104030 - Introduction to Partial Differential Equations

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Abstract

1 Introduction

PDE In PDE, the solution is a function of a couple of variables $u(x_1, x_2, \ldots x_m)$ such that:

$$F(x_1, x_2, \ldots x_m, u_{x_1}, u_{x_2}, \ldots, , u_{x_m}, u_{x_1 x_1}, \ldots) = 0$$

Notation is

$$u_{x_i} = \frac{\partial u}{\partial x_i}$$

Usually $m = 2$. For example

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_2 x_2}, u_{x_1 x_0}) = 0$$

Is PDE of two variables of order 2.

Linear PDE PDE is linear if F is linear in u and its derivatives. First order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Second order linear PDE is

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2)u_{x_1 x_1} + B(x_1, x_2)u_{x_1 x_2} + C(x_1, x_2)u_{x_2 x_2} + a(x_1, x_2)u_{x_1} + b(x_1, x_2)u_{x_2} + c(x_1, x_2)u + d(x_1, x_2) = 0$$

Quasilinear PDE Quasilinear PDE is linear only in highest order derivative. First order quasilinear PDE:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}) = a(x_1, x_2, u)u_{x_1} + b(x_1, x_2, u)u_{x_2} + c(x_1, x_2, u) = 0$$

And second order one:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1 x_1}, u_{x_1 x_2}, u_{x_2 x_2}) =$$
$$= A(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_1} + B(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_1 x_2} + C(x_1, x_2, u, u_{x_1}, u_{x_2})u_{x_2 x_2} + g(x_1, x_2, u, u_{x_1}, u_{x_2}) = 0$$

For homogeneous linear PDE solution always exist. In addition, if u_1, u_2, then any linear combination of those $\lambda_1 u_1 + \lambda_2 u_2$ will also be a solution. Thus set of solutions of linear homogeneous PDE is vector space.

Autonomous PDE If F is independent on x_i, then if $u(x_1, \ldots, x_i, \ldots, x_m)$ is solution then $u(x_1, \ldots, x_i + \lambda, \ldots x_m)$ is solution too.
In particular if u is independent on all x_i, then $u(x_1 + \lambda_1, \ldots, x_i + \lambda_i, \ldots, x_m + \lambda_m)$.

1.1 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

Solution describes movement of wave.
Lets start ODE describing harmonic oscillator, would be

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = k(x - x_0)$$

Now suppose that we have N such masses and position of mass is $\bar{x}_i = x_i + u(x_i, t)$, where u is displacement of mass and $x_i - x_{i-1} = \Delta$. Then the position of mass is described as

$$\frac{\partial^2 \bar{x}_i}{\partial t^2} = m\frac{\partial^2}{\partial t^2}u(x_i, t) = k(\bar{x}_{i+1} - \bar{x}_i) + k(\bar{x}_i - \bar{x}_{i-1})$$

Thus

$$m\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = k\left[u(x_{i+1}, t) - 2u(x_i, t) + u(x_{i-1}, t)\right]$$

In limit $\Delta \to 0$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}u(x,t) = c^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x,t)$$

Where

$$c^2 = \lim_{\Delta \to 0}\frac{\Delta^2 k_\Delta}{m_\Delta}$$

Possible solutions For each function f in $\mathcal{C}^2$, $u = f(x - ct)$ is a solution of wave equation:

$$\begin{cases} u_{xx} = f''(x - ct) \\ u_{tt} = c^2 f''(x - ct) \end{cases}$$

This solution is moving wave, because it moves along x axis with constant velocity c. Since c can be negative too, we have solution

$$u(x,t) = f(x + ct) + g(x - ct)$$

1.2 Heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Here, u means amount of heat in point x at time t.
Amount of heat in interval $[a, b]$ is

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

And heat flux in point x at time t is $k\frac{\partial u}{\partial x}$
Then flux out of interval is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

Thus

$$\int_a^b \frac{\partial}{\partial t}u(x,t)\,\mathrm{d}x = k\frac{\partial u}{\partial x} - k\frac{\partial u}{\partial x}$$

In limit $b \to a$ we get

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Example solution

$$u(x,t) = e^{-kst}\sin\left(\sqrt{s}x\right)$$

for some parameter s. Here we also can add some constant to x and acquire additional solution:

$$U(x,t) = e^{-kst}\sin\left(\sqrt{s}(x + \lambda)\right) = \cos\left(\sqrt{s}\lambda\right)e^{-kst}\sin\left(\sqrt{s}x\right) + \sin\left(\sqrt{s}\lambda\right)e^{-kst}\cos\left(\sqrt{s}x\right)$$

Thus

$$w(x,t) = e^{-kst}\cos\left(\sqrt{s}x\right)$$

is solution too.

1.3 Diffusion equation

Suppose $u(x_1, x_2, x_3, t)$ describes concentration of material in space. From continuity:

$$\frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot (\vec{v}u) = 0$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x_1}(\vec{v}u) + \frac{\partial}{\partial x_2}(\vec{v}u) + \frac{\partial}{\partial x_3}(\vec{v}u) = 0$$

for some vector field v independent on u.

1.4 Elliptic PDEs

Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0$$

Poisson equation

$$\nabla^2 u = f(x_1, x_2)$$

2 First-order PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

We can easily guess solution similarly to wave equation: $u(x, t) = f(x - ct)$ for some differentiable f.
Suppose we have initial conditions $u(x, 0) = u_0(x)$. Is it determines uniquely a solution of equation? Obviously, $u(x, t) = u_0(x - ct)$ is a solution.
Lets show it's unique. Take a look at parametrization $x(t) = s_1 + ct$.

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x(t), t) = c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

Thus u is constant on every line of form $x(t) = s + ct$. Such lines, along which the PDE becomes an ordinary differential equation, are called characteristic curves or just characteristics. Thus if we know a value of u in some point on a line, we know it on the whole line.

Is it possible to find a solution if we are given initial conditions for some curve $x(t)$ for $t \in [a, b]$. So we want to find a solution such that the surface of solution comprises a given curve in 3D.
The solution exists if the curve of initial conditions doesn't merges with characteristic line, we have a unique solution. If it does, either there is no solution, or there are infinite number of solution.

2.1 Quasilinear first-order equations

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

where a, b, c are continuously differentiable in some neighborhood of point (x_0, y_0, z_0).
Take a look at

$$f(x, y, z) = z - u(x, y)$$

$$\nabla f = \left(-\frac{\partial u}{\partial x}, -\frac{\partial u}{\partial y}, 1\right)$$

and

$$\nabla f \cdot (a, b, c) = -a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + c = 0$$

Thus vector (a, b, c) is tangent to solution surface.
Now define curve such that

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = a(x(t), y(t), z(t)) \\ \frac{\mathrm{d}y}{\mathrm{d}t} = b(x(t), y(t), z(t)) \\ \frac{\mathrm{d}z}{\mathrm{d}t} = c(x(t), y(t), z(t)) \end{cases}$$

The curve $(x(t), y(t), z(t))$ is characteristic curve of PDE.
If there is no dependence on z (i.e. equation is linear) we can take a look on 2-dimensional curve in xy-plane.

Theorem If characteristic curve intersects solution surface of quasilinear first-order PDE at some point, it is contained in the surface.

Proof Let $(x(t), y(t), z(t))$ characteristic curve of PDE and suppose for some t_0

$$u(x(t_0), y(t_0)) = z(t_0)$$

Define

$$w(t) = z(t) - u(x(t), y(t))$$

Note that $w(t_0) = 0$. Now

$$G\big(x(t), y(t), w(t)\big) = \frac{dw}{dt} = \frac{dz}{dt} - \frac{\partial u}{\partial x}\big(x(t), y(t)\big)\frac{dx}{dt} - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)\frac{dy}{dt} = c\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) -$$
$$- \frac{\partial u}{\partial x}\big(x(t), y(t)\big)a\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big) - \frac{\partial u}{\partial y}\big(x(t), y(t)\big)b\Big(x(t), y(t), w(t) + u\big(x(t), y(t)\big)\Big)$$

If we substitute $w = 0$, we get

$$G\big(x(t), y(t), 0\big) = c\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial x}a\big(x(t), y(t), u\big(x(t), y(t)\big)\big) - \frac{\partial u}{\partial y}b\big(x(t), y(t), u\big(x(t), y(t)\big)\big) = 0$$

That means that $w = 0$ is a solution of ODE, and since $a, b, c \in \mathcal{C}^1$, te solution is unique, i.e. $w = 0$ is the only solution, and thus characteristic curve is contained in the solution surface.

2.2 Existence and uniquness theorem for first-order quasilinear PDE

Existence and uniquness theorem for first-order quasilinear PDE Suppose we have initial curve $\Gamma(s) = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$ which around some point s_0 is continuously differentiable. Suppose also

$$a(x_0, y_0, z_0)\dot{\bar{y}}(s_0) - b(x_0, y_0, z_0)\dot{\bar{x}}(s_0) \neq 0$$

(transversality condition).
Then in neighborhood of s_0 exists unique solution of PDE.

Proof Define functions $x(s,t)$, $y(s,t)$, $z(s,t)$ around $(s_0, 0)$ such that

$$\begin{cases} x(s,0) = \bar{x}(s) \\ y(s,0) = \bar{y}(s) \\ z(s,0) = \bar{z}(s) \end{cases}$$

and

$$\begin{cases} \frac{\partial x}{\partial t} = a\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial y}{\partial t} = b\big(x(s,t), y(s,t), z(s,t)\big) \\ \frac{\partial z}{\partial t} = c\big(x(s,t), y(s,t), z(s,t)\big) \end{cases}$$

From uniquness and existance of ODE, exists unique solution (x, y, z) in neighbourhood of s_0.
Lets try to find s, t, as a function of x,y. It is possible if conditions of inverse function theorem are fulfilled, i.e.

$$\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \neq 0$$

in $(s_0, 0)$.
Now define $u(x,y) = z\big(s(x,y), t(x,y)\big)$.

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}\right] + b\left[\frac{\partial z}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial y}\right] = \frac{\partial z}{\partial t}\left[a\frac{\partial t}{\partial x} + b\frac{\partial t}{\partial y}\right] + \frac{\partial z}{\partial s}\left[a\frac{\partial s}{\partial x} + b\frac{\partial s}{\partial y}\right] =$$
$$= \frac{\partial z}{\partial t}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial y}\right]}_{\frac{\mathrm{d}t}{\mathrm{d}t}} + \frac{\partial z}{\partial s}\underbrace{\left[\frac{\partial x}{\partial t}\frac{\partial s}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial s}{\partial y}\right]}_{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{\partial s}{\partial t} = c$$

If crossing conditions are not fulfilled we have a couple of options:

- If initial curve is characteristic curve, we have infinite number of solutions.
- If initial curve is not characteristic curve, but their projection on xy-plane is same, we have no solution, since each solution includes characteristic curve.

In other cases, if for example initial curve is tangent to characteristic curve and their projection on xy-plane are different, there are different possibilities.

Example

$$yu_x - xu_y = 0$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = c \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s) \end{cases}$$

Now we want to find s, t as a function of x,y:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right)$$

Note that characteristic curves are rings.

Example

$$yu_x - xu_y = u$$

with initial curve $(s, 0, H(s))$ and $0 < \alpha \leq s \leq \beta$

Solution

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \\ \dot{z} = u \end{cases} \Rightarrow \begin{cases} \ddot{x} = -x \\ \ddot{y} = -y \\ z = C(s)e^t \end{cases} \Rightarrow \begin{cases} x = A(s)\sin(t) + B(s)\cos(t) \\ y = A(s)\cos(t) - B(s)\sin(t) \\ z = C(s)e^t \end{cases}$$

Now from initial curve

$$A(s) = s \quad B(s) = 0$$

$$\begin{cases} x = s\cos(t) \\ y = -s\sin(t) \\ z = h(s)e^t \end{cases}$$

Now we want to find s, t as a function of x,y:

$$x^2 + y^2 = s^2 \Rightarrow s = \sqrt{x^2 + y^2}$$

Now

$$\tan t = -\frac{y}{x} \Rightarrow t = \arctan\left(-\frac{y}{x}\right)$$

$$u(x, y) = h\left(\sqrt{x^2 + y^2}\right) e^{\arctan\left(-\frac{y}{x}\right)}$$

2.3 Burgers' equation

$$u_y + uu_x = 0$$

(which is partial case of equation of form

$$\frac{\partial u}{\partial y} + \frac{\partial}{\partial y} F(u) = 0$$

for $F = \frac{1}{2}u^2$)
Note that

$$\frac{u_y}{u_x} = -u \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = -u \Rightarrow u = \frac{\mathrm{d}x}{\mathrm{d}y}$$

Here y denotes time.
To solve it, we take integral:

$$\int_a^b \left[\frac{\partial u(x,y)}{\partial y} + \frac{\partial}{\partial x} F(u(x,y))\right] \mathrm{d}x = 0$$

$$\frac{\partial}{\partial y} \underbrace{\int_a^b u \,\mathrm{d}x}_{Q(y)} + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big)$$

$$\frac{\mathrm{d}Q}{\mathrm{d}y} = F\Big(u(a,y)\Big) - F\Big(u(b,y)\Big)$$

Now as for any quasilinear PDE:

$$\begin{cases} \dot{x} = z \\ \dot{y} = 1 \\ \dot{z} = 0 \end{cases} \Rightarrow \begin{cases} x = c_2 t + c_3 \\ y = t + c_1 \\ \dot{z} = c_2 \end{cases}$$

For initial conditions $\Big(s, 0, h(s)\Big)$:

$$\begin{cases} x = h(s)t + s \\ y = t \\ \dot{z} = h(s) \end{cases}$$

Now

$$s = x - yu \Rightarrow u = h(x - yu)$$

Checking transversality condition:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} \cdot 1 - \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \cdot h(s) = 1 \neq 0$$

Since

$$\frac{\partial u}{\partial x} = h'(x - yu) \cdot \left(1 - y\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} = \frac{h'(x - yu)}{1 + h'(x - yu) \cdot y}$$

even if we start from $\mathcal{C}^\infty$ function we can get $1 + h'(x - yu) \cdot y = 0$ and thus undefined derivative.
Geometrically, the slope of projections of characteristic curves is equal to $h(s)$ thus they can cross in some point.

Weak solutions We define a weak solution of equation, function u fulfilling the equation:

$$\forall a, b \quad \frac{\partial}{\partial y} u(x,y) \,\mathrm{d}x + F\Big(u(b,y)\Big) - F\Big(u(a,y)\Big) = 0$$

Intuitively, F is flux, and u is density, thus change in number of particles (integral) is difference between particles going in and out.
Suppose for solution $u(x,y)$ exists curve of non-continuousness γ, i.e, u is not continuous in each point of curve:

$$u(y) = \begin{cases} u^+(y) & y < \gamma(y) \\ u^-(y) & y > \gamma(y) \end{cases}$$

$$Q_{a,b}(y) = \int_a^b u(x,y)\,\mathrm{d}x = \int_a^{\gamma(y)} u^+(x,y)\,\mathrm{d}x + \int_{\gamma(y)}^b u^-(x,y)\,\mathrm{d}x$$

$$\begin{aligned}\frac{\partial Q}{\partial y} &= \int_a^{\gamma(y)} \frac{\partial u^+(x,y)}{\partial y}\,\mathrm{d}x + u^+(x,\gamma(y))\cdot\gamma'(y) + \int_{\gamma(y)}^b \frac{\partial u^-(x,y)}{\partial y}\,\mathrm{d}x - u^-(x,\gamma(y))\cdot\gamma'(y) = \\ &= -\int_a^{\gamma(y)} \frac{\mathrm{d}F(u^+)}{\mathrm{d}x}\,\mathrm{d}x - \int_{\gamma(y)}^b \frac{\mathrm{d}F(u^-)}{\mathrm{d}x}\,\mathrm{d}x + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = \\ = -\left[F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^+(a,y)\Big)\right] &- \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right]\end{aligned}$$

Meaning

$$-\left[F\Big(u^+(\gamma(y),y)\Big) - F(u^+(a,y))\right] - \left[F\Big(u^-(b,y)\Big) - F\Big(u^+(\gamma(y),y)\Big)\right] + \gamma'(y)\left[u^+(x,\gamma(y)) - u^-(x,\gamma(y))\right] = F\Big(u^-(a,y)\Big) - F\Big(u^+(b,y)\Big)$$

$$\gamma'(y)\left[u^+\Big(x,\gamma(y)\Big) - u^-\Big(x,\gamma(y)\Big)\right] = F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)$$

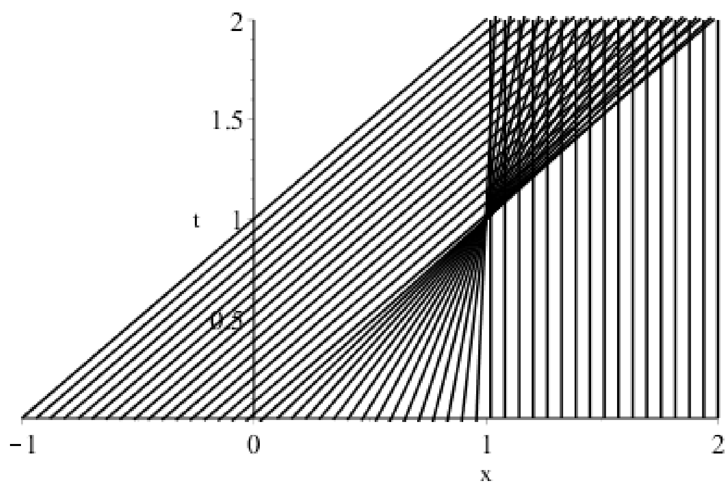
$$\gamma' = \frac{F\Big(u^+(\gamma(y),y)\Big) - F\Big(u^-(\gamma(y),y)\Big)}{u^+\big(x,\gamma(y)\big) - u^-\big(x,\gamma(y)\big)}$$

This equation is called Rankine–Hugoniot conditions. If $F(u) = \frac{1}{2}u^2$, we get $\gamma'(y) = \frac{1}{2}\left(u^+ + u^-\right)$

Example Suppose we have initial conditions $u(x,0) = h(x)$ for

$$h(x) = \begin{cases} 1 & x < 0 \\ 0 & x > \alpha \\ 1 - \frac{x}{\alpha} & 0 \le x \le \alpha \end{cases}$$

For $0 < y < 1$ we have a triangle Δ ($0 < x < \alpha$ and $y < \frac{x}{\alpha}$) for which there is intersection of two solution:



