104030 - Introduction to Partial Differential Equations

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Abstract

1 Introduction

PDE In PDE, the solution is a function of a couple of variables $u(x_1, x_2, \dots x_m)$ such that:

$$F(x_1, x_2, \dots x_m, u_{x_1}, u_{x_2}, \dots, , u_{x_m}, u_{x_1 x_1}, \dots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually $m = 2$. For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_1 x_0}) = 0$$

Is PDE of two variables of order 2.

Linear PDE PDE is linear if F is linear in u and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2)u_{x_1 x_1} + B(x_1, x_2)u_{x_1 x_2} + C(x_1, x_2)u_{x_2 x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Quasilinear PDE Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2 x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0$$

For homogeneous linear PDE solution always exist. In addition, if u_1, u_2, then any linear combination of those $\lambda_1 u_1 + \lambda_2 u_2$ will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

Autonomous PDE If F is independent on x_i, then if $u(x_1, \dots, x_i, \dots, x_m)$ is solution then $u(x_1, \dots, x_i + \lambda, \dots x_m)$ is solution too.
In particular if u is independent on all x_i, then $u(x_1 + \lambda_1, \dots, x_i + \lambda_i, \dots, x_m + \lambda_m)$.

1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.
Lets start ODE describing harmonic oscillator, would be

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = k(x - x_0)$$

Now suppose that we have N such masses and position of mass is $\bar{x}_i = x_i + u(x_i, t)$, where u is displacement of mass and $x_i - x_{i-1} = \Delta$. Then the position of mass is described as

$$\frac{\partial^2 \bar{x}_i}{\partial t^2} = m\frac{\partial^2}{\partial t^2}u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = k\left[u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)\right]$$

In limit $\Delta \to 0$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x,t)$$

Where

$$c^2 = \lim_{\Delta \to 0}\frac{\Delta^2 k_\Delta}{m_\Delta}$$

Possible solutions For each function f in $\mathcal{C}^2$, $u = f(x - ct)$ is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along x axis with constant velocity c. Since c can be negative too, we have solution

$$u(x,t) = f(x + ct) + g(x - ct)$$

1.2 Heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Here, u means amount of heat in point x at time t.
Amount of heat in interval $[a, b]$ is

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

And heat flux in point x at time t is $k\frac{\partial u}{\partial x}$
Then flux out of interval is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

Thus

$$\int_a^b \frac{\partial}{\partial t}u(x,t)\,\mathrm{d}x = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

In limit $b \to a$ we get

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Example solution

$$u(x,t) = e^{-kst}\sin\left(\sqrt{s}x\right)$$

for some parameter s. Here we also can add some constant to x and acquire additional solution:

$$U(x,t) = e^{-kst}\sin\left(\sqrt{s}(x + \lambda)\right) = \cos\left(\sqrt{s}\lambda\right)e^{-kst}\sin\left(\sqrt{s}x\right) + \sin\left(\sqrt{s}\lambda\right)e^{-kst}\cos\left(\sqrt{s}x\right)$$

Thus

$$w(x,t) = e^{-kst}\cos\left(\sqrt{s}x\right)$$

is solution too.

1.3 Diffusion equation

Suppose $u(x_1, x_2, x_3, t)$ describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field v independent on u.

1.4 Elliptic PDEs

Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

Poisson equation

$$\nabla^2 u = f(x_1, x_2)$$

2 First-order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation: $u(x,t) = f(x - ct)$ for some differentiable f.
Suppose we have initial conditions $u(x, 0) = u_0(x)$. Is it determines uniquely a solution of equation? Obviously, $u(x,t) = u_0(x-ct)$ is a solution.
Lets show it's unique. Take a look at parametrization $x(t) = s_1 + ct$.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t), t) = c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thus u is constant on every line of form $x(t) = s+ct$. Such lines, along which the PDE becomes an ordinary differential equation, are called characteristic curves or just characteristics. Thus if we know a value of u in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve $x(t)$ for $t \in [a, b]$. So we want to find a solution such that the surface of solution comprises a given curve in 3D.
The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where a, b, c are continuously differentiable in some neighborhood of point (x_0, y_0, z_0).
Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$$

and

$$\nabla f \cdot (a, b, c) = -a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + c = 0$$

Thus vector (a, b, c) is tangent to solution surface.
Now define curve such that

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), z(t)) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), z(t)) \\ \frac{\mathrm{d}z}{\mathrm{d}t} = c(x(t), y(t), z(t)) \end{cases}$$

The curve $(x(t), y(t), z(t))$ is characteristic curve of PDE.
If there is no dependence on z (i.e. equation is linear) we can take a look on 2-dimensional curve in xy-plane.

Theorem If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

Proof Let $(x(t), y(t), z(t))$ characteristic curve of PDE and suppose for some t_0

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that $w(t_0) = 0$. Now

$$G\big(x(t), y(t), w(t)\big) = \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}\big(x(t), y(t)\big)\frac{dx}{dt} - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)\frac{dy}{dt} = c\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) -$$
$$-\frac{\partial u}{\partial x}\big(x(t), y(t)\big)a\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)b\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\big)\Big)$$

If we substitute $w = 0$, we get

$$G\big(x(t), y(t), 0\big) = c\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial x}a\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial y}b\big(x(t), y(t), u\big(x(t), y(t)\big)\big)\big) = 0$$

That means that $w = 0$ is a solution of ODE, and since $a, b, c \in \mathcal{C}^1$, te solution is unique, i.e. $w = 0$ is the only solution, and thus characteristic curve is contained in the solution surface.

2.2 Existence and uniquness theorem for first-order quasilinear PDE

Existence and uniquness theorem for first-order quasilinear PDE Suppose we have initial curve $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$ which around some point s_0 is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0)\dot{\bar{y}}(s_0) - b(x_0, y_0, z_0)\dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).
Then in neighborhood of s_0 exists unique solution of PDE.

Proof Define functions $x(s,t)$, $y(s,t)$, $z(s,t)$ around $(s_0, 0)$ such that

$$\begin{cases} x(s,0) = \bar{x}(s) \\ y(s,0) = \bar{y}(s) \\ z(s,0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial y}{\partial t} = b\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial z}{\partial t} = c\big(x(s,t), y(s,t), z(s,t)\big) \end{cases}$$

From uniquness and existance of ODE, exists unique solution (x, y, z) in neighbourhood of s_0.
Lets try to find s, t, as a function of x,y. It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in $(s_0, 0)$.
Now define $u(x,y) = z\big(s(x,y), t(x,y)\big)$.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right] + b\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right] = \frac{\partial z}{\partial t}\left[a\frac{\partial t}{\partial x} + b\frac{\partial t}{\partial y}\right] + \frac{\partial z}{\partial s}\left[a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y}\right] =$$
$$= \frac{\partial z}{\partial t}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial y}\right]}_{\frac{\mathrm{d}t}{\mathrm{d}t}} + \frac{\partial z}{\partial s}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial s}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial s}{\partial y}\right]}_{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{\partial s}{\partial t} = c$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on xy-plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on xy-plane are different, there are different possibilities.

Example

$$yu_x - xu_y = 0$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s) \end{cases}$$

Now we want to find s, t as a function of x,y:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

Example

$$yu_x - xu_y = u$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ z = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find s, t as a function of x,y:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y} F(u) = 0$$

for $F = \frac{1}{2}u^2$)
Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = -u \Rightarrow u = \frac{\mathrm{d}x}{\mathrm{d}y}$$

Here y denotes time.
To solve it, we take integral:

$$\int_a^b \left[\frac{\partial u(x,y)}{\partial y} + \frac{\partial}{\partial x} F(u(x,y))\right] \mathrm{d}x = 0$$

$$\frac{\partial}{\partial y} \underbrace{\int_a^b u \,\mathrm{d}x}_{Q(y)} + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big)$$

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = F\Big(u(a,y)\Big) - F\Big(u(b,y)\Big)$$

Now as for any quasilinear PDE:

$$\begin{cases} \dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = c_2 t + c_3 \\ y = t + c_1 \\ \dot{z} = c_2 \end{cases}$$

For initial conditions $\Big(s, 0, h(s)\Big)$:

$$\begin{cases} x = h(s)t + s \\ y = t \\ \dot{z} = h(s) \end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} \cdot 1 - \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \cdot h(s) = 1 \neq 0$$

Since

$$\frac{\partial u}{\partial x} = h'(x - yu) \cdot \left(1 - y\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{h'(x - yu)}{1 + h'(x - yu) \cdot y}$$

even if we start from $\mathcal{C}^\infty$ function we can get $1 + h'(x - yu) \cdot y = 0$ and thus undefined derivative.
Geometrically, the slope of projections of characteristic curves is equal to $h(s)$ thus they can cross in some point.

