104165 - Real functions

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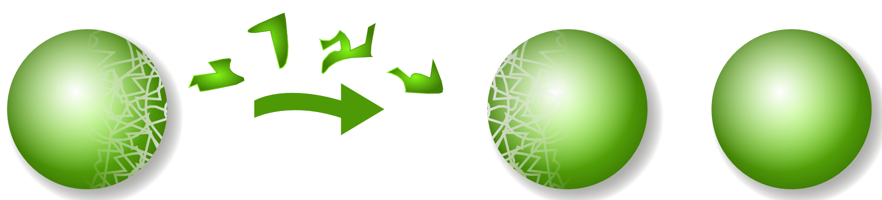
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1,r_2,\ldots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \ldots$$

$$\forall x \quad |g(x) - f(x)| \le \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \le \lambda(A) \le \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:
$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$
- If $A \subseteq B$, then $\lambda(A) \le \lambda(B)$.

Theorem Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

Proof Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0,1]$$

Thus

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0,1]) = 1$$

On the other hand

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$

Thus $\lambda(E) = 0$. However,

$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$

From sigma-additivity

$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$

But $\lambda(\mathbb{R}) \geq \lambda([0,1])$, in contradiction.

Reqirements for measure in $\mathbb{R}$

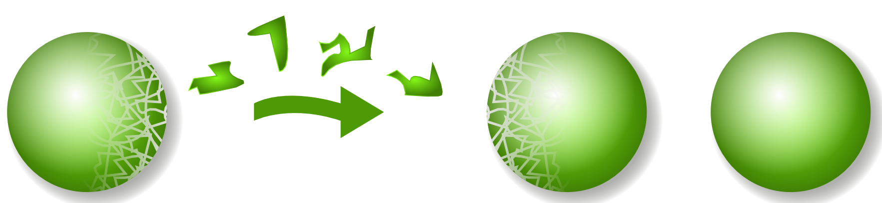
1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n} (b_i - a_i)$
4. If $A = \dot\bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In $\mathbb{R}^3$ it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach–Tarski paradox Denote B – unit ball in $\mathbb{R}^3$. We can write

$$B = \dot{\bigcup_{i=1}^{5}} A_i$$

and find C_i by rotation or translation of A_i such that $\dot\bigcup_{i=1}^{5} C_i$ is two unit balls.



2.1 Construction of λ

Special boxes Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^{n} (b_i - a_i)$$

Special polygons is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.
Let P is special polygon written as $P = \bigcap_{i=1}^{k} A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^{k} \lambda(A_i)$$

Claim

1. The definition is independent on choice of A_i.
2. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.
3. If P_1, P_2 are special polygons with disjoint interior then
$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$
4. For all $x \in \mathbb{R}^n$
$$\lambda(x + P) = \lambda(P)$$

Proof

1. Let $P = \bigcap A_i = \bigcap B_i$.
   If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus
$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$
2. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1.
3. Find A_i which divides both P_1 and P_2.
4. ...

Alternative proof For special boxes
$$\lambda(E) = \lim_{N \to \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N}\mathbb{Z}^n \right|$$
For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim
$$b - a = \lim_{N \to \infty} \frac{1}{N} \left| E \cap \frac{1}{N}\mathbb{Z} \right|$$
First of all
$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$
To find $\left|[a, b] \cap \frac{1}{2}\mathbb{Z}\right|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means
$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2}\mathbb{Z} \right| \leq 2b - 2a + 1$$
And for any N:
$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N}\mathbb{Z} \right| \leq Nb - Na + 1$$
$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N}\mathbb{Z} \right| \leq b - a + \frac{1}{N}$$
By sandwich rule, we get the equality.
We can do the same for higher dimension and for open sets, and then we can easily proof the claim.
If P is special polygon and we take $\lim_{N \to \infty} \frac{1}{N^n} \left| P \cap \frac{1}{N}\mathbb{Z}^n \right| = \sum \lambda(A_i)$ when $P = \bigcap A_i$

Open sets

Definition G is open if $\forall x \in G$ exists ball $B(x,r)$ such that $B \subset G$. Alternatively we can replace ball with special box. Thus for any open $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

Claim

1. $$0 \leq \lambda(G) \leq \infty$$
2. $$\lambda(G) = 0 \iff G = \emptyset$$
3. $$\lambda(\mathbb{R}^n) = \infty$$
4. $$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$
5. $$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum \lambda(G_k)$$
6. $$\lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right) = \sum \lambda(G_k)$$
7. $$\lambda(P) = \lambda(\operatorname{int} P) \inf \{\lambda(G) : P \subseteq G\}$$
8. $$\lambda(x + G) = \lambda(G)$$

Lemma Let $K \subseteq \mathbb{R}^n$ compact set and $\{G_i\}_{i \in I}$ open cover ($K \subseteq \bigcup G_i$). Then exists $\epsilon > 0$ such that $\forall x \in K$ exists $i \in I$ such that $B(x, \epsilon) \subseteq G_i$.

Claim

1. Obvious
2. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$and thus $\lambda(G) \leq \lambda(P) > 0$
3. Any box is subset of $\mathbb{R}^n$ thus $\lambda(\mathbb{R}^n) = \infty$
4. Obvious
5. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

   $$P = \bigcup_{j=1}^{N} I_j$$

   finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^{N} = I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j, then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$.
   If this is possible, for such P denote

   $$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

   Obviously $\bigcup P_k = P$ and union is finite since for some m, for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j, and also internals of P_k are disjoint.

Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P, thus

$$\lambda\Big(\bigcup(G_k)\Big) = \sup\Big\{\lambda(P) | P \subseteq \bigcup(G_k)\Big\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

6. ...

7. ...

8. Obvious since it's right for polygons