

104030 - Introduction to Partial Differential Equations

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Abstract

1 Introduction

PDE In PDE, the solution is a function of a couple of variables $u(x_1, x_2, \dots, x_m)$ such that:

$$F(x_1, x_2, \dots, x_m, u_{x_1}, u_{x_2}, \dots, u_{x_m}, u_{x_1 x_1}, \dots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually $m = 2$. For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_1 x_2}) = 0$$

Is PDE of two variables of order 2.

Linear PDE PDE is linear if F is linear in u and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$\begin{aligned} F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) = \\ = A(x_1, x_2)u_{x_1 x_1} + B(x_1, x_2)u_{x_1 x_2} + C(x_1, x_2)u_{x_2 x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0 \end{aligned}$$

Quasilinear PDE Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$\begin{aligned} F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) = \\ = A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2 x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0 \end{aligned}$$

For homogeneous linear PDE solution always exist. In addition, if u_1, u_2 , then any linear combination of those $\lambda_1 u_1 + \lambda_2 u_2$ will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

Autonomous PDE If F is independent on x_i , then if $u(x_1, \dots, x_i, \dots, x_m)$ is solution then $u(x_1, \dots, x_i + \lambda, \dots, x_m)$ is solution too.

In particular if u is independent on all x_i , then $u(x_1 + \lambda_1, \dots, x_i + \lambda_i, \dots, x_m + \lambda_m)$.

1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.

Lets start ODE describing harmonic oscillator, would be

$$m \frac{d^2 x}{dt^2} = k(x - x_0)$$

Now suppose that we have N such masses and positionn of mass is $\bar{x}_i = x_i + u(x_i, t)$, where u is displacement of mass and $x_i - x_{i-1} = \Delta$. Then the position of mass is described as

$$\frac{\partial \bar{x}_i}{\partial t^2} = m \frac{\partial^2}{\partial t^2} u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m \frac{d}{dt^2} u(x, t) = k [u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)]$$

In limit $\Delta \rightarrow 0$:

$$\frac{d}{dt^2} u(x, t) = c^2 \frac{d}{dx^2} u(x, t)$$

Where

$$c^2 = \lim_{\Delta \rightarrow 0} \frac{\Delta^2 k_{\Delta}}{m_{\Delta}}$$

Possible solutions For each function f in \mathcal{C}^2 , $u = f(x - ct)$ is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along x axis with constant velocity c . Since c can be negative too, we have solution

$$u(x, t) = f(x + ct) + g(x - ct)$$

1.2 Heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Here, u means amount of heat in point x at time t .

Amount of heat in interval $[a, b]$ is

$$Q(t) = \int_a^b u(x, t) dx$$

And heat flux in point x at time t is $k \frac{\partial u}{\partial x}$

Then flux out of interval is

$$\frac{dQ}{dt} = k \frac{\partial u}{\partial x}(b, t) - k \frac{\partial u}{\partial x}(a, t)$$

Thus

$$\int_a^b \frac{\partial}{\partial t} u(x, t) dx = k \frac{\partial u}{\partial x}(b, t) - k \frac{\partial u}{\partial x}(a, t)$$

In limit $b \rightarrow a$ we get

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Example solution

$$u(x, t) = e^{-kst} \sin(\sqrt{s}x)$$

for some parameter s . Here we also can add some constant to x and acquire additional solution:

$$U(x, t) = e^{-kst} \sin(\sqrt{s}(x + \lambda)) = \cos(\sqrt{s}\lambda) e^{-kst} \sin(\sqrt{s}x) + \sin(\sqrt{s}\lambda) e^{-kst} \cos(\sqrt{s}x)$$

Thus

$$w(x, t) = e^{-kst} \cos(\sqrt{s}x)$$

is solution too.

1.3 Diffusion equation

Suppose $u(x_1, x_2, x_3, t)$ describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \vec{\nabla} \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field v independent on u .

1.4 Elliptic PDEs

Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

Poisson equation

$$\Delta u = f(x_1, x_2)$$

2 First-order PDE

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation: $u(x, t) = f(x - ct)$ for some differentiable f .

Suppose we have initial conditions $u(x, 0) = u_0(x)$. Is it determines uniquely a solution of equation? Obviously, $u(x, t) = u_0(x - ct)$ is a solution.