Weak solutions We define a weak solution of equation, function u fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y} u(x,y)\,\mathrm{d}x + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big) = 0$$

Intuitively, F is flux, and u is density, thus change in number of particles (integral) is difference between particles going in and out.
Suppose for solution $u(x,y)$ exists curve of non-continuousness γ, i.e, u is not continuous in each point of curve:

$$u(y) = \begin{cases} u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y) \end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y)\,\mathrm{d}x = \int_a^{\gamma(y)} u^+(x,y)\,\mathrm{d}x + \int_{\gamma(y)}^b u^-(x,y)\,\mathrm{d}x$$

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y}\,\mathrm{d}x + u^+(x,\gamma(y))\cdot\gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y}\,\mathrm{d}x - u^-(x,\gamma(y))\cdot\gamma'(y) = \\ &= -\int_a^{\gamma(y)} \frac{\mathrm{d}F(u^+)}{\mathrm{d}x}\,\mathrm{d}x - \int_{\gamma(y)}^b \frac{\mathrm{d}F(u^-)}{\mathrm{d}x}\,\mathrm{d}x + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = \\ &= -\left[F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^+(a,y)\Big)\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right]\end{aligned}$$

Meaning

$$-\left[F\Big(u^+(\gamma(y),y)\Big) - F(u^+(a,y))\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+(x,\gamma(y)) - u^-(x,\gamma(y))\right] = F\Big(u^-(a,y)\Big) - F\Big(u^+(b,y)\Big)$$

$$\gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)$$

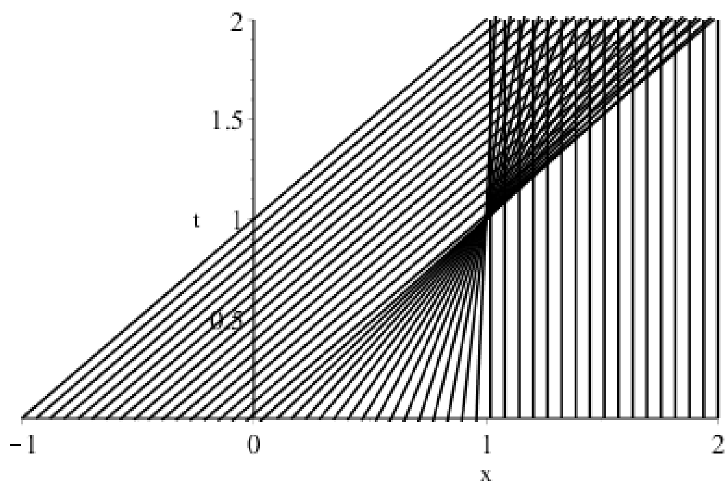
$$\gamma' = \frac{F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)}{u^+\big(x,\gamma(y)\big) - u^-\big(x,\gamma(y)\big)}$$

This equation is called Rankine–Hugoniot conditions. If $F(u) = \frac{1}{2}u^2$, we get $\gamma'(y) = \frac{1}{2}\left(u^+ + u^-\right)$

Example Suppose we have initial conditions $u(x,0) = h(x)$ for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

For $0 < y < 1$ we have a triangle Δ ($0 < x < \alpha$ and $y < \frac{x}{\alpha}$) for which there is intersection of two solution:



In point x, y we have slope $u(x,y)$ thus the charecteristic curve crosses x-axis at $x_0 = x - uy$ and from initial conditions, $u = 1 - \frac{x_0}{\alpha}$. Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

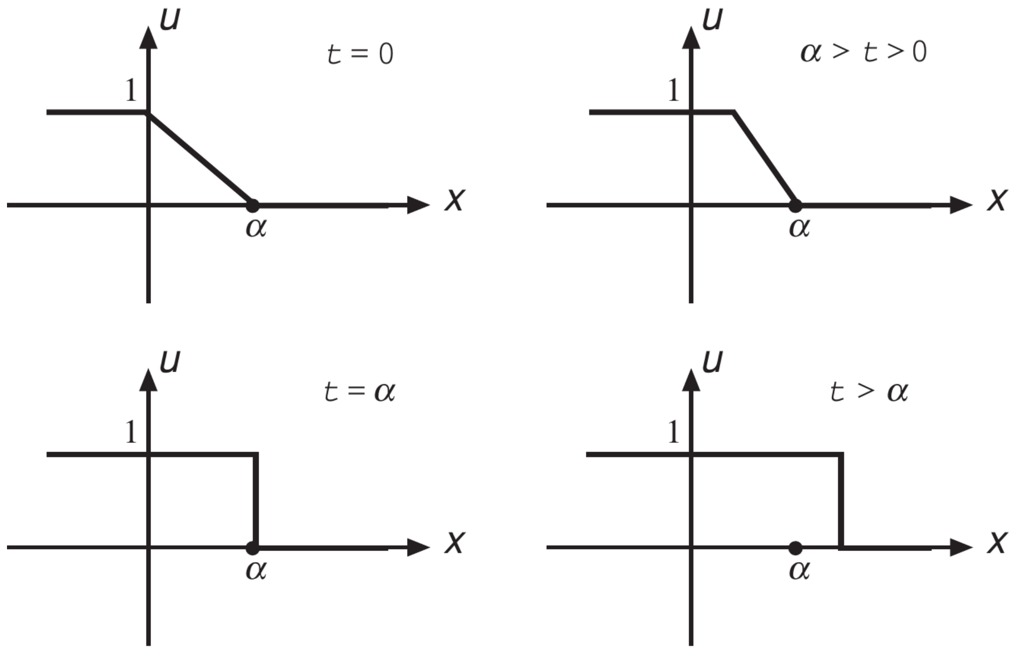
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for $y > 1$ from Rankine–Hugoniot conditions

$$u(x,y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y-\alpha) \\ 0 & x > \alpha + \frac{1}{2}(y-\alpha) \end{cases}$$

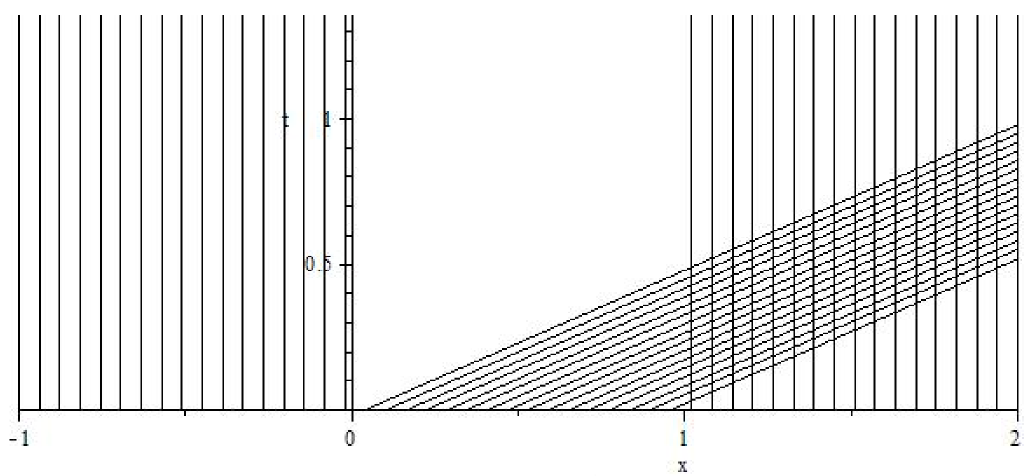
Such a solution is called a shock wave.



Example For

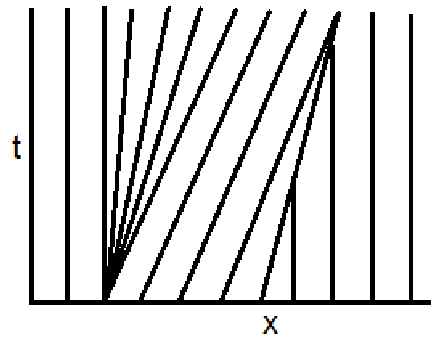
$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

Now there is no place where characteristic curves meet



In the region without characteristic curves ($0 \le x \le y$) we get the following: the solution starts from some point $x_0 = x - uy$, and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$



What happens if $\alpha \to 0$? We get $u = \frac{x}{y}$ for $0 \leq x \leq y$. We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.
However, also shock wave along $y = x$ is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we cant reproduce the solution for some $y < y_0$ even if I know the values for $y = y_0$.

