

104165 - Real functions

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Abstract

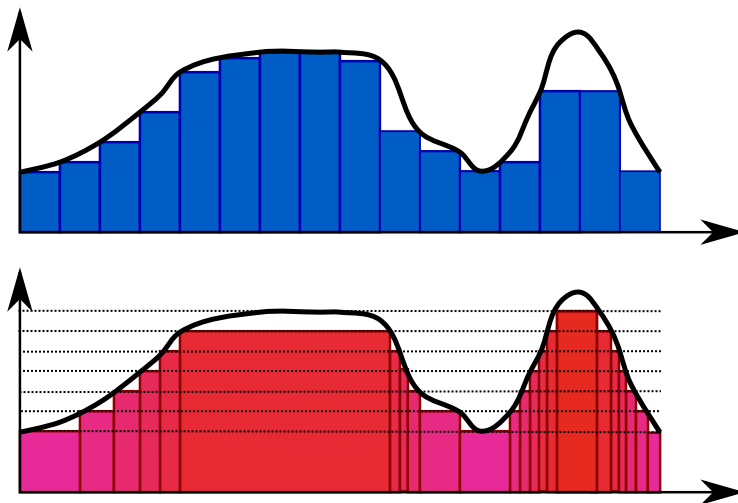
1 Introduction

If $\forall x \quad f_n(x) \rightarrow f(x)$ (pointwise) does $\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$?

Define $f_n(x) = \chi_{r_1, r_2, \dots, r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0, 1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q} \cap [0, 1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x) dx = \lim \sum f(x'_i)(x_{i+1} - x_i)$$



By dividing on y , we bound the error by the size of each interval, ϵ :

$$g(x) = s\chi_{A_1} + (s + \epsilon)\chi_{A_2} + \dots$$

$$\forall x \quad |g(x) - f(x)| \leq \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ ?

1. $\lambda([a, b]) = b - a$
2. $0 \leq \lambda(A) \leq \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i, j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : s \in A\}$.

From those properties we get additional properties:

- Additivity:

$$A = \bigcup_{i=1}^n A_i \Rightarrow \lambda(A) = \sum_{i=1}^n \lambda(A_i)$$

- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem 2.1. Function λ fulfilling 1-5 and defined on every subset of \mathbb{R} doesn't exist.

Proof. Suppose there exists such λ .

Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $[0, \frac{1}{2}]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.

From definition $E \subset [0, \frac{1}{2}]$. Take a look at

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E \right) \subseteq [0, 1]$$

Thus

$$\lambda \left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E \right) \right) \leq \lambda([0, 1]) = 1$$

On the other hand

$$\lambda \left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E \right) \right) = \sum_{k=2}^{\infty} \lambda \left(\frac{1}{k} + E \right) = \lambda(E)$$

Thus $\lambda(E) = 0$. However,

$$\mathbb{R} = \bigcup_{r \in \mathbb{Q}} r + E$$

From sigma-additivity

$$\lambda(\mathbb{R}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$

But $\lambda(\mathbb{R}) \geq \lambda([0, 1])$, in contradiction. □

Requirements for measure in \mathbb{R}

1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i)$
4. If $A = \bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In \mathbb{R}^3 it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach–Tarski paradox Denote B – unit ball in \mathbb{R}^3 . We can write

$$B = \bigcup_{i=1}^5 A_i$$

and find C_i by rotation or translation of A_i such that $\bigcup_{i=1}^5 C_i$ is two unit balls.



2.1 Construction of λ

Definition 2.1 (Special boxes). Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^n (b_i - a_i)$$

Definition 2.2 (Special polygons). is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.

Let P is special polygon written as $P = \bigcap_{i=1}^k A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^k \lambda(A_i)$$

Proposition 2.2. The definition is independent on choice of A_i .

Proof. Let $P = \bigcap A_i = \bigcap B_i$.

If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus

$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$

□

Proposition 2.3. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.

Proof. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1 .

□

Proposition 2.4. If P_1, P_2 are special polygons with disjoint interior then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$

Proof. Find A_i which divides both P_1 and P_2 .