Lets show it's unique. Take a look at parametrization $x(t) = s_1 + ct$.

$$\frac{d}{dt}u(x(t), t) = c \frac{\partial u}{\partial x}(x(t), t) + \frac{\partial u}{\partial t}(x(t), t) = 0$$

Thus u is constant on every line of form $x(t) = s + ct$. Such lines are called characteristic curves or just characteristics. Thus if we know a value of u in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve $x(t)$ for $t \in [a, b]$. So we want to find a solution such that the surface of solution comprises a given curve in 3D.

The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where a, b, c are continuously differentiable in some neighborhood of point (x_0, y_0, z_0) .

Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1 \right)$$

and

$$\nabla f \cdot (a, b, c) = -a \frac{\partial u}{\partial x} - b \frac{\partial u}{\partial y} + c = 0$$

Thus vector (a, b, c) is tangent to solution surface.

Now define curve such that

$$\begin{cases} \frac{dx}{dt} = a(x(t), y(t), z(t)) \\ \frac{dy}{dt} = b(x(t), y(t), z(t)) \\ \frac{dz}{dt} = c(x(t), y(t), z(t)) \end{cases}$$

The curve $(x(t), y(t), z(t))$ is characteristic curve of PDE.

If there is no dependence on z (i.e. equation is linear) we can take a look on 2-dimensional curve in xy -plane.

Theorem If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

Proof Let $(x(t), y(t), z(t))$ characteristic curve of PDE and suppose for some t_0

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that $w(t_0) = 0$. Now

$$\begin{aligned} G(x(t), y(t), w(t)) &= \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}(x(t), y(t)) \frac{dx}{dt} - \frac{\partial u}{\partial y}(x(t), y(t)) \frac{dy}{dt} = c \left(x(t), y(t), w(t) + u(x(t), y(t)) \right) - \\ &\quad - \frac{\partial u}{\partial x}(x(t), y(t)) a \left(x(t), y(t), w(t) + u(x(t), y(t)) \right) - \frac{\partial u}{\partial y}(x(t), y(t)) b \left(x(t), y(t), w(t) + u(x(t), y(t)) \right) \end{aligned}$$

If we substitute $w = 0$, we get

$$G(x(t), y(t), 0) = c(x(t), y(t), u(x(t), y(t))) - \frac{\partial u}{\partial x}(x(t), y(t)) a(x(t), y(t), u(x(t), y(t))) - \frac{\partial u}{\partial y}(x(t), y(t)) b(x(t), y(t), u(x(t), y(t))) = 0$$

That means that $w = 0$ is a solution of ODE, and since $a, b, c \in C^1$, the solution is unique, i.e. $w = 0$ is the only solution, and thus characteristic curve is contained in the solution surface.

2.2 Existence and uniqueness theorem for first-order quasilinear PDE

Existence and uniqueness theorem for first-order quasilinear PDE Suppose we have initial curve $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$ which around some point s_0 is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0) \dot{\bar{y}}(s_0) - b(x_0, y_0, z_0) \dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).

Then in neighborhood of s_0 exists unique solution of PDE.

Proof Define functions $x(s, t), y(s, t), z(s, t)$ around $(s_0, 0)$ such that

$$\begin{cases} x(s, 0) = \bar{x}(s) \\ y(s, 0) = \bar{y}(s) \\ z(s, 0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a(x(s, t), y(s, t), z(s, t)) \\ \frac{\partial y}{\partial t} = b(x(s, t), y(s, t), z(s, t)) \\ \frac{\partial z}{\partial t} = c(x(s, t), y(s, t), z(s, t)) \end{cases}$$

From uniqueness and existence of ODE, exists unique solution (x, y, z) in neighbourhood of s_0 .

Lets try to find s, t , as a function of x, y . It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in $(s_0, 0)$.

Now define $u(x, y) = z(s(x, y), t(x, y))$.

$$\begin{aligned} a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} &= a \left[\frac{\partial z}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial x} \right] + b \left[\frac{\partial z}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial z}{\partial t} \frac{\partial t}{\partial y} \right] = \frac{\partial z}{\partial t} \left[a \frac{\partial t}{\partial x} + b \frac{\partial t}{\partial y} \right] + \frac{\partial z}{\partial s} \left[a \frac{\partial s}{\partial x} + b \frac{\partial s}{\partial y} \right] = \\ &= \frac{\partial z}{\partial t} \underbrace{\left[\frac{\partial x}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial t}{\partial y} \right]}_{\frac{dt}{dt}} + \frac{\partial z}{\partial s} \underbrace{\left[\frac{\partial x}{\partial t} \frac{\partial s}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial s}{\partial y} \right]}_{\frac{ds}{dt}} = \frac{\partial z}{\partial t} = c \end{aligned}$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on xy -plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on xy -plane are different, there are different possibilities.