Entory principle Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

2.4 Fully non-linear equations

Hamilton-Jacoby equation

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial F}{\partial p} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial F}{\partial q} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \\ \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} \\ \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$
We need to find $\bar{p}$ and $\bar{q}$. We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by s

$$\frac{\partial u}{\partial x}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

$$\bar{p}(s)\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \bar{q}(s)\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

Now we can find p and q.
Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with $u = 0$, then characteristic curves are perpendicular to initial curve. We get $u(x, y)$ equal to distance from initial curve, since absolute value of gradient of u is 1 due to equation.
If we have $u = \phi(s)$ on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

Higher dimension We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

3 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + du_x + eu_y + fu = g$$

Definition Equation is called hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$ and elliptic if $b^2 - ac < 0$. Wave equation is hyperbolic in the whole space.
We want to simplify the equation: we are searching for $\xi(x,y)$ and $\eta(x,y)$ such that

$$\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} \neq 0$$

and solution $u(x,y) = w\big(\xi(x,y), \eta(x,y)\big)$.
Derivatives of u are

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi \xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta \eta_{yy}$$

$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}$$

Now we can get equation of form

$$A(\xi,\eta)w_{\xi\xi} + 2B(\xi,\eta)w_{\xi\eta} + C(\xi,\eta)w_{\eta\xi} + D(\xi,\eta)w_{\eta\eta} + F(\xi,\eta) = 0$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{\xi\eta} = 0$$

i.e.,

$$w(\xi,\eta) = f(\xi) + g(\eta)$$

If we substitute derivatives back into general equation

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} &= a\big[w_{\xi\xi}\xi_x^2 + w_{\xi\eta}\xi_x\eta_x + w_\xi\xi_{xx} + w_{\eta\xi}\eta_x\xi_x + w_{\eta\eta}\eta_x^2 + w_\eta\eta_{xx}\big] + \\ &+2b[w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}] + \\ &+c\big[w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta\eta_{yy}\big] = \\ = \Big(a\xi_x^2 + 3b\xi_x\xi_y + c\xi_y^2\Big)w_{\xi\xi} + 2\Big(a\xi_x\eta_x + c\eta_y\xi_y + b(\xi_x\eta_y + \xi_y\eta_x)\Big)w_{\xi\eta} &+ \Big(a\eta_x^2 + 3b\eta_x\eta_y + c\eta_y^2\Big)w_{\eta\eta} + \dots \end{aligned}$$

We can rewrite it in matrix form as

$$\begin{pmatrix}\xi_x & \xi_y \\ \eta_x & \eta_y\end{pmatrix}\begin{pmatrix}a & b \\ b & c\end{pmatrix}\begin{pmatrix}\xi_x & \eta_x \\ \xi_y & \eta_y\end{pmatrix} = \begin{pmatrix}A & B \\ B & C\end{pmatrix}$$

$$\begin{vmatrix}A & B \\ B & C\end{vmatrix} = \begin{vmatrix}a & b \\ b & c\end{vmatrix} \cdot \begin{vmatrix}\xi_x & \eta_x \\ \xi_y & \eta_y\end{vmatrix}^2$$

Since the determinant is exactly $ac - b^2$, under the variable substitution the sign of $b^2 - ac$ is conserved.

Canonical form The form $w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$, where ℓ_1 is first-order differential operator is called canonical form of hyperbolic equation.

Theorem Each hyperbolic equation can be written in canonical form

Proof We want to show that

$$\begin{cases} A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

i.e., that equation $a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0$ has two independent solutions.
Dividing by ψ_y^2:

$$a\left(\frac{\psi_x}{\psi_y}\right)^2 + 2b\frac{\psi_x}{\psi_y} + c = 0$$

This is algebric equation, with solutions

$$\frac{\psi_x}{\psi_y} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \lambda_\pm$$

We acquired a pair of equations

$$\psi_x - \lambda_\pm \psi_y = 0$$

And those are two independent solutions which result in $A = 0$ and $C = 0$.

Wave equation canonical form

$$u_{tt} - c^2 u_{xx} = 0$$

The canonical change of coordinates is

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u_t = -cw_\xi + cw_\eta$$

$$u_x = w_\xi + w_\eta$$

$$u_{tt} = c^2 w_{\xi\xi} - 2c^2 w_{\xi\eta} + c^2 w_{\eta\eta}$$

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

Then

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$$

The solution of canonical equation $w_{\xi\eta} = 0$ is $w(\xi, \eta) = F(\xi) + G(\eta)$, thus solution of wave equation:

$$u(x,t) = F(x + ct) + G(x - ct)$$

An example for physical object fulfilling wave equation is infinite string. To find a solution we need initial conditions, for example, velocity and location at time $t = 0$:

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

, where $f \in \mathcal{C}^2$, $g \in \mathcal{C}^1$.

Theorem Exists unique solution of wave equation with those initial conditions.

Proof Substituting initial conditions into general solutions:

$$\begin{cases} u(x,0) = F(x) + G(x) = f(x) \\ u_t(x,0) = c[F'(x) - G'(x)] = g(x) \end{cases}$$

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{g(x)}{c} \end{cases} \Rightarrow \begin{cases} F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_1 \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_2 \end{cases}$$

Now, since $F(x) + G(x) = f(x)$, thus $D_1 + D_2 = 0$.
Substituting into solution, we acquire what is called d'Alembert's formula:

$$u(x,t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

From construction, the solution is unique.

Example

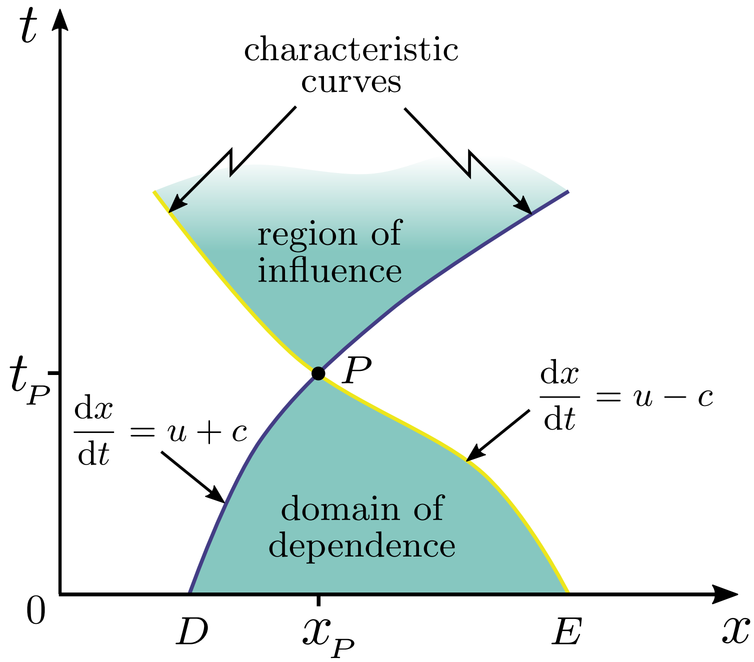
$$\begin{cases} g(x) = 0 \\ f(x) = e^{-x^2} \end{cases}$$

$$u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x+ct)^2}$$

Standing wave To get standing wave we want $G = 0$, i.e.,

$$\begin{cases} f(x) = F(x) \\ g(x) = cF'(x) \end{cases} \quad \Rightarrow g(x) = cf'(x)$$

Domain of dependence and region of influence Domain of dependence of u in point (x_0, t_0) is a characteristic triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$, (x_0, t_0). Any point outside of triangle doesn't affect the value of u in point.
Region of influence of point x_0 is cone bounded by condition $x_0 - ct < x < x_0 + ct$.



Weak solution

$$\begin{cases} g(x) = 0 \\ f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}$$

The weak solution

$$u(x,t) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct)$$

is not differentiable, but is solves the equation in some sense.

3.1 Generalization of d'Alembert's formula for non-homogeneous equations

Consider non-homogeneous equation

$$u_{tt} - c^2 u_{xx} = \varphi(x,t)$$

Remember Green's theorem, for differentiable P and Q defined in Ω:

$$\iint_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t} \, dx \, dt = \oint_{\partial\Omega} P(x,t) \, dx + Q(x,t) \, dt$$

Lets define $Q = c^2 u_x$ and $P = u_t$, and choose $\Omega(x_0, t_0)$ to be characteristic triangle.