□

Proposition 2.5. For all $x \in \mathbb{R}^n$

$$\lambda(x + P) = \lambda(P)$$

Alternative proof. For special boxes

$$\lambda(E) = \lim_{N \rightarrow \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim

$$b - a = \lim_{N \rightarrow \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find $|[a, b] \cap \frac{1}{2} \mathbb{Z}|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any N :

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq b - a + \frac{1}{N}$$

By sandwich rule, we get the equality.

We can do the same for higher dimension and for open sets, and then we can easily proof the claim.

If P is special polygon and we take $\lim_{N \rightarrow \infty} \frac{1}{N^n} |P \cap \frac{1}{N} \mathbb{Z}^n| = \sum \lambda(A_i)$ when $P = \bigcap A_i$

□

Open sets

Definition 2.3. G is open if $\forall x \in G$ exists ball $B(x, r)$ such that $B \subset G$. Alternatively we can replace ball with special box.

Thus for any open $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

Lemma 2.1. Let $K \subseteq \mathbb{R}^n$ compact set and $\{G_i\}_{i \in I}$ open cover ($K \subseteq \bigcup G_i$). Then exists $\epsilon > 0$ such that $\forall x \in K$ exists $i \in I$ such that $B(x, \epsilon) \subseteq G_i$.

Lemma 2.2. For all polygon of dimension P

$$\lambda(P) = \inf \{\lambda(G) : P \subset G\}$$

Proof.

$$P \subseteq G \Rightarrow \lambda(P) \leq \lambda(G)$$

Infimum would give

$$\lambda(P) \leq \inf \{\lambda(G) : P \subset G\}$$

Write $P = \bigcup_{k=1}^N I_k$. Then

$$\lambda(P) = \sum_{k=1}^N \lambda(I_k)$$

For ϵ find I_k^ϵ such that

$$\begin{cases} \text{int } I_k^\epsilon \supseteq I_k \\ \lambda(I_k^\epsilon) \leq \lambda(I_k) + \frac{\epsilon}{N} \end{cases}$$

Denote $G = \bigcup_{k=1}^N \text{int}(I_k^\epsilon)$, then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^N \lambda(\text{int } I_k^\epsilon) = \sum_{k=1}^N \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^N \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

□

Proposition 2.6.

$$0 \leq \lambda(G) \leq \infty$$

Proof. Obvious

□

Proposition 2.7.

$$\lambda(G) = 0 \iff G = \emptyset$$

Proof. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$ and thus $\lambda(G) \geq \lambda(P) > 0$

□

Proposition 2.8.

$$\lambda(\mathbb{R}^n) = \infty$$

Proof. Any box is subset of \mathbb{R}^n thus $\lambda(\mathbb{R}^n) = \infty$

□

Proposition 2.9.

$$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$

Proof. Obvious

□

Proposition 2.10.

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum \lambda(G_k)$$

Proof. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

$$P = \bigcup_{j=1}^N I_j$$

finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^N I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j , then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$. If this is possible, for such P denote

$$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

Obviously $\bigcup P_k = P$ and union is finite since for some m , for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j , and also internals of P_k are disjoint.

Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P , thus

$$\lambda\left(\bigcup (G_k)\right) = \sup \left\{ \lambda(P) | P \subseteq \bigcup (G_k) \right\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

□

Proposition 2.11.

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum \lambda(G_k)$$

Proof. Since we have sigma-subadditivity, we need only one direction of inequality:

$$\lambda(G_k) = \sup \{ \lambda(P) : P \subseteq G_k \}$$

For any N

$$\sum_{k=1}^N \lambda(G_k) = \sup \left\{ \sum_{k=1}^N \lambda(P_k) : P_k \subseteq G_k \right\} = \sup \left\{ \lambda\left(\bigcup_{k=1}^N P_k\right) : P_k \subseteq G_k \right\} \leq \lambda\left(\bigcup_{k=1}^N G_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$$

i.e.,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$$

□

Proposition 2.12.

$$\lambda(P) = \lambda(\text{int } P) = \inf \{ \lambda(G) : P \subseteq G \}$$

Proof. First, proof that $\lambda(P) = \lambda(\text{int } P)$. If $I = P$ is non-empty special box $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. For any $\epsilon > 0$, $I_{\epsilon} = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$. $I_{\epsilon} \subseteq \text{int } I$.