Example

$$yu_x - xu_y = 0$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s) \sin(t) + B(s) \cos(t) \\ y = A(s) \cos(t) - B(s) \sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s \cos(t) \\ y = -s \sin(t) \\ z = h(s) \end{cases}$$

Now we want to find s, t as a function of x, y :

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

Example

$$yu_x - xu_y = u$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s) \sin(t) + B(s) \cos(t) \\ y = A(s) \cos(t) - B(s) \sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s \cos(t) \\ y = -s \sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find s, t as a function of x, y :

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y} F(u) = 0$$

for $F = \frac{1}{2}u^2$)

Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{dx}{dy} = -u \Rightarrow u = \frac{dx}{dy}$$

Here y denotes time.

To solve it, we take integral:

$$\begin{aligned} \int_a^b \left[\frac{\partial u(x, y)}{\partial y} + \frac{\partial}{\partial x} F(u(x, y)) \right] dx &= 0 \\ \frac{\partial}{\partial y} \underbrace{\int_a^b u dx}_{Q(y)} + F(u(b, y)) - F(u(a, y)) &= 0 \\ \frac{dQ}{dy} &= F(u(a, y)) - F(u(b, y)) \end{aligned}$$

Now as for any quasilinear PDE:

$$\begin{cases} \dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = c_2 t + c_3 \\ y = t + c_1 \\ z = c_2 \end{cases}$$

For initial conditions $(s, 0, h(s))$:

$$\begin{cases} x = h(s)t + s \\ y = t \\ z = h(s) \end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{d\bar{x}}{ds} \cdot 1 - \frac{d\bar{y}}{ds} \cdot h(s) = 1 \neq 0$$

Since

$$\begin{aligned} \frac{\partial u}{\partial x} &= h'(x - yu) \cdot \left(1 - y \frac{\partial u}{\partial x} \right) \\ \frac{\partial u}{\partial x} &= \frac{h'(x - yu)}{1 + h'(x - yu) \cdot y} \end{aligned}$$

even if we start from \mathcal{C}^∞ function we can get $1 + h'(x - yu) \cdot y = 0$ and thus undefined derivative.

Geometrically, the slope of projections of characteristic curves is equal to $h(s)$ thus they can cross in some point.

Weak solutions We define a weak solution of equation, function u fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y} u(x, y) dx + F(u(b, y)) - F(u(a, y)) = 0$$

Intuitively, F is flux, and u is density, thus change in number of particles (integral) is difference between particles going in and out.

Suppose for solution $u(x, y)$ exists curve of non-continuousness γ , i.e, u is not continuous in each point of curve:

$$u(y) = \begin{cases} u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y) \end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y)dx = \int_a^{\gamma(y)} u^+(x,y)dx + \int_{\gamma(y)}^b u^-(x,y)dx$$

$$\begin{aligned} \frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y} dx + u^+(x, \gamma(y)) \cdot \gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y} dx - u^-(x, \gamma(y)) \cdot \gamma'(y) = \\ &= - \int_a^{\gamma(y)} \frac{dF(u^+)}{dx} dx - \int_{\gamma(y)}^b \frac{dF(u^-)}{dx} dx + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = \\ &= - [F(u^+(\gamma(y), y)) - F(u^+(a, y))] - [F(u^-(b, y)) - F(u^-(\gamma(y), y))] + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] \end{aligned}$$

Meaning

$$- [F(u^+(\gamma(y), y)) - F(u^+(a, y))] - [F(u^-(b, y)) - F(u^-(\gamma(y), y))] + \gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = F(u^-(a, y)) - F(u^+(b, y))$$

$$\gamma'(y) [u^+(x, \gamma(y)) - u^-(x, \gamma(y))] = F(u^+(\gamma(y), y)) - F(u^-(\gamma(y), y))$$