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\Omega(x_0,t_0)} u_{tt} - c^2 u_{xx}\,\mathrm{d}x\,\mathrm{d}t = \oint P(x,t)\,\mathrm{d}x - Q(x,t)\,\mathrm{d}t = -\left[\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t\right]$$

Lets divide the curve integral into three integrals along each of lines. For first line $\mathrm{d}t = 0$, for second $\mathrm{d}x + c\,\mathrm{d}t = 0$ and for third $\mathrm{d}x - c\,\mathrm{d}t = 0$.

$$\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} \underbrace{u_t}_{g(x)\text{ in }t=0}\,\mathrm{d}x - \int_{(x_0+ct_0,0)}^{(x_0,t_0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x + \int_{(x_0,t_0)}^{(x_0-ct_0,0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s - c\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u + c\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u$$

Since

$$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u = u(x_0,t_0) - f(x_0+ct_0)$$

$$\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u = f(x_0-ct_0) - u(x_0,t_0)$$

we get

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s + 2cu(x_0,t_0) - cf(x_0+ct_0) - cf(x_0-ct_0)$$

from which we get the solution

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

We've got a cadidate for the solution. Let's check that u is actually solving PDE. Define v, w, such that w is a solution of homogeneous PDE and $v = u - w$, i.e.,

$$v(x,t) = \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Let's show that v solves PDE. Rewrite v as double integral:

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Define

$$H(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi$$

and then

$$v(x,t) = \frac{1}{2c}\int_0^t H(x,t,\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial t} = \frac{1}{2c}\underbrace{H(x,t,t)}_{0} + \frac{1}{2c}\int_0^t \frac{\partial H}{\partial t}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial t} = c[\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 H}{\partial t^2} = c^2[\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2c}\int_0^t \frac{\partial^2 H}{\partial t^2}\,\mathrm{d}\tau = \varphi(x,t) + \frac{c}{2}\int_0^t \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial x} = \frac{1}{2c}\int_0^t \frac{\partial H}{\partial x}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial x} = \varphi(x+c(t-\tau),\tau) - \varphi(x-c(t-\tau),\tau)$$

$$\frac{\partial^2 H}{\partial x^2} = \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2c}\int_0^t \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\,\mathrm{d}\tau$$

Thus we got

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = \varphi(x,t)$$

Suppose we have two solutions u_1 and u_2 then $u = u_1 - u_2$ is solution of homogeneous equation with 0 initial conditions, and thus $u = 0$. That means the solution is unique.
The presented initial condition problem has 3 properties:

1. Solution exist
2. It's unique
3. It's stable

Stability of wave equation For all $\tau > 0$, $\epsilon > 0$, exists $\delta > 0$ such that if

$$\begin{cases} |f(x) - \tilde{f}(x)| < \delta \\ |g(x) - \tilde{g}(x)| < \delta \\ |\varphi(x) - \tilde{\varphi}(x)| < \delta \end{cases}$$

For all $-\infty < x < \infty$ and $0 \le t \le \tau$ and if u, $\tilde{u}$ are solutions of corresponding wave equations, then

$$|u(x,t) - \tilde{u}(x,t)| < \epsilon$$

Proof From the general solution:

$$u(x,t) - \tilde{u}(x,t) =$$

$$= \left| \frac{f(x+ct)+f(x-ct)}{2} - \frac{\tilde{f}(x+ct)+\tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)-\tilde{g}(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)-\tilde{\varphi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \right| \le$$

$$\le \left|\frac{f(x+ct)-\tilde{f}(x+ct)}{2}\right| + \left|\frac{f(x-ct)-\tilde{f}(x-ct)}{2}\right| + \frac{1}{2c}\int_{x-ct}^{x+ct} |g(s)-\tilde{g}(s)|\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} |\varphi(\xi,\eta)-\tilde{\varphi}(\xi,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \le$$

$$\le \frac{\delta}{2} + \frac{\delta}{2}\frac{1}{2c}\cdot 2c\cdot\delta + \frac{1}{2c}\frac{ct}{2} = 2\delta + \frac{\delta t}{4} \le 2\delta + \frac{\delta\tau}{4} \le \epsilon$$

Thus we choose $\delta < \frac{\epsilon}{2+\frac{\tau}{4}}$.

3.2 Wave equation with bound conditions

Half-infinite string Suppose string is fixed in one of its ends, at $x = 0$: $u(0,t) = 0$. This is called Dirichlet boundary condition. We want to solve the PDE for $x > 0$.

Property of wave equation If $u(x,t)$ is solution, then $u(-x,t)$ is also solution:

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$u(-x,t) = F(-x+ct) + G(-x-xt) = \bar{F}(x+ct) + \bar{G}(x-ct)$$

Where $\bar{F}(s) = G(-s)$ and $\bar{G}(s) = F(-s)$.

Lets extend f and g on the whole plane in odd way:

$$\bar{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(x) & x < 0 \end{cases}$$

and same for g.
Note that initial conditions have to be consistent, i.e., $f(0) = 0$, $g(0) = 0$, else the solution is discontinuous in 0. Lets use D'Lambert solution:

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(s)\,\mathrm{d}s$$

Then the solution of half-infinite string is

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

Neumann boundary condition In this case, instead of giving boundary condition on u, we give boundary condition of u_x: $u_x(0,t) = 0$. Physical meaning is that there is no force in this point. In this case we will extend function in even way (derivative of even function in 0 is 0). Then we get

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{c}\int_{0}^{ct-x} g(s)\,\mathrm{d}s + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

Uniquness Suppose we have two solutions, by subtracting them, we get

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = u_x(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

We acquire $u(x,t) = 0$. Since $u(x,t)$ is of form $F(x+ct) + G(x-ct)$, we get that twobsolutions are identical.

Wave equation with finite string Suppose we have string from a to b:

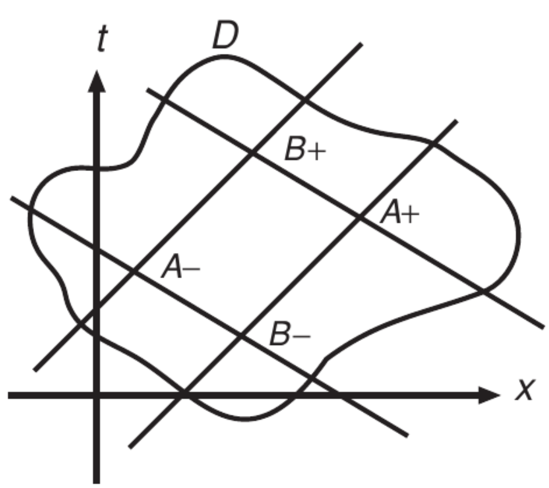
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & a \le x \le b \\ u(x,0) = f(x) \\ u_t(a,t) = h(t) \\ u_t(b,t)b = q(t) \end{cases}$$

Here the consistency conditions are

$$\begin{cases} h(0) = f(a) & q(0) = f(b) \\ h'(0) = g(a) & q'(0) = g(b) \end{cases}$$

Here we could have conditions derivatives instead of values as well.

Parallelogram identity



$u \in \mathcal{C}^2$ is the solution of wave equation iff for any parallelogram with sides parallel to characteristic lines with vertices A_-, A_+, B_-, B_+

$$u(A_-) + u(A_+) = u(B_-) + u(B_+)$$

Proof One direction is simple, since value of solution is constant along characteristic curves.
For second direction lets switch to canonical coordinates

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

So

$$w(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

w is solution of wave equation iff $w_{\xi\eta} = 0$.
Note that parallelogram turned into rectangular in new coordinates, i.e.,

$$w(\xi_0, \eta_1) + w(\xi_1, \eta_0) = w(\xi_1, \eta_1) + w(xi_0, \eta_0)$$

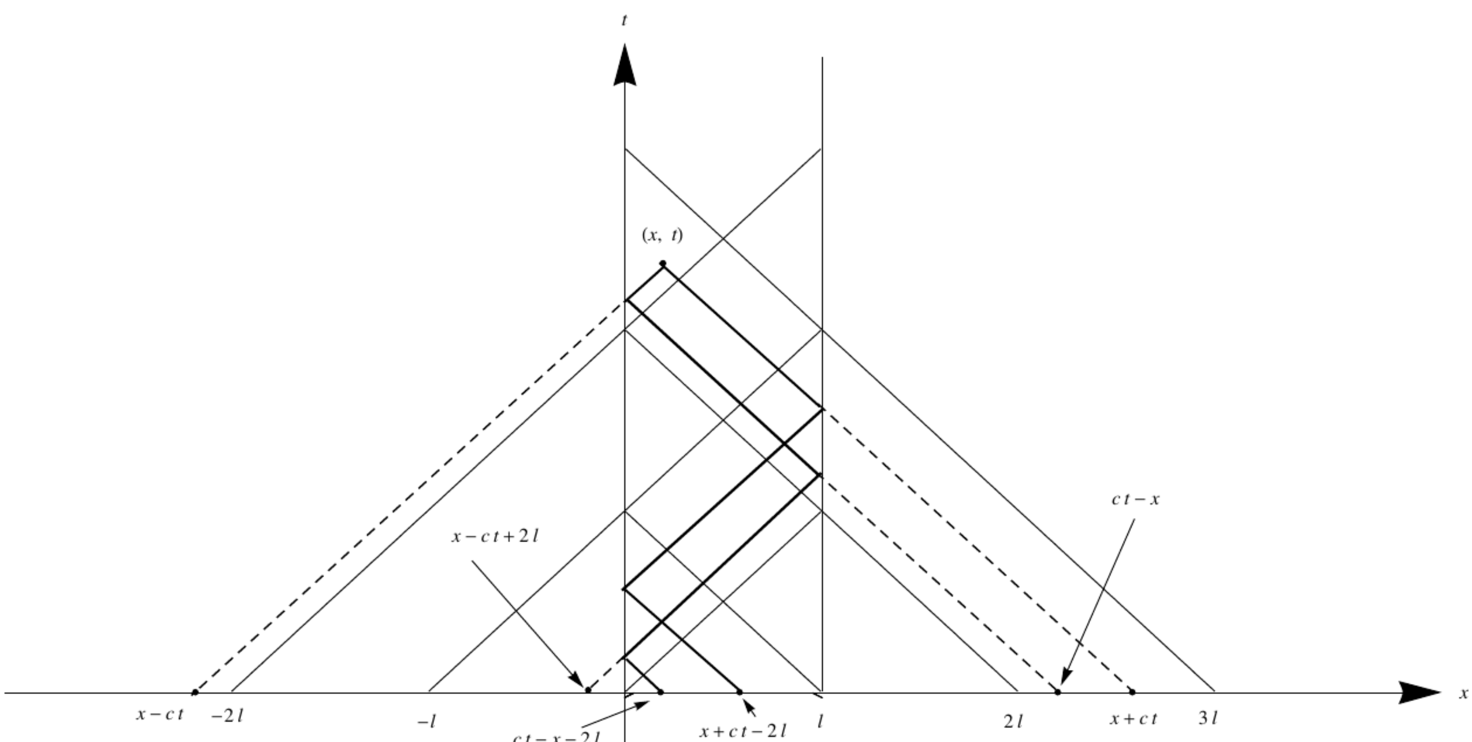
Dividing by $(\xi_1 - \xi_0)(\eta_1 - \eta_0)$:

$$\frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} - \frac{w(\xi_1, \eta_1) + w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = 0$$

Taking limit:

$$\lim_{\xi_1 \to \xi_0} \lim_{\eta_1 \to \eta_0} \frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0) - w(\xi_1, \eta_1) - w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = w_{\xi\eta} = 0$$

In this way we can solve wave equation on finite range:



Using this method we can get solution for homogeneous wave equation for finite string.

Non-homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = \varphi(x, t)$$

In this case parallelogram identity doesn't work.
Lets extend φ to the half plane $x > 0$ to some function $\tilde{\varphi} \in \mathcal{C}^1$.
Lets solve non-homogeneous equation with 0 initial conditions: $w = \frac{1}{2c} \int\limits_{\triangle} \tilde{\varphi}$.

with D'Lambert formula. Lets solve homogeneous equation in the interval:

$$\begin{cases} v(a,t) = h(t) + w(a,t) \\ v(b,t) = q(t) + w(b,t) \\ v(x,0) = f(x) \\ v_t(x,0) = g(x) \end{cases}$$

Then the solution is

$$u(x,t) = w(x,t) + v(x,t)$$

Checking the solution:

$$u_{tt} - c^2 u_{xx} = w_{tt} - c^2 w_{xx} + v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

which is $\varphi(x,t)$ in our interval.

Energy method Define

$$E(t) = \int_a^b \left[u_t^2(x,t) + c^2 u_x^2(x,t)\right] \mathrm{d}x$$

If $u \in \mathcal{C}^2$ we can differentiate it:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \left[2u_t u_{tt} + 2c^2 u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b \left[u_t u_{xx} + u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b (u_x u_t)_x \,\mathrm{d}x = 2c^2[u_x(b,t)u_t(b,t) - u_x(a,t)u_t(a,t)]$$

If any combination of Dirichlet and Neumann conditions is fulfilled, the integral is 0, i.e., energy is conserved. Thus

$$E(t) = E(0) = \int_a^b g^2(x) + c^2 (f'(x))^2 \,\mathrm{d}x$$

From that we can conclude the solution is unique. As usual, suppose there are two solutions, u and v. Subtracting we get a solution for homogeneous equation with homogeneous initial conditions $w = u - v$. Then $E_w(t) = E_w(0) = 0$, thus $w_x = w_t = 0$ and $w(x,t) = 0$.
Also, from energy difference, we can conclude the solutions are stable.

3.3 Variable separation

Lets guess solution of form

$$U(x,t) = A(x)B(t)$$

Substituting into wave equation:

$$u_{tt} - c^2 u_{xx} = A(x)B''(t) - c^2 A''(x)B(t) = 0$$

Dividing by $A(x)B(t)$ (assume they are not zero):

$$\frac{B''(t)}{B(t)} = c^2 \frac{A''(x)}{A(x)} = \mu$$

That means

$$\begin{cases} A'' = \frac{\mu}{c^2} A = -\lambda A \\ B'' = \mu B \end{cases}$$

Back to initial conditions $u(0,t) = u(1,t) = 0$, that means $A(0) = A(1) = 0$. The question is when

$$A'' + \lambda A = 0$$

If $\lambda < 0$, the solution is

$$A = \alpha e^{-\sqrt{-\lambda}x} + \beta e^{\sqrt{\lambda}x}$$

Substituting initial conditions we get

$$\begin{cases} \alpha + \beta = 0 \\ \alpha e^{-\sqrt{-\lambda}} + \beta e^{\sqrt{-\lambda}} = 0 \end{cases}$$

Since $\lambda \neq 0$, we conclude $\alpha = \beta = 0$ which is trivial solution. If $\lambda = 0$ we get $A = \alpha x + \beta$, which is also trivially $A = 0$. If $\lambda > 0$,

$$A = \alpha \sin\left(\sqrt{\lambda}x\right) + \beta \cos\left(\sqrt{\lambda}x\right)$$

Since $A(0) = 0$, $\beta = 0$. Since $A(1) = 0$, $\sqrt{\lambda} = k\pi$ for some $k \in \mathbb{N}$, i.e., $\lambda_k = k^2\pi^2$. The solution is

$$A_k = \sin(k\pi x)$$

Back to B:

$$\frac{B''}{B} = -c^2k^2\pi^2$$

i.e.,

$$B_k(t) = a_k \sin(ck\pi t) + b_k \cos(ck\pi t)$$

Thus the solution of wave equation

$$u_k(x,t) = a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

(note, that by using trigonometric identities we can get it to canonical form).
Define

$$u(x,t) \sim \sum_{k=1}^{\infty} a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

Substituting $t = 0$:

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f(x)$$

$$u_t(x,0) = c\pi \sum_{k=1}^{\infty} k a_k \sin(k\pi x) \ = g(x)$$

If we'll find a_k, b_k fulfilling those conditions, then we have a "solution". How we find them? Look at following integral:

$$\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x = \int_0^1 \left[\sum_{k=1}^{\infty} b_k \sin(k\pi x)\right] \sin(n\pi x)\,\mathrm{d}x = \sum_{k=1}^{\infty} b_k \underbrace{\int_0^1 \sin(k\pi x)\sin(n\pi x)\,\mathrm{d}x}_{\frac{\delta_{nk}}{2}} = \frac{b_n}{2}$$

Thus,

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x$$

Exactly in the same way we can get

$$a_n = \frac{2}{c\pi n}\int_0^1 g(x)\sin(n\pi x)\,\mathrm{d}x$$

Convergence If $\sum |ka_k| < \infty$ and $\sum |b_k| < \infty$, our series converge uniformly. If also $\sum |k^2 a_k| < \infty$ and $\sum |kb_k| < \infty$, then $u(x,t) \in \mathcal{C}^1$. Analogously, if $\sum |k^3 a_k| < \infty$ and $\sum |k^2 b_k| < \infty$, $u \in \mathcal{C}^2$.
Suppose $\max\limits_{(0,1)} |f| < M_0$, then

$$|b_n| \le 2\int_0^1 |f(x)\sin(n\pi x)|\,\mathrm{d}x \le 2M_0$$

. Suppose also that $f \in \mathcal{C}^1$ and $\max\limits_{(0,1)} |f'| < M_1$ then

$$b_n = -\frac{2}{n\pi}\int_0^1 f(x)(\cos(n\pi x))'\,\mathrm{d}x$$

Integrating by parts and using the fact $f(0) = f(1) = 0$:

$$b_n = \frac{2}{n\pi}\int_0^1 f'(x)\cos(n\pi x)\,\mathrm{d}x \le \frac{2}{n\pi}M_1$$

To show that $b_n < \frac{B_2}{n^2}$, we need $f \in \mathcal{C}^2$, $\max\limits_{(0,1)} |f''| < M_2$ and $f'(0) + f'(1) = 0$.
In general, if $f \in \mathcal{C}^l$ and sum $f^{(l-1)}(0) + f^{(l-1)}(1) = 0$, we can bound $|b_n| < \frac{B_l}{n^l}$.