In point x, y we have slope $u(x,y)$ thus the charecteristic curve crosses x-axis at $x_0 = x - uy$ and from initial conditions, $u = 1 - \frac{x_0}{\alpha}$. Thus

$$u = 1 - \frac{x - uy}{\alpha}$$

$$\alpha u = \alpha - x + uy$$

$$(\alpha - y)u = \alpha - x$$

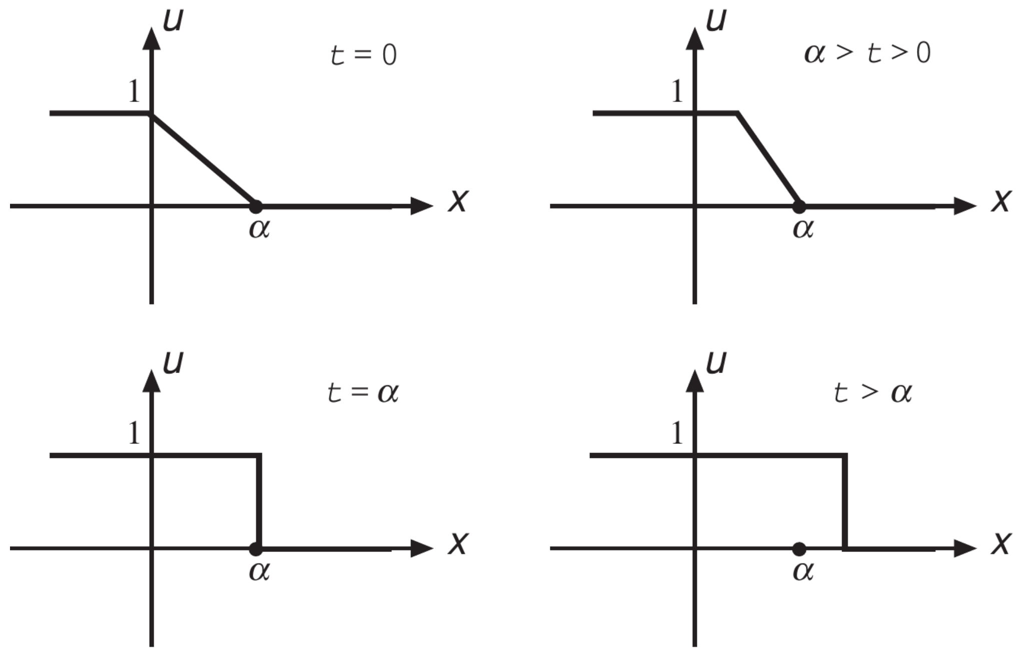
Acquiring

$$u = \frac{x - \alpha}{y - \alpha}$$

And now for $y > 1$ from Rankine–Hugoniot conditions

$$u(x,y) = \begin{cases} 1 & x < \alpha + \frac{1}{2}(y-\alpha) \\ 0 & x > \alpha + \frac{1}{2}(y-\alpha) \end{cases}$$

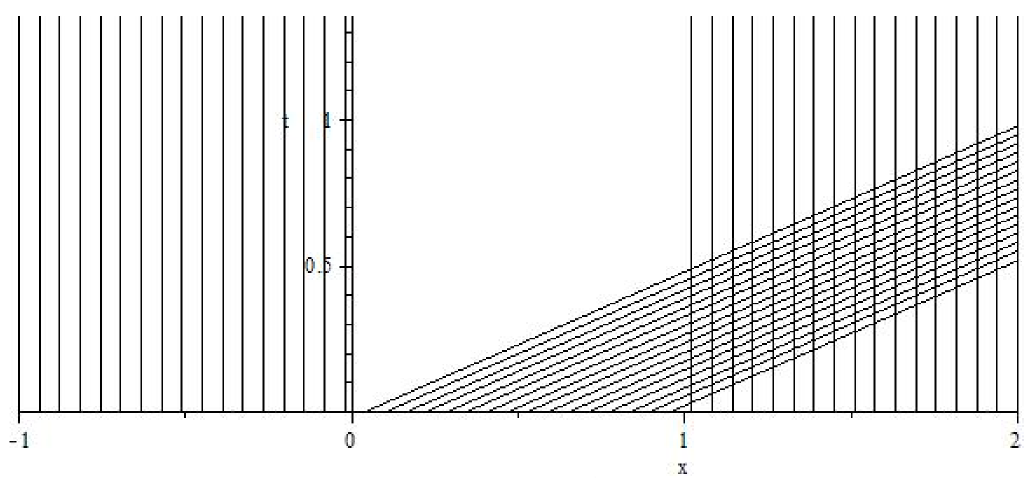
Such a solution is called a shock wave.



Example For

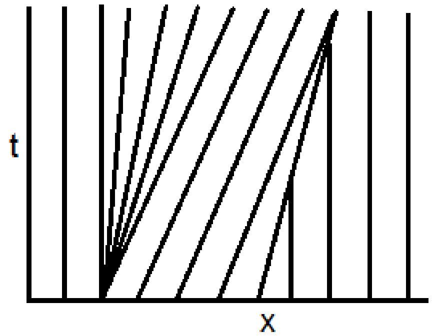
$$h(x) = \begin{cases} 0 & x > \alpha \\ 1 & x < 0 \\ \frac{x}{\alpha} & 0 \leq x \leq \alpha \end{cases}$$

Now there is no place where characteristic curves meet



In the region without characteristic curves ($0 \leq x \leq y$) we get the following: the solution starts from some point $x_0 = x - uy$, and similarly to the previous case, from initial conditions,

$$u = \frac{x}{\alpha + y}$$



What happens if $\alpha \to 0$? We get $u = \frac{x}{y}$ for $0 \leq x \leq y$. We acquired rarefaction wave - starting from something non-continuous we got continuous solution. This is weak solution.
However, also shock wave along $y = x$ is also solution of initial conditions. This solution is worse, because shock wave loses information, which means we cant reproduce the solution for some $y < y_0$ even if I know the values for $y = y_0$.

Entory principle Weak solution is unique if characteristic curves meet shock wave from direction of increasing time.

2.4 Fully non-linear equations

Hamilton-Jacoby equation

$$u_x^2 + u_y^2 = 1$$

can we generalize the method of solution of quasilinear equations to fully non-linear equations? Yes. We have some

$$F(x, y, u, u_x, u_y) = 0$$

. In our case

$$F(x, y, u, p, q) = p^2 + q^2 - 1$$

Characteristic equations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial F}{\partial p} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial F}{\partial q} \\ \frac{\mathrm{d}z}{\mathrm{d}t} = p\frac{\partial F}{\partial p} + q\frac{\partial F}{\partial q} \\ \frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} \\ \frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} \end{cases}$$

Suppose we have initial curve $\Gamma = (\bar{x}(s), \bar{y}(s), \bar{z}(s))$
We need to find $\bar{p}$ and $\bar{q}$. We have two additional conditions:

$$F(x, y, u, u_x, u_y) = 0$$

also

$$u(\bar{x}(s), \bar{y}(s)) = \bar{z}(s)$$

Differentiating by s

$$\frac{\partial u}{\partial x}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \frac{\partial u}{\partial y}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

$$\bar{p}(s)\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + \bar{q}(s)\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\bar{z}}{\mathrm{d}s}$$

Now we can find p and q.
Back to our equation:

$$\begin{cases} \dot{x} = 2p \\ \dot{y} = 2q \\ \dot{z} = 2(p^2 + q^2) \\ \dot{p} = \dot{q} = 0 \end{cases}$$

In case we have initial curve with $u = 0$, then characteristic curves are perpendicular to initial curve. We get $u(x, y)$ equal to distance from initial curve, since absolute value of gradient of u is 1 due to equation.
If we have $u = \phi(s)$ on initial curve, we acquire

$$u(x, y) = \min(x - \bar{x}(s))^2 + (y - \bar{y}(s))^2 + \phi(s)$$

Higher dimension We can trivially extend quasilinear equations to more dimensions. In this case we have initial surface instead of curve.

3 Wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

More generally the equation is

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + du_x + eu_y + fu = g$$

Definition Equation is called hyperbolic if $b^2 - ac > 0$, parabolic if $b^2 - ac = 0$ and elliptic if $b^2 - ac < 0$. Wave equation is hyperbolic in the whole space.
We want to simplify the equation: we are searching for $\xi(x,y)$ and $\eta(x,y)$ such that

$$\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial x} \neq 0$$

and solution $u(x,y) = w\big(\xi(x,y), \eta(x,y)\big)$.
Derivatives of u are

$$u_y = w_\xi \xi_y + w_\eta \eta_y$$

$$u_{yy} = w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi \xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta \eta_{yy}$$

$$u_{xy} = \frac{\partial}{\partial x}\frac{\partial u}{\partial y} = w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}$$

Now we can get equation of form

$$A(\xi,\eta)w_{\xi\xi} + 2B(\xi,\eta)w_{\xi\eta} + C(\xi,\eta)w_{\eta\xi} + D(\xi,\eta)w_{\eta\eta} + F(\xi,\eta) = 0$$

If we can find variable substitution such that

$$A = C = D = F = 0$$

Then

$$Bw_{\xi\eta} = 0$$

i.e.,

$$w(\xi,\eta) = f(\xi) + g(\eta)$$

If we substitute derivatives back into general equation

$$\begin{aligned} au_{xx} + 2bu_{xy} + cu_{yy} &= a\big[w_{\xi\xi}\xi_x^2 + w_{\xi\eta}\xi_x\eta_x + w_\xi\xi_{xx} + w_{\eta\xi}\eta_x\xi_x + w_{\eta\eta}\eta_x^2 + w_\eta\eta_{xx}\big] + \\ &+ 2b[w_{\xi\xi}\xi_x\xi_y + w_{\xi\eta}\xi_y\eta_x + w_{\xi\eta}\xi_x\eta_y + w_{\eta\xi}\eta_x\eta_y + w_\xi\xi_{xy} + w_\eta\eta_{xy}] + \\ &+ c\big[w_{\xi\xi}\xi_y^2 + w_{\xi\eta}\xi_y\eta_y + w_\xi\xi_{yy} + w_{\eta\xi}\eta_y\xi_y + w_{\eta\eta}\eta_y^2 + w_\eta\eta_{yy}\big] = \\ = \Big(a\xi_x^2 + 3b\xi_x\xi_y + c\xi_y^2\Big)w_{\xi\xi} + 2\Big(a\xi_x\eta_x + c\eta_y\xi_y &+ b(\xi_x\eta_y + \xi_y\eta_x)\Big)w_{\xi\eta} + \Big(a\eta_x^2 + 3b\eta_x\eta_y + c\eta_y^2\Big)w_{\eta\eta} + \dots \end{aligned}$$

We can rewrite it in matrix form as

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix}\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} a & b \\ b & c \end{vmatrix} \cdot \begin{vmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{vmatrix}^2$$

Since the determinant is exactly $ac - b^2$, under the variable substitution the sign of $b^2 - ac$ is conserved.

Canonical form The form $w_{\xi\eta} + \ell_1[w] = G(\xi,\eta)$, where ℓ_1 is first-order differential operator is called canonical form of hyperbolic equation.

Theorem Each hyperbolic equation can be written in canonical form

Proof We want to show that

$$\begin{cases} A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

i.e., that equation $a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 = 0$ has two independent solutions.
Dividing by ψ_y^2:

$$a\left(\frac{\psi_x}{\psi_y}\right)^2 + 2b\frac{\psi_x}{\psi_y} + c = 0$$

This is algebric equation, with solutions

$$\frac{\psi_x}{\psi_y} = \frac{-b \pm \sqrt{b^2 - ac}}{a} = \lambda_\pm$$

We acquired a pair of equations

$$\psi_x - \lambda_\pm \psi_y = 0$$

And those are two independent solutions which result in $A = 0$ and $C = 0$.

Wave equation canonical form

$$u_{tt} - c^2 u_{xx} = 0$$

The canonical change of coordinates is

$$\begin{cases} \xi = x - ct \\ \eta = x + ct \end{cases}$$

$$u_t = -cw_\xi + cw_\eta$$

$$u_x = w_\xi + w_\eta$$

$$u_{tt} = c^2 w_{\xi\xi} - 2c^2 w_{\xi\eta} + c^2 w_{\eta\eta}$$

$$u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta}$$

Then

$$u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$$

The solution of canonical equation $w_{\xi\eta} = 0$ is $w(\xi, \eta) = F(\xi) + G(\eta)$, thus solution of wave equation:

$$u(x,t) = F(x + ct) + G(x - ct)$$

An example for physical object fulfilling wave equation is infinite string. To find a solution we need initial conditions, for example, velocity and location at time $t = 0$:

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

, where $f \in \mathcal{C}^2$, $g \in \mathcal{C}^1$.

Theorem Exists unique solution of wave equation with those initial conditions.

Proof Substituting initial conditions into general solutions:

$$\begin{cases} u(x,0) = F(x) + G(x) = f(x) \\ u_t(x,0) = c[F'(x) - G'(x)] = g(x) \end{cases}$$

$$\begin{cases} F'(x) + G'(x) = f'(x) \\ F'(x) - G'(x) = \frac{g(x)}{c} \end{cases} \Rightarrow \begin{cases} F'(x) = \frac{f'(x)}{2} + \frac{g(x)}{2c} \\ G'(x) = \frac{f'(x)}{2} - \frac{g(x)}{2c} \end{cases} \Rightarrow \begin{cases} F(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_1 \\ G(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(s)\,\mathrm{d}s + D_2 \end{cases}$$

Now, since $F(x) + G(x) = f(x)$, thus $D_1 + D_2 = 0$.
Substituting into solution, we acquire what is called d'Alembert's formula:

$$u(x,t) = \frac{f(x - ct) + f(x + ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s$$

From construction, the solution is unique.

Example

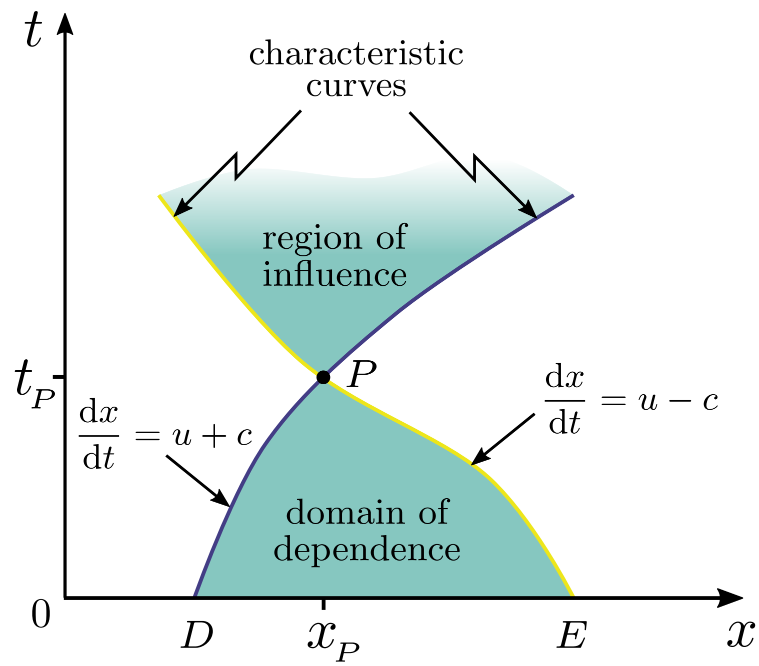
$$\begin{cases} g(x) = 0 \\ f(x) = e^{-x^2} \end{cases}$$

$$u(x,t) = \frac{1}{2}e^{-(x+ct)^2} + \frac{1}{2}e^{-(x+ct)^2}$$

Standing wave To get standing wave we want $G = 0$, i.e.,

$$\begin{cases} f(x) = F(x) \\ g(x) = cF'(x) \end{cases} \quad \Rightarrow g(x) = cf'(x)$$

Domain of dependence and region of influence Domain of dependence of u in point (x_0, t_0) is a characteristic triangle with vertices $(x_0 - ct_0, 0)$, $(x_0 + ct_0, 0)$, (x_0, t_0). Any point outside of triangle doesn't affect the value of u in point.
Region of influence of point x_0 is cone bounded by condition $x_0 - ct < x < x_0 + ct$.



Weak solution

$$\begin{cases} g(x) = 0 \\ f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}$$

The weak solution

$$u(x,t) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct)$$

is not differentiable, but is solves the equation in some sense.

3.1 Generalization of d'Alembert's formula for non-homogeneous equations

Consider non-homogeneous equation

$$u_{tt} - c^2 u_{xx} = \varphi(x,t)$$

Remember Green's theorem, for differentiable P and Q defined in Ω:

$$\iint\limits_{\Omega} \frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t}\,\mathrm{d}x\,\mathrm{d}t = \oint\limits_{\partial\Omega} P(x,t)\,\mathrm{d}x + Q(x,t)\,\mathrm{d}t$$

Lets define $Q = c^2 u_x$ and $P = u_t$, and choose $\Omega(x_0, t_0)$ to be characteristic triangle.