That means that $\lambda(I_{\epsilon}) \leq \lambda(\text{int } I)$. Obviously, $\lambda(I_{\epsilon}) \rightarrow \lambda(I)$, i.e. $\lambda(I) \leq \lambda(\text{int } I)$.

Generally, for $P = \bigcup_{k=1}^N I_k$,

$$\text{int } P \supseteq \bigcup_{k=1}^N \text{int } I_k$$

thus

$$\lambda(\text{int } P) \geq \lambda\left(\bigcup_{k=1}^N \text{int } I_k\right) = \sum_{k=1}^N \lambda(\text{int } I_k) \geq \sum_{k=1}^N \lambda(I_k) = \lambda(P)$$

For any P

$$\lambda(\text{int } P) \geq \lambda P$$

However

$$\lambda(\text{int } P) = \sum \{ \lambda(Q) : Q \subseteq \text{int } P \}$$

$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\text{int } P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2.

□

Proposition 2.13.

$$\lambda(x + G) = \lambda(G)$$

Proof. Obvious since it's right for polygons

□

2.2 Compact sets

Definition 2.4. For compact $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf \{ \lambda(G) : K \subseteq G \text{ } G \text{ is open} \}$$

Proposition 2.14.

$$0 \leq \lambda(K) < \infty$$

Proof. Each K is subset of open box A and $\lambda(A) < \infty$ □

Proposition 2.15.

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

Proof. Obvious □

Proposition 2.16. Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

Proof.

$$K_i \subseteq G_i$$

$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$

$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

Proposition 2.17.

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

Proof. For K_1, K_2 exists $\epsilon > 0$ such that $\forall x \in K_1 y \in K_2, d(x, y) \geq \epsilon$. Denote

$$U_i = \bigcup_{x \in K_i} B\left(x, \frac{\epsilon}{2}\right) \supset K_i$$

Let $K_1 \cup K_2 \subset G_i$, since $K_i \subset U_i$,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since $U_1 \cap U_2 = \emptyset$ (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$

$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda((G \cap U_1) \cup (G \cap U_2)) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subset G, \text{ open} \}$$

$$\lambda_*(A) = \sup \{ \lambda(K) : A \supset K, \text{ compact} \}$$

Proposition 2.18.

$$\lambda_*(A) \leq \lambda^*(A)$$

Proof. If G is open and K compact and $K \subset A \subset G$ then $K \subset G$, i.e. $\lambda(K) \leq \lambda(G)$. From that, taking supremum on K and infimum on G , we get the required result. \square

Proposition 2.19.

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

Proof. Obvious. \square

Proposition 2.20.

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof.

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subset G, \text{ open} \}$$

Thus exists G_k such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty} \left(\lambda^*(A_k) + \frac{\epsilon}{2^k} \right) = \lambda^*(A_k) + \epsilon$$

\square

Proposition 2.21. For disjoint A_k

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. For all i choose $K_i \subseteq A_i$. Choose some N , then

$$\bigcup_{k=1}^N K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since $\bigcup_{k=1}^N K_k$ is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \lambda\left(\bigcup_{k=1}^N K_k\right) = \sum_{k=1}^N \lambda(K_k)$$

By taking supremum on K_i , we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^N \lambda_*(A_k)$$

\square

Proposition 2.22. If A is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Proof. If A is compact, obviously $\lambda_*(A) = \lambda(A)$, and $\lambda^*(A) = \lambda(A)$ by definition.

For open A , obviously $\lambda(A) = \lambda^*(A)$. In addition, for any special polygon $P \subset A$, $\lambda(P) \leq \lambda_*(A)$. However

$$\lambda^*(A) = \lambda(A) = \sup \{ \lambda(P) : P \subseteq A \} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

\square

Denote

$$\mathcal{L}_0 = \{A \subset \mathbb{R}^n :: \lambda^*(A) = \lambda_*(A) < \infty\}$$

All compact sets and all open set with finite measure are in \mathcal{L}_0 .