Generalization

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(0) \\ u(0,t) = h(t) \\ u(1,t) = q(t) \end{cases}$$

Define $w(x,t)$ such that $w(0,t) = h(t)$ and $w(1,t) = q(t)$ and $v = u - w$. Then

$$v_{tt} - c^2 v_{xx} = u_{tt} - c^2 u_{xx} - w_{tt} + c^2 w_{xx} = \varphi(x,t) - w_{tt} + c^2 w_{xx} = \tilde{\varphi}(x,t)$$

Thus we can assume bound conditions are 0, as soon as we can solve non-homogeneous equation.

$$v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

Guess solution

$$v(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\pi x)$$

Substituting:

$$\sum_{k=1}^{\infty} \left(q_k''(t) + k^2 c^2 \pi^2 q_k(t)\right) \sin(k\pi x) = \tilde{\varphi}(x,t)$$

Suppose we can expand

$$\tilde{\varphi}(x,t) = \sum_{k=1}^{\infty} p_k(t) \sin(k\pi x)$$

.

$$\int_0^1 \tilde{\varphi} \sin(n\pi x)\,\mathrm{d}x = \sum_{k=1}^{\infty} p_k(t) \int_0^1 \sin(k\pi x)\sin(n\pi x)\,\mathrm{d}x = \frac{p_n(t)}{2}$$

Thus

$$p_n(t) = 2\int_0^1 \tilde{\varphi} \sin(n\pi x)\,\mathrm{d}x$$

By coefficient comparison:

$$q_k''(t) + k^2 c^2 \pi^2 q_k(t) = 2\int_0^1 \tilde{\varphi} \sin(n\pi x)\,\mathrm{d}x$$

Since we know that

$$\begin{cases} v(x,0) = \sum_{k=1}^{\infty} q_k(0) \sin(k\pi x) = f(x) \\ v_t(x,0) = \sum_{k=1}^{\infty} q_k'(0) \sin(k\pi x) = g(x) \end{cases}$$

i.e.,

$$q_k(0) = 2\int_0^1 f(x) \sin(k\pi x)\,\mathrm{d}x$$

$$q_k'(0) = 2\int_0^1 g(x) \sin(k\pi x)\,\mathrm{d}x$$

Meaning we can solve ODE, and get solution for wave equation.

Neumann bound conditions Once again guessing the solution

$$\begin{cases} u(x,t) = A(x)B(t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ A'(0) = A'(1) = 0 \end{cases}$$

We get once again

$$A(x) = a\sin\left(\sqrt{\lambda}x\right) + b\cos\left(\sqrt{\lambda}x\right)$$

$$A'(x) = \sqrt{\lambda}a\cos\left(\sqrt{\lambda}x\right) - \sqrt{\lambda}b\sin\left(\sqrt{\lambda}x\right)$$

Substituting initial conditions:

$$A'(0) = \sqrt{\lambda}a \Rightarrow a = 0$$
$$A'(1) = \sqrt{\lambda}b\sin\left(\sqrt{\lambda}x\right)$$

Thus we get the same $\lambda_k = k\pi$, however the series contains cosines instead of sines:

$$A_k(x) = \cos(k\pi x)$$

and we can solve in a similar way.

Operator of hyperbolic equation We can define linear operator

$$L(u) = \left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x\partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\right)$$

Then the equation is

$$L(u) = h$$

We can turn it into canonical form:

$$L'(u) = \left(\frac{\partial^2}{\partial\xi\partial\eta} + d'\frac{\partial}{\partial x} + f'\frac{\partial}{\partial y} + g'\right)$$

Cauchy problem for hyperbolic equation Given a curve in space $\vec{\mathbf{r}}(s) = \left(\bar{x}(s), \bar{y}(s)\right)$, ge define initial conditions

$$\begin{cases} u\big(x(s), y(s)\big) = h(s) \\ u_x\big(x(s), y(s)\big) = \varphi(s) \\ u_y\big(x(s), y(s)\big) = \psi(s) \end{cases}$$

However, since we need two conditions, there is consistency requirement on those functions:

$$\frac{\mathrm{d}h}{\mathrm{d}s} = u_x\big(x(s), y(s)\big)\frac{\partial\bar{x}}{\partial s} + u_y\big(x(s), y(s)\big)\frac{\partial\bar{x}}{\partial s} == \varphi(s)\frac{\partial\bar{x}}{\partial s} + \psi(s)\frac{\partial\bar{x}}{\partial s}$$

Suppose we have equation of form

$$\begin{cases} au_{xx} + 2bu_{xy} + cu_{yy} = d \\ u_{xx}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{xy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s} \\ u_{xy}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{yy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}s} \end{cases}$$

To have an opportunity to evaluate second derivatives, we need to find solution of this linear system, i.e., we need that

$$\begin{vmatrix} a & 2b & c \\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} & 0 \\ 0 & \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \end{vmatrix} \neq 0$$

or

$$a\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}s}\right)^2 - 2b\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} + c\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\right)^2 \neq 0$$

meaning the direction of tangent line is not in direction of characteristic lines.
We can derive the system once again and thus find third-order derivatives, doing it up to infinity, we get all the partial derivative.

Cauchy–Kowalevski theorem If the coefficients and initial curve are analytic functions, then exists unique analytic solution.

4 Heat equation

For some positive k

$$u_t - ku_{xx} = 0$$

Temperature $u(x, t)$ fulfills heat equation.

Dirichlet bound conditions

$$u(a, 0) = u(b, 0) = 0$$

Neumann bound conditions The meaning of Neumann bound condition is that there is heat isolation in interval bounds:

$$u_x(a,t) = u_x(b,t) = 0$$

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_a^b \frac{\partial Q}{\partial t}\,\mathrm{d}x = k\int_a^b \frac{\partial^2 u}{\partial x^2}(x,t)\,\mathrm{d}x = k\left[\frac{\partial u}{\partial x}(,t) - \frac{\partial u}{\partial x}(a,t)\right] = 0$$

Thus $Q(t)$ is constant.

$$Q(t) = Q(0) = \int_a^b f(x)\,\mathrm{d}x$$

Solution of heat equation Suppose, for Dirichlet bound condition, that solution is series of sines.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t)\sin(n\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -(n\pi)^2 a_n(t)\sin(n\pi x)$$

Then we get

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\left[a_n'(t) + k(n\pi)^2 a_n(t)\right]\sin(n\pi x) = 0$$

Thus, by coefficient comparison

$$a_n'(t) + k(n\pi)^2 a_n(t) = 0$$

$$a_n(t) = a_n(0)e^{-k(n\pi)^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(0)e^{-k(n\pi)^2 t}\sin(n\pi x)$$

From initial conditions we can find $a_n(0)$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\sin(n\pi x)$$

Our series is infinite differentiable if $t > 0$.
If $t = 0$, we need

$$\lim_{t\to 0^-} u(x,t) = f(x)$$

For $t < 0$, coefficients diverge, and thus we can't find solutions for $t < 0$. Physically we can see it from entropy grows.

Example

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = 1 \\ u(0,t) = u(1,t) = 0 \quad t > 0 \end{cases}$$

We acquire

$$a_n = \frac{1}{2}\int_0^1 1\cdot\sin(n\pi x)\,\mathrm{d}x$$

$$|a_n| \le \int_0^1 \mathrm{d}x = \frac{1}{2}$$

Thus

$$|a_n| \le \frac{1}{2}e^{-k(n\pi)^2 t}$$

Thus the series absolutely converges for $t > 0$.
In limit $t \to \infty$, $a_n \to 0$, and thus $u(x,t) \to 0$.

Stability For each $\epsilon > 0$ exists $\delta > 0$ such that if

$$\max_{a \leq x \leq b} |u(x,0)| < \delta$$

then

$$\max_{a \leq x \leq b} |u(x,t)| < \epsilon$$

Proof We'll proof the weaker version of the theorem, with condition that $\sum |a_n| < \delta$. Then coefficients of $u(x,t)$ are bounded by

$$a_n(t) \leq e^{-k(n\pi)^2 t} \frac{\delta}{2}$$

i.e.

$$|u(x,t)| \leq \sum_{n=1}^{\infty} |a_n| e^{-k(n\pi)^2 t} < \sum |a_n| < \delta$$

5 Potential equation

$$u_{xx} + u_{yy} = 0$$

Bound conditions are

$$\begin{cases} u(x,0) = f(x) \\ u_x(x,0) = g(x) \end{cases}$$

By variable separation we get

$$u_n = A_n(x) B_n(y)$$

we know that

$$A_n(x) = \sin(n\pi x)$$

Since

$$(n\pi)^2 = \frac{A_n''}{A_n} = -\frac{B_n''}{B_n}$$

$$B_n(y) = \alpha_n \sinh(n\pi y) + \beta_n \cosh(n\pi y)$$

Stability Is potential equation stable? Suppose $\max |f(x)| < \delta$ and $\max |g(x)| < \delta$
No. For example

$$u(x,y) = \frac{1}{n^3} e^{(n\pi)^2 y} \sin(n\pi x)$$

Then

$$u(x,0) = \frac{\sin(n\pi x)}{n^2}$$

$$u_y(x,0) = \frac{\sin(n\pi x)}{n}$$

This doesn't fulfills stability condition

$$|u_n(x,y)| < \epsilon$$

for large enough n.

Laplace equation

$$\nabla^2 u = 0$$

If u fulfills $\nabla^2 u = 0$, it is called harmonic function.

Poisson equation

$$\nabla^2 u = f$$

Where f describes mass/charge distribution in space.