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \iint\limits_{\Omega(x_0,t_0)} u_{tt} - c^2 u_{xx}\,\mathrm{d}x\,\mathrm{d}t = \oint P(x,t)\,\mathrm{d}x - Q(x,t)\,\mathrm{d}t = -\left[\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t\right]$$

Lets divide the curve integral into three integrals along each of lines. For first line $\mathrm{d}t = 0$, for second $\mathrm{d}x + c\,\mathrm{d}t = 0$ and for third $\mathrm{d}x - c\,\mathrm{d}t = 0$.

$$\oint\limits_{\partial\Omega(x_0,t_0)} u_t\,\mathrm{d}x + c^2 u_x\,\mathrm{d}t =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} \underbrace{u_t}_{g(x)\text{ in }t=0}\,\mathrm{d}x - \int_{(x_0+ct_0,0)}^{(x_0,t_0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x + \int_{(x_0,t_0)}^{(x_0-ct_0,0)} cu_t\,\mathrm{d}t + u_x\,\mathrm{d}x =$$

$$= \int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s - c\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u + c\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u$$

Since

$$\int_{(x_0+ct_0,0)}^{(x_0,t_0)} \mathrm{d}u = u(x_0,t_0) - f(x_0+ct_0)$$

$$\int_{(x_0,t_0)}^{(x_0-ct_0,0)} \mathrm{d}u = f(x_0-ct_0) - u(x_0,t_0)$$

we get

$$\iint\limits_{\Omega(x_0,t_0)} \varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s + 2cu(x_0,t_0) - cf(x_0+ct_0) - cf(x_0-ct_0)$$

from which we get the solution

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

We've got a cadidate for the solution. Let's check that u is actually solving PDE. Define v, w, such that w is a solution of homogeneous PDE and $v = u - w$, i.e.,

$$v(x,t) = \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

Let's show that v solves PDE. Rewrite v as double integral:

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau$$

Define

$$H(x,t,\tau) = \int_{x-c(t-\tau)}^{x+c(t-\tau)} \varphi(\xi,\tau)\,\mathrm{d}\xi$$

and then

$$v(x,t) = \frac{1}{2c}\int_0^t H(x,t,\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial t} = \frac{1}{2c}\underbrace{H(x,t,t)}_{0} + \frac{1}{2c}\int_0^t \frac{\partial H}{\partial t}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial t} = c[\varphi(x+c(t-\tau),\tau) + \varphi(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 H}{\partial t^2} = c^2[\varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)]$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{1}{2c}\int_0^t \frac{\partial^2 H}{\partial t^2}\,\mathrm{d}\tau = \varphi(x,t) + \frac{c}{2}\int_0^t \varphi_x(x+c(t-\tau),\tau) - \varphi_x(x-c(t-\tau),\tau)\,\mathrm{d}\tau$$

$$\frac{\partial v}{\partial x} = \frac{1}{2c}\int_0^t \frac{\partial H}{\partial x}\,\mathrm{d}\tau$$

$$\frac{\partial H}{\partial x} = \varphi(x + c(t-\tau),\tau) - \varphi(x - c(t-\tau),\tau)$$

$$\frac{\partial^2 H}{\partial x^2} = \varphi_x(x + c(t-\tau),\tau) - \varphi_x(x - c(t-\tau),\tau)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{2c}\int_0^t \varphi_x(x + c(t-\tau),\tau) - \varphi_x(x - c(t-\tau),\tau)\,\mathrm{d}\tau$$

Thus we got

$$\frac{\partial^2 v}{\partial t^2} - c^2\frac{\partial^2 v}{\partial x^2} = \varphi(x,t)$$

Suppose we have two solutions u_1 and u_2 then $u = u_1 - u_2$ is solution of homogeneous equation with 0 initial conditions, and thus $u = 0$. That means the solution is unique.
The presented initial condition problem has 3 properties:

1. Solution exist
2. It's unique
3. It's stable

Stability of wave equation For all $\tau > 0$, $\epsilon > 0$, exists $\delta > 0$ such that if

$$\begin{cases} |f(x) - \tilde{f}(x)| < \delta \\ |g(x) - \tilde{g}(x)| < \delta \\ |\varphi(x) - \tilde{\varphi}(x)| < \delta \end{cases}$$

For all $-\infty < x < \infty$ and $0 \le t \le \tau$ and if u, $\tilde{u}$ are solutions of corresponding wave equations, then

$$|u(x,t) - \tilde{u}(x,t)| < \epsilon$$

Proof From the general solution:

$$u(x,t) - \tilde{u}(x,t) =$$

$$= \left| \frac{f(x+ct) + f(x-ct)}{2} - \frac{\tilde{f}(x+ct) + \tilde{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s) - \tilde{g}(s)\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} \varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta \right| \le$$

$$\le \left| \frac{f(x+ct) - \tilde{f}(x+ct)}{2} \right| + \left| \frac{f(x-ct) - \tilde{f}(x-ct)}{2} \right| + \frac{1}{2c}\int_{x-ct}^{x+ct} |g(s) - \tilde{g}(s)|\,\mathrm{d}s + \frac{1}{2c}\iint\limits_{\Omega(x_0,t_0)} |\varphi(\xi,\eta) - \tilde{\varphi}(\xi,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta \le$$

$$\le \frac{\delta}{2} + \frac{\delta}{2}\frac{1}{2c}\cdot 2c\cdot\delta + \frac{1}{2c}\frac{ct}{2} = 2\delta + \frac{\delta t}{4} \le 2\delta + \frac{\delta\tau}{4} \le \epsilon$$

Thus we choose $\delta < \frac{\epsilon}{2+\frac{\tau}{4}}$.

3.2 Wave equation with bound conditions

Half-infinite string Suppose string is fixed in one of its ends, at $x = 0$: $u(0,t) = 0$. This is called Dirichlet boundary condition. We want to solve the PDE for $x > 0$.

Property of wave equation If $u(x,t)$ is solution, then $u(-x,t)$ is also solution:

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$u(-x,t) = F(-x+ct) + G(-x-xt) = \bar{F}(x+ct) + \bar{G}(x-ct)$$

Where $\bar{F}(s) = G(-s)$ and $\bar{G}(s) = F(-s)$.

Lets extend f and g on the whole plane in odd way:

$$\bar{f}(x) = \begin{cases} f(x) & x > 0 \\ -f(x) & x < 0 \end{cases}$$

and same for g.
Note that initial conditions have to be consistent, i.e., $f(0) = 0$, $g(0) = 0$, else the solution is discontinuous in 0. Lets use D'Lambert solution:

$$\bar{u}(x,t) = \frac{\bar{f}(x+ct) + \bar{f}(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \bar{g}(s)\,\mathrm{d}s$$

Then the solution of half-infinite string is

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)-f(ct-x)}{2} + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

Neumann boundary condition In this case, instead of giving boundary condition on u, we give boundary condition of u_x: $u_x(0,t) = 0$. Physical meaning is that there is no force in this point. In this case we will extend function in even way (derivative of even function in 0 is 0). Then we get

$$u(x,t) = \begin{cases} \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s & x > ct \\ \frac{f(x+ct)+f(ct-x)}{2} + \frac{1}{c}\int_{0}^{ct-x} g(s)\,\mathrm{d}s + \frac{1}{2c}\int_{ct-x}^{x+ct} g(s)\,\mathrm{d}s & x < ct \end{cases}$$

Uniquness Suppose we have two solutions, by subtracting them, we get

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = u_x(x,0) = 0 \\ u(0,t) = 0 \end{cases}$$

We acquire $u(x,t) = 0$. Since $u(x,t)$ is of form $F(x+ct) + G(x-ct)$, we get that twobsolutions are identical.

Wave equation with finite string Suppose we have string from a to b:

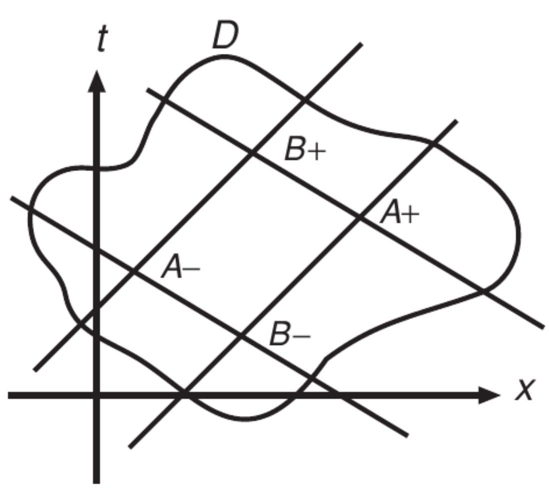
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & a \le x \le b \\ u(x,0) = f(x) \\ u_t(a,t) = h(t) \\ u_t(b,t)b = q(t) \end{cases}$$

Here the consistency conditions are

$$\begin{cases} h(0) = f(a) & q(0) = f(b) \\ h'(0) = g(a) & q'(0) = g(b) \end{cases}$$

Here we could have conditions derivatives instead of values as well.

Parallelogram identity



$u \in \mathcal{C}^2$ is the solution of wave equation iff for any parallelogram with sides parallel to characteristic lines with vertices A_-, A_+, B_-, B_+

$$u(A_-) + u(A_+) = u(B_-) + u(B_+)$$

Proof One direction is simple, since value of solution is constant along characteristic curves.
For second direction lets switch to canonical coordinates

$$\begin{cases} \xi = x + ct \\ \eta = x - ct \end{cases}$$

So

$$w(\xi, \eta) = u\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2}\right)$$

w is solution of wave equation iff $w_{\xi\eta} = 0$.
Note that parallelogram turned into rectangular in new coordinates, i.e.,

$$w(\xi_0, \eta_1) + w(\xi_1, \eta_0) = w(\xi_1, \eta_1) + w(xi_0, \eta_0)$$

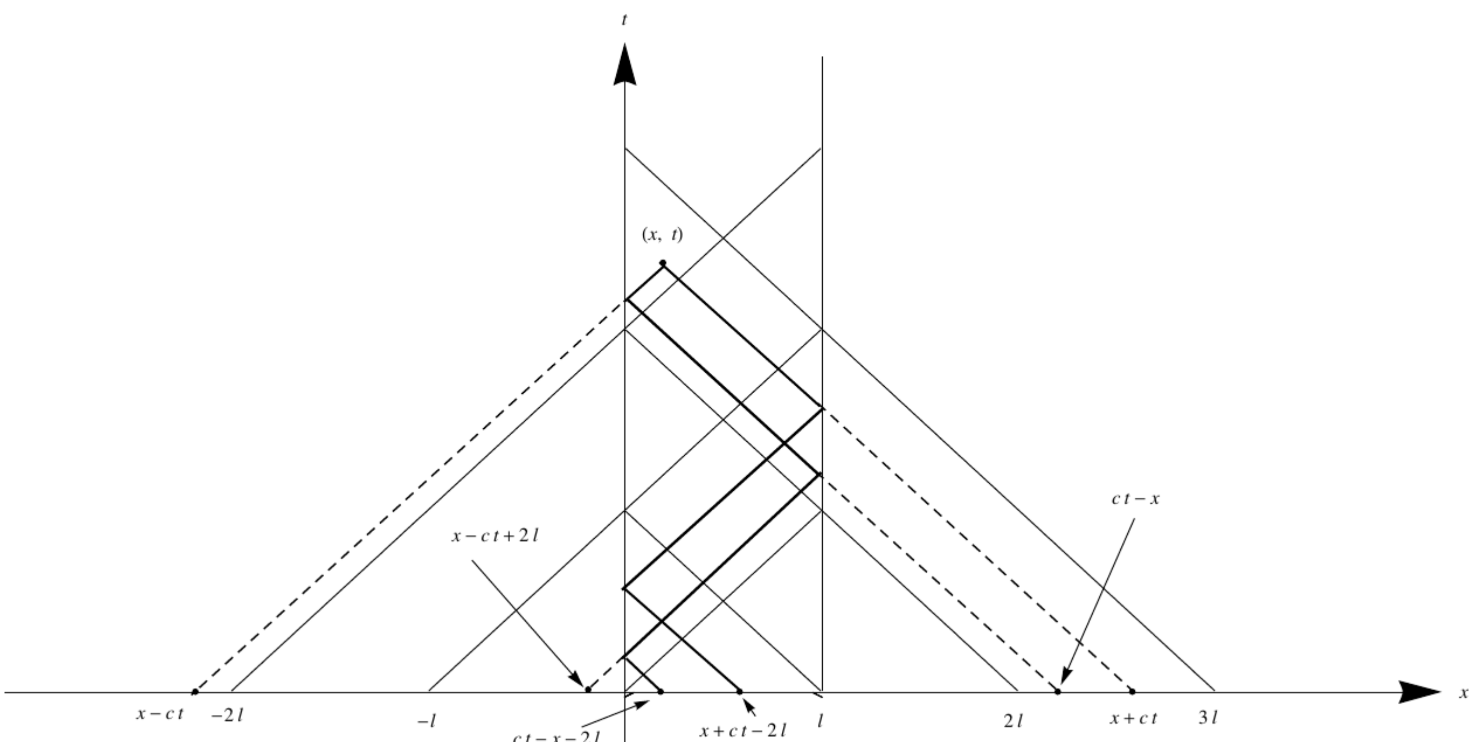
Dividing by $(\xi_1 - \xi_0)(\eta_1 - \eta_0)$:

$$\frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} - \frac{w(\xi_1, \eta_1) + w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = 0$$

Taking limit:

$$\lim_{\xi_1 \to \xi_0} \lim_{\eta_1 \to \eta_0} \frac{w(\xi_0, \eta_1) + w(\xi_1, \eta_0) - w(\xi_1, \eta_1) - w(xi_0, \eta_0)}{(\xi_1 - \xi_0)(\eta_1 - \eta_0)} = w_{\xi\eta} = 0$$

In this way we can solve wave equation on finite range:



Using this method we can get solution for homogeneous wave equation for finite string.

Non-homogeneous wave equation

$$u_{tt} - c^2 u_{xx} = \varphi(x, t)$$

In this case parallelogram identity doesn't work.
Lets extend φ to the half plane $x > 0$ to some function $\tilde{\varphi} \in \mathcal{C}^1$.
Lets solve non-homogeneous equation with 0 initial conditions: $w = \frac{1}{2c} \int\limits_{\triangle} \tilde{\varphi}$.

with D'Lambert formula. Lets solve homogeneous equation in the interval:

$$\begin{cases} v(a,t) = h(t) + w(a,t) \\ v(b,t) = q(t) + w(b,t) \\ v(x,0) = f(x) \\ v_t(x,0) = g(x) \end{cases}$$

Then the solution is

$$u(x,t) = w(x,t) + v(x,t)$$

Checking the solution:

$$u_{tt} - c^2 u_{xx} = w_{tt} - c^2 w_{xx} + v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

which is $\varphi(x,t)$ in our interval.

Energy method Define

$$E(t) = \int_a^b \left[u_t^2(x,t) + c^2 u_x^2(x,t)\right] \mathrm{d}x$$

If $u \in \mathcal{C}^2$ we can differentiate it:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_a^b \left[2u_t u_{tt} + 2c^2 u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b \left[u_t u_{xx} + u_x u_{xt}\right] \mathrm{d}x = 2c^2 \int_a^b (u_x u_t)_x \, \mathrm{d}x = 2c^2 [u_x(b,t)u_t(b,t) - u_x(a,t)u_t(a,t)]$$

If any combination of Dirichlet and Neumann conditions is fulfilled, the integral is 0, i.e., energy is conserved. Thus

$$E(t) = E(0) = \int_a^b g^2(x) + c^2 (f'(x))^2 \, \mathrm{d}x$$

From that we can conclude the solution is unique. As usual, suppose there are two solutions, u and v. Subtracting we get a solution for homogeneous equation with homogeneous initial conditions $w = u - v$. Then $E_w(t) = E_w(0) = 0$, thus $w_x = w_t = 0$ and $w(x,t) = 0$.
Also, from energy difference, we can conclude the solutions are stable.

3.3 Variable separation

Lets guess solution of form

$$U(x,t) = A(x)B(t)$$

Substituting into wave equation:

$$u_{tt} - c^2 u_{xx} = A(x)B''(t) - c^2 A''(x)B(t) = 0$$

Dividing by $A(x)B(t)$ (assume they are not zero):

$$\frac{B''(t)}{B(t)} = c^2 \frac{A''(x)}{A(x)} = \mu$$

That means

$$\begin{cases} A'' = \frac{\mu}{c^2} A = -\lambda A \\ B'' = \mu B \end{cases}$$

Back to initial conditions $u(0,t) = u(1,t) = 0$, that means $A(0) = A(1) = 0$. The question is when

$$A'' + \lambda A = 0$$

If $\lambda < 0$, the solution is

$$A = \alpha e^{-\sqrt{-\lambda}x} + \beta e^{\sqrt{\lambda}x}$$

Substituting initial conditions we get

$$\begin{cases} \alpha + \beta = 0 \\ \alpha e^{-\sqrt{-\lambda}} + \beta e^{\sqrt{-\lambda}} = 0 \end{cases}$$

Since $\lambda \neq 0$, we conclude $\alpha = \beta = 0$ which is trivial solution. If $\lambda = 0$ we get $A = \alpha x + \beta$, which is also trivially $A = 0$. If $\lambda > 0$,

$$A = \alpha \sin\left(\sqrt{\lambda}x\right) + \beta \cos\left(\sqrt{\lambda}x\right)$$

Since $A(0) = 0$, $\beta = 0$. Since $A(1) = 0$, $\sqrt{\lambda} = k\pi$ for some $k \in \mathbb{N}$, i.e., $\lambda_k = k^2\pi^2$. The solution is

$$A_k = \sin(k\pi x)$$

Back to B:

$$\frac{B''}{B} = -c^2k^2\pi^2$$

i.e.,

$$B_k(t) = a_k \sin(ck\pi t) + b_k \cos(ck\pi t)$$

Thus the solution of wave equation

$$u_k(x,t) = a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

(note, that by using trigonometric identities we can get it to canonical form).
Define

$$u(x,t) \sim \sum_{k=1}^{\infty} a_k \sin(k\pi x)\sin(ck\pi t) + b_k \sin(k\pi x)\cos(ck\pi t)$$

Substituting $t = 0$:

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) = f(x)$$

$$u_t(x,0) = c\pi \sum_{k=1}^{\infty} k a_k \sin(k\pi x) \ = g(x)$$

If we'll find a_k, b_k fulfilling those conditions, then we have a "solution". How we find them? Look at following integral:

$$\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x = \int_0^1 \left[\sum_{k=1}^{\infty} b_k \sin(k\pi x)\right]\sin(n\pi x)\,\mathrm{d}x = \sum_{k=1}^{\infty} b_k \underbrace{\int_0^1 \sin(k\pi x)\sin(n\pi x)\,\mathrm{d}x}_{\frac{\delta_{nk}}{2}} = \frac{b_n}{2}$$

Thus,

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,\mathrm{d}x$$

Exactly in the same way we can get

$$a_n = \frac{2}{c\pi n}\int_0^1 g(x)\sin(n\pi x)\,\mathrm{d}x$$

Convergence If $\sum |ka_k| < \infty$ and $\sum |b_k| < \infty$, our series converge uniformly. If also $\sum |k^2 a_k| < \infty$ and $\sum |kb_k| < \infty$, then $u(x,t) \in \mathcal{C}^1$. Analogously, if $\sum |k^3 a_k| < \infty$ and $\sum |k^2 b_k| < \infty$, $u \in \mathcal{C}^2$.
Suppose $\max\limits_{(0,1)} |f| < M_0$, then

$$|b_n| \le 2\int_0^1 |f(x)\sin(n\pi x)|\,\mathrm{d}x \le 2M_0$$

. Suppose also that $f \in \mathcal{C}^1$ and $\max\limits_{(0,1)} |f'| < M_1$ then

$$b_n = -\frac{2}{n\pi}\int_0^1 f(x)(\cos(n\pi x))'\,\mathrm{d}x$$

Integrating by parts and using the fact $f(0) = f(1) = 0$:

$$b_n = \frac{2}{n\pi}\int_0^1 f'(x)\cos(n\pi x)\,\mathrm{d}x \le \frac{2}{n\pi}M_1$$

To show that $b_n < \frac{B_2}{n^2}$, we need $f \in \mathcal{C}^2$, $\max\limits_{(0,1)} |f''| < M_2$ and $f'(0) + f'(1) = 0$.
In general, if $f \in \mathcal{C}^l$ and sum $f^{(l-1)}(0) + f^{(l-1)}(1) = 0$, we can bound $|b_n| < \frac{B_l}{n^l}$.