Proposition 2.23.

$$\lambda_*(A) = \lambda_*(A + x)$$

$$\lambda^*(A) = \lambda^*(A + x)$$

Definition 2.5. For set in \mathcal{L}_0 , $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Lemma 2.3. If $A, B \in \mathcal{L}_0$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{L}_0$ and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

Proof.

$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) = \lambda(A) + \lambda(B) = \lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) \leq \lambda^*(A \cup B)$$

□

Theorem 2.24. $A \subseteq \mathbb{R}^n$ with $\lambda^*(A) < \infty$. $A \in \mathcal{L}_0$ iff for all $\epsilon > 0$ exists compact K and open G , $K \subseteq A \subseteq G$ and $\lambda(G \setminus K) < \epsilon$

Proof. \Rightarrow :

Let $A \in \mathcal{L}_0$. We can find compact K and open G , $K \subseteq A \subseteq G$ such that

$$\lambda(G) < \lambda^*(A) + \frac{\epsilon}{2}$$

$$\lambda(K) > \lambda_*(A) - \frac{\epsilon}{2}$$

Note that, by lemma

$$\lambda(G) = \lambda(K) + \lambda(G \setminus K)$$

$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \epsilon$$

\Leftarrow :

$$\lambda^*(A) \leq \lambda(G) = \lambda(K) + \lambda(G \setminus K) < \lambda(K) + \epsilon \leq \lambda_*(A) + \epsilon$$

Thus $\lambda^*(A) = \lambda_*(A)$ and $A \in \mathcal{L}_0$.

□

Collary 2.1. If $A, B \in \mathcal{L}_0$, then $A \cup B, A \cap B, A \setminus B \in \mathcal{L}_0$

Proof. First, show that $A \setminus B \in \mathcal{L}_0$. Take $K_1 \subseteq A \subseteq G_1$ and $K_2 \subseteq B \subseteq G_2$.

$$\lambda(G_1 \setminus K_1) < \frac{\epsilon}{2}$$

$$\lambda(G_2 \setminus K_2) < \frac{\epsilon}{2}$$

Denote $K = K_1 \setminus G_2$ and $G = G_1 \setminus K_2$.

$$K \subseteq A \setminus B \subseteq G$$

$$G \setminus K = (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$$

$$\lambda(G \setminus K) \leq \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) < \epsilon$$

Now

$$A \cup B = (A \setminus B) \cup B \in \mathcal{L}_0$$

$$A \cap B = A \setminus (A \setminus B) \in \mathcal{L}_0$$

□

Theorem 2.25. Let $\{A_k\}$ set in \mathcal{L}_0 and $A = \bigcup_{k=1}^{\infty} A_k$ such that $\lambda^*(A) < \infty$ then $A \in \mathcal{L}_0$ and

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

In addition, if $A_i \cap A_j = \emptyset$,

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Suppose $\{A_k\}$ are disjoint.

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda_*(A_k) \leq \lambda_*(A)$$

Thus $A \in \mathcal{L}_0$ and

$$\lambda(A) = \lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Now generally, define

$$B_1 = A_1 \in \mathcal{L}_0$$

$$B_2 = A_2 \setminus A_1$$

and so on:

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \in \mathcal{L}_0$$

Now $\{B_k\}$ are disjoint and $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{L}_0$. Thus

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Note Any ball $B(0, R)$ is in \mathcal{L} , since it is inside special box large enough.

Definition 2.6. Let $A \subseteq \mathbb{R}^n$, we say A is Lebesgue measurable if $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$. It's measure equals

$$\lambda(A) = \sup \{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Denote a set of all such sets as \mathcal{L} .

Proposition 2.26. If $\lambda^*(A) < \infty$, $A \in \mathcal{L} \iff A \in \mathcal{L}_0$. For those sets λ definitions are equivalent.