Elliptic PDEs For elliptic equation $b^2 - 4ac < 0$. In this case, we can get the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + L_1(\xi, \eta) = f$$

where L_1 is first-order differential operator.

5.1 Laplace equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \nabla^2 u = 0$$

Gauss law For vector field $\vec{\mathbf{w}} \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega}$

$$\iint\limits_{\Omega} \boldsymbol{\nabla} \cdot \vec{\mathbf{w}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \vec{\mathbf{w}} \cdot \hat{\mathbf{n}} \, \mathrm{d}s$$

Thus

$$\iiint\limits_{\Omega} \nabla^2 \vec{\mathbf{u}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s$$

i.e., if function is harmonic,

$$\oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s = 0$$

Conclusion The equation $\nabla^2 u = f$ in Ω for u fulfilling $\frac{\partial u}{\partial n} = g$ in each point of $\partial\Omega$ there is no solution if

$$\iiint\limits_{\Omega} \neq \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

The necessary condition for solution of Neumann problem

$$\iiint\limits_{\Omega} = \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

For Dirichlet problem, there is no such constraint.

Examples of harmonic functions In $n = 1$, linear functions are harmonic.
In $n = 2$, any real or imaginary part of analytic function is harmonic, e.g., $e^x \sin y$.

The mean value property of harmonic function If u is harmonic in Ω which contains $B_R(x) = \{y||x - y| < R\}$ then

$$u(x) = \frac{1}{|\partial B_R(x)|} \oint\limits_{\partial\Omega} u \, \mathrm{d}s$$

and

$$u(x) = \frac{1}{|B_R(x)|} \int\limits_{\Omega} u(y) \, \mathrm{d}^n y$$

Proof Suppose $x = 0$. Rewrite $y \in B_R(x)$ as $y = \rho\alpha$ for $\alpha = \frac{y}{\|y\|}$ and $\rho = \|y\|$.
For each $\rho \in [0, R]$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n}(\rho\alpha) \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n}(y) \, \mathrm{d}s_y = \iint \nabla^2 u \, \mathrm{d}^n x = 0$$

With variable substitution $\mathrm{d}s_y = \rho^{n-1} \, \mathrm{d}s_\alpha$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n}(\rho\alpha) \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n}(\rho\alpha)\rho^{n-1} \, \mathrm{d}s_\alpha = \rho^{n-1} \int\limits_{B_\rho(0)} \frac{\partial u}{\partial \rho}(\rho\alpha) \, \mathrm{d}s_\alpha = \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha) \, \mathrm{d}s_\alpha$$

Thus

$$\frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha)\, \mathrm{d}s_\alpha = 0$$

meaning

$$H(\rho) = \int\limits_{B_\rho(0)} u(\rho\alpha)\, \mathrm{d}s_\alpha = \text{const}$$

Denote volume of unit ball as ω_n, then $|B_n| = \omega_n R^n$ and $|\partial B_n| = \omega_{n-1} R^{n-1}$

$$H(0) = \omega_n \cdot u(0)$$

And since $H(1) = H(0)$:

$$u(0) = \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(0)} \mathrm{d}s$$

as required.

$$\omega_n \rho^{n-1} u(0) = \int\limits_{\partial B_\rho(0)} u(y)\, \mathrm{d}s_y$$

$$\int_0^R n\omega_n \rho^{n-1} u(0)\, \mathrm{d}\rho = \int_0^R \mathrm{d}\rho \int\limits_{\partial B_\rho(0)} \mathrm{d}s_y\, u(y)$$

$$\omega_n R^n u(0) = \iint\limits_{B_r(0)} u(y)\, \mathrm{d}^n y$$

i.e.,

$$u(0) = \frac{1}{|B_r(0)|} \iint\limits_{B_r(0)} u(y)\, \mathrm{d}^n y$$

Strong maximum principle If u is harmonic and it acquires maximum or minimum, then it is constant.

Subharmonic and superharmonic functions Subharmonic function is function for which $\nabla^2 u \leq 0$ and superharmonic is one for which $\nabla^2 u \geq 0$.
In this case, strong maximum principle applies only in one direction (maximum for subharmonic, minimum for superharmonic).
For mean value theorem we get inequality instead of equality.

Proof let u subharmonic, and $m = u(x) = \max\limits_{\Omega} u$. The set $W = \{y : u(y) = m\}$ is closed relatively to Ω. Let $z \in W$, $B_R(z) \in \Omega$.

$$m = u(z) \leq \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\, \mathrm{d}^n y = m$$

Thus for all $z \in W$

$$u(z) = \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\, \mathrm{d}^n y$$

That means

$$\int\limits_{B_r(z)} u(x) - m\, \mathrm{d}^n x = 0$$

Thus $u(x) = m$ for all $x \in B_R(z)$, which turns W is open set. Thus W is both open and closed, i.e. $W = \Omega$.

Conclusion Poisson equation $\nabla^2 u = f$ in Ω with bound condition $u_{\partial\Omega} = y$ has not more than one solution.

Proof Suppose there are two solutions, $u_1 = u_2$, define $v = u_1 - u_2$ is harmonic function with ound condition $v_{\partial\Omega} = 0$. The function v is harmonic. If $v \neq 0$, it has either maximum or minimum, in contradiction with strong maximum principle.

Weak maximum principle For compact connected Ω and $u = \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$
If $\nabla^2 u \leq 0$,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

If $\nabla^2 u \geq 0$,

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$$

Harnack's inequality If u harmonic and non-negative in interval Ω and $\bar{\Omega}' \subsetneq \Omega$, then exists constant $c(\Omega, \Omega')$ independent on u such that

$$\sup_{\Omega'} u \leq c \inf_{\Omega} u$$

i.e., for any two points $x, y \in \Omega'$

$$u(x) \leq cu(y)$$

Proof Let $\Omega = B_{4R}(y)$ and $\Omega' = B_R(y)$. Choose $x_1, x_2 \in \Omega'$.
Since $B_R(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_1) = \frac{1}{\omega_n R^n} \int\limits_{B_R(x_1)} \mathrm{d}x\, u \leq \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u$$

Similarly, since $B_{3R}(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int\limits_{B_{3R}(x_2)} \mathrm{d}x\, u \geq \frac{1}{\omega_n (3R)^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u = \frac{1}{3^n} \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u \geq u(x_1)$$

We got

$$3^n u(x_2) \leq u(x_1)$$

Since $\bar{\Omega}' \subsetneq \Omega$, there exists $R > 0$ such that distance from any point of Ω' to any point of Ω^c is greater than $4R$.
For any pair of points $x_1, x_2 \in \Omega$, the path between then can be covered by m balls B_j of radius R, such that intersection of each pair of consequentive balls is non-empty.
So, let $y_j \in B_j \cap B_{j+1}$ and $y_1 = p$, $y_m = q$, then $u(y_j) \leq 3^n u(y_{j+1})$, since $y_j, y_{j+1} \in_{j+1}$ and $B_{4R}(y_{j+1}) \subset \Omega$. Than means that

$$u(q) \leq 3^{nm} u(p)$$

Radial harmonic functions Lets search for harmonic functions of form $u(x) = f(r)$ for f defined on $\mathbb{R}^+$.

$$|x| = \sqrt{\sum x_i} = r$$

is harmonic on $\mathbb{R}^n \setminus 0$ and radial.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} - \frac{x_i^2}{r^2}$$

Thus

$$\frac{\partial u}{\partial x_i} = f'(r) \frac{\partial r}{\partial x_i} = f'(r) \frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i^2} = f''(r) \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^2}\right)$$

$$\nabla^2 u = f''(r) \sum \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{n}{r} - \frac{\sum x_i^2}{r^2}\right) = f''(r) + \frac{n-1}{n} f'(r) = 0$$

This is Euler equation with solution For $n = 2$

$$f(r) = c_1 \ln r + c_2$$

For $n > 2$:

$$f(r) = \frac{c_1}{r^{n-2}} + c_2$$

Fundamental solution Define fundamental solution of Laplace equation:

$$\begin{cases} \Gamma(r) = \frac{1}{2\pi}\ln(r) & n = 2 \\ \Gamma(r) = \frac{1}{n(2-n)\omega_n} r^{2-n} & n > 2 \end{cases}$$

