104165 - Real functions

Baruch Solel

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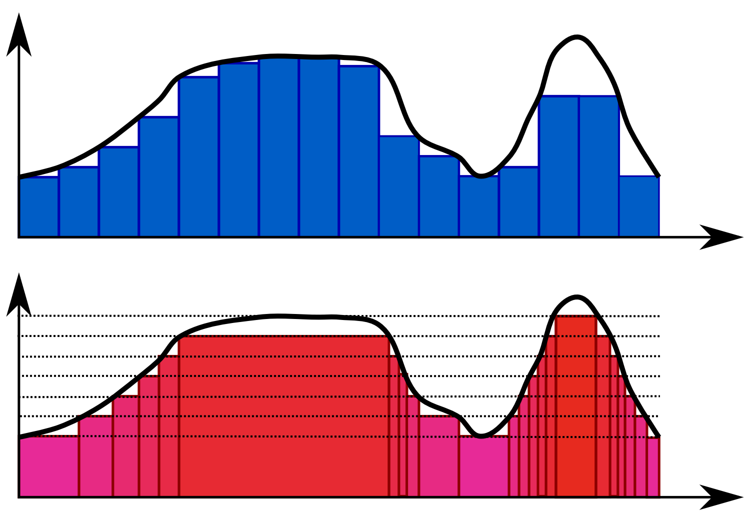
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1, r_2, \ldots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \ldots$$

$$\forall x \quad |g(x) - f(x)| \le \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \le \lambda(A) \le \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:
$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$
- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

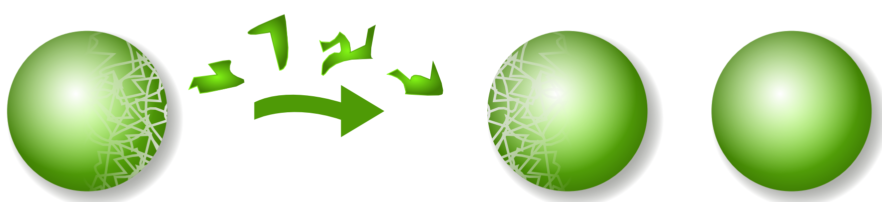
Proof Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at
$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0, 1]$$
Thus
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0, 1]) = 1$$
On the other hand
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$
Thus $\lambda(E) = 0$. However,
$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$
From sigma-additivity
$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$
But $\lambda(\mathbb{R}) \geq \lambda([0, 1])$, in contradiction.

Reqirements for measure in $\mathbb{R}$

1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$
4. If $A = \dot{\bigcup}_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In $\mathbb{R}^3$ it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach–Tarski paradox Denote B – unit ball in $\mathbb{R}^3$. We can write
$$B = \dot{\bigcup_{i=1}^{5}} A_i$$
and find C_i by rotation or translation of A_i such that $\dot{\bigcup}_{i=1}^{5} C_i$ is two unit balls.



2.1 Construction of λ

Special boxes Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^{n} (b_i - a_i)$$

Special polygons is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.
Let P is special polygon written as $P = \bigcap_{i=1}^{k} A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^{k} \lambda(A_i)$$

Claim

1. The definition is independent on choice of A_i.
2. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.
3. If P_1, P_2 are special polygons with disjoint interior then
$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$
4. For all $x \in \mathbb{R}^n$
$$\lambda(x + P) = \lambda(P)$$

Proof

1. Let $P = \bigcap A_i = \bigcap B_i$.
If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus
$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$
2. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1.
3. Find A_i which divides both P_1 and P_2.
4. ...

