# 104030 - Introduction to Partial Differential Equations

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**Abstract**

## 1 Introduction

**PDE** In PDE, the solution is a function of a couple of variables $u(x_1, x_2, \dots x_m)$ such that:

$$F(x_1, x_2, \dots x_m, u_{x_1}, u_{x_2}, \dots, , u_{x_m}, u_{x_1 x_1}, \dots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually $m = 2$. For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_1 x_0}) = 0$$

Is PDE of two variables of order 2.

**Linear PDE** PDE is linear if $F$ is linear in $u$ and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2)u_{x_1 x_1} + B(x_1, x_2)u_{x_1 x_2} + C(x_1, x_2)u_{x_2 x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

**Quasilinear PDE** Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2 x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0$$

For homogeneous linear PDE solution always exist. In addition, if $u_1$, $u_2$, then any linear combination of those $\lambda_1 u_1 + \lambda_2 u_2$ will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

**Autonomous PDE** If $F$ is independent on $x_i$, then if $u(x_1, \dots, x_i, \dots, x_m)$ is solution then $u(x_1, \dots, x_i + \lambda, \dots x_m)$ is solution too.
In particular if $u$ is independent on all $x_i$, then $u(x_1 + \lambda_1, \dots, x_i + \lambda_i, \dots, x_m + \lambda_m)$.

## 1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.
Lets start ODE describing harmonic oscillator, would be

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = k(x - x_0)$$

Now suppose that we have $N$ such masses and position of mass is $\bar{x}_i = x_i + u(x_i, t)$, where $u$ is displacement of mass and $x_i - x_{i-1} = \Delta$. Then the position of mass is described as

$$\frac{\partial^2 \bar{x}_i}{\partial t^2} = m\frac{\partial^2}{\partial t^2}u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = k\left[u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)\right]$$

In limit $\Delta \to 0$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x,t)$$

Where

$$c^2 = \lim_{\Delta \to 0}\frac{\Delta^2 k_\Delta}{m_\Delta}$$

**Possible solutions** For each function $f$ in $\mathcal{C}^2$, $u = f(x - ct)$ is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along $x$ axis with constant velocity $c$. Since $c$ can be negative too, we have solution

$$u(x,t) = f(x + ct) + g(x - ct)$$

## 1.2 Heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Here, $u$ means amount of heat in point $x$ at time $t$.
Amount of heat in interval $[a, b]$ is

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

And heat flux in point $x$ at time $t$ is $k\frac{\partial u}{\partial x}$
Then flux out of interval is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

Thus

$$\int_a^b \frac{\partial}{\partial t}u(x,t)\,\mathrm{d}x = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

In limit $b \to a$ we get

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

**Example solution**

$$u(x,t) = e^{-kst}\sin\left(\sqrt{s}x\right)$$

for some parameter $s$. Here we also can add some constant to $x$ and acquire additional solution:

$$U(x,t) = e^{-kst}\sin\left(\sqrt{s}(x+\lambda)\right) = \cos\left(\sqrt{s}\lambda\right)e^{-kst}\sin\left(\sqrt{s}x\right) + \sin\left(\sqrt{s}\lambda\right)e^{-kst}\cos\left(\sqrt{s}x\right)$$

Thus

$$w(x,t) = e^{-kst}\cos\left(\sqrt{s}x\right)$$

is solution too.

## 1.3 Diffusion equation

Suppose $u(x_1, x_2, x_3, t)$ describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field $v$ independent on $u$.

## 1.4 Elliptic PDEs

### Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

### Poisson equation

$$\nabla^2 u = f(x_1, x_2)$$

# 2 First-order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation: $u(x,t) = f(x - ct)$ for some differentiable $f$.
Suppose we have initial conditions $u(x, 0) = u_0(x)$. Is it determines uniquely a solution of equation? Obviously, $u(x,t) = u_0(x-ct)$ is a solution.
Lets show it's unique. Take a look at parametrization $x(t) = s_1 + ct$.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t), t) = c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thus $u$ is constant on every line of form $x(t) = s + ct$. Such lines, along which the PDE becomes an ordinary differential equation, are called characteristic curves or just characteristics. Thus if we know a value of $u$ in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve $x(t)$ for $t \in [a, b]$. So we want to find a solution such that the surface of solution comprises a given curve in 3D.
The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

## 2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where $a, b, c$ are continuously differentiable in some neighborhood of point $(x_0, y_0, z_0)$.
Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$$

and

$$\nabla f \cdot (a, b, c) = -a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + c = 0$$

Thus vector $(a, b, c)$ is tangent to solution surface.
Now define curve such that

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), z(t)) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), z(t)) \\ \frac{\mathrm{d}z}{\mathrm{d}t} = c(x(t), y(t), z(t)) \end{cases}$$

The curve $(x(t), y(t), z(t))$ is characteristic curve of PDE.
If there is no dependence on $z$ (i.e. equation is linear) we can take a look on 2-dimensional curve in xy-plane.