We conclude that $\Gamma(|x-y|)$ is harmonic function in $\mathbb{R}^n \setminus \{y\}$.
For $n = 2$

$$\lim_{r\to\infty} \Gamma(n) = \infty$$

and for $n > 2$

$$\lim_{r\to\infty} \Gamma(n) = 0$$

Also, for any $n \geq 2$

$$\lim_{r\to 0} \Gamma(n) = -\infty$$

Homogenity of Γ For $n > 2$

$$\frac{\partial \Gamma(x-y)}{\partial x_i} = \frac{1}{n\omega_n}\frac{x_i - y_i}{|x-y|^n}$$

$$\frac{\partial^2 \Gamma}{\partial x_i \partial x_j} = \frac{1}{n\omega_n}\left[|x-y|^2\delta_{ij} - n(x_i - y_i)(x_j - y_j)\right]|x-y|^{-n-2}$$

$$\left|\frac{\partial \Gamma(x-y)}{\partial x_i}\right| \leq \frac{1}{n\omega_n}|x-y|^{1-n}$$

$$\left|\frac{\partial^2 \Gamma(x-y)}{\partial x_i \partial x_j}\right| \leq \frac{1}{n\omega_n}|x-y|^{-n}$$

Also

$$\begin{cases} \Gamma'(r) = \frac{1}{n\omega_n r} & n = 2 \\ \Gamma'(r) = \frac{1}{n\omega_n} r^{1-n} \end{cases}$$

Green identities If $\Omega \subset \mathbb{R}^n$ bounded set with boundcin $\mathcal{C}^2$ and $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$\int\limits_{\Omega} v\nabla^2 u + \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}s$$

$$\int\limits_{\Omega} v\nabla^2 u - u\nabla^2 v \,\mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\,\mathrm{d}s$$

If $\Omega \subset \mathbb{R}^n$ bounded set with bound in $\mathcal{C}^1$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$u(y) = \int\limits_{\Omega} \Gamma(x-y)\nabla^2 u \,\mathrm{d}x + \int\limits_{\partial\Omega} \left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

If u harmonic

$$u(y) = \int\limits_{\partial\Omega} \left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

Proof $\Gamma(x-y)$ is harmonic in $\Omega \setminus B_\rho(y) = \Omega_\delta$. Choose $v(x) = \Gamma(x-y)$,

$$\int\limits_{\Omega_\delta} \Gamma\nabla^2 u \,\mathrm{d}x = \int\limits_{\partial\Omega} \Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\,\mathrm{d}s_x - \int\limits_{\partial B_\rho(y)} \Gamma\frac{\partial u}{\partial n} - u\frac{\partial \Gamma}{\partial n}\,\mathrm{d}s$$

Now

$$\int B_\rho(y)\Gamma(|x-y|)\nabla^2 u(x)\,\mathrm{d}x \leq \max_{B_\rho(y)} \cdot \int B_\rho(y)\Gamma(|x-y|)\,\mathrm{d}x = \frac{\omega_n}{n\omega_n}\int_0^\rho r^{2-n} r^{n-1}\,\mathrm{d}r = \frac{1}{n}\int_0^\rho r\,\mathrm{d}r = \frac{\rho^2}{2n} \xrightarrow{\rho\to 0} 0$$

$$\left|\frac{\partial u}{\partial n}\right| = \left|\nabla^2 u \cdot n\right| \leq C$$

$$\left| \int_{\partial B_\rho(y)} \Gamma \frac{\partial u}{\partial n} \right| \leq C \int_{B_\rho(y)} |\Gamma| = \frac{c}{n\omega_n} \rho^{2-n} n\omega_n \rho^{n-1} = C\rho \to 0$$

$$\int_{B_\rho(y)} u \frac{\partial \Gamma}{\partial n} \,\mathrm{d}s_x = \Gamma'(\rho) \int_{B_\rho(y)} u \,\mathrm{d}s_x = \frac{1}{n\omega_n \rho^{n-1}} \int_{B_\rho(y)} u \,\mathrm{d}s_x \xrightarrow{\rho \to 0} u(y)$$

Substituting it back into equation we get exactly what was needed:

$$u(y) = \int_\Omega \Gamma(x-y) \nabla^2 u \,\mathrm{d}x + \int_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] \mathrm{d}s_x$$

Green function For all $y \in \Omega$ define $h^y(x)$ such that

- $h^y(x)$ is harmonic in Ω
- $h^y(x) = -\Gamma(|x-y|)$ for all $x \in \partial\Omega$

(suppose there exists one).
Now define Green function $G(x,y) = \Gamma(|x-y|) + h^y(x)$. Properties of G:

1. G harmonic for $x \neq y$
2. $G(x,y) = 0$ for all $x \in \partial\Omega$, $y \in \Omega$

Using second Green identity

$$\int_\Omega h \nabla^2 u \,\mathrm{d}x = \int_{\partial\Omega} h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n} \,\mathrm{d}s$$

and summing it with $u(y)$ we get:

$$u(y) = \int_\Omega G(x,y) \nabla^2 u \,\mathrm{d}x + \int_{\partial\Omega} u(x) \frac{\partial}{\partial n_x} G(x,y) \,\mathrm{d}s_x$$

Conclusion If $\nabla^2 u = f$ on Ω and $u = y$ on $\partial\Omega$ the solution

$$u(y) = \int_\Omega G(x,y) f(x) \,\mathrm{d}x + \int_{\partial\Omega} g(x) \frac{\partial G}{\partial n} \,\mathrm{d}s_x$$

Lemma For all $x \neq y$, $x, y \in \Omega$

$$G(x,y) = G(y,x)$$

In particular, for constant x, G is harmonic in y.

Proof Define

$$V_x(z) = G(z,x)$$

and

$$W_y(z) = G(z,y)$$

We want to show that $V_x(y) = W_y(x)$ on $\Omega \setminus (B_\epsilon(x) \cap B_\epsilon(y))$

$$0 = \int_{\Omega_\epsilon} V_x \nabla^2 W_y - V_y \nabla^2 W_x \,\mathrm{d}z = \int_{\partial\Omega_\epsilon} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s =$$

$$= \underbrace{\int_{\partial B_\epsilon(x)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_1} + \underbrace{\int_{\partial B_\epsilon(y)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_2} = I_1(\epsilon) + I_2(\epsilon)$$

$$W_y(x) = \int_{B_\epsilon(x)} V_x \nabla^2 W_y \,\mathrm{d}z - \int_{\partial B_\epsilon(x)} \left[W_x \frac{\partial V_y}{\partial n} - V_y \frac{\partial W_x}{\partial n} \right] \mathrm{d}s = I_1(\epsilon)$$

$$V_x(z) = \Gamma(|z-x|) + h^y(z)$$

Now

$$V_x(y) = \int\limits_{B_\epsilon(y)} W_y \nabla^2 V_x \,\mathrm{d}z - \int\limits_{\partial B_\epsilon(y)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n}\right] \mathrm{d}s = -I_2(\epsilon)$$

Thus

$$W_y(x) - V_x(y) = I_1(\epsilon) + I_2(\epsilon) = 0 \Rightarrow W_y(x) = V_x(y)$$

Green function in ball Take a look at $y \in B_R(0)$, define reflection point $y^* = \frac{R^2}{y^2} y$. Note that

$$|y^*||y| = R^2$$

and that if $y \to \partial B_R(0)$, $y^* \to \partial B_R(0)$. Define

$$h^y = -\Gamma(|x-y^*|)\left(\frac{|y|}{R}\right)^{2-n}$$

Then

$$G(x,y) = \Gamma(|x-y|) - \Gamma(|x-y^*|)\left(\frac{|y|}{R}\right)^{2-n}$$

From harmony of Γ, G is harmonic in x. For $x \in \partial B_r$:

$$(x-y^*)^2 = \left(x - \frac{R^2}{y^2}y\right)^2 = |x|^2 - 2x\frac{R^2}{y^2}y + \frac{R^4}{|y|^2} = \frac{R^2}{y^2}\left[R^2 - 2xy + |x|^2\right] = \frac{R^2}{y^2}(x-y)^2$$

$$|x-y^*| = \frac{R}{|y|}|x-y|$$

$$h^y(x) = -\left(\frac{|y|}{R}\right)^{2-n} \frac{1}{n(2-n)\omega_n}(x-y^*)^{2-n} = -\frac{1}{n(2-n)\omega_n}(x-y)^{2-n} = -\Gamma(|x-y|)$$

i.e., $h^y(x)$ we defined fulfills conditions on $h^y(x)$.
For $n=2$

$$h^y(x) = \begin{cases} -\frac{1}{2\pi}\ln|x-y^*| + \frac{1}{2\pi}\ln\frac{R}{|y|} & y \neq 0 \\ -\frac{1}{2\pi}\ln R & y = 0 \end{cases}$$

For $n>2$

$$h^y(x) = \begin{cases} -\left(\frac{|y|}{R}\right)^{n-2} \frac{1}{n\omega_n}|x-y^*|^{2-n} & y \neq 0 \\ -\Gamma(R) & y = 0 \end{cases}$$

And thus for $n=2$

$$G(x,y) = \frac{1}{2\pi}\ln\left[\left(\frac{R}{|y|}\right)\frac{|x-y|}{|x-y^*|}\right]$$

For $n>2$:

$$G(x,y) = \frac{1}{n\omega_n}\left[|x-y|^{2-n} - \left(\frac{|y|}{R}\right)^{n-2}|x-y^*|^{2-n}\right]$$