Generalization

$$\begin{cases} u_{tt} - c^2 u_{xx} = \varphi(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(0) \\ u(0,t) = h(t) \\ u(1,t) = q(t) \end{cases}$$

Define $w(x,t)$ such that $w(0,t) = h(t)$ and $w(1,t) = q(t)$ and $v = u - w$. Then

$$v_{tt} - c^2 v_{xx} = u_{tt} - c^2 u_{xx} - w_{tt} + c^2 w_{xx} = \varphi(x,t) - w_{tt} + c^2 w_{xx} = \tilde{\varphi}(x,t)$$

Thus we can assume bound conditions are 0, as soon as we can solve non-homogeneous equation.

$$v_{tt} - c^2 v_{xx} = \tilde{\varphi}(x,t)$$

Guess solution

$$v(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\pi x)$$

Substituting:

$$\sum_{k=1}^{\infty} \left(q_k''(t) + k^2 c^2 \pi^2 q_k(t) \right) \sin(k\pi x) = \tilde{\varphi}(x,t)$$

Suppose we can expand

$$\tilde{\varphi}(x,t) = \sum_{k=1}^{\infty} p_k(t) \sin(k\pi x)$$

.

$$\int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x = \sum_{k=1}^{\infty} p_k(t) \int_0^1 \sin(k\pi x) \sin(n\pi x) \, \mathrm{d}x = \frac{p_n(t)}{2}$$

Thus

$$p_n(t) = 2 \int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x$$

By coefficient comparison:

$$q_k''(t) + k^2 c^2 \pi^2 q_k(t) = 2 \int_0^1 \tilde{\varphi} \sin(n\pi x) \, \mathrm{d}x$$

Since we know that

$$\begin{cases} v(x,0) = \sum_{k=1}^{\infty} q_k(0) \sin(k\pi x) = f(x) \\ v_t(x,0) = \sum_{k=1}^{\infty} q_k'(0) \sin(k\pi x) = g(x) \end{cases}$$

i.e.,

$$q_k(0) = 2 \int_0^1 f(x) \sin(k\pi x) \, \mathrm{d}x$$

$$q_k'(0) = 2 \int_0^1 g(x) \sin(k\pi x) \, \mathrm{d}x$$

Meaning we can solve ODE, and get solution for wave equation.

Neumann bound conditions Once again guessing the solution

$$\begin{cases} u(x,t) = A(x)B(t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \\ A'(0) = A'(1) = 0 \end{cases}$$

We get once again

$$A(x) = a \sin\left(\sqrt{\lambda} x\right) + b \cos\left(\sqrt{\lambda} x\right)$$

$$A'(x) = \sqrt{\lambda} a \cos\left(\sqrt{\lambda} x\right) - \sqrt{\lambda} b \sin\left(\sqrt{\lambda} x\right)$$

Substituting initial conditions:

$$A'(0) = \sqrt{\lambda} a \Rightarrow a = 0$$
$$A'(1) = \sqrt{\lambda} b \sin\left(\sqrt{\lambda} x\right)$$

Thus we get the same $\lambda_k = k\pi$, however the series contains cosines instead of sines:

$$A_k(x) = \cos(k\pi x)$$

and we can solve in a similar way.

Operator of hyperbolic equation We can define linear operator

$$L(u) = \left(a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + f\frac{\partial}{\partial y} + g\right)$$

Then the equation is

$$L(u) = h$$

We can turn it into canonical form:

$$L'(u) = \left(\frac{\partial^2}{\partial \xi \partial \eta} + d'\frac{\partial}{\partial x} + f'\frac{\partial}{\partial y} + g'\right)$$

Cauchy problem for hyperbolic equation Given a curve in space $\vec{\mathbf{r}}(s) = \big(\bar{x}(s), \bar{y}(s)\big)$, ge define initial conditions

$$\begin{cases} u\big(x(s), y(s)\big) = h(s) \\ u_x\big(x(s), y(s)\big) = \varphi(s) \\ u_y\big(x(s), y(s)\big) = \psi(s) \end{cases}$$

However, since we need two conditions, there is consistency requirement on those functions:

$$\frac{\mathrm{d}h}{\mathrm{d}s} = u_x\big(x(s), y(s)\big)\frac{\partial \bar{x}}{\partial s} + u_y\big(x(s), y(s)\big)\frac{\partial \bar{x}}{\partial s} == \varphi(s)\frac{\partial \bar{x}}{\partial s} + \psi(s)\frac{\partial \bar{x}}{\partial s}$$

Suppose we have equation of form

$$\begin{cases} au_{xx} + 2bu_{xy} + cu_{yy} = d \\ u_{xx}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{xy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\varphi}{\mathrm{d}s} \\ u_{xy}\frac{\mathrm{d}\bar{x}}{\mathrm{d}s} + u_{yy}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} = \frac{\mathrm{d}\psi}{\mathrm{d}s} \end{cases}$$

To have an opportunity to evaluate second derivatives, we need to find solution of this linear system, i.e., we need that

$$\begin{vmatrix} a & 2b & c \\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} & 0 \\ 0 & \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} & \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} \end{vmatrix} \neq 0$$

or

$$a\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}s}\right)^2 - 2b\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s} + c\left(\frac{\mathrm{d}\bar{x}}{\mathrm{d}s}\right)^2 \neq 0$$

meaning the direction of tangent line is not in direction of characteristic lines.
We can derive the system once again and thus find third-order derivatives, doing it up to infinity, we get all the partial derivative.

Cauchy–Kowalevski theorem If the coefficients and initial curve are analytic functions, then exists unique analytic solution.

4 Heat equation

For some positive k

$$u_t - ku_{xx} = 0$$

Temperature $u(x,t)$ fulfills heat equation.

Dirichlet bound conditions

$$u(a, 0) = u(b, 0) = 0$$

Neumann bound conditions The meaning of Neumann bound condition is that there is heat isolation in interval bounds:

$$u_x(a,t) = u_x(b,t) = 0$$

$$Q(t) = \int_a^b u(x,t)\,\mathrm{d}x$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_a^b \frac{\partial Q}{\partial t}\,\mathrm{d}x = k\int_a^b \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = k\left[\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right] = 0$$

Thus $Q(t)$ is constant.

$$Q(t) = Q(0) = \int_a^b f(x)\,\mathrm{d}x$$

Solution of heat equation Suppose, for Dirichlet bound condition, that solution is series of sines.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin(n\pi x)$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} a_n'(t)\sin(n\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} -(n\pi)^2 a_n(t)\sin(n\pi x)$$

Then we get

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\left[a_n'(t) + k(n\pi)^2 a_n(t)\right]\sin(n\pi x) = 0$$

Thus, by coefficient comparison

$$a_n'(t) + k(n\pi)^2 a_n(t) = 0$$

$$a_n(t) = a_n(0)e^{-k(n\pi)^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n(0)e^{-k(n\pi)^2 t}\sin(n\pi x)$$

From initial conditions we can find $a_n(0)$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n(0)\sin(n\pi x)$$

Our series is infinite differentiable if $t > 0$.
If $t = 0$, we need

$$\lim_{t\to 0^-} u(x,t) = f(x)$$

For $t < 0$, coefficients diverge, and thus we can't find solutions for $t < 0$. Physically we can see it from entropy grows.

Example

$$\begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = 1 \\ u(0,t) = u(1,t) = 0 \quad t > 0 \end{cases}$$

We acquire

$$a_n = \frac{1}{2}\int_0^1 1\cdot\sin(n\pi x)\,\mathrm{d}x$$

$$|a_n| \le \int_0^1 \mathrm{d}x = \frac{1}{2}$$

Thus

$$|a_n| \le \frac{1}{2}e^{-k(n\pi)^2 t}$$

Thus the series absolutely converges for $t > 0$.
In limit $t \to \infty$, $a_n \to 0$, and thus $u(x,t) \to 0$.

Stability For each $\epsilon > 0$ exists $\delta > 0$ such that if

$$\max_{a \le x \le b} |u(x,0)| < \delta$$

then

$$\max_{a \le x \le b} |u(x,t)| < \epsilon$$

Proof We'll proof the weaker version of the theorem, with condition that $\sum |a_n| < \delta$. Then coefficients of $u(x,t)$ are bounded by

$$a_n(t) \le e^{-k(n\pi)^2 t} \frac{\delta}{2}$$

i.e.

$$|u(x,t)| \le \sum_{n=1}^{\infty} |a_n| e^{-k(n\pi)^2 t} < \sum |a_n| < \delta$$

5 Potential equation

$$u_{xx} + u_{yy} = 0$$

Bound conditions are

$$\begin{cases} u(x,0) = f(x) \\ u_x(x,0) = g(x) \end{cases}$$

By variable separation we get

$$u_n = A_n(x) B_n(y)$$

we know that

$$A_n(x) = \sin(n\pi x)$$

Since

$$(n\pi)^2 = \frac{A_n''}{A_n} = -\frac{B_n''}{B_n}$$

$$B_n(y) = \alpha_n \sinh(n\pi y) + \beta_n \cosh(n\pi y)$$

Stability Is potential equation stable? Suppose $\max |f(x)| < \delta$ and $\max |g(x)| < \delta$
No. For example

$$u(x,y) = \frac{1}{n^3} e^{(n\pi)^2 y} \sin(n\pi x)$$

Then

$$u(x,0) = \frac{\sin(n\pi x)}{n^2}$$

$$u_y(x,0) = \frac{\sin(n\pi x)}{n}$$

This doesn't fulfills stability condition

$$|u_n(x,y)| < \epsilon$$

for large enough n.

Laplace equation

$$\nabla^2 u = 0$$

If u fulfills $\nabla^2 u = 0$, it is called harmonic function.

Poisson equation

$$\nabla^2 u = f$$

Where f describes mass/charge distribution in space.

Elliptic PDEs For elliptic equation $b^2 - 4ac < 0$. In this case, we can get the canonical form

$$u_{\xi\xi} + u_{\eta\eta} + L_1(\xi, \eta) = f$$

where L_1 is first-order differential operator.

5.1 Laplace equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} u = \nabla^2 u = 0$$

Gauss law For vector field $\vec{\mathbf{w}} \in \mathcal{C}^1(\Omega) \cap \mathcal{C}(\bar{\Omega}$

$$\iint\limits_{\Omega} \boldsymbol{\nabla} \cdot \vec{\mathbf{w}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \vec{\mathbf{w}} \cdot \hat{\mathbf{n}} \, \mathrm{d}s$$

Thus

$$\iiint\limits_{\Omega} \nabla^2 \vec{\mathbf{u}} \, \mathrm{d}^3 x = \oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s$$

i.e., if function is harmonic,

$$\oiint\limits_{\partial\Omega} \frac{\partial u}{\partial n} \, \mathrm{d}s = 0$$

Conclusion The equation $\nabla^2 u = f$ in Ω for u fulfilling $\frac{\partial u}{\partial n} = g$ in each point of $\partial\Omega$ there is no solution if

$$\iiint\limits_{\Omega} \neq \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

The necessary condition for solution of Neumann problem

$$\iiint\limits_{\Omega} = \oiint\limits_{\partial\Omega} g \, \mathrm{d}s$$

For Dirichlet problem, there is no such constraint.

Examples of harmonic functions In $n = 1$, linear functions are harmonic.
In $n = 2$, any real or imaginary part of analytic function is harmonic, e.g., $e^x \sin y$.

The mean value property of harmonic function If u is harmonic in Ω which contains $B_R(x) = \{y||x - y| < R\}$ then

$$u(x) = \frac{1}{|\partial B_R(x)|} \oint\limits_{\partial\Omega} u \, \mathrm{d}s$$

and

$$u(x) = \frac{1}{|B_R(x)|} \int\limits_{\Omega} u(y) \, \mathrm{d}^n y$$

Proof Suppose $x = 0$. Rewrite $y \in B_R(x)$ as $y = \rho\alpha$ for $\alpha = \frac{y}{\|y\|}$ and $\rho = \|y\|$.
For each $\rho \in [0, R]$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \iint \nabla^2 u \, \mathrm{d}^n x = 0$$

With variable substitution $\mathrm{d}s_y = \rho^{n-1} \, \mathrm{d}s_\alpha$:

$$\int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \, \mathrm{d}s_y = \int\limits_{B_\rho(0)} \frac{\partial u}{\partial n} \rho^{n-1} \, \mathrm{d}s_\alpha = \rho^{n-1} \int\limits_{B_\rho(0)} \frac{\partial u}{\partial \rho} \, \mathrm{d}s_\alpha = \rho^{n-1} \frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha) \, \mathrm{d}s_\alpha$$

Thus
$$\frac{\partial}{\partial \rho} \int\limits_{B_\rho(0)} u(\rho\alpha)\,\mathrm{d}s_\alpha = 0$$
meaning
$$H(\rho) = \int\limits_{B_\rho(0)} u(\rho\alpha)\,\mathrm{d}s_\alpha = \text{const}$$
Denote volume of unit ball as ω_n, then $|B_n| = \omega_n R^n$ and $|\partial B_n| = \omega_{n-1} R^{n-1}$
$$H(0) = \omega_n \cdot u(0)$$
And since $H(1) = H(0)$:
$$u(0) = \frac{1}{n\omega_n R^{n-1}} \int\limits_{\partial B_R(0)} \mathrm{d}s$$
as required.
$$\omega_n \rho^{n-1} u(0) = \int\limits_{\partial B_\rho(0)} u(y)\,\mathrm{d}s_y$$
$$\int_0^R n\omega_n \rho^{n-1} u(0)\,\mathrm{d}\rho = \int_0^R \mathrm{d}\rho \int\limits_{\partial B_\rho(0)} \mathrm{d}s_y\, u(y)$$
$$\omega_n R^n u(0) = \iint\limits_{B_r(0)} u(y)\,\mathrm{d}^n y$$
i.e.,
$$u(0) = \frac{1}{|B_r(0)|} \iint\limits_{B_r(0)} u(y)\,\mathrm{d}^n y$$

Strong maximum principle If u is harmonic and it acquires maximum or minimum, then it is constant.

Subharmonic and superharmonic functions Subharmonic function is function for which $\nabla^2 u \leq 0$ and superharmonic is one for which $\nabla^2 u \geq 0$.
In this case, strong maximum principle applies only in one direction (maximum for subharmonic, minimum for superharmonic).
For mean value theorem we get inequality instead of equality.

Proof let u subharmonic, and $m = u(x) = \max\limits_{\Omega} u$. The set $W = \{y : u(y) = m\}$ is closed relatively to Ω. Let $z \in W$, $B_R(z) \in \Omega$.
$$m = u(z) \leq \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\,\mathrm{d}^n y = m$$
Thus for all $z \in W$
$$u(z) = \frac{1}{|B_R(z)|} \int\limits_{B_r(z)} \int u(y)\,\mathrm{d}^n y$$
That means
$$\int\limits_{B_r(z)} u(x) - m\,\mathrm{d}^n x = 0$$
Thus $u(x) = m$ for all $x \in B_R(z)$, which turns W is open set. Thus W is both open and closed, i.e. $W = \Omega$.

Conclusion Poisson equation $\nabla^2 u = f$ in Ω with bound condition $u_{\partial\Omega} = y$ has not more than one solution.

Proof Suppose there are two solutions, $u_1 = u_2$, define $v = u_1 - u_2$ is harmonic function with ound condition $v_{\partial\Omega} = 0$. The function v is harmonic. If $v \neq 0$, it has either maximum or minimum, in contradiction with strong maximum principle.

Weak maximum principle For compact connected Ω and $u = \mathcal{C}^2(\Omega) \cap \mathcal{C}(\partial\Omega)$
If $\nabla^2 u \le 0$,

$$\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$$

If $\nabla^2 u \ge 0$,

$$\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$$

Harnack's inequality If u harmonic and non-negative in interval Ω and $\bar{\Omega}' \subsetneq \Omega$, then exists constant $c(\Omega, \Omega')$ independent on u such that

$$\sup_{\Omega'} u \le c \inf_{\Omega} u$$

i.e., for any two points $x, y \in \Omega'$

$$u(x) \le cu(y)$$

Proof Let $\Omega = B_{4R}(y)$ and $\Omega' = B_R(y)$. Choose $x_1, x_2 \in \Omega'$.
Since $B_R(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_1) = \frac{1}{\omega_n R^n} \int\limits_{B_R(x_1)} \mathrm{d}x\, u \le \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u$$

Similarly, since $B_{3R}(x_1) \subset B_{4R}(y)$, we can use the mean value property:

$$u(x_2) = \frac{1}{\omega_n (3R)^n} \int\limits_{B_{3R}(x_2)} \mathrm{d}x\, u \ge \frac{1}{\omega_n (3R)^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u = \frac{1}{3^n} \frac{1}{\omega_n R^n} \int\limits_{B_{2R}(y)} \mathrm{d}x\, u \ge u(x_1)$$

We got

$$3^n u(x_2) \le u(x_1)$$

Since $\bar{\Omega}' \subsetneq \Omega$, there exists $R > 0$ such that distance from any point of Ω' to any point of Ω^c is greater than $4R$.
For any pair of points $x_1, x_2 \in \Omega$, the path between then can be covered by m balls B_j of radius R, such that intersection of each pair of consequentive balls is non-empty.
So, let $y_j \in B_j \cap B_{j+1}$ and $y_1 = p$, $y_m = q$, then $u(y_j) \le 3^n u(y_{j+1})$, since $y_j, y_{j+1} \in_{j+1}$ and $B_{4R}(y_{j+1}) \subset \Omega$. Than means that

$$u(q) \le 3^{nm} u(p)$$

Radial harmonic functions Lets search for harmonic functions of form $u(x) = f(r)$ for f defined on $\mathbb{R}^+$.

$$|x| = \sqrt{\sum x_i} = r$$

is harmonic on $\mathbb{R}^n \setminus 0$ and radial.