**Theorem** If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

**Proof** Let $(x(t), y(t), z(t))$ characteristic curve of PDE and suppose for some $t_0$

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that $w(t_0) = 0$. Now

$$G\big(x(t), y(t), w(t)\big) = \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}\big(x(t), y(t)\big)\frac{dx}{dt} - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)\frac{dy}{dt} = c\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) -$$
$$-\frac{\partial u}{\partial x}\big(x(t), y(t)\big)a\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)b\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\big)\Big)$$

If we substitute $w = 0$, we get

$$G\big(x(t), y(t), 0\big) = c\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial x}a\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial y}b\big(x(t), y(t), u\big(x(t), y(t)\big)\big)\big) = 0$$

That means that $w = 0$ is a solution of ODE, and since $a, b, c \in \mathcal{C}^1$, te solution is unique, i.e. $w = 0$ is the only solution, and thus characteristic curve is contained in the solution surface.

## 2.2 Existence and uniquness theorem for first-order quasilinear PDE

**Existence and uniquness theorem for first-order quasilinear PDE** Suppose we have initial curve $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$ which around some point $s_0$ is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0)\dot{\bar{y}}(s_0) - b(x_0, y_0, z_0)\dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).
Then in neighborhood of $s_0$ exists unique solution of PDE.

**Proof** Define functions $x(s,t)$, $y(s,t)$, $z(s,t)$ around $(s_0, 0)$ such that

$$\begin{cases} x(s,0) = \bar{x}(s) \\ y(s,0) = \bar{y}(s) \\ z(s,0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial y}{\partial t} = b\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial z}{\partial t} = c\big(x(s,t), y(s,t), z(s,t)\big) \end{cases}$$

From uniquness and existance of ODE, exists unique solution $(x, y, z)$ in neighbourhood of $s_0$.
Lets try to find $s$, $t$, as a function of $x$,$y$. It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in $(s_0, 0)$.
Now define $u(x,y) = z\big(s(x,y), t(x,y)\big)$.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right] + b\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right] = \frac{\partial z}{\partial t}\left[a\frac{\partial t}{\partial x} + b\frac{\partial t}{\partial y}\right] + \frac{\partial z}{\partial s}\left[a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y}\right] =$$
$$= \frac{\partial z}{\partial t}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial y}\right]}_{\frac{\mathrm{d}t}{\mathrm{d}t}} + \frac{\partial z}{\partial s}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial s}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial s}{\partial y}\right]}_{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{\partial s}{\partial t} = c$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on $xy$-plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on $xy$-plane are different, there are different possibilities.

**Example**

$$yu_x - xu_y = 0$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

**Solution**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s) \end{cases}$$

Now we want to find $s$, $t$ as a function of $x$,$y$:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

**Example**

$$yu_x - xu_y = u$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

**Solution**

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ z = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find $s$, $t$ as a function of $x$,$y$:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

## 2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y} F(u) = 0$$

for $F = \frac{1}{2}u^2$)
Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = -u \Rightarrow u = \frac{\mathrm{d}x}{\mathrm{d}y}$$

Here $y$ denotes time.
To solve it, we take integral:

$$\int_a^b \left[\frac{\partial u(x,y)}{\partial y} + \frac{\partial}{\partial x} F(u(x,y))\right] \mathrm{d}x = 0$$

$$\frac{\partial}{\partial y} \underbrace{\int_a^b u \,\mathrm{d}x}_{Q(y)} + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big)$$

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = F\Big(u(a,y)\Big) - F\Big(u(b,y)\Big)$$

Now as for any quasilinear PDE:

$$\begin{cases} \dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = c_2 t + c_3 \\ y = t + c_1 \\ \dot{z} = c_2 \end{cases}$$

For initial conditions $\Big(s, 0, h(s)\Big)$:

$$\begin{cases} x = h(s)t + s \\ y = t \\ \dot{z} = h(s) \end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} \cdot 1 - \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \cdot h(s) = 1 \neq 0$$

Since

$$\frac{\partial u}{\partial x} = h'(x - yu) \cdot \left(1 - y\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{h'(x-yu)}{1 + h'(x-yu) \cdot y}$$

even if we start from $\mathcal{C}^\infty$ function we can get $1 + h'(x - yu) \cdot y = 0$ and thus undefined derivative.
Geometrically, the slope of projections of characteristic curves is equal to $h(s)$ thus they can cross in some point.

