi)=\int\limits_{\Omega}|\boldsymbol{\nabla}(u_0+\epsilon\phi)|^2=\int\limits_{\Omega}|\boldsymbol{\nabla} u_0|^2+2\epsilon\int\limits_{\Omega}\boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi+\epsilon^2\int\limits_{\Omega}|\boldsymbol{\nabla}\phi|^2\geq I(u_0)=\int\limits_{\Omega}|\boldsymbol{\nabla} u_0|^2$$

Thus

$$\int\limits_{\Omega}\boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi\geq 0$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} u_0\phi)=\boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi+\nabla^2 u_0\phi$$

$$\int\limits_{\Omega}\boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi=-\int\nabla^2 u_0\phi+\int\boldsymbol{\nabla}(\boldsymbol{\nabla} u_0\phi)=-\int\nabla^2 u_0\phi+\oint\limits_{\partial\Omega}\phi\boldsymbol{\nabla} u_0\geq 0$$

Thus

$$\int\nabla^2 u_0\phi=0$$

for any $\phi\in\mathcal{C}(\bar{\Omega})$ thus

$$\nabla^2 u_0=0$$

## 5.4 Neumann bound condition

Is there solution of Neumann bound condition

$$\begin{cases}\nabla^2 u(x)=0 & x\in\Omega\\ \frac{\partial u}{\partial n}=g & y\in\partial\Omega\end{cases}$$

**Proof** Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Define functional

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \oint_{\partial\Omega} ug$$

If there exists minimum $u_0$, take a look on $u_0 + \epsilon\phi$ for $\phi \in \mathcal{C}(\bar{\Omega})$.

$$0 \leq I(u_0 + \epsilon\phi) - I(u_0) = \epsilon \iiint_{\Omega} + \epsilon \int_{\partial\Omega} \phi g + \frac{1}{2}\epsilon^2 |\nabla u_0|^2$$

Thus

$$\iiint_{\Omega} \nabla u_i \nabla \phi - \oint_{\partial\Omega} \phi g = 0$$

For any $\phi \in \mathcal{C}^2 \cap \bar{\otimes}$.

$$-\int \nabla^2 u_0 \phi + \int_{\partial\Omega} \phi \frac{\partial u}{\partial n} - \int_{\partial\Omega} = 0$$

thus

$$\oint \left( \frac{\partial u}{\partial n} - g \right) \phi = 0$$

i.e.

$$\frac{\partial u}{\partial n} = g$$

If $\oint_{\partial\Omega} g > 0$ obviously there is no minimum of $I$: for $u = \lambda$

$$I(u) = -\lambda \oint_{\partial\Omega} g$$

Thus $\oint g = 0$ is necessary condition for $I$ to be bounded. By Gauss

$$0 = \int_{\Omega} \nabla^2 u = \oint_{\partial\Omega} \frac{\partial u}{\partial n} = \oint_{\partial\Omega} g$$

# 6 Heat equation in higher dimensions

$$\frac{\partial u}{\partial t} = \nabla_x^2 u$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.
Note the equation is unchanged under transformation

$$\begin{cases} x \to ax = x' \\ t \to a^2 t = t' \end{cases}$$

Thus if $u(x,t)$ is solution then $u(ax, a^2t)$ is solution. Thus, lets search for solutions dependent on $\xi = \frac{|x|}{\sqrt{t}}$:

$$u(x,t) = w\left(\frac{|x|}{\sqrt{t}}\right)$$

Define heat as

$$Q(t) = \int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^n} w\left(\frac{|x|}{\sqrt{t}}\right)\mathrm{d}x = \int w(\xi) t^{\frac{n}{2}}\,\mathrm{d}\xi = t^{\frac{n}{2}} \int w(\xi)\,\mathrm{d}\xi = \text{const}$$

So, to keep $Q$ constant we want

$$u(x,t) = t^{-\frac{n}{2}} w\left(\frac{|x|}{\sqrt{t}}\right)$$

Substituting into equation:

$$(\xi^{n-1} w')' + \frac{1}{2}(\xi^n w)' = 0$$

$$\xi^{n-1}w' + \frac{1}{2}\xi^n w = c$$

Substituting $\xi = 0$ we get $c = 0$:

$$\xi^{n-1}w' + \frac{1}{2}\xi^n w = 0$$

$$w' + \frac{1}{2}\xi w = 0$$

$$w(\xi) = ce^{-\frac{|\xi|^2}{4}}$$

If we want to normalize

$$\int u(x,t)\,\mathrm{d}x = 1$$

We get

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}}$$

Which is $n$ dimensional Gaussian.