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

$$\frac{\partial^2 r}{\partial x_i^2} = \frac{1}{r} - \frac{x_i^2}{r^2}$$

Thus

$$\frac{\partial u}{\partial x_i} = f'(r) \frac{\partial r}{\partial x_i} = f'(r) \frac{x_i}{r}$$

$$\frac{\partial^2 u}{\partial x_i^2} = f''(r) \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{1}{r} - \frac{x_i^2}{r^2}\right)$$

$$\nabla^2 u = f''(r) \sum \left(\frac{x_i}{r}\right)^2 - f'(r) \left(\frac{n}{r} - \frac{\sum x_i^2}{r^2}\right) = f''(r) + \frac{n-1}{n} f'(r) = 0$$

This is Euler equation with solution For $n = 2$

$$f(r) = c_1 \ln r + c_2$$

For $n > 2$:

$$f(r) = \frac{c_1}{r^{n-2}} + c_2$$

Fundamental solution Define fundamental solution of Laplace equation:

$$\begin{cases} \Gamma(r) = \frac{1}{2\pi}\ln(r) & n=2 \\ \Gamma(r) = \frac{1}{n(2-n)\omega_n} r^{2-n} & n>2 \end{cases}$$

We conclude that $\Gamma(|x-y|)$ is harmonic function in $\mathbb{R}^n \setminus \{y\}$.
For $n=2$

$$\lim_{r\to\infty}\Gamma(n) = \infty$$

and for $n>2$

$$\lim_{r\to\infty}\Gamma(n) = 0$$

Also, for any $n \geq 2$

$$\lim_{r\to 0}\Gamma(n) = -\infty$$

Homogenity of Γ For $n>2$

$$\frac{\partial\Gamma(x-y)}{\partial x_i} = \frac{1}{n\omega_n}\frac{x_i-y_i}{|x-y|^n}$$

$$\frac{\partial^2\Gamma}{\partial x_i \partial x_j} = \frac{1}{n\omega_n}\left[|x-y|^2\delta_{ij} - n(x_i-y_i)(x_j-y_j)\right]|x-y|^{-n-2}$$

$$\left|\frac{\partial\Gamma(x-y)}{\partial x_i}\right| \leq \frac{1}{n\omega_n}|x-y|^{1-n}$$

$$\left|\frac{\partial^2\Gamma(x-y)}{\partial x_i\partial x_j}\right| \leq \frac{1}{n\omega_n}|x-y|^{-n}$$

Also

$$\begin{cases} \Gamma'(r) = \frac{1}{n\omega_n r} & n=2 \\ \Gamma'(r) = \frac{1}{n\omega_n} r^{1-n} \end{cases}$$

Green identities If $\Omega \subset \mathbb{R}^n$ bounded set with boundcin $\mathcal{C}^2$ and $u, v \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$\int\limits_{\Omega} v\nabla^2 u + \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v \,\mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n}\,\mathrm{d}s$$

$$\int\limits_{\Omega} v\nabla^2 u - u\nabla^2 v\,\mathrm{d}x = \int\limits_{\partial\Omega} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\,\mathrm{d}s$$

If $\Omega \subset \mathbb{R}^n$ bounded set with bound in $\mathcal{C}^1$ and $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$

$$u(y) = \int\limits_{\Omega}\Gamma(x-y)\nabla^2 u\,\mathrm{d}x + \int\limits_{\partial\Omega}\left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

If u harmonic

$$u(y) = \int\limits_{\partial\Omega}\left[u(x)\frac{\partial}{\partial n_x}\Gamma(x-y) - \Gamma(x-y)\frac{\partial u}{\partial n_x}\right]\mathrm{d}s_x$$

Proof $\Gamma(x-y)$ is harmonic in $\Omega \setminus B_\rho(y) = \Omega_\delta$. Choose $v(x) = \Gamma(x-y)$,

$$\int\limits_{\Omega_\delta}\Gamma\nabla^2 u\,\mathrm{d}x = \int\limits_{\partial\Omega}\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\,\mathrm{d}s_x - \int\limits_{\partial B_\rho(y)}\Gamma\frac{\partial u}{\partial n} - u\frac{\partial\Gamma}{\partial n}\,\mathrm{d}s$$

Now

$$\int B_\rho(y)\Gamma(|x-y|)\nabla^2 u(x)\,\mathrm{d}x \leq \max_{B_\rho(y)} \cdot \int B_\rho(y)\Gamma(|x-y|)\,\mathrm{d}x = \frac{\omega_n}{n\omega_n}\int_0^\rho r^{2-n}r^{n-1}\,\mathrm{d}r = \frac{1}{n}\int_0^\rho r\,\mathrm{d}r = \frac{\rho^2}{2n} \xrightarrow{\rho\to 0} 0$$

$$\left|\frac{\partial u}{\partial n}\right| = \left|\nabla^2 u \cdot n\right| \leq C$$

$$\left| \int\limits_{\partial B_\rho(y)} \Gamma \frac{\partial u}{\partial n} \right| \leq C \int\limits_{B_\rho(y)} |\Gamma| = \frac{c}{n\omega_n} \rho^{2-n} n\omega_n \rho^{n-1} = C\rho \to 0$$

$$\int\limits_{B_\rho(y)} u \frac{\partial \Gamma}{\partial n} \, \mathrm{d}s_x = \Gamma'(\rho) \int\limits_{B_\rho(y)} u \, \mathrm{d}s_x = \frac{1}{n\omega_n \rho^{n-1}} \int\limits_{B_\rho(y)} u \, \mathrm{d}s_x \xrightarrow{\rho \to 0} u(y)$$

Substituting it back into equation we get exactly what was needed:

$$u(y) = \int\limits_{\Omega} \Gamma(x-y) \nabla^2 u \, \mathrm{d}x + \int\limits_{\partial\Omega} \left[u(x) \frac{\partial}{\partial n_x} \Gamma(x-y) - \Gamma(x-y) \frac{\partial u}{\partial n_x} \right] \mathrm{d}s_x$$

Green function Green function can be understood as inverse of Laplacian operator in a sense that if $\nabla^2 u = f$, $u(c) = \int G(x,y) f(y) \, \mathrm{d}y$. For all $y \in \Omega$ define $h^y(x)$ such that

- $h^y(x)$ is harmonic in Ω
- $h^y(x) = -\Gamma(|x-y|)$ for all $x \in \partial\Omega$

(suppose there exists one).
Now define Green function $G(x,y) = \Gamma(|x-y|) + h^y(x)$. Properties of G:

1. G harmonic for $x \neq y$
2. $G(x,y) = 0$ for all $x \in \partial\Omega$, $y \in \Omega$

Using second Green identity

$$\int\limits_{\Omega} h \nabla^2 u \, \mathrm{d}x = \int\limits_{\partial\Omega} h \frac{\partial u}{\partial n} - u \frac{\partial h}{\partial n} \, \mathrm{d}s$$

and summing it with $u(y)$ we get:

$$u(y) = \int\limits_{\Omega} G(x,y) \nabla^2 u \, \mathrm{d}x + \int\limits_{\partial\Omega} u(x) \frac{\partial}{\partial n_x} G(x,y) \, \mathrm{d}s_x$$

Conclusion If $\nabla^2 u = f$ on Ω and $u = g$ on $\partial\Omega$ the solution

$$u(y) = \int\limits_{\Omega} G(x,y) f(x) \, \mathrm{d}x + \int\limits_{\partial\Omega} g(x) \frac{\partial G}{\partial n} \, \mathrm{d}s_x$$

Lemma For all $x \neq y$, $x, y \in \Omega$

$$G(x,y) = G(y,x)$$

In particular, for constant x, G is harmonic in y.

Proof Define

$$V_x(z) = G(z,x)$$

and

$$W_y(z) = G(z,y)$$

We want to show that $V_x(y) = W_y(x)$ on $\Omega \setminus (B_\epsilon(x) \cap B_\epsilon(y))$

$$0 = \int\limits_{\Omega_\epsilon} V_x \nabla^2 W_y - V_y \nabla^2 W_x \, \mathrm{d}z = \int\limits_{\partial\Omega_\epsilon} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s =$$

$$= \underbrace{\int\limits_{\partial B_\epsilon(x)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_1} + \underbrace{\int\limits_{\partial B_\epsilon(y)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n} \right] \mathrm{d}s}_{I_2} = I_1(\epsilon) + I_2(\epsilon)$$

$$W_y(x) = \int\limits_{B_\epsilon(x)} V_x \nabla^2 W_y \,\mathrm{d}z - \int\limits_{\partial B_\epsilon(x)} \left[W_x \frac{\partial V_y}{\partial n} - V_y \frac{\partial W_x}{\partial n}\right] \mathrm{d}s = I_1(\epsilon)$$

$$V_x(z) = \Gamma(|z-x|) + h^y(z)$$

Now

$$V_x(y) = \int\limits_{B_\epsilon(y)} W_y \nabla^2 V_x \,\mathrm{d}z - \int\limits_{\partial B_\epsilon(y)} \left[V_y \frac{\partial W_x}{\partial n} - W_x \frac{\partial V_y}{\partial n}\right] \mathrm{d}s = -I_2(\epsilon)$$

Thus

$$W_y(x) - V_x(y) = I_1(\epsilon) + I_2(\epsilon) = 0 \Rightarrow W_y(x) = V_x(y)$$

Green function in ball Take a look at $y \in B_R(0)$, define reflection point $y^* = \frac{R^2}{y^2} y$. Note that

$$|y^*||y| = R^2$$

and that if $y \to \partial B_R(0)$, $y^* \to \partial B_R(0)$. Define

$$h^y = -\Gamma(|x-y^*|)\left(\frac{|y|}{R}\right)^{2-n}$$

Then

$$G(x,y) = \Gamma(|x-y|) - \Gamma(|x-y^*|)\left(\frac{|y|}{R}\right)^{2-n}$$

From harmony of Γ, G is harmonic in x. For $x \in \partial B_r$:

$$(x-y^*)^2 = \left(x - \frac{R^2}{y^2} y\right)^2 = |x|^2 - 2x\frac{R^2}{y^2} y + \frac{R^4}{|y|^2} = \frac{R^2}{y^2}\left[R^2 - 2xy + y^2\right] = \frac{R^2}{y^2}(x-y)^2$$

$$|x-y^*| = \frac{R}{|y|}|x-y|$$

$$h^y(x) = -\left(\frac{|y|}{R}\right)^{2-n} \frac{1}{n(2-n)\omega_n}(x-y^*)^{2-n} = -\frac{1}{n(2-n)\omega_n}(x-y)^{2-n} = -\Gamma(|x-y|)$$

i.e., $h^y(x)$ we defined fulfills conditions on $h^y(x)$.
For $n=2$

$$h^y(x) = \begin{cases} -\frac{1}{2\pi}\ln|x-y^*| + \frac{1}{2\pi}\ln\frac{R}{|y|} & y \neq 0 \\ -\frac{1}{2\pi}\ln R & y = 0 \end{cases}$$

For $n > 2$

$$h^y(x) = \begin{cases} -\left(\frac{|y|}{R}\right)^{n-2} \frac{1}{n\omega_n}|x-y^*|^{2-n} & y \neq 0 \\ -\Gamma(R) & y = 0 \end{cases}$$

And thus for $n=2$

$$G(x,y) = \frac{1}{2\pi}\ln\left[\left(\frac{R}{|y|}\right)\frac{|x-y|}{|x-y^*|}\right]$$

For $n > 2$:

$$G(x,y) = \frac{1}{n\omega_n}\left[|x-y|^{2-n} - \left(\frac{|y|}{R}\right)^{n-2}|x-y^*|^{2-n}\right]$$

If $u \in \bar{B}_R$ harmonic,

$$u(y) = \oint\limits_{\partial B_R} u(x)\frac{\partial G}{\partial n_x}\,\mathrm{d}s_x$$

Definition $K(x,y) = \frac{\partial G}{\partial \hat{\mathbf{n}}_x}$ is called Poisson kernel:

$$\frac{\partial}{\partial x_i} G(x,y) = \frac{\partial}{\partial x_i}\Gamma(x-y) - \frac{\partial}{\partial x_i}\Gamma\left(\frac{|y|}{R}|x-y^*|\right)$$

$$\frac{\partial}{\partial x_i}\Gamma(x-y) = \frac{1}{n\omega_n}\frac{x_i-y_i}{|x-y|}|x-y|^{-n+1} = \frac{1}{n\omega_n}\frac{x_i-y_i}{|x-y|^n}$$

$$\frac{\partial}{\partial x_i}\Gamma\left(\frac{|y|}{R}|x-y^*|\right) = \frac{1}{n\omega_n}\left(\frac{|y|}{R}\right)^2 (x_i - y_i^*)|x-y|^{-n}$$

Since we are on ball, $\hat{\mathbf{n}}_x = \sum \frac{x_i}{R}\hat{\mathbf{x}}_\mathbf{i}$:

$$\frac{\partial}{\partial n_x} G(x,y) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Gamma(x-y) = \frac{1}{n\omega_n}|x-y|^n \sum_{i=1}^{n}\left[\frac{x_i}{R}(x_i-y_i) - \frac{x_i}{R}\left(\frac{|y|^2}{R^2}(x_i-y_i)\right)\right] =$$

$$= \frac{1}{n\omega_n}\frac{1}{R}|x-y|^n \sum_{i=1}^{n}\left[x_i^2 - x_i y_i - \frac{x_i^2|y|^2}{R^2} + x_i y_i\right] = \frac{1}{n\omega_n}\frac{1}{R}|x-y|^n(R^2-|y|^2)$$

Conclusion If $u \in \bar{B}_R$ harmonic,

$$u(y) = \oint_{\partial B_R} u(x)K(x,y)\,\mathrm{d}s_x$$

In case $y=0$ we get $K(x,0) = \frac{1}{n\omega_n R^{n-1}}$ and acquire the mean value theorem.

Claim K fulfills following conditions

1. $K(x,y) > 0$, $y \in B_R$, $x \in \partial B_R$.
2. $\nabla_y^2 K(x,y) = 0$, $y \in B_R$
3. $\oint\limits_{\partial B_R} K(x,y)\,\mathrm{d}s_x = 1$ for all $y \in B_R$

Theorem If $g \in \mathcal{C}(\partial B_R)$ then

$$u(y) = \oint_{\partial B_R} K(x,y)g(x)\,\mathrm{d}s_x$$

is harmonic function and $u(x) = g(x)$ for all $x \in \partial B_R$.

Proof Note that

$$K(x,y) = \frac{1}{Rn\omega_n}|x-y|^n(R^2-|y|^2)$$

is harmonic in y in B_R for $x \in \partial B_R$

$$\nabla_y^2 u(y) = \oint_{\partial B_R} \nabla_y^2 K(x,y)g(x)\,\mathrm{d}s_x = 0$$

Lets show that $\lim_{y\to y_0\in\partial B_R} u(y) = g(y_0)$. Lets choose $\epsilon > 0$. We choose δ_1 such that $|g(x)-g(y_0)| < \epsilon$ if $x \in \partial B_R \cap B_{\delta_1}(y_0)$

$$\oint_{\partial B_R} g(x)K(x,y)\,\mathrm{d}s_x - g(y_0) = \oint_{\partial B_R} (g(x)-g(y_0))K(x,y)\,\mathrm{d}s_x =$$

$$= \underbrace{\oint_{\partial B_R \setminus \partial B_R \cap B_{\delta_1}(y_0)} (g(x)-g(y_0))K(x,y)\,\mathrm{d}s_x}_{I_1} + \underbrace{\oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(x)-g(y_0))K(x,y)\,\mathrm{d}s_x}_{I_2}$$

$$|I_2| \le \oint_{\partial B_R \cap B_{\delta_1}(y_0)} (g(y)-g(y_0))K(x,y)\,\mathrm{d}s_y \le \epsilon \oint_{\partial B_R} K(x,y) \le \epsilon$$

$|x-y_0| \geq \delta_1$ and $|y-y_0| > \frac{\delta_1}{2}$, thus

$$|x-y| \geq |x-y_0| - |y-y_0| > \delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2}$$

$$K(x,y) \leq \frac{1}{Rn\omega_n}\left(\frac{\delta_1}{2}\right)^{-n}(R^2-|y|^2) \overset{y\to y_0}{\longrightarrow} 0$$

$$|I_1(y)| \leq (R^2-|y|^2)\max_{\partial B_R}|y|\frac{\delta_1^{-n}}{Rn\omega_n} \overset{y\to y_0}{\longrightarrow} 0$$

Conclusion u is continuous in $y_0 \in \partial B_R$ and $\lim_{y\to y_0\in\partial B_R} u(y) = g(y_0)$, i.e. Dirichlet problem has solution if the bound is continuous.

Theorem Continuous function in Ω is harmonic iff it fulfills mean value theorem.

Proof Suppose u fulfills mean value theorem, take a look at $B_R(x_0) \subset \Omega$ for some $x_0 \in \Omega$.
Let v harmonic function that equals to u on ∂B_R. Since v fulfills mean value theorem, then $u-v$ also fulfills it. However, $u-v=0$ on ball bound, and thus from weak extremum theorem we get that $u-v=0$ in the whole ball, i.e. $u=v$.

Conclusion If u_n a sequence of harmonic functions uniformly converging to u, u is harmonic.

Proof If $u_n \to u$ uniformly in Ω, then for all $x_0 \in \Omega$, exists $r>0$ such that $B_r(x_0) \subset \Omega$, and since $u_n \to u$ uniformly in $\partial B_r(x_0)$ thus

$$\lim_{n\to\infty} \frac{1}{|\partial B_r(x_0)|} \int\limits_{\partial B_r(x_0)} u_n(x)\,\mathrm{d}s = \int\limits_{\partial B_r(x_0)} u(x)\,\mathrm{d}s$$

and also

$$\lim_{n\to\infty} u_n(x_0) = u(x_0)$$

However, from harmony,

$$u_n(x_0) = \frac{1}{|\partial B_r(x_0)|} \int\limits_{\partial B_r(x_0)} u_n(x)\,\mathrm{d}s$$

i.e.

$$u(x_0) = \frac{1}{|\partial B_r(x_0)|} \int\limits_{\partial B_r(x_0)} u(x)\,\mathrm{d}s$$

and thus u is harmonic.

Conclusion If u is harmonic in Ω, $u \in C^\infty(\Omega)$.