**Weak solutions** We define a weak solution of equation, function $u$ fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y} u(x,y)\,\mathrm{d}x + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big) = 0$$

Intuitively, $F$ is flux, and $u$ is density, thus change in number of particles (integral) is difference between particles going in and out.
Suppose for solution $u(x,y)$ exists curve of non-continuousness $\gamma$, i.e, $u$ is not continuous in each point of curve:

$$u(y) = \begin{cases} u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y) \end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y)\,\mathrm{d}x = \int_a^{\gamma(y)} u^+(x,y)\,\mathrm{d}x + \int_{\gamma(y)}^b u^-(x,y)\,\mathrm{d}x$$

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y}\,\mathrm{d}x + u^+(x,\gamma(y))\cdot\gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y}\,\mathrm{d}x - u^-(x,\gamma(y))\cdot\gamma'(y) = \\ &= -\int_a^{\gamma(y)} \frac{\mathrm{d}F(u^+)}{\mathrm{d}x}\,\mathrm{d}x - \int_{\gamma(y)}^b \frac{\mathrm{d}F(u^-)}{\mathrm{d}x}\,\mathrm{d}x + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = \\ &= -\left[F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^+(a,y)\Big)\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right]\end{aligned}$$

Meaning

$$-\left[F\Big(u^+(\gamma(y),y)\Big) - F(u^+(a,y))\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+(x,\gamma(y)) - u^-(x,\gamma(y))\right] = F\Big(u^-(a,y)\Big) - F\Big(u^+(b,y)\Big)$$

$$\gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)$$

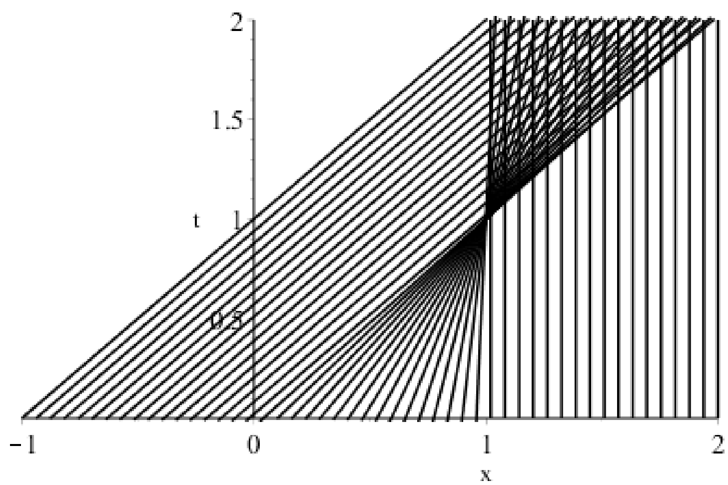
$$\gamma' = \frac{F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)}{u^+\big(x,\gamma(y)\big) - u^-\big(x,\gamma(y)\big)}$$

This equation is called Rankine–Hugoniot conditions. If $F(u) = \frac{1}{2}u^2$, we get $\gamma'(y) = \frac{1}{2}(u^+ + u^-)$

**Example** Suppose we have initial conditions $u(x,0) = h(x)$ for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

For $0 < y < 1$ we have a triangle $\Delta$ ($0 < x < \alpha$ and $y < \frac{x}{\alpha}$) for which there is intersection of two solution:




In point $x, y$ we have slope $u(x,y)$ thus the charecteristic curve crosses $x$-axis at $x_0 = x - uy$ and from initial conditions, $u = 1 - \frac{x_0}{\alpha}$. Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