Proof. If $A \in \mathcal{L}_0$ in, then $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$, thus $A \in \mathcal{L}$.

Now, if $A \in \mathcal{L}$ and $\lambda^*(A) < \infty$. For all $N \in \mathbb{N}$,

$$A \cap B(0, N) \in \mathcal{L}_0$$

However

$$A = \bigcup_{N=1}^{\infty} [A \cap B(0, N)]$$

And $\lambda^*(A) < \infty$, thus $A \in \mathcal{L}_0$.

Denote

$$\tilde{\lambda}(A) = \sup \{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Obviously, $\tilde{\lambda}(A) \geq \lambda(A)$ (take $M = A$). On the other side,

$$\forall M \in \mathcal{L}_0 \quad \lambda(A \cap M) \leq \lambda(A)$$

thus $\tilde{\lambda}(A) = \lambda(A)$

□

Proposition 2.27.

$$\emptyset \in \mathcal{L}$$

Proof.

$$\emptyset \in \mathcal{L}_0 \Rightarrow \emptyset \in \mathcal{L}$$

□

Proposition 2.28.

$$A \in \mathcal{L} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$(\mathbb{R}^n \cap A) \cap M = M \setminus A = M \setminus (A \cap M) \in \mathcal{L}_0$$

□

Proposition 2.29.

$$\{A_i\}_{i=1}^{\infty} \in \mathcal{L} \Rightarrow A = \bigcup A_i \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$\begin{aligned} A \cap M &= \bigcup_{i=1}^{\infty} (A_i \cap M) \\ \lambda^*(A \cap M) &\leq \lambda(M) \end{aligned}$$

Thus

$$A \cap M \in \mathcal{L}_0$$

□

Proposition 2.30. If $\forall N \in \mathbb{N}$, $A \cap B(0, N) \in \mathcal{L}_0$, then $A \in \mathcal{L}$.

Definition 2.7. For some set X , set M of its subsets is called σ -algebra if

1. $\emptyset \in M$
2. $A \in M \Rightarrow X \setminus A \in M$
3. $\{A_i\}_{i=1}^{\infty} \in M \Rightarrow A = \bigcup A_i \in M$

Examples

1. 2^X for any X is σ -algebra
2. All subsets of \mathbb{R} that are countable or their complement is countable.
3. All open sets in \mathbb{R} is not σ -algebra.

Proposition 2.31. If M is σ -algebra and $\{A_k\}_{k=1}^{\infty} \subset M$, then

$$\bigcap_{k=1}^{\infty} A_k \in M$$

Proof.

$$X \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \setminus A_k) \in M$$

□

Proposition 2.32. All open and closed sets are in \mathcal{L}

Proof. Let A some open set. Then $A \cap B(0, N) \in \mathcal{L}_0$. Since \mathcal{L} is closed for complementation, also closed sets are in \mathcal{L} .

□

Proposition 2.33. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Denote $A = \bigcup_{k=1}^{\infty} A_k$. For $M \in \mathcal{L}_0$

$$\lambda(A \cap M) = \lambda\left(\bigcup_{k=1}^{\infty} (A_k \cap M)\right) \leq \sum_{k=1}^{\infty} \lambda(A_k \cap M) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Since it right for any M ,

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Proposition 2.34. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ and $A_i \cap A_j = \emptyset$ then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. For some $N \in \mathbb{N}$, choose $\{M_p \in \mathcal{L}_0\}_{p=1}^N$. Define $\mathcal{L}_0 \ni M = \bigcup_{p=1}^N M_p$.

$$\lambda(A) \geq \lambda(A \cap M) = \sum_{k=1}^{\infty} \lambda(A_k \cap M) \geq \sum_{k=1}^N \lambda(A_k \cap M) \geq \sum_{k=1}^N \lambda(A_k \cap M_k)$$

Thus

$$\lambda_A \geq \sup \left\{ \sum_{k=1}^N \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0 \right\} = \sum_{k=1}^N \sup \{ \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0 \} = \sum_{k=1}^N \lambda(A_k)$$

Since it's right for any N ,

$$\lambda_A \geq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Theorem 2.35. The defined λ fulfills properties of measure.