Proof If u harmonic, $u(y) = \int\limits_{\partial B_R} K(x,y)u(x)\,\mathrm{d}s_x$ and K is infinitely differentiable by $y \in B_R$:

$$\frac{\partial^m}{\partial y_j^m}u(y) = \int\limits_{\partial B_R} \frac{\partial^m}{\partial y_j^m}K(x,y)u(x)\,\mathrm{d}s_x$$

Conclusion If u_n monotonic sequence of harmonic functions in Ω and exists $y \in \mathbb{R}$ such that $\{u_n(y)\}$ is bounded then in every bounded interval $\bar{\Omega}' \subsetneq \Omega$ sequence u_n converges uniformly to harmonic function.

Proof $u_{n+1}(x) \geq u_n(x)$ for all $x \in \Omega$, lets show that for all $x \in \Omega'$ sequence converges to finite limit. Denote $\omega_{mn} = u_m - u_n$.
By Harnack's identity for $m > n$

$$\sup_{\Omega'} \omega_{mn} \leq C \inf_{\Omega} \omega_{mn}$$

$$u_m(x) - u_n(x) = \omega_{mn}(x) \leq C[u_m(y_0) - u_n(y_0)] < K$$

K is constant independent on n and m, in particular, u_n is bounded for all $x \in \Omega'$, thus exists $u(x) = \lim_{n\to\infty} u_n(x)$ in Ω'.
It's left to proof sequence converges uniformly and thus u is harmonic.

Definition A sequence of function u_n is equicontinuous in x_0 if for all $\epsilon > 0$ exists $\delta > 0$, $N > 0$ such that if $x - x_0 < \delta$ and $n > N$,

$$|u_n(x) - u_n(x_0)| < \epsilon$$

The particular case is when derivatives are bounded, since if $|\boldsymbol{\nabla} u_n| < K$,

$$|u_n(x) - u_n(x_0)| < K|x - x_0|$$

Arzelá–Ascoli theorem If u_n sequence of functions equicontinuous in compact interval and bounded, then exists uniformly converging subsequence.

Note The only bounded harmonic function in $\mathbb{R}^n$ is constant one.

Theorem If u harmonic in $\Omega \subset \mathbb{R}^n$ then

$$\left|\frac{\partial u}{\partial x_j}\right| \le \sup_{\Omega} |u| \cdot \frac{n}{d(x, \partial\Omega)}$$

Proof

$$u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} u(y) \, \mathrm{d}y$$

Since u is harmonic and in particular $\frac{\partial u}{\partial x_i}$ is harmonic

$$\nabla^2 u = 0 \Rightarrow \nabla^2 \frac{\partial}{\partial x_i} u = \frac{\partial}{\partial x_i} \nabla^2 u = 0$$

we get

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} \frac{\partial}{\partial x_j} u(y) \, \mathrm{d}y$$

Consider function

$$v_i(y) = \begin{cases} 0 & i \neq j \\ u(y) & i = j \end{cases}$$

then

$$\frac{\partial}{\partial x_j} u(x) = \frac{1}{\omega_n r^n} \int\limits_{B_r(x)} \boldsymbol{\nabla} \cdot v \, \mathrm{d}y = \frac{1}{\omega_n r^n} \oint\limits_{\partial B_r(x)} u(y) n_j(y) \, \mathrm{d}s_y$$

where $n_j(y) = \hat{\mathbf{n}} \cdot \hat{\mathbf{x}}_j$, and thus $|n_j| \le 1$:

$$\left|\frac{\partial}{\partial x_j} u(x)\right| \le \frac{1}{\omega_n r^n} \left| \oint\limits_{\partial B_r(x)} u(y) n_j(y) \, \mathrm{d}s_y \right| \le \frac{1}{\omega_n r^n} \oint\limits_{\partial B_r(x)} |u(y)| \, \mathrm{d}s_y < \frac{1}{\omega_n r^n} \sup |u| n \omega_n r^{n-1} = \sup |u| \frac{n}{r}$$

for all $r < d(x, \partial\Omega)$.

Theorem If $\{u_n\}$ sequence of bounded harmonic functions in Ω then on each compact set $K \subset \Omega$ exists subsequence uniformly converging to u harmonic on Ω.

Proof By previous theorem each compact set $K \subset \Omega$ fulfills $d(x, \partial\Omega) < C(K)$ for all $x \in K$ and thus partial derivatives are bounded $|\partial u_n x_i| < C(K)$ thus $\{u_n\}$ is equicontinuous on K and thus it has subsequence uniformly converging to u harmonic in K.
Lets choose compact sets

$$K_j = \left\{ x \in \Omega, \ |x| \le j, \ d(x, \Omega^C) \ge \frac{1}{j} \right\}$$

such that $\bigcup_{n=1}^{\infty} K_n = \Omega$. Suppose u_{nj} is converging subsequence in K_n thus exists subsequence of u_{nj} converging in K_{n+1}. Diagonal sequence (of mutual indices) converges uniformly for all K and its limit u is harmonic.

5.2 Dirichlet problem in arbitrary interval

Let Ω bounded interval and $g \in \mathcal{C}(\partial\Omega)$ We want to solve the following problem:

$$\begin{cases} \nabla^2 u = 0 & u \in \mathcal{C}(\bar{\Omega}) \\ u(x) = g(x) & x \in \partial\Omega \end{cases}$$

Definition u is generalized subharmonic function iff u is continuous, for any ball $\bar{B} \subsetneq \Omega$ and for all harmonic and continuous h such that $u \leq h$ in B, $u \leq h$ on ∂B.

Example $u(x) = |x - x_0|$ is generalized subharmonic function in $\mathbb{R}^n$.

Conclusion Generalized subharmonic function fulfills mean value inequality:

$$u(x_0) \leq \frac{1}{|\partial B_r(x_0)|} \oint\limits_{\partial B_r(x_0)} u$$

Also, if function fulfills mean value inequality it is generalized subharmonic function.

Theorem If $\nabla^2 u \geq 0$ on Ω, u isgeneralized subharmonic function on Ω.

Lemma Let u generalized subharmonic function in Ω, v generalized superharmonic function in Ω and $u, v \in \mathcal{C}(\bar{\Omega})$ and $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω

Proof Directly from weak extermum.

Lemma If u,v generalized subharmonic functions, then $\max(u, v)(x) = \max(u(x), v(x))$ is generalized subharmonic function. This is right for any finite amount of functions.

Proof If h harmonic on ball and $h \geq \max(u, v)(x)$ on ∂B, in particular $h \geq u$, $h \geq v$ on ∂B and thus $h \geq u$, $h \geq v$ on B and thus $h \geq \max(u, v)(x)$ on B.

Definition Let $\Omega \subset \mathbb{R}^n$ bounded. Let g bounded on $\partial\Omega$, define S_g as set of all generalized subharmonic functions continuous on $\bar{\Omega}$ such that for all $x \in \partial\Omega$ $u(x) \leq g(x)$.

5.3 Perron method

Lemma Let u generalized subharmonic function in Ω, $\bar{B} \subsetneq \Omega$, define $U(x) = u(x)$ for $x \in \Omega \setminus B$ such that $\nabla^2 U(x) = 0$ for $x \in B$ and $U(x) = u(x)$ on ∂B. U is generalized subharmonic function. $U(x) > u(x)$ for all x. $U(x)$ is called harmonic lifting of u and is unique from maximum principle.

Proof Let h harmonic on B such that $h(x) \geq U(x) \geq u(x)$ on ∂B_1 and thus $h(x) \geq u(x) = U(x)$ in $x \in B_1 \setminus B$.

$$\partial(B \cap B_1) = (\partial B \cap \bar{B}_1) \cup (\partial B_1 \cap \bar{B})$$

For $\partial B \cap \bar{B}_1$ $U = u < h$. For $\partial B_1 \cap \bar{B}$ $h \geq U$. Thus $H \geq U$ on $\partial(B \cap B_1)$ however $h - U$ is nonnegative and harmonic on $\partial(B \cap B_1)$ and thus $H \geq U$ in $B \cap B_1$.

Theorem $u(x) = \sup\limits_{w \in Sg} w(x)$ is harmonic.

Proof Constant M is harmonic (constant) and it fulfills $M \geq y$ on $\partial\Omega$ thus $w \leq M$ for all $w \in S_g$ in particular $w(x) \leq M$ for all $x \in \Omega$ and $w \in S_g$.
Let $y \in \Omega$, there exists a sequence of functions $v_n \in S_g$ such that

$$\lim_{n\to\infty} v_n(y) = u(y)$$

Let $\bar{B}_r(y) \subsetneq \Omega$ and define V_n that is harmonic lifting of v_n. Since $V_n(x) \geq v_n(x)$ for all $x \in \Omega$ and $V_n \in S_g$ (it is generalized subharmonic and equals to v_n on $\partial\Omega$),

$$v_n(y) \leq V_n(y) \leq u(y)$$

for all $y \in \Omega$ thus $V_n(y) \to u(y)$. Thus there is subsequence of V_n uniformly converging to v in $B_\rho(y) \subset B_r(y)$. In B_ρ $v \leq u$.
Suppose exists $z \in B_\rho(y)$ such that $v(z) < u(z)$. Exists $\bar{u} \in S_g$ such that $v(z) < \bar{u}(z) \leq u(z)$. Then

$$\bar{u} < w_j = \bar{u} \vee V_n \in S_g$$

Let W_j harmonic lift of w_j on $B_r(y)$. $W_j \in S_g$ thus once again exists subsequence uniformly converging to w in B_ρ. We know that

$$v \leq w \leq u$$

and $v(y) = w(y) = u(y)$. Thus, since both are harmonic, $w - v$ is harmonic and non-negative, thus $w - v = 0$. We got the contradiction, thus $v = u$.

Conclusion If Direchelet problem

$$\begin{cases} \nabla^2 w(x) = 0 & x \in \Omega \\ w(y) = g(y) & x \in \partial\Omega \end{cases}$$

has solution, then $u(x) = \sup\limits_{v \in Sg} v(x)$ is solution.

Solution $u(x)$ is harmonic and $g \geq u$ on $\partial\Omega$, thus $w \geq u$ on Ω.
On the other hand, w is harmonic and in particular subharmonic, thus $w \in S_g$ and $w \leq u$. We conclude $w = u$.

Exercise

$$\Omega = B_1 \setminus \{0\}$$

Then

$$\partial\Omega = \partial B_1 \cup \{0\}$$

For bound condition $u(\partial B_1) = 1$ and $u(0) = 0$ there is no solution.

Definition $y \in \partial\Omega$ is called regular if exists generalized subharmonic function w on $\bar{\Omega}$ such that $w(y) = 0$ and $w(\bar{\Omega} \setminus \{0\}) < 0$.

Exercise If there exist a ball $B_\epsilon(z)$ in Ω^C tangent to $\partial\Omega$ in y, then y is regular.

Theorem If Ω is bounded interval, $\partial\Omega$ is regular in every point, $g \in \mathcal{C}(\partial\Omega)$ exists $u \in \mathcal{C}(\bar{\Omega})$ such that $u = g$ on $\partial\Omega$ and $\boldsymbol{\nabla} u = 0$.

Proof We need

$$\lim_{x \to y \in \partial\Omega} u(x) = g(y)$$

Define

$$\begin{cases} v_1(x) = g(y) - \epsilon + hw(x) \\ v_2(x) = g(y) + \epsilon - hw(x) \end{cases}$$

where $h > 0$ is constant to be determined later and w is the function from definition of regular point.
v_1 is subharmonic and v_2 is superharmonic. For h big enough we know that $v_1(x) < g(x)$ and $v_2(x) > g(x)$ for $x \in \partial\Omega$.
$0 > -m \geq w(x)$ if $|x - y| > \delta$. Thus

$$v_1(x) \leq g(x) - \epsilon - hm$$

Choose h such that

$$g(x) + \epsilon - hm < \min_{x \in \partial\Omega} g(x)$$

Since $|g(x)-g(y)|<\epsilon$ if $|x-y|<\delta$ thus $v_1(x)<g(x)$ if $|x-y|\leq\delta$ and thus $v_1(x)\leq g$ in $\partial\Omega$ and similarly $v_2(x)\geq g$ in $\partial\Omega$. Thus

$$v_2\geq u\geq v_1$$

for $u=\sup_{w\in S_g} w(x)$.

$$g(x)+\epsilon-hw(x)\geq u(x)\geq g(x)-\epsilon+hw(x)$$

$$-\epsilon\leq\liminf_{x\to y} u(x)-g(x)\leq\limsup_{x\to y} u(x)-g(x)<\epsilon$$

Theorem Regularity is sufficient condition for solution existence.

Proof Suppose there exists solution of Dirichlet problem in interval with bound condition $g(x)=-|x-y|$ for $y\in\partial\Omega$. Thus exists harmonic function h which equals to g on $\partial\Omega$, then $h<0$ in Ω and $h(y)=0$, i.e., y is regular.

Hilbert proof of Direchlet problem For all $u\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\partial\Omega)$ and $u=g$ on $\partial\Omega$.
Define Direchlet integral- functional

$$I(u)=\int\limits_\Omega |\boldsymbol{\nabla} u|^2$$

Suppose there exists minimum of I , u_0 (this is not easy to show). Then

$$I(u_0)\leq\int\limits_\Omega |\boldsymbol{\nabla} u|^2$$

Let $\phi\in\mathcal{C}^2(\Omega)\cap\mathcal{C}(\partial\Omega)$ such that $\phi=0$ on $\partial\Omega$. Then $u_0+\epsilon\phi$ fulfills the condition.

$$I(u_0+\epsilon\phi)=\int\limits_\Omega |\boldsymbol{\nabla}(u_0+\epsilon\phi)|^2=\int\limits_\Omega |\boldsymbol{\nabla} u_0|^2+2\epsilon\int\limits_\Omega \boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi+\epsilon^2\int\limits_\Omega |\boldsymbol{\nabla}\phi|^2\geq I(u_0)=\int\limits_\Omega |\boldsymbol{\nabla} u_0|^2$$

Thus

$$\int\limits_\Omega \boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi\geq 0$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} u_0\phi)=\boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi+\nabla^2 u_0\phi$$

$$\int\limits_\Omega \boldsymbol{\nabla} u_0\boldsymbol{\nabla}\phi=-\int\nabla^2 u_0\phi+\int\boldsymbol{\nabla}(\boldsymbol{\nabla} u_0\phi)=-\int\nabla^2 u_0\phi+\oint\limits_{\partial\Omega}\phi\boldsymbol{\nabla} u_0\geq 0$$

Thus

$$\int\nabla^2 u_0\phi=0$$

for any $\phi\in\mathcal{C}(\bar{\Omega})$ thus

$$\nabla^2 u_0=0$$

5.4 Neumann bound condition

Is there solution of Neumann bound condition

$$\begin{cases}\nabla^2 u(x)=0 & x\in\Omega\\ \frac{\partial u}{\partial n}=g & y\in\partial\Omega\end{cases}$$

Proof Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Define functional

$$I(u) = \frac{1}{2} \int\limits_{\Omega} |\boldsymbol{\nabla} u|^2 + \oint\limits_{\partial\Omega} ug$$

If there exists minimum u_0, take a look on $u_0 + \epsilon\phi$ for $\phi \in \mathcal{C}(\bar{\Omega})$.

$$0 \leq I(u_0 + \epsilon\phi) - I(u_0) = \epsilon \iiint\limits_{\Omega} + \epsilon \int\limits_{\partial\Omega} \phi g + \frac{1}{2}\epsilon^2 |\boldsymbol{\nabla} u_0|^2$$

Thus

$$\iiint\limits_{\Omega} \boldsymbol{\nabla} u_i \boldsymbol{\nabla} \phi - \oint\limits_{\partial\Omega} \phi g = 0$$

For any $\phi \in \mathcal{C}^2 \cap \bar{\otimes}$.

$$-\int \nabla^2 u_0 \phi + \int\limits_{\partial\Omega} \phi \frac{\partial u}{\partial n} - \int\limits_{\partial\Omega} = 0$$

thus

$$\oint \left(\frac{\partial u}{\partial n} - g \right) \phi = 0$$

i.e.

$$\frac{\partial u}{\partial n} = g$$

If $\oint\limits_{\partial\Omega} g > 0$ obviously there is no minimum of I: for $u = \lambda$

$$I(u) = -\lambda \oint\limits_{\partial\Omega} g$$

Thus $\oint g = 0$ is necessary condition for I to be bounded. By Gauss

$$0 = \int\limits_{\Omega} \nabla^2 u = \oint\limits_{\partial\Omega} \frac{\partial u}{\partial n} = \oint\limits_{\partial\Omega} g$$

6 Heat equation in higher dimensions

$$\frac{\partial u}{\partial t} = \nabla_x^2 u$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}^+$.
Note the equation is unchanged under transformation

$$\begin{cases} x \to ax = x' \\ t \to a^2 t = t' \end{cases}$$

Thus if $u(x,t)$ is solution then $u(ax, a^2t)$ is solution. Thus, lets search for solutions dependent on $\xi = \frac{|x|}{\sqrt{t}}$:

$$u(x,t) = w\left(\frac{|x|}{\sqrt{t}}\right)$$

Define heat as

$$Q(t) = \int\limits_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x = \int\limits_{\mathbb{R}^n} w\left(\frac{|x|}{\sqrt{t}}\right) \mathrm{d}x = \int w(\xi) t^{\frac{n}{2}}\,\mathrm{d}\xi = t^{\frac{n}{2}} \int w(\xi)\,\mathrm{d}\xi = \text{const}$$

So, to keep Q constant we want

$$u(x,t) = t^{-\frac{n}{2}} w\left(\frac{|x|}{\sqrt{t}}\right)$$

Substituting into equation:

$$(\xi^{n-1} w')' + \frac{1}{2}(\xi^n w)' = 0$$

$$\xi^{n-1}w' + \frac{1}{2}\xi^n w = c$$

Substituting $\xi = 0$ we get $c = 0$:

$$\xi^{n-1}w' + \frac{1}{2}\xi^n w = 0$$

$$w' + \frac{1}{2}\xi w = 0$$

$$w(\xi) = ce^{-\frac{|\xi|^2}{4}}$$

If we want to normalize

$$\int u(x,t)\,\mathrm{d}x = 1$$

We get

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

Which is n dimensional Gaussian.