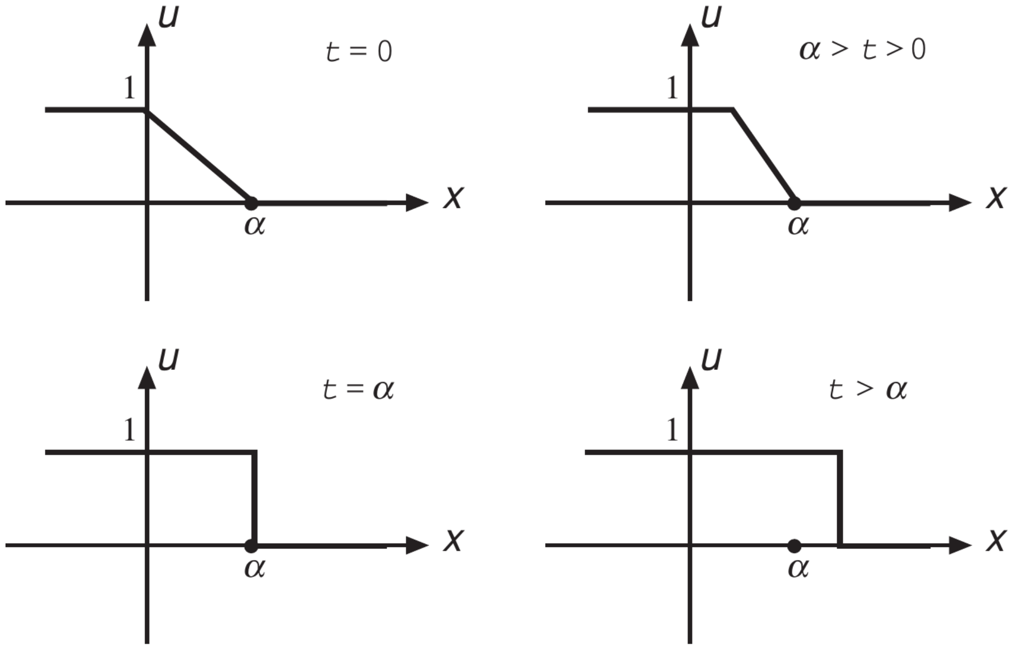
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for $y > 1$ from Rankine–Hugoniot conditions

$$u(x,y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y-\alpha) \\ 0 & x > \alpha + \frac{1}{2}(y-\alpha) \end{cases}$$

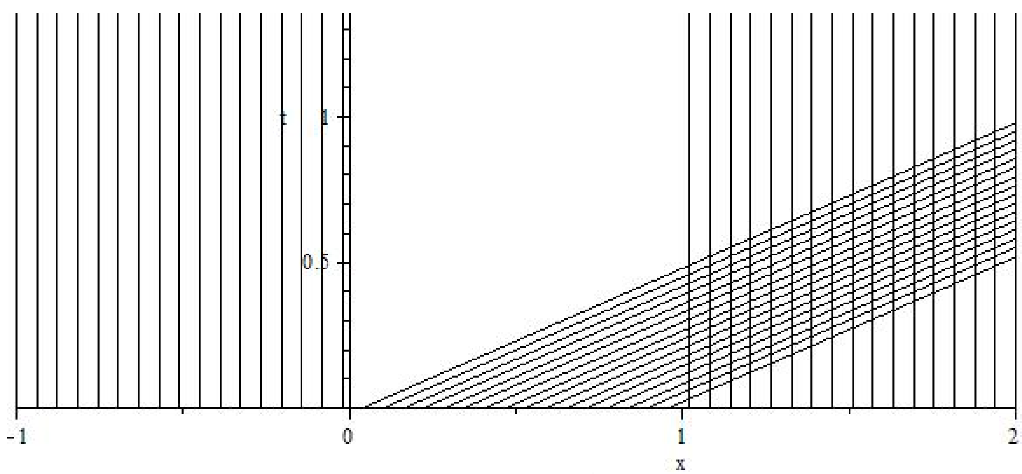
Such a solution is called a shock wave.




**Example** For

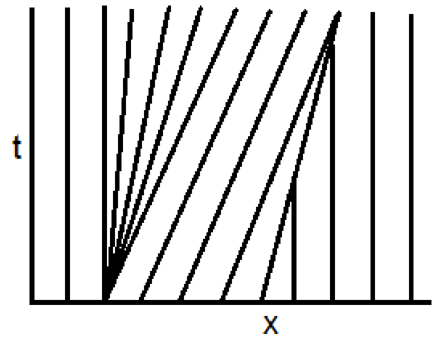
$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

Now there is no place where characteristic curves meet




In the region without characteristic curves ($0 \le x \le y$) we get the following: the solution starts from some point $x_0 = x - uy$, and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$




**What happens if $\alpha \to 0$?** We get $u = \frac{x}{y}$ for $0 \le x \le y$. We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.
However, also shock wave along $y = x$ is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we cant reproduce the solution for some $y < y_0$ even if I know the values for $y = y_0$.