Alternative proof For special boxes

$$\lambda(E) = \lim_{N \to \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim

$$b - a = \lim_{N \to \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find $\left|[a, b] \cap \frac{1}{2}\mathbb{Z}\right|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any N:

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b-a-\frac{1}{N}\leq \frac{1}{N}\left|[a,b]\cap\frac{1}{N}\mathbb{Z}\right|\leq b-a+\frac{1}{N}$$

By sandwich rule, we get the equality.
We can do the same for higher dimension and for open sets, and then we can easily proof the claim.
If P is special polygon and we take $\lim_{N\to\infty}\frac{1}{N^n}\left|P\cap\frac{1}{N}\mathbb{Z}^n\right|=\sum\lambda(A_i)$ when $P=\bigcap A_i$

Open sets

Definition G is open if $\forall x\in G$ exists ball $B(x,r)$ such that $B\subset G$. Alternatively we can replace ball with special box. Thus for any open $G\neq\emptyset$

$$G=\bigcup\{P\text{ special polygon}\}$$

And we can define

$$\lambda(G)=\sup\{\lambda(P)|P\subseteq G\}$$

Claim

1. $$0\leq\lambda(G)\leq\infty$$
2. $$\lambda(G)=0\iff G=\emptyset$$
3. $$\lambda(\mathbb{R}^n)=\infty$$
4. $$G_1\subseteq G_2\Rightarrow\lambda(G_1)\leq\lambda(G_2)$$
5. $$\lambda\left(\bigcup_{k=1}^{\infty}G_k\right)\leq\sum\lambda(G_k)$$
6. $$\lambda\left(\dot{\bigcup_{k=1}^{\infty}}G_k\right)=\sum\lambda(G_k)$$
7. $$\lambda(P)=\lambda(\operatorname{int}P)=\inf\{\lambda(G):P\subseteq G\}$$
8. $$\lambda(x+G)=\lambda(G)$$

Lemma 2.1. Let $K\subseteq\mathbb{R}^n$ compact set and $\{G_i\}_{i\in I}$ open cover ($K\subseteq\bigcup G_i$). Then exists $\epsilon>0$ such that $\forall x\in K$ exists $i\in I$ such that $B(x,\epsilon)\subseteq G_i$.

Lemma 2.2. For all polygon of dimension P

$$\lambda(P)=\inf\{\lambda(G):P\subset G\}$$

Proof.

$$P\subseteq G\Rightarrow\lambda(P)\leq\lambda(G)$$

Infimum would give

$$\lambda(P)\leq\inf\{\lambda(G):P\subset G\}$$

Write $P=\bigcup_{k=1}^{N}I_k$. Then

$$\lambda(P)=\sum_{k=1}^{N}\lambda(I_k)$$

For ϵ find I_k^ϵ such that

$$\begin{cases}\operatorname{int}I_k^\epsilon\supseteq I_k\\ \lambda(I_k^\epsilon)\leq\lambda(I_k)+\frac{\epsilon}{N}\end{cases}$$

Denote $G = \bigcup_{k=1}^{N} \operatorname{int}(I_k^\epsilon)$, then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^{N} \lambda(\operatorname{int} I_k^\epsilon) = \sum_{k=1}^{N} \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^{N} \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

$\square$

Proof

1. Obvious
2. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$and thus $\lambda(G) \leq \lambda(P) > 0$
3. Any box is subset of $\mathbb{R}^n$ thus $\lambda(\mathbb{R}^n) = \infty$
4. Obvious
5. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

   $$P = \bigcup_{j=1}^{N} I_j$$

   finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^{N} = I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j, then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$.
   If this is possible, for such P denote

   $$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

   Obviously $\bigcup P_k = P$ and union is finite since for some m, for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j, and also internals of P_k are disjoint.
   Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P, thus

   $$\lambda\Big(\bigcup (G_k)\Big) = \sup \Big\{ \lambda(P) | P \subseteq \bigcup (G_k) \Big\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