Kernel of heat

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4t}}$$

Then

$$\frac{\partial K}{\partial t} = \nabla_x^2 K$$

Also note that $K > 0$ and that, from normalization,

$$\int\limits_{\mathbb{R}^4} K(x,y,t)\,\mathrm{d}^n x = 1$$

Moreover, for all $\delta > 0$ and for all $y \in \mathbb{R}^n$

$$\lim_{t\to 0} \int\limits_{|x-y|\geq\delta} K(x,y,t)\,\mathrm{d}^n x = 0$$

i.e.,

$$\lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x-y|\geq\delta} e^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}^n x = 0$$

Proof By variable substitution $\xi = \frac{x}{\sqrt{t}}$:

$$\lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x-y|\geq\delta} e^{-\frac{|x-y|^2}{4t}}\,\mathrm{d}^n x = \lim_{t\to 0} \frac{1}{(4\pi t)^{\frac{n}{2}}} \int\limits_{|x|\geq\delta} e^{-\frac{x^2}{4t}}\,\mathrm{d}^n x = \lim_{t\to 0} \frac{1}{(4\pi)^{\frac{n}{2}}} \int\limits_{|\xi|\geq\frac{\delta}{\sqrt{t}}} e^{-\frac{\xi^2}{4}}\,\mathrm{d}^n \xi = 0$$

Define

$$u(x,t) = \int\limits_{\mathbb{R}^n} K(x,y,t) f(y)\,\mathrm{d}^n y$$

Conclusion If $u_t = \nabla^2 u$ then $\min\limits_{\Omega} u$, $\max\limits_{\Omega} u$ are acquired on $\partial_p\Omega$. In particular there is unique solution of heat equation Dirichlet problem in $\Omega = D \times [0,T]$: if u_1, u_2 are solutions of heat equation in Ω and $u_1 = u_2$ on $\partial_p\Omega$ then $u_1 = u_2$ on Ω.

Theorem If $|f|$ is bounded and continuous, $u(x,t)$ is solution of the heat equation in $\mathbb{R}^n \times (0,\infty)$ fulfilling $\lim_{t\to 0} u(x,t) = f(x)$.

Proof

$$\frac{\partial u}{\partial t} = \iint\limits_{\mathbb{R}^n} \frac{\partial K}{\partial t} f(x)\,\mathrm{d}x$$

$$\nabla_x^2 u = \iint\limits_{\mathbb{R}^n} \nabla_x^2 K f(x)\,\mathrm{d}x$$

To be allowed to switch between integral and derivative we need to show that

$$u(x,t+h) - u(x,t) = \int\limits_{\mathbb{R}^n} [K(x,y,t+h) - K(x,y,h)] f(x)\,\mathrm{d}x$$

is bounded by integrable function. Since K is decaying exponent, it is true.
Lets show that $\lim_{t\to 0} u(x,t) = f(x)$.

$$u(x,t) = \int\limits_{|x-y|\le\delta} K(x,y,t) f(y)\,\mathrm{d}^n y + \int\limits_{|x-y|>\delta} K(x,y,t) f(y)\,\mathrm{d}^n y$$

Since integral of K is 1

$$f(x) = \iiint\limits_{\mathbb{R}^n} K(x,y,t) f(x)\,\mathrm{d}^n x$$

and thus

$$u(x,t) = \int\limits_{|x-y|\le\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y + \int\limits_{|x-y|>\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y$$

Since f is bounded:

$$\int\limits_{|x-y|>\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y \le 2c \int\limits_{|x-y|>\delta} K(x,y,t) \to 0$$

Always exists ϵ such that $|f(x) - f(y)| < \epsilon$

$$\int\limits_{|x-y|\le\delta} K(x,y,t)\big(f(y) - f(x)\big)\,\mathrm{d}^n y \le \int\limits_{|x-y|\le\delta} K(x,y,t)|(f(y) - f(x)|\,\mathrm{d}^n y \le \epsilon \int\limits_{|x-y|\le\delta} K(x,y,t)\,\mathrm{d}^n y \le \epsilon \int\limits_{\mathbb{R}^n} K(x,y,t)\,\mathrm{d}^n y \le \epsilon$$

Thus

$$\limsup_{t\to 0} |u(x,y) - f(x)| = \epsilon$$

and

$$\lim_{t\to 0} u(x,t) = f(x)$$

Note that temperature has infinite velocity - even if we start with f with compact support, for any $t > 0$ for any point we have $u > 0$.

Heat eqation in compact region Let $\Omega = D \times [0,T]$ for some compact $D \subsetneq \mathbb{R}^n$. We call $\partial_p\Omega\,\{D \times \{0\}\} \cup \{\partial D \times (0,T]\}$ parabolic bound of Ω.

Theorem If $u \in \mathcal{C}(\bar{\Omega})$, $u_t, u_{x_i x_j} \in \mathcal{C}(\Omega)$ and $u_t - \nabla^2 u \le 0$ on Ω then

$$\max_{(x,t)\in\bar{\Omega}} u(x,t) = \max_{(x,t)\in\partial_p\Omega} u(x,t)$$

Proof Suppose $u_t - \nabla^2 u < 0$ for all $(x,t) \in \Omega$. Suppose $(x_0, t_0) \in \Omega$ is maximum of u. Then

$$\nabla^2 u \leq 0$$

$$u_t = 0$$

i.e., $u_t - \nabla^2 u \geq 0$ in contradiction.
If (x_0, T) is maximum, then

$$\nabla^2 u \leq 0$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=T} = T$$

thus still $u_t - \nabla^2 u \geq 0$ in contradiction.
If we suppose that $u_t - \nabla^2 u \leq 0$ define $v(x,t) = u(x,t) - \delta t$. Then v fulfills $v_t - \nabla^2 v < 0$.
Suppose $\max u(x,t) = u(x_0, t_0) = m$, then

$$\max_{\Omega} v \geq m - \delta T$$

is acquired in $(x_\delta, t_\delta) \in \partial_p \Omega$. Looking on sequence $\delta_n \to 0$, since $\partial_p \Omega$ is compact, $(x_{\delta_n}, t_{\delta_n}) \to (x_0, t_0) \in \partial_p \Omega$, i.e.

$$\lim_{\delta to 0} v(x_\delta, t_\delta) = u(x_0, t_0) = m$$

Thus maximum is acquired on parabolic boundary.

Theorem If f is bounded on $\mathbb{R}^n$ and $u(x,t)$ fulfills

$$u_t - \nabla^2 u \geq 0$$

and $u \in \mathcal{C}^2(\mathbb{R}^n \times \mathbb{R}^+) \cap \mathcal{C}(\mathbb{R}^n \times \bar{\mathbb{R}}^+)$, $u(x,0) = f$ and in addition

$$u(x,t) \leq M e^{a|x|^2}$$

for some $M, a > 0$ then

$$u(x,t) < \sup f(x)$$

for all x, t.
Note that it's enough to show theorem for $0 < t \leq T$ for some constant $T > 0$, since then we can use $u(x,T)$ as new initial conditions to get to arbitrary t.

$$K(x,y,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

is solution of heat equation. So is

$$K(x,y,t+T) = \frac{1}{(4\pi (t+T))^{\frac{n}{2}}} e^{-\frac{|x|^2}{4(t+T)}}$$

However $K(x,y,-t)$ is not solution of heat equation. But

$$K'(x,y,-t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} e^{+\frac{|x|^2}{-4t}}$$

is solution of heat equation. It can be seen by extending to complex plane, we can note that under transformation

$$\begin{cases} t \to -t \\ x \to ix \end{cases}$$

heat equation is preserved.
So lets look at

$$w_{y,T}(x,t) = K'(x,y,T-t) = \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T-t)}}$$

with

$$w_{y,T}(x,0) = \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{|x|^2}{4T}}$$

for $0 < t \leq T$. This function solves heat equation, however this function is increasing and goes to infinity in finite time. This happens due to value of $w_{y,T}(x,0)$ in infinity - it is infinite, i.e. there is infinite amount of heat in $x \to \infty$.

Proof Define

$$v_\mu(x,t) = u(x,t) = \mu w_{T,y}(x,t)$$

Then

$$\frac{\partial}{\partial t} v_\mu(x,t) - \nabla^2 v_\mu \le 0$$

Define

$$\Omega_{\rho,T} = \{(x,t) \,:\, |x-y| < \rho,\; 0 \le t < T\}$$

We know that

$$\max_{\bar{\Omega}_{\rho,T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho,T}} v_\mu(t)$$

In particular, for $x = y$

$$v_\mu(y,t) \le \max_{\bar{\Omega}_{\rho,T}} v_\mu(t) = \max_{\partial_p \Omega_{\rho,T}} v_\mu(t)$$

In ball $B_\rho(y)$

$$v_\mu(x,0) \le u(x,0)$$

For $0 \le t \le \frac{T}{2}$

$$v_\mu(x,t) = u(x,t) - \mu w(x,t) \le Me^{a|x|^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} \le Me^{a(|y|+\rho)^2} - \mu \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\frac{\rho^2}{4T}} = e^{a\rho^2} \left[Me^{2|y|\rho + a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2 \left(\frac{1}{4T} - a\right)} \right]$$

Since we are free to choose T and ρ, we can choose them such that $\frac{1}{4T} - a > 0$ and then for rho big enough

$$v_\mu(x,t) \le e^{a\rho^2} \left[Me^{2|y|\rho + a|y|^2} - \frac{1}{(4\pi T)^{\frac{n}{2}}} e^{\rho^2 \left(\frac{1}{4T} - a\right)} \right] < C$$

for any C. Thus, $v_\mu(x,t)$ can acquire maximum only in $t = 0$:

$$\max_{\substack{|x-y| \le \rho \\ 0 \le t \le \frac{T}{2}}} v_\mu(x,t) = \max_{|x-y| \le \rho} v_\mu(x,0)$$

Thus, taking μ to 0:

$$u(x,t) \le \max_{\mathbb{R}^n} u(x,0)$$

for $y \in \mathbb{R}^n$ and $0 \le t \le \frac{2}{a}$.

Conclusion If $u_t - \nabla^2 u = 0$ then

$$\sup_{x \in \mathbb{R}^n} |u(x,t)| \le \sup_{x \in \mathbb{R}^n} |u(x,0)|$$

Conclusion If u_1, u_2 fulfill heat equation such that $u_1(x,0) = u_2(x,0)$ and in addition

$$\begin{cases} |u_1(x,t)| < Me^{a|x|^2} \\ |u_2(x,t)| < Me^{a|x|^2} \end{cases}$$

then $u_1(x,t) = u_2(x,t)$ for all $x, t \in \mathbb{R}^n \times \mathbb{R}^+$.
Take a look on time integral of heat kernel,

$$G(x,y) = \int_0^\infty K(x,y,t)\,\mathrm{d}t$$

If $n \ge 3$ the integral converges both in infinity and in 0, for $x \ne y$.
Also

$$u(x,t) = \int_{\mathbb{R}^n} \mathrm{d}y\, K(x,y,t) f(y)$$

Assume f has compact support (we don't have to, but it makes things easier). We see that

$$u = \mathcal{O}\left(\frac{1}{t^{\frac{n}{2}}}\right)$$

$$\int_0^\infty \mathrm{d}t\, u(x,t) = \int_0^\infty \int_{\mathbb{R}^n} \mathrm{d}y\, K(x,y,t) f(y) = \int_{\mathbb{R}^n} \mathrm{d}y\, f(y) \int_0^\infty K(x,y,t) = \int_{\mathbb{R}^n} \mathrm{d}y\, f(y) G(x,y)$$

Integrating heat equation

$$\int_0^\infty \frac{\partial u}{\partial t} = u(x,\infty) - u(x,0) = -f(x)$$

since $u(x,0) = f(x)$ and $u(x,\infty) = 0$. on the other hand

$$\int_0^\infty \nabla_x^2 u(x,t)\,\mathrm{d}t = \nabla_x^2 \int_0^\infty u(x,t)\,\mathrm{d}t$$

Denote

$$V = \int_0^\infty u(x,t)\,\mathrm{d}t$$

then

$$\nabla^2 V = -f$$

This is Laplace equation, with $n \geq 3$, i.e.

$$V(x) = -\int \Gamma(|x-y|) f(y)\,\mathrm{d}y$$

where $\Gamma(r) = \frac{1}{(2-r)\omega_n} r^{2-n}$.

7 Wave equation in $\mathbb{R}^n \times \mathbb{R}$

We look at

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \nabla_x^2 u \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Lets assume radial solution: $u(x,t) = w(|x|,t)$ for some $w = w(r,t)$ Then we get

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r}\frac{\partial w}{\partial r}$$

Define $v(r) = rw$, then

$$v_r = w + w_r r$$

$$v_{rr} = 2w_r + w_{rr} r$$

$$\frac{1}{r} v_{rr} = w_{rr} + \frac{2}{r} w_r$$

which exactly Laplacian for $n = 3$.

$$\frac{\partial^2 v}{\partial t^2} = r\frac{\partial^2 w}{\partial t^2} = c^2\left(r\frac{\partial^2 w}{\partial r^2} + (n-1)\frac{\partial w}{\partial r}\right)$$

For $n = 3$:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2}{\partial r^2} v$$

which is one-dimensional wave equation for $r > 0$. Since

$$\frac{\partial}{\partial x_i} w(|x|,t) = w_r(0,t)\frac{\partial |x|}{\partial x_i}$$

and $\frac{\partial |x|}{\partial x_i}$ is not continuous in 0, we require

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial r} w_t(0) = 0$$

Assume initial condition of form $w(r,0) = f(r)$, $w_t(r,0) = g(r)$ and $f'(0) = g'(0) = 0$. Define extension of f and g on $\mathbb{R}$: $f(|r|)$ and $g(|r|)$.
Then, for $v(r,0) = rf(|r|)$ and $v_t(r,0) = rg(|r|)$. The D'Alembert solution is

$$rw(r,t) = v(r,t) = \frac{(r+ct)f(|r+ct|) + (r-ct)f(|r-ct|)}{2} + \frac{1}{2ct}\int_{r-ct}^{r+ct} sg(|s|)\,\mathrm{d}s$$

i.e.,

$$w(r,t) = \frac{(r+ct)f(|r+ct|) + (r-ct)f(|r-ct|)}{2r} + \frac{1}{2ctr}\int_{r-ct}^{r+ct} sg(|s|)\,\mathrm{d}s$$

Note that if g and f has compact support, exist time t_0 such that $w(r, t > t_0) = 0$
In 1D wave equation we required $f \in \mathcal{C}^2$ and $g \in \mathcal{C}^1$, however in 3D it's not enough.
For example, assuming $t = 0$

$$\lim_{r\to 0} w(r,t) = f(ct) + ctf'(ct) = w(0,t)$$

Sperical average For $h(x)$, $x \in \mathbb{R}^n$, spherical average of h is

$$M_h(x,r) = \frac{1}{|\partial B_r(x)|}\int\limits_{|\partial B_r(x)|} h(y)\,\mathrm{d}s_y$$

Note that

$$M_h(x,0) = 0$$

With variable substitution $x + r\xi = y$ we get

$$M_h(x,r) = \frac{1}{n\omega_x r^{n-1}}\int\limits_{\partial B_1(0)} r^{n-1}h(x+r\xi)\,\mathrm{d}s_\xi = \frac{1}{n\omega_x}\int\limits_{\partial B_1(0)} h(x+r\xi)\,\mathrm{d}s_\xi$$

Now $M_h(x,r)$ is even in r.

$$\begin{aligned}\frac{\partial}{\partial r}M_h(x,r) &= \frac{1}{n\omega_x}\int\limits_{\partial B_1(0)}\sum_{x_i}\xi_i\frac{\partial h}{\partial x_i}(x+r\xi)\,\mathrm{d}s_\xi = \frac{1}{n\omega_x r}\int\limits_{\partial B_1(0)}\vec{\xi}\cdot\boldsymbol{\nabla}_\xi h(x+r\xi)\,\mathrm{d}s_\xi \overset{\text{Guass}}{=} \frac{1}{n\omega_x r}\int\limits_{B_1(0)}\nabla_\xi^2 h(x+r\xi)\,\mathrm{d}\xi = \\ &= \frac{r}{n\omega_x}\int\limits_{B_1(0)}\nabla_x^2 h(x+r\xi)\,\mathrm{d}^n\xi = \frac{1}{n\omega_n r^{n-1}}\int\limits_{B_r(x)}\nabla_x^2 h(y)\,\mathrm{d}^n y = \frac{1}{n\omega_n r^{n-1}}\nabla^2\int_0^r\mathrm{d}\rho\int\limits_{\partial B_\rho(x)} h(y)\,\mathrm{d}s_y = \\ &= \frac{1}{r^{n-1}}\nabla^2\int_0^r\mathrm{d}\rho\,\rho^{n-1}\underbrace{\frac{1}{n\omega_n\rho^{n-1}}\int\limits_{\partial B_\rho(x)} h(y)\,\mathrm{d}s_y}_{M_h(x,\rho)} = r^{1-n}\nabla_x^2\int_0^r\mathrm{d}\rho\,\rho^{n-1}M_h(x,\rho)\end{aligned}$$

Since radial Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} = r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right)$$

$$\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}M_h(x,r)\right] = \frac{\partial}{\partial r}\nabla_x^2\int_0^r\mathrm{d}\rho\,\rho^{n-1}M_h(x,\rho) = \nabla_x^2 r^{n-1}M_h(x,r) == r^{n-1}\nabla_x^2 M_h(x,r) = r^{n-1}M_{\nabla^2 h}(x,r)$$

Thus

$$r^{1-n}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}M_h(x,r)\right] = \frac{\partial^2}{\partial r^2}M_h(x,r) + \frac{n-1}{r}M_h(x,r) = \nabla_x^2 M_h(x,r) = M_{\nabla^2 h}(x,r)$$

Let u solution of wave equation, then

$$M_u(x,r,t) = \frac{1}{n\omega_n}\int\limits_{|\xi|=1} u(x+r\xi,t)\,\mathrm{d}s_\xi$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla_x^2 u$$

$$\frac{\partial^2}{\partial t^2}M_u(x,r,t) = c^2 M_{\nabla^2 u}(x,r,t) = c^2\left[\frac{\partial^2 M_u}{\partial r^2} + \frac{n-2}{r}\frac{\partial}{\partial r}M_u\right]$$

COnclusion Spherical average of wave equation in x is acquired as solution of wave equation for $M_u(x,r,t)$ as a function of r and t.
For $n=3$:

$$M_u(x,r,t) = \frac{1}{2r}[(ct+r)M_f(x,ct+r) - (ct-r)M_f(x,ct-r)] + \frac{1}{c}\frac{1}{2r}\int +ct - r^{ct+r}\rho M_g(x,\rho)\,\mathrm{d}\rho$$