1. $0 \leq \lambda(A) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i)$
4. If $A = \bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Definition 2.8 (Measure). For some set X , measure of X is function μ defined on σ -algebra M of subsets of X and fulfills

1. $0 \leq \mu(A) \leq \infty$
2. $\mu(\emptyset) = 0$
3. If $A = \bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.

We denote measure space as (X, μ, M) .

Theorem 2.36. Let (X, μ, M) measure space.

1. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k A_k \subset A_{k+1}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

2. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k A_k \supset A_{k+1}$ and $\mu(A_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

Proof.

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup \left[\bigcup_{k=1}^{\infty} A_{k+1} \setminus A_k \right]$$

Since those sets are disjoint

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) = \lim_{N \rightarrow \infty} \mu(A_1) + \sum_{k=1}^N \mu(A_{k+1} \setminus A_k) = \lim_{N \rightarrow \infty} \mu\left(A_1 \cup \left[\bigcup_{k=1}^N A_{k+1} \setminus A_k \right]\right) = \lim_{N \rightarrow \infty} \mu(A_{N+1})$$

□

Proposition 2.37. If $\lambda^*(A) = 0$, $A \in \mathcal{L}$ and for any $B \subset A$, $B \in \mathcal{L}$ and $\lambda(B) = 0$.

Proof.

$$\lambda_*(A) \leq \lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}_0$$

Monotonicity of upper measure □

Theorem 2.38. A is measurable iff $\forall \epsilon > 0$ exist open G and closed F such that

$$F \subseteq A \subseteq G$$

and

$$\lambda(G \setminus F) \leq \epsilon$$

Proof. \Leftarrow :

Suppose exist such G and K . For all k choose G_k and F_k such that

$$\lambda(G_k \setminus F_k) < \frac{1}{k}$$

Denote

$$B = \bigcup_{k=1}^{\infty} F_k$$

$$\lambda^*(A \setminus B) = 0$$

and

$$A \setminus B \subseteq G_k \setminus B \subseteq G_k \setminus F_k$$

Thus

$$\lambda^*(A \setminus B) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

Thus $\lambda^*(A \setminus B) = 0$ and $A \setminus B \in \mathcal{L}$.

However $B \in \mathcal{L}$ and $A = B \cup (A \setminus B)$, thus $A \in \mathcal{L}$.

\Rightarrow :

Suppose $A \in \mathcal{L}$. Denote $E_k = B(0, k) \setminus B(0, k-1)$. This is partition of \mathbb{R}^n . $E_k \in \mathcal{L}_0$ and so is $A \cap E_k \in \mathcal{L}$. Thus for all k there is

$$K_k \subseteq A \cap E_k \subseteq G_k$$

such that $\lambda(G_k \setminus K_k) < \frac{\epsilon}{2^k}$. Denote

$$F = \bigcup_{k=1}^{\infty} K_k$$

$$G = \bigcup_{k=1}^{\infty} G_k$$

$$\lambda(G \setminus F) = \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus F)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus K_k)\right) \leq \sum_{k=1}^{\infty} \lambda(G_k \setminus K_k) < \epsilon$$

Now, F is closed. Let $F \ni x_k \rightarrow x$. The sequence converges and thus bounded, and thus exists N such that $\{x_k\} \cup \{x\} \in B(0, N)$.

Thus $\{x_k\} \subseteq \left(\bigcup_{i=1}^N E_i\right) \cap F$ and $\{x_k\} \subseteq \bigcup_{i=1}^N K_i$ and thus $\{x_k\} \cup \{x\} \in F$. □

Proposition 2.39. If A is measurable then $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Proof. If $\lambda^*(A) < \infty$ we've already seen this.

Suppose $\lambda^*(A) = \infty$. Thus $\inf \{\lambda(G) : A \subseteq G\} = \infty$. By previous theorem exists closed F such that $F \subseteq A \subseteq G$ and $\lambda(G \setminus A) \leq 1$.

$$\infty = \lambda(G) = \lambda(G \setminus A) + \lambda(A) \leq \lambda(G \setminus A) + \lambda(A) \leq 1 + \lambda(A)$$

Thus, $\lambda(A) = \infty$.