6. Since we have sigma-subadditivity, we need only one direction of inequality:

   $$\lambda(G_k) = \sup \{\lambda(P) : P \subseteq G_k\}$$

   For any N

   $$\sum_{k=1}^{N} \lambda(G_k) = \sup \left\{ \sum_{k=1}^{N} \lambda(P_k) : P_k \subseteq G_k \right\} = \sup \left\{ \lambda\left(\dot{\bigcup_{k=1}^{N}} P_k \right) : P_k \subseteq G_k \right\} \leq \lambda\left(\dot{\bigcup_{k=1}^{N}} G_k \right) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k \right)$$

   i.e.,

   $$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k \right)$$

7. First, proof that $\lambda(P) = \lambda(\operatorname{int} P)$. If $I = P$ is non-empty special box $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. For any $\epsilon > 0$, $I_\epsilon = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$. $I_\epsilon \subseteq \operatorname{int} I$.
   That means that $\lambda(I_\epsilon) \leq \lambda(\operatorname{int} I)$. Obviously, $\lambda(I_\epsilon) \to \lambda(I)$, i.e. $\lambda(I) \leq \lambda(\operatorname{int} I)$.
   Generally, for $P = \bigcup_{k=1}^{N} I_k$,

   $$\operatorname{int} P \geq \bigcup_{k=1}^{N} \operatorname{int} I_k$$

thus

$$\lambda(\operatorname{int} P) \geq \lambda\left(\bigcup_{k=1}^{N} \operatorname{int} I_k\right) = \sum_{k=1}^{N} \lambda(\operatorname{int} I_k) \geq \sum_{k=1}^{N} \lambda(I_k) = \lambda(P)$$

For any P

$$\lambda(\operatorname{int} P) \geq \lambda P$$

However

$$\lambda(\operatorname{int} P) = \sum \{\lambda(Q) : Q \subseteq \operatorname{int} P\}$$
$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\operatorname{int} P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2.

8. Obvious since it's right for polygons

2.2 Compact sets

Definition 2.1. For compact $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf \{\lambda(G) : K \subseteq G \quad G \text{ is open}\}$$

Proposition 2.1.

$$0 \leq \lambda(K) < \infty$$

Proof. Each K is subset of open box A and $\lambda(A) < \infty$ □

Proposition 2.2.

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

Proof. Obvious □

Proposition 2.3. Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

Proof.

$$K_i \subseteq G_i$$
$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$
$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

Proposition 2.4.

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

Proof. For K_1, K_2 exists $\epsilon > 0$ such that $\forall\, x \in K_1\ y \in K_2$, $d(x, y) \geq \epsilon$. Denote

$$U_i = \bigcup_{x \in K_i} B\left(x, \frac{\epsilon}{2}\right) \supset K_i$$

Let $K_1 \cup K_2 \subset G_i$, since $K_i \subset U_i$,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since $U_1 \cap U_2 = \emptyset$ (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$
$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda\big((G \cap U_1) \cap (G \cap U_2)\big) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

$$\lambda_*(A) = \sup\{\lambda(K) : A \supset G, \text{ compact}\}$$

Proposition 2.5.

$$\lambda_*(A) \leq \lambda^*(A)$$

Proof. If G is open and K compact and $K \subset A \subset G$ then $K \subset G$, i.e. $\lambda(K) \leq \lambda(G)$. From that, taking supremum on K and infimum on G, we get the required result. □

Proposition 2.6.

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

Proof. Obvious. □

Proposition 2.7.

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof.

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

Thus exists G_k such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty}\left(\lambda^*(A_k) + \frac{\epsilon}{2^k}\right) = \lambda^*(A_k) + \epsilon$$

□

Proposition 2.8. For disjoint A_k

$$\lambda^*\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. For all i choose $K_i \subseteq A_i$. Choose some N, then

$$\bigcup_{k=1}^{N} K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since $\bigcup_{k=1}^{N} K_k$ is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \lambda\left(\bigcup_{k=1}^{N} K_k\right) = \sum_{k=1}^{N} \lambda(K_k)$$

By taking supremum on K_i, we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \sum_{k=1}^{N} \lambda_*(A_n)$$

□

Proposition 2.9. If A is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Proof. If A is compact, obviously $\lambda_*(A) = \lambda(A)$, and $\lambda^*(A) = \lambda(A)$ by definition.
For open A, obviously $\lambda(A) = \lambda^*(A)$. In addition, for any special polygon $P \subset A$, $\lambda(P) \leq \lambda_*(A)$. However

$$\lambda^*(A) = \lambda(A) = \sup\{\lambda(P) : P \subseteq A\} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

□