Then

$$u(x,t) = \lim_{r\to 0} M_u(x,r,t) = \frac{\partial}{\partial\rho}\rho M_f(x,\rho)\Big|_{\rho=ct} + \frac{1}{c}\rho M_g(x,\rho)\Big|_{\rho=ct} = M_f(x,ct) + ct\frac{\partial}{\partial\rho}M_f(x,\rho)\Big|_{\rho=ct} + tM_g(x,ct) =$$

$$= \frac{\partial}{\partial t}(tM_f(x,ct)) + tM_g(x,ct) = \left(\frac{t}{n\omega_n}\int\limits_{|\xi|=1} f(x+ct\xi)\,\mathrm{d}s_\xi\right) + \frac{t}{n\omega_n}\int\limits_{|\xi|=1} g(x+ct\xi)\,\mathrm{d}s_\xi =$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int\limits_{|y-x|=ct} f(x)\,\mathrm{d}s_y\right) + \frac{1}{4\pi c^2 t}\int\limits_{|y-x|=ct} g(x)\,\mathrm{d}s_y =$$

$$= \frac{1}{4\pi}\int\limits_{\xi=1} f(x+ct\xi)\,\mathrm{d}s_\xi + \frac{ct}{4\pi}\int\limits_{\xi=1} (\boldsymbol{\nabla}_{x+ct\xi}f)\cdot\xi\,\mathrm{d}s_\xi + \frac{t}{4\pi}\int\limits_{\xi=1} g(x+ct\xi)\,\mathrm{d}s_\xi$$

2D wave equation We can reduce dimensionality by with initial conditions. If

$$\begin{cases} u(x_1,x_2,x_3,0) = f(x_1,x_2) \\ u_t(x_1,x_2,x_3,0) = g(x_1,x_2) \end{cases}$$

Thus $u = u(x_1,x_2,t)$ is solution of 2D wave equation. Take a look at sphere around $(x_1,x_2,0)$:

$$|y-x| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + y_3^2} = ct$$

$$y_3 = \sqrt{c^2t^2 - (x_1-y_1)^2 - (x_2-y_2)^2}$$

And thus

$$\mathrm{d}s_y = \sqrt{1 + \left(\frac{\partial y_3}{\partial y_1}\right)^2 + \left(\frac{\partial y_3}{\partial y_2}\right)^2}\,\mathrm{d}y_1\,\mathrm{d}y_2 = \left|\frac{ct}{y_3}\right|\mathrm{d}y_1\,\mathrm{d}y_2$$

Define

$$r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$$

and

$$u(x_1,x_2,t) = \frac{\partial}{\partial t}\left(\frac{2}{4\pi c^2 t}\iint\limits_{r<ct} f(y_1,y_2)\left|\frac{ct}{y_3}\right|\mathrm{d}y_1\,\mathrm{d}y_2\right) + \frac{2}{4\pi c^2 t}\iint\limits_{r<ct} g(y_1,y_2)\left|\frac{ct}{y_3}\right|\mathrm{d}y_1\,\mathrm{d}y_2 =$$

$$= \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{r<ct}\frac{f(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2\right] + \frac{1}{2\pi c}\iint\limits_{r<ct}\frac{g(y_1,y_2)}{\sqrt{(ct)^2-r^2}}\,\mathrm{d}y_1\,\mathrm{d}y_2$$

8 Algebric theory of PDEs

Take a look at heat equation

$$\frac{\partial u}{\partial t} = \nabla_x^2 u$$

Define $L(u) = \nabla^2 u$. L is linear operator on functions in $\mathcal{C}^2$. However since for $f\in\mathcal{C}^k$, $L(f)\in\mathcal{C}^{k-2}$, we can say that $L:\mathcal{C}^\infty\to\mathcal{C}^\infty$ and then

$$\frac{\partial u}{\partial t} = L(u)$$

In case of wave equation Define $v = \frac{\partial u}{\partial t}$ and then

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}u\\v\end{pmatrix} = L\begin{pmatrix}u\\v\end{pmatrix}$$

Then wave equation is

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ \nabla^2 u \end{pmatrix}$$

In case of Poisson equation

$$Lu = f$$

i.e. $u = L^{-1}f$.
To solve such equations we want to diagonalize L, for that we'd like L to be symmetric.

8.1 Sturm-Liouville problem

$$\frac{\partial u}{\partial t} = s(x)u_{xx} + b(x)u_x c(x)u$$

Given interval $[a, b]$ with initial conditions we want to rewrite the equation as

$$\frac{\partial u}{\partial t} = (pu_x)_x + qu$$

for some $p(x)$ $q(x)$. It's always possible to find $r(x)$ such that multiplication of first equation by $r(x)$ will give the second one for $v = ru$.
Define operator

$$L(v) = -\frac{1}{r(x)}[(p(x)v_x)_x + qv]$$

Then we write equation as

$$L(v) = \frac{\partial v}{\partial t}$$

And we want to diagonalize the operator, i.e. find v and λ such that

$$Lv - \lambda v = 0$$

L is defined over vector space $\mathcal{C}^2[a, b]$ and fulfills a pair of bound conditions:

$$\begin{cases} B_a(u) = \alpha u(a) + \beta u'(a) = 0 & |\alpha| + |\beta| > 0 \\ B_b(u) = \gamma u(b) + \delta u'(b) = 0 & |\gamma| + |\delta| > 0 \end{cases}$$

Denote

$$X = \left\{f \in \mathcal{C}^2[a, b],\ B_a(f) = B_b(f) = 0\right\}$$

Alternatively,

$$X = \left\{f \in \mathcal{C}^2[a, b],\ u(a) = u(b),\ u'(a) = u'(b)\right\}$$

Note that for $f \in X$ in's possible that $L(f) \notin X$.

$$\langle u, v\rangle = \int_a^b \int_a^b u(x)v(x)r(x)\,\mathrm{d}x$$

This is innter product, since it's linear, symmetric and positive.

Definition L is symmetric operator relatively to inner product iff

$$\langle u, Lv\rangle = \langle Lu, v\rangle$$

Let's check if L we defined is symmetric:

$$uLv - vLu = \frac{1}{r}\Big[u[(pv_x)_x + qv] - v[(pu_x)_x + qu]\Big] = \frac{1}{r}[pu_x v - pv_x u]_x$$

$$\langle u, Lv\rangle - \langle Lu, v\rangle = \int_a^b [pu_x v - pv_x u]_x\,\mathrm{d}x = \Big[pu_x v - pv_x u\Big|_a^b = 0$$

Theorem Sturm–Liouville has real eigenvalues and orthogonal eigenfunctions. In case of divided bound conditions X, each eigenvalue is simple. In any case, multiplicity of each eigenvalue is not greater than 2.

Proof For each there are two independent solutions u_1^λ, u_2^λ. If λ is eigenvalue, exist $A, B \in \mathbb{R}$ such that

$$\begin{cases} B_a(Au_1^\lambda + Bu_2^\lambda) = 0 \\ B_b(Au_1^\lambda + Bu_2^\lambda) = 0 \end{cases}$$

$$\begin{cases} A\Big(\alpha u_1^\lambda(a) + \beta {u_1^\lambda}'(a)\Big) + B\Big(\alpha u_2^\lambda(a) + \beta {u_2^\lambda}'(a)\Big) = 0 \\ A\Big(\gamma u_1^\lambda(b) + \delta {u_1^\lambda}'(b)\Big) + B\Big(\gamma u_2^\lambda(b) + \delta {u_2^\lambda}'(b)\Big) = 0 \end{cases}$$

If this linear system has solution,

$$\Big(\alpha u_1^\lambda(a) + \beta {u_1^\lambda}'(a)\Big)\Big(\gamma u_2^\lambda(b) + \delta {u_2^\lambda}'(b)\Big) - \Big(\alpha u_2^\lambda(a) + \beta {u_2^\lambda}'(a)\Big)\Big(\gamma u_1^\lambda(b) + \delta {u_1^\lambda}'(b)\Big) = 0$$

This is actually equation in λ, that has infinite number of solution.
For $\lambda_1 \neq \lambda_2$,

$$Lu_1 = \lambda_1 u_1$$

$$Lu_2 = \lambda_2 u_2$$

$$\langle u_2, Lu_1 \rangle = \lambda_1 \langle u_1, u_2 \rangle$$

$$\langle u_2, Lu_1 \rangle = \langle u_1, Lu_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle$$

$$(\lambda_1 - \lambda_2)\langle u_1, u_2 \rangle = 0$$

Thus if $\lambda_1 \neq \lambda_2$, $\langle u_1, u_2 \rangle = 0$.
Suppose $\lambda \in \mathbb{C}$:

$$Lu = \lambda u$$

$$\overline{Lu} = \overline{\lambda u}$$

$$\bar{L}\bar{u} = L\bar{u} = \bar{\lambda}\bar{u}$$

Thus both u and $\bar{u}$ are eigenfunctions, with different eigenvalues, i.e., $\langle u, \bar{u} \rangle = 0$, meaning

$$\int_a^b |u|^2 \, \mathrm{d}x = 0$$

$$u = 0$$

$$Lu = Lu_1 + iLu_2$$

$$\overline{Lu} = Lu_1 - iLu_2 = L(u_1 - iu_2) = L\bar{u}$$

Now

$$Lu = \lambda u$$

$$Lv = \lambda v$$

$$0 = vLu - uLv = \frac{1}{2}[[(pu')' + qu]v - [(pv')' + qv]u] = \frac{1}{2}[p \cdot (u'v - v'u)]' = 0$$

Thus

$$p \cdot (u'v - v'u) = c = \text{const}$$

$$u'v - v'u = \frac{c}{p}$$

Since $p > 0$ and $u'v - v'u$ is 0 on bound, $c = 0$, thus

$$u'v = v'u$$

i.e., $\left(\frac{u}{v}\right)'$ or $\left(\frac{v}{u}\right)'$ is 0, i.e., u and v are linearly dependent.

Example

$$Lu = u''$$

with periodic bound conditions on $[-\pi, \pi]$. Solutions are $\sin(\theta)$ $\cos(\theta)$.

Suppose u_i are normalized eigenfunctions of L, and

$$V_n = \operatorname{span}(u_1, \ldots, u_n)$$

If $f \in X$, what is closest function to f in V_n:

$$\min_{w \in V_N} \|w - f\|$$

We call w a projection of f on V_N.

Claim

$$w = \sum_i^N \langle f, u_i \rangle u_i \in V_N$$

Proof

$$w \in V_N \Rightarrow w = \sum_1^N \alpha_i u_i$$

$$\min_{\alpha_1, \ldots, \alpha_N} \left\| f - \sum_1^N \alpha_i u_i \right\|^2$$

$$\left\| f - \sum_1^N \alpha_i u_i \right\|^2 = \left\langle f - \sum_1^N \alpha_i u_i, f - \sum_1^N \alpha_i u_i \right\rangle = \langle f, f \rangle - 2 \sum \alpha_i \langle f, u_i \rangle + \sum_{i,j} \alpha_i \alpha_j \langle u_i, u_j \rangle = \|f\|^2 - 2\alpha_i \langle f, u_i \rangle + \sum \alpha_i^2 = H(\alpha_i, \ldots, \alpha_N)$$

$$\frac{\partial H}{\partial \alpha_i} = 2[\alpha_i - \langle f, u_i \rangle] = 0$$

Thus

$$\alpha_i = \langle f, u_i \rangle$$

Theorem Let $\{u_i\}$ is infinite sequence of orthonormal functions in X, for all $f \in X$

$$\sum_i |\langle f, u_i \rangle|^2 \leq \|f\|^2$$

this is Bessel equation.

$$0 \leq \left\| f - \sum_1^N \langle f, u_i \rangle u_i \right\|^2 = \|f\|^2 - \sum_{i=1}^N |\langle f, u_i \rangle|^2$$

If

$$\sum_{i=1}^\infty |\langle f, u_i \rangle|^2 = \|f\|^2$$

Then

$$\lim_{N \to \infty} \left\| f - \sum_1^N \langle f, u_i \rangle u_i \right\| = 0$$

and we say that series

$$\sum_i^\infty \langle f, u_i \rangle u_i = f$$

Note this is not pointwise convergence:

$$\sum_i^\infty \langle f, u_i \rangle u_i(x) \neq f(x)$$

but rather norm convergence:

$$\lim_{N \to \infty} \int_a^b \left| f(x) - \sum_i^\infty \langle f, u_i \rangle u_i(x) \right|^2 r(x) \, \mathrm{d}x$$

Definition Orthonormal sequence is called total if for all $f \in X$

$$\sum_{i=1}^{\infty} |\langle f, u_i \rangle|^2 = \|f\|^2$$

(Parseval's identity).

Theorem

1. Sequence of eigenfunctions of Sturm–Liouville is total sequence.
2. Eigenvalues are $\lambda_1 < \lambda_2 < \dots < \lambda_n < \dots$ and $\lim_{n\to\infty} \lambda_n = \infty$.

Conclusion If $f \in X$ then series

$$\sum_{n=1}^{\infty} \langle f, u_i \rangle \, u_i = \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\int_0^{\pi} f(y) \sin(ny) \, \mathrm{d}y \right] \sin(nx)$$

$$\langle f, u_i \rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x$$

$$a_n(f) = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, \mathrm{d}x$$

$$f(x) = \sum a_n \sin(nx)$$

Theorem

$$l = \int_a^b \sqrt{\frac{r(x)}{p(x)}} \, \mathrm{d}x$$

$$\lambda_n \sim \left(\frac{n\pi}{l} \right)^2$$

which means

$$\lim_{n\to\infty} \lambda_n \sim \left(\frac{l}{n\pi} \right)^2 = 1$$

In addition, exist constants

$$|u_n| < c$$

$$|u_n'| < cn$$

$$|u_n''| < cn^2$$

Generalized heat equation

$$\begin{cases} r(x) u_t = (p u_x)_x + qu \\ B_a(u) = B_b(u) = 0 \\ u(x, 0) = f \end{cases}$$

$$(p u_x^{(i)})_x + q u^{(i)} + r \lambda_i^{(i)} = 0$$

Lets search the solution of form

$$u(x, t) = \sum_{j=1}^{\infty} a_j(t) u^{(j)}(x)$$

$$r(x) \sum_{j=1}^{\infty} \dot{a}_j(t) u^{(i)}(x) = r(x) \sum_{j=1}^{\infty} a_j(t) \lambda_j u^{(j)}(x)$$

i.e.,

$$\dot{a}_j(t) = \lambda_j a_j(t)$$

$$a_j(t) = a_j(0) e^{-\lambda_j t}$$

Thus

$$u(x,t) = \sum_{j=1}^{\infty} a_j(0) e^{-\lambda_j t} u^{(j)}(x)$$

$a_j(0)$ are generalized Fourier coefficients of f:

$$a_j(0) = \int_a^b r f u^{(j)} \,\mathrm{d}x$$

$$u(x,t) = \sum_{j=1}^{\infty} \left(\int_a^b r(y) f(y) u^{(j)}(y) \,\mathrm{d}y \right) e^{-\lambda_j t} u^{(j)}(x) = \int_a^b \left[\sum_{j=1}^{\infty} r(y) u^{(j)}(y) e^{-\lambda_j t} u^{(j)}(x) \right] f(y) \,\mathrm{d}y$$

Define kernel of heat

$$K(x,y,t) = r(y) \sum_{j=1}^{\infty} u^{(j)}(x) u^{(j)}(y) e^{-\lambda_j t}$$

And then

$$u(x,t) = \int_a^b K(x,y,t) f(y) \,\mathrm{d}y$$

Note that K is symmetric:

$$K(x,y,t) = K(y,x,t)$$

We can say that

$$K(x,y,t) \approxeq e^{Lt}$$

For operator L with eigenvalues λ_n and eigenfunctions u_n, define

$$\mathcal{L}(x,y) = \sum_{n=1}^{\infty} \lambda_n u_n(x) u_n(y)$$

Then

$$Lu = \int \mathcal{L}(x,y) u(y) \,\mathrm{d}y = \sum_{n=1}^{\infty} \lambda_n \langle u, u_n \rangle u_n(x)$$

i.e.,

$$L u_n = \lambda_n u_n$$

Thus

$$e^{\mathcal{L}t} = \sum e^{-\lambda_n t} u_n(x) u_n(y) = K$$

The last question is whether the series converge:

$$\frac{\partial u}{\partial t} = -\sum_{j=1}^{\infty} \lambda_j e^{-\lambda_j t} a_j(f) u_j(x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{j=1}^{\infty} e^{-\lambda_j t} a_j(f) u_j''(x)$$

$$a_j(f) = \int_a^b f(x) u_j(x) r(x) \,\mathrm{d}x$$

$$|a_j(f)| < c$$

and

$$\sum_{j=1}^{\infty} a_j^2(f) - \|f\|^2 < \infty$$

Thus

$$\lim_{j \to \infty} a_j(f) = 0$$

(this is Riemann–Lebesgue lemma).
Since

$$\left| \lambda_j e^{-\lambda_j t} a_j(f) \right| < j^2 e^{-j^2 t} c$$

If $t > 0$ coefficient series absolutely converges and thus the solution is in $\mathcal{C}^{\infty}$.

Non-homogeneous equations

$$u_t = (pu_x)_x + qu + f(x,t)$$

First of all, let's solve the following ODE:

$$\dot{x} = ax + f(t)$$

The homogeneous solution is

$$x(t) = e^{at}$$

thus

$$x(t) = a_0 e^{at} + e^{at}\int_0^t e^{-as} f(s)\,\mathrm{d}s$$

We want to solve

$$\dot{a}_j = -\lambda_j a_j + f_j(t)$$

where

$$f_j(t) = \int_a^b f(t) u_j(x)\,\mathrm{d}x$$

thus

$$a_j(t) = \underbrace{a_j(0)}_{0} + e^{-\lambda_j t}\int_0^t e^{\lambda_j s} f_j(s)\,\mathrm{d}s$$

$$|\dot{a}_j(t)||f_j(t)| + |\lambda_j||a_j|$$

Since $|a_j| \sim e^{-j^2}$,we want only

$$\sum_{j=1}^{\infty} |f_j(t)| < \infty$$

and then

$$\frac{\partial u}{\partial t} = \sum \dot{a}_j u_j(x) < c$$

If f is differentiable, $\sum_{j=1}^{\infty} |f_j(t)| < \infty$:

Note