**Kernel of heat**

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x-y|^2}{4t}}$$

Then

$$\frac{\partial K}{\partial t} = \nabla_x^2 K$$

Also note that $K > 0$ and that, from normalization,

$$\int\limits_{\mathbb{R}^4} K(x,y,t)\,\mathrm{d}^n x = 1$$

Moreover, for all $\delta > 0$ and for all $y \in \mathbb{R}^n$

$$\lim_{t\to 0} \int\limits_{|x-y|\geq\delta} K(x,y,t)\,\mathrm{d}^n x = 0$$

i.e.,

$$\lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x-y|\geq\delta} e^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}^n x = 0$$

**Proof** By variable substitution $\xi = \frac{x}{\sqrt{t}}$:

$$\lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x-y|\geq\delta} e^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}^n x = \lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x|\geq\delta} e^{-\frac{x^2}{4t}}\,\mathrm{d}^n x = \lim_{t\to 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \int\limits_{|\xi|\geq\frac{\delta}{\sqrt{t}}} e^{-\frac{\xi^2}{4}}\,\mathrm{d}^n \xi = 0$$

Define

$$u(x,t) = \int\limits_{\mathbb{R}^n} K(x,y,t)f(y)\,\mathrm{d}^n y$$

**Conclusion** If $u_t = \nabla^2 u$ then $\min_\Omega u$, $\max_\Omega u$ are acquired on $\partial_p\Omega$. In particular there is unique solution of heat equation Dirichlet problem in $\Omega = D \times [0,T]$: if $u_1$, $u_2$ are solutions of heat equation in $\Omega$ and $u_1 = u_2$ on $\partial_p\Omega$ then $u_1 = u_2$ on $\Omega$.

**Theorem** If $|f|$ is bounded and continuous, $u(x,t)$ is solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$ fulfilling $\lim_{t\to 0} u(x,t) = f(x)$.

**Proof**

$$\frac{\partial u}{\partial t} = \iint\limits_{\mathbb{R}^n} \frac{\partial K}{\partial t} f(x)\,\mathrm{d}x$$

$$\nabla_x^2 u = \iint\limits_{\mathbb{R}^n} \nabla_x^2 K f(x)\,\mathrm{d}x$$

To be allowed to switch between integral and derivative we need to show that

$$u(x,t+h) - u(x,t) = \int\limits_{\mathbb{R}^n} [K(x,y,t+h) - K(x,y,h)] f(x)\,\mathrm{d}x$$

is bounded by integrable function. Since $K$ is decaying exponent, it is true.
Lets show that $\lim_{t\to 0} u(x,t) = f(x)$.

$$u(x,t) = \int\limits_{|x-y|\le\delta} K(x,y,t) f(y)\,\mathrm{d}^n y + \int\limits_{|x-y|>\delta} K(x,y,t) f(y)\,\mathrm{d}^n y$$

Since integral of $K$ is 1

$$f(x) = \iint\limits_{\mathbb{R}^n} K(x,y,t) f(x)\,\mathrm{d}^n x$$

and thus

$$u(x,t) = \int\limits_{|x-y|\le\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y + \int\limits_{|x-y|>\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y$$

Since $f$ is bounded:

$$\int\limits_{|x-y|>\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y \le 2c \int\limits_{|x-y|>\delta} K(x,y,t) \to 0$$

Always exists $\epsilon$ such that $|f(x) - f(y)| < \epsilon$

$$\int\limits_{|x-y|\le\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y \le \int\limits_{|x-y|\le\delta} K(x,y,t)|(f(y) - f(x)|\,\mathrm{d}^n y \le \epsilon \int\limits_{|x-y|\le\delta} K(x,y,t)\,\mathrm{d}^n y \le \epsilon \int\limits_{\mathbb{R}^n} K(x,y,t)\,\mathrm{d}^n y \le \epsilon$$

Thus

$$\limsup_{t\to 0} |u(x,y) - f(x)| = \epsilon$$

and

$$\lim_{t\to 0} u(x,t) = f(x)$$

Note that temperature has infinite velocity - even if we start with $f$ with compact support, for any $t > 0$ for any point we have $u > 0$.

**Heat eqaution in compact region** Let $\Omega = D \times [0,T]$ for some compact $D \subsetneq \mathbb{R}^n$. We call $\partial_p \Omega\,\{D \times \{0\}\} \cup \{\partial D \times (0,T]\}$ parabolic bound of $\Omega$.

**Theorem** If $u \in \mathcal{C}(\bar{\Omega})$, $u_t, u_{x_i x_j} \in \mathcal{C}(\Omega)$ and $u_t - \nabla^2 u \le 0$ on $\Omega$ then

$$\max_{(x,t)\in\bar{\Omega}} u(x,t) = \max_{(x,t)\in\partial_p\Omega} u(x,t)$$

**Proof** Suppose $u_t - \nabla^2 u < 0$ for all $(x,t) \in \Omega$. Suppose $(x_0, t_0) \in \Omega$ is maximum of $u$. Then

$$\nabla^2 u \le 0$$

$$u_t = 0$$

i.e., $u_t - \nabla^2 u \ge 0$ in contradiction.
If $(x_0, T)$ is maximum, then

$$\nabla^2 u \le 0$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=T} = T$$

thus still $u_t - \nabla^2 u \ge 0$ in contradiction.
If we suppose that $u_t - \nabla^2 u \le 0$ define $v(x,t) = u(x,t) - \delta t$. Then $v$ fulfills $v_t - \nabla^2 v < 0$.
Suppose $\max u(x,t) = u(x_0,t_0) = m$, then

$$\max_{\Omega} v \ge m - \delta T$$

is acquired in $(x_\delta, t_\delta) \in \partial_p \Omega$. Looking on sequence $\delta_n \to 0$, since $\partial_p \Omega$ is compact, $(x_{\delta_n}, t_{\delta_n}) \to (x_0, t_0) \in \partial_p \Omega$, i.e.