**Entory principle** Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

## 2.4 Fully non-linear equations

**Hamilton-Jacoby equation**

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial F}{\partial p} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial F}{\partial q} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \\ \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} \\ \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$
We need to find $\bar{p}$ and $\bar{q}$. We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by $s$

$$\frac{\partial u}{\partial x}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

$$\bar{p}(s)\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \bar{q}(s)\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

Now we can find $p$ and $q$.
Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with $u = 0$, then characteristic curves are perpendicular to initial curve. We get $u(x, y)$ equal to distance from initial curve, since absolute value of gradient of $u$ is 1 due to equation.
If we have $u = \phi(s)$ on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

**Higher dimension** We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

## 3 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + du_x + eu_y + fu = g$$

**Definition** Equation is called hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$ and elliptic if $b^2 - ac < 0$. Wave equation is hyperbolic in the whole space.
We want to simplify the equation: we are searching for $\xi(x,y)$ and $\eta(x,y)$ such that

$$\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} \neq 0$$

and solution $u(x,y) = w\big(\xi(x,y), \eta(x,y)\big)$.
Derivatives of $u$ are

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi \xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta \eta_{yy}$$

$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi \xi_{xy} + w_\eta \eta_{xy}$$

Now we can get equation of form

$$A(\xi,\eta)w_{\xi\xi} + 2B(\xi,\eta)w_{\xi\eta} + C(\xi,\eta)w_{\eta\xi} + D(\xi,\eta)w_{\eta\eta} + F(\xi,\eta) = 0$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{\xi\eta} = 0$$

i.e.,

$$w(\xi,\eta) = f(\xi) + g(\eta)$$

If we substitute derivatives back into general equation

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} = a\big[w_{\xi\xi}\xi_x^2 + w_{\xi\eta}\xi_x\eta_x + w_\xi\xi_{xx} + w_{\eta\xi}\eta_x\xi_x + w_{\eta\eta}\eta_x^2 + w_\eta\eta_{xx}\big] + \\ +2b[w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}] + \\ +c\big[w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta\eta_{yy}\big] = \\ = \Big(a\xi_x^2 + 3b\xi_x\xi_y + c\xi_y^2\Big)w_{\xi\xi} + 2\Big(a\xi_x\eta_x + c\eta_y\xi_y + b(\xi_x\eta_y + \xi_y\eta_x)\Big)w_{\xi\eta} + \Big(a\eta_x^2 + 3b\eta_x\eta_y + c\eta_y^2\Big)w_{\eta\eta} + \dots \end{aligned}$$

We can rewrite it in matrix form as

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \cdot \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}^2$$

Since the determinant is exactly $ac - b^2$, under the variable substitution the sign of $b^2 - ac$ is conserved.

**Canonical form** The form $w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$, where $\ell_1$ is first-order differential operator is called canonical form of hyperbolic equation.

**Theorem** Each hyperbolic equation can be written in canonical form

**Proof** We want to show that

$$\begin{cases} A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

i.e., that equation $a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0$ has two independent solutions.
Dividing by $\psi_y^2$:

$$a\left(\frac{\psi_x}{\psi_y}\right)^2 + 2b\frac{\psi_x}{\psi_y} + c = 0$$

This is algebric equation, with solutions

$$\frac{\psi_x}{\psi_y} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \lambda_\pm$$

We acquired a pair of equations

$$\psi_x - \lambda_\pm\psi_y = 0$$

And those are two independent solutions which result in $A = 0$ and $C = 0$.