Now, take a look at $\{A \cap B(0, N)\}_N$.

$$\infty = \lambda(A) = \lambda\left(\bigcup_N (A \cap B(0, N))\right) = \lim_{N \rightarrow \infty} \lambda(A \cap B(0, N))$$

$$\infty \leftarrow \lambda(A \cap B(0, N)) = \lambda_*(A \cap B(0, N)) \leq \lambda_*(A)$$

□

Reminder We've built $E \subseteq [0, \frac{1}{2}]$ such that $q + E : q \in \mathbb{Q}$ is disjoint. And

$$\forall k \in \mathbb{N} \quad \frac{1}{k} + E \subseteq [0, 1]$$

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

Proposition 2.40. E is not measurable

Proof.

$$\begin{aligned} \bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E \right) &\subseteq [0, 1] \\ 1 = \lambda_* \left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E \right) \right) &\geq \sum_{k=2}^{\infty} \lambda_* \left(\frac{1}{k} + E \right) \end{aligned}$$

i.e., $\lambda_*(E) = 0$. On the other hand

$$\infty = \lambda^*(\mathbb{R}) = \lambda^* \left(\bigcup_{q \in \mathbb{Q}} q + E \right) \leq \sum_q \lambda^*(q + E) = \sum_q \lambda^*(E)$$

Thus $\lambda^*(E) > 0$, i.e., E is not measurable. □

Proposition 2.41. For any measurable $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$, exists non-measurable $B \subseteq A$.

Proof. We've seen that

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

thus

$$\begin{aligned} A &= \bigcup_{q \in \mathbb{Q}} A \cap (q + E) \\ 0 \leq \lambda^*(A) &= \lambda^* \left(\bigcup_{q \in \mathbb{Q}} A \cap (q + E) \right) \leq \sum_q \lambda^*(A \cap (q + E)) \end{aligned}$$

Thus exists q_0 such that $0 < \lambda^*(A \cap (q_0 + E))$, denote

$$B = A \cap (q_0 + E)$$

$$\lambda_*(B) \leq \lambda_*(q_0 + E) = \lambda_*(E) = 0$$

i.e. $B \notin \mathcal{L}$. □

Proposition 2.42. B measurable, $A \subseteq B$, then

$$\lambda^*(A) + \lambda_*(B \setminus A) = \lambda(B)$$

Proof.

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subseteq G \}$$

$$\lambda(G) + \lambda_*(B \setminus A) \geq \lambda(G) + \lambda_*(B \setminus G) = \lambda(G) + \lambda(B \setminus G) \geq \lambda(B)$$

On the other hand, for any $K \subseteq B \setminus A$

$$\lambda^*(A) + \lambda(K) \leq \lambda(B \setminus K) + \lambda(K) = \lambda(B)$$

By taking supremum on K , we get

$$\lambda^*(A) + \lambda(B \setminus A) \leq \lambda(B)$$

□

Proposition 2.43 (Carathéodory's condition).

$$A \subseteq \mathbb{R}^n \text{ measurable} \iff \forall E \subseteq \mathbb{R}^n \quad \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Proof. \Rightarrow :

Let A measurable set. Choose general E . For open $G \supset E$,

$$\lambda(G) = \lambda(G \cap A) + \lambda(G \setminus A) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Since it's right for any G , by taking infimum:

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

And by subadditivity

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

i.e.,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

\Leftarrow :

Suppose the condition is right for A . Let $M \in \mathcal{L}_0$, then

$$\lambda(M) = \lambda^*(M \cap A) + \lambda^*(M \setminus A)$$

From previous proposition

$$\lambda(M) = \lambda^*(M \cap A) + \lambda_*(M \setminus A)$$

Thus

$$\lambda_*(M \setminus A) = \lambda^*(M \setminus A)$$

and thus $M \setminus A \in \mathcal{L}_0$, i.e. $A \in \mathcal{L}$. □