$$\lim_{\delta to 0} v(x_\delta, t_\delta) = u(x_0, t_0) = m$$

Thus maximum is acquired on parabolic boundary.

**Theorem** If $f$ is bounded on $\mathbb{R}^n$ and $u(x,t)$ fulfills

$$u_t - \nabla^2 u \ge 0$$

and $u \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^+) \cap \mathcal{C}(\mathbb{R}^n \times \bar{\mathbb{R}}^+)$, $u(x,0) = f$ and in addition

$$u(x,t) \le M e^{a|x|^2}$$

for some $M, a > 0$ then

$$u(x,t) < \sup f(x)$$

for all $x$, $t$.
Note that it's enough to show theorem for $0 < t \le T$ for some constant $T > 0$, since then we can use $u(x,T)$ as new initial conditions to get to arbitrary $t$.

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

is solution of heat equation. So is

$$K(x,y,t+T) = \frac{1}{(4\pi(t+T))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+T)}}$$

However $K(x,y,-t)$ is not solution of heat equation. But

$$K'(x,y,-t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{+\frac{|x|^2}{-4t}}$$

is solution of heat equation. It can be seen by extending to complex plane, we can note that under transformation

$$\begin{cases} t \to -t \\ x \to ix \end{cases}$$

heat equation is preserved.
So lets look at

$$w_{y,T}(x,t) = K'(x,y,T-t) = \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T-t)}}$$

with

$$w_{y,T}(x,0) = \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{|x|^2}{4T}}$$

for $0 < t \le T$. This function solves heat equation, however this function is increasing and goes to infinity in finite time. This happens due to value of $w_{y,T}(x,0)$ in infinity - it is infinite, i.e. there is infinite amount of heat in $x \to \infty$.

**Proof** Define

$$v_\mu(x,t) = u(x,t) = \mu w_{T,y}(x,t)$$

Then

$$\frac{\partial}{\partial t} v_\mu(x,t) - \nabla^2 v_\mu \leq 0$$

Define

$$\Omega_{\rho,T} = \{(x,t) \,:\, |x-y| < \rho,\ 0 \leq t < T\}$$

We know that

$$\max_{\bar{\Omega}_{\rho,T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho,T}} v_\mu(t)$$

In particular, for $x = y$

$$v_\mu(y,t) \leq \max_{\bar{\Omega}_{\rho,T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho,T}} v_\mu(t)$$

In ball $B_\rho(y)$

$$v_\mu(x,0) \leq u(x,0)$$

For $0 \leq t \leq \frac{T}{2}$

$$v_\mu(x,t) = u(x,t) - \mu w(x,t) \leq Me^{a|x|^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} \leq Me^{a(|y|+\rho)^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} = e^{a\rho^2}\left[Me^{2|y|\rho + a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2\left(\frac{1}{4T} - a\right)}\right]$$

Since we are free to choose $T$ and $\rho$, we can choose them such that $\frac{1}{4T} - a > 0$ and then for rho big enough

$$v_\mu(x,t) \leq e^{a\rho^2}\left[Me^{2|y|\rho + a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2\left(\frac{1}{4T} - a\right)}\right] < C$$

for any $C$. Thus, $v_\mu(x,t)$ can acquire maximum only in $t = 0$:

$$\max_{\substack{|x-y|\leq\rho \\ 0\leq t\leq \frac{T}{2}}} v_\mu(x,t) = \max_{|x-y|\leq\rho} v_\mu(x,0)$$

Thus, taking $\mu$ to 0:

$$u(x,t) \leq \max_{\mathbb{R}^n} u(x,0)$$

for $y \in \mathbb{R}^n$ and $0 \leq t \leq \frac{2}{a}$.

**Conclusion** If $u_t - \nabla^2 u = 0$ then

$$\sup_{x\in\mathbb{R}^n} |u(x,t)| \leq \sup_{x\in\mathbb{R}^n} |u(x,0)|$$

**Conclusion** If $u_1$, $u_2$ fulfill heat equation such that $u_1(x,0) = u_2(x,0)$ and in addition

$$\begin{cases} |u_1(x,t)| < Me^{a|x|^2} \\ |u_2(x,t)| < Me^{a|x|^2} \end{cases}$$

then $u_1(x,t) = u_2(x,t)$ for all $x,t \in \mathbb{R}^n \times \mathbb{R}^+$.