**Wave equation canonical form**

$$u_{tt} - c^2u_{xx} = 0$$

The canonical change of coordinates is

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u_t = -cw_\xi + cw_\eta$$

$$u_x = w_\xi + w_\eta$$

$$u_{tt} = c^2w_{\xi\xi} - 2c^2w_{\xi\eta} + c^2w_{\eta\eta}$$

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

Then

$$u_{tt} - c^2u_{xx} = -4c^2w_{\xi\eta}$$

The solution of canonical equation $w_{\xi\eta} = 0$ is $w(\xi, \eta) = F(\xi) + G(\eta)$, thus solution of wave equation:

$$u(x, t) = F(x + ct) + G(x - ct)$$

An example for physical object fulfilling wave equation is infinite string. To find a solution we need initial conditions, for example, velocity and location at time $t = 0$:

$$\begin{cases} u(x, 0) = f(x) \\ u_t(x, 0) = g(x) \end{cases}$$

, where $f \in \mathcal{C}^2$, $g \in \mathcal{C}^1$.

**Theorem** Exists unique solution of wave equation with those initial conditions.

**Proof** Substituting initial conditions into general solutions:

$$\begin{cases} u(x, 0) = F(x) + G(x) = f(x) \\ u_t(x, 0) = c[F'(x) - G'(x)] = g(x) \end{cases}$$

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{g(x)}{c} \end{cases} \Rightarrow \begin{cases} F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_1 \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_2 \end{cases}$$

Now, since $F(x) + G(x) = f(x)$, thus $D_1 + D_2 = 0$.
Substituting into solution, we acquire what is called d'Alembert's formula:

$$u(x, t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

From construction, the solution is unique.

**Example**

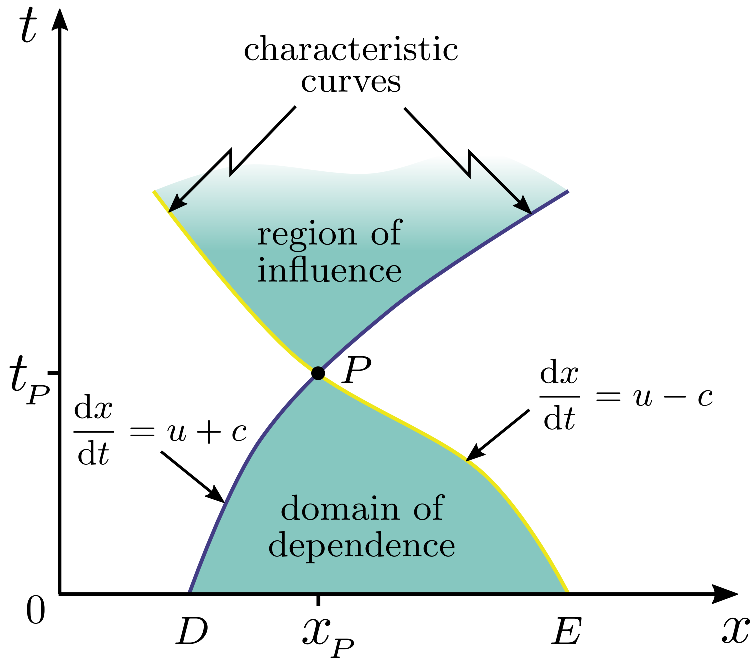
$$\begin{cases} g(x) = 0 \\ f(x) = e^{-x^2} \end{cases}$$

$$u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x+ct)^2}$$

**Standing wave** To get standing wave we want $G = 0$, i.e.,

$$\begin{cases} f(x) = F(x) \\ g(x) = cF'(x) \end{cases} \quad \Rightarrow g(x) = cf'(x)$$

**Domain of dependence and region of influence** Domain of dependence of $u$ in point $(x_0, t_0)$ is a characteristic triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$, $(x_0, t_0)$. Any point outside of triangle doesn't affect the value of $u$ in point.
Region of influence of point $x_0$ is cone bounded by condition $x_0 - ct < x < x_0 + ct$.




**Weak solution**

$$\begin{cases} g(x) = 0 \\ f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}$$

The weak solution

$$u(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)$$

is not differentiable, but is solves the equation in some sense.

## 3.1 Generalization of d'Alembert's formula for non-homogeneous equations

Consider non-homogeneous equation

$$u_{tt} - c^2 u_{xx} = \varphi(x,t)$$

Remember Green's theorem, for differentiable $P$ and $Q$ defined in $\Omega$:

$$\iint\limits_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t}\, \mathrm{d}x\, \mathrm{d}t = \oint\limits_{\partial\Omega} P(x,t)\, \mathrm{d}x + Q(x,t)\, \mathrm{d}t$$

Lets define $Q = c^2 u_x$ and $P = u_t$, and choose $\Omega(x_0, t_0)$ to be characteristic triangle.