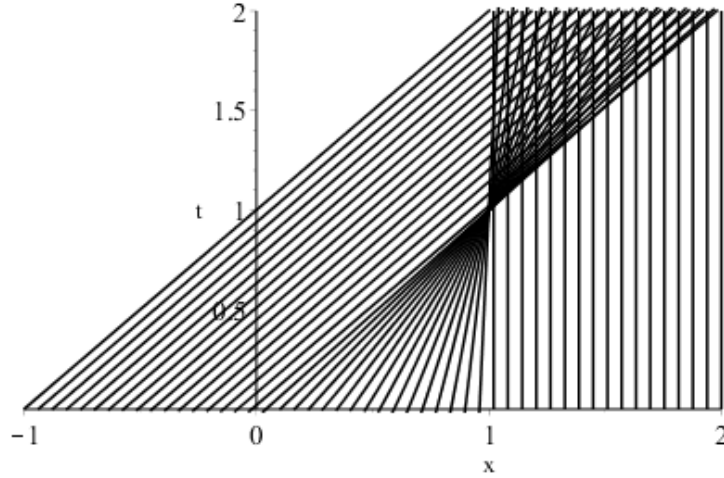
$$\gamma' = \frac{F(u^+(\gamma(y), y)) - F(u^-(\gamma(y), y))}{u^+(x, \gamma(y)) - u^-(x, \gamma(y))}$$

This equation is called Rankine–Hugoniot conditions. If $F(u) = \frac{1}{2}u^2$, we get $\gamma'(y) = \frac{1}{2}(u^+ + u^-)$

Example Suppose we have initial conditions $u(x, 0) = h(x)$ for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \leq x \leq \alpha \end{cases}$$

For $0 < y < 1$ we have a triangle Δ ($0 < x < \alpha$ and $y < \frac{x}{\alpha}$) for which there is intersection of two solution:



In point x, y we have slope $u(x, y)$ thus the charecteristic curve crosses x -axis at $x_0 = x - uy$ and from initial conditions, $u = 1 - \frac{x_0}{\alpha}$. Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

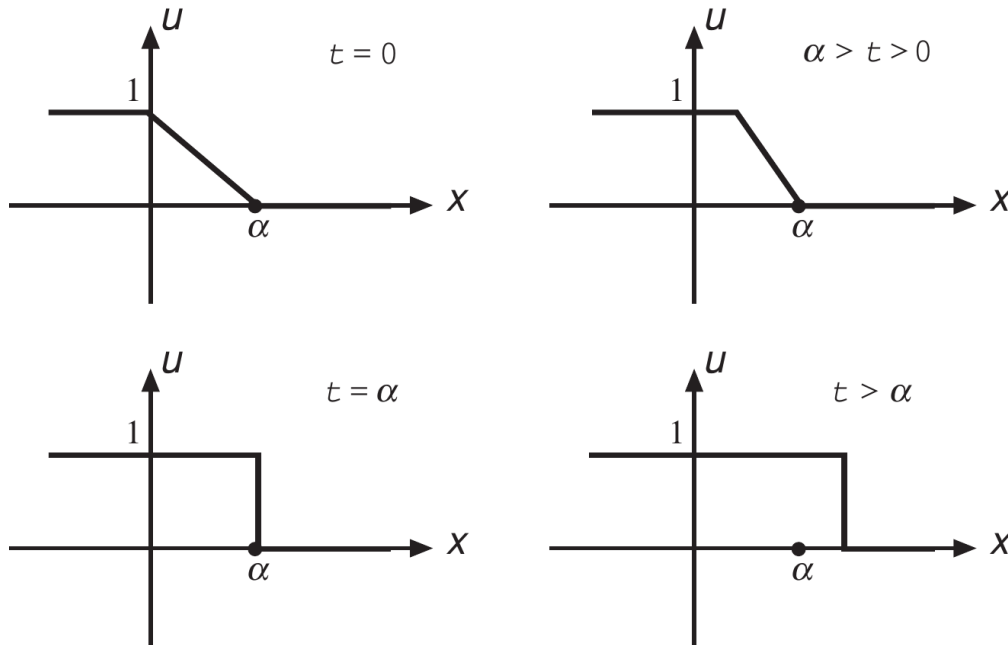
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for $y > 1$ from Rankine–Hugoniot conditions

$$u(x, y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y - \alpha) \\ 0 & x > \alpha + \frac{1}{2}(y - \alpha) \end{cases}$$