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\Omega(x_0,t_0)} u_{tt} - c^2 u_{xx}\,\mathrm{d}x\,\mathrm{d}t = \oint P(x,t)\,\mathrm{d}x - Q(x,t)\,\mathrm{d}t = -\left[\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t\right]$$

Lets divide the curve integral into three integrals along each of lines. For first line $\mathrm{d}t = 0$, for second $\mathrm{d}x + c\,\mathrm{d}t = 0$ and for third $\mathrm{d}x - c\,\mathrm{d}t = 0$.

$$\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} \underbrace{u_t}_{g(x) \text{ in } t=0}\,\mathrm{d}x - \int_{(x_0+ct_0,0)}^{(x_0,t_0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x + \int_{(x_0,t_0)}^{(x_0-ct_0,0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s - c\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u + c\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u$$

Since

$$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u = u(x_0,t_0) - f(x_0+ct_0)$$

$$\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u = f(x_0-ct_0) - u(x_0,t_0)$$

we get

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s + 2cu(x_0,t_0) - cf(x_0+ct_0) - cf(x_0-ct_0)$$

from which we get the solution

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

We've got a cadidate for the solution. Let's check that $u$ is actually solving PDE. Define $v$, $w$, such that $w$ is a solution of homogeneous PDE and $v = u - w$, i.e.,

$$v(x,t) = \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Let's show that $v$ solves PDE. Rewrite $v$ as double integral:

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Define

$$H(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi$$

and then

$$v(x,t) = \frac{1}{2c}\int_0^t H(x,t,\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial t} = \frac{1}{2c}\underbrace{H(x,t,t)}_{0} + \frac{1}{2c}\int_0^t \frac{\partial H}{\partial t}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial t} = c[\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 H}{\partial t^2} = c^2[\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2c}\int_0^t \frac{\partial^2 H}{\partial t^2}\,\mathrm{d}\tau = \varphi(x,t) + \frac{c}{2}\int_0^t \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial x} = \frac{1}{2c}\int_0^t \frac{\partial H}{\partial x}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial x} = \varphi(x + c(t-\tau),\tau) - \varphi(x - c(t-\tau),\tau)$$

$$\frac{\partial^2 H}{\partial x^2} = \varphi_x(x + c(t-\tau),\tau) - \varphi_x(x - c(t-\tau),\tau)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2c}\int_0^t \varphi_x(x + c(t-\tau),\tau) - \varphi_x(x - c(t-\tau),\tau)\,\mathrm{d}\tau$$

Thus we got

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = \varphi(x,t)$$

Suppose we have two solutions $u_1$ and $u_2$ then $u = u_1 - u_2$ is solution of homogeneous equation with 0 initial conditions, and thus $u = 0$. That means the solution is unique.
The presented initial condition problem has 3 properties:

1. Solution exist
2. It's unique
3. It's stable

**Stability of wave equation** For all $\tau > 0$, $\epsilon > 0$, exists $\delta > 0$ such that if

$$\begin{cases} |f(x) - \tilde{f}(x)| < \delta \\ |g(x) - \tilde{g}(x)| < \delta \\ |\varphi(x) - \tilde{\varphi}(x)| < \delta \end{cases}$$

For all $-\infty < x < \infty$ and $0 \le t \le \tau$ and if $u$, $\tilde{u}$ are solutions of corresponding wave equations, then

$$|u(x,t) - \tilde{u}(x,t)| < \epsilon$$

**Proof** From the general solution:

$$u(x,t) - \tilde{u}(x,t) =$$

$$= \left|\frac{f(x+ct)+f(x-ct)}{2} - \frac{\tilde{f}(x+ct)+\tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) - \tilde{g}(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\right| \le$$

$$\le \left|\frac{f(x+ct)-\tilde{f}(x+ct)}{2}\right| + \left|\frac{f(x-ct)-\tilde{f}(x-ct)}{2}\right| + \frac{1}{2c}\int_{x-ct}^{x+ct} |g(s) - \tilde{g}(s)|\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} |\varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \le$$

$$\le \frac{\delta}{2} + \frac{\delta}{2}\frac{1}{2c}\cdot 2c\cdot\delta + \frac{1}{2c}\frac{ct}{2} = 2\delta + \frac{\delta t}{4} \le 2\delta + \frac{\delta\tau}{4} \le \epsilon$$

Thus we choose $\delta < \frac{\epsilon}{2+\frac{\tau}{4}}$.