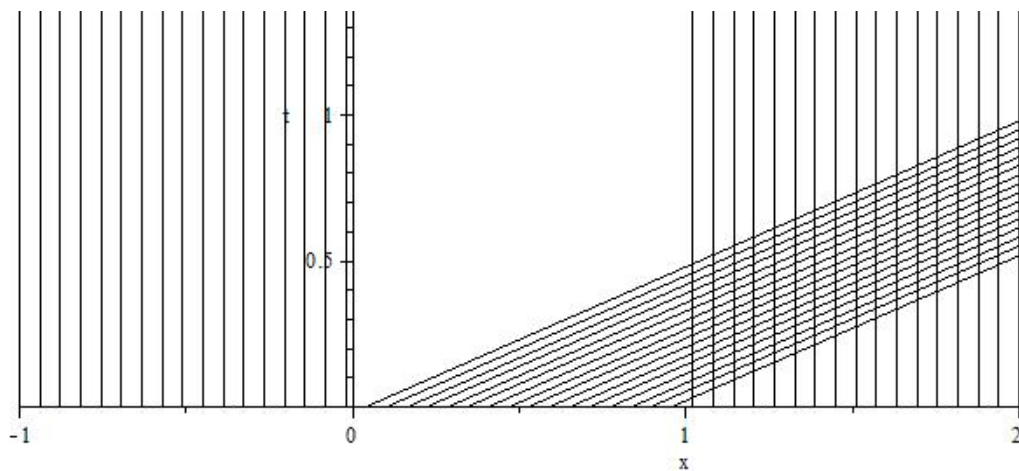
Such a solution is called a shock wave.



Example For

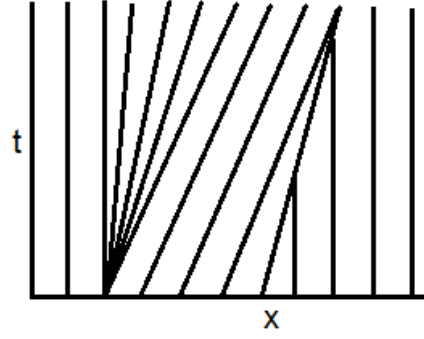
$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \leq x \leq \alpha \end{cases}$$

Now there is no place where characteristic curves meet



In the region without characteristic curves ($0 \leq x \leq y$) we get the following: the solution starts from some point $x_0 = x - uy$, and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$



What happens if $\alpha \rightarrow 0$? We get $u = \frac{x}{y}$ for $0 \leq x \leq y$. We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.

However, also shock wave along $y = x$ is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we can't reproduce the solution for some $y < y_0$ even if I know the values for $y = y_0$.

Entropy principle Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

2.4 Fully non-linear equations

Hamilton-Jacobi equation

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial p} \\ \frac{dy}{dt} = \frac{\partial F}{\partial q} \\ \frac{dz}{dt} = p \frac{\partial F}{\partial p} + q \frac{\partial F}{\partial q} \\ \frac{dp}{dt} = -\frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} \\ \frac{dq}{dt} = -\frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$

We need to find \bar{p} and \bar{q} . We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by s

$$\begin{aligned} \frac{\partial u}{\partial x} \frac{d\bar{x}}{ds} + \frac{\partial u}{\partial y} \frac{d\bar{y}}{ds} &= \frac{d\bar{z}}{ds} \\ \bar{p}(s) \frac{d\bar{x}}{ds} + \bar{q}(s) \frac{d\bar{y}}{ds} &= \frac{d\bar{z}}{ds} \end{aligned}$$

Now we can find p and q .

Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with $u = 0$, then characteristic curves are perpendicular to initial curve. We get $u(x, y)$ equal to distance from initial curve, since absolute value of gradient of u is 1 due to equation.

If we have $u = \phi(s)$ on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

Higher dimension We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

3 Wave equation

$$u_{yy} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + du + f$$

We want to simplify the equation: we are searching for $s(x, y)$ and $t(x, y)$ and solution $u(x, y) = w(s(x, y), t(x, y))$. Derivatives of u are

$$u_y = w_s s_y + w_t t_y$$

$$u_{yy} = w_{ss} s_y^2 + w_{st} s_y t_y + w_s s_{yy} + w_{ts} t_y s_y + w_{tt} t_y^2 + w_t t_{yy}$$

Now we can get equation of form

$$A(s, t)w_{ss} + 2B(s, t)w_{st} + C(s, t)w_{tt} + D(s, t)w_{tt} + F(s, t)$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{st} = 0$$

i.e.,

$$w(s, t) = f(s) + g(t)$$