

# 104165 - Real functions

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## Abstract

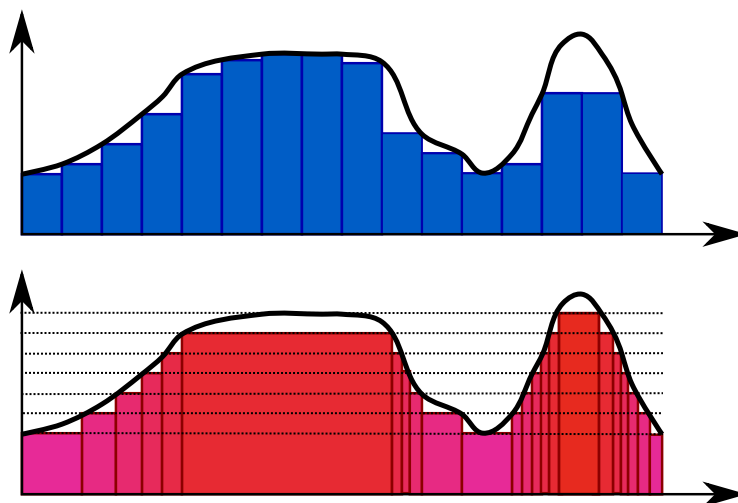
## 1 Introduction

If  $\forall x \quad f_n(x) \rightarrow f(x)$  (pointwise) does  $\int_0^1 f_n(x) dx \rightarrow \int_0^1 f(x) dx$ ?

Define  $f_n(x) = \chi_{r_1, r_2, \dots, r_n}$ , where  $\{r_i\} = \mathbb{Q} \cap [0, 1]$ , i.e., first  $n$  rational numbers. Those functions are integrable since they are non-zero in finite number of points. However,  $f(x) = \chi_{\mathbb{Q} \cap [0, 1]}$  is not integrable.

**Riemann integral: limit** We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x) dx = \lim \sum f(x'_i)(x_{i+1} - x_i)$$



By dividing on  $y$ , we bound the error by the size of each interval,  $\epsilon$ :

$$g(x) = s\chi_{A_1} + (s + \epsilon)\chi_{A_2} + \dots$$

$$\forall x \quad |g(x) - f(x)| \leq \epsilon$$

## 2 Measure

For  $A \subseteq \mathbb{R}$  we want to define size of  $A$  which we will denote  $\lambda(A)$ . What do we require from  $\lambda$ ?

1.  $\lambda([a, b]) = b - a$
2.  $0 \leq \lambda(A) \leq \infty$
3.  $\lambda(\emptyset) = 0$
4. If  $A = \bigcup_{k=1}^{\infty} A_k$  and  $\forall i, j \quad A_i \cap A_j = \emptyset$ , then  $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$ .
5.  $\lambda(A + x) = \lambda(A)$ , where  $A + x = \{s + x : s \in A\}$ .

From those properties we get additional properties:

- Additivity:

$$A = \bigcup_{i=1}^n A_i \Rightarrow \lambda(A) = \sum_{i=1}^n \lambda(A_i)$$

- If  $A \subseteq B$ , then  $\lambda(A) \leq \lambda(B)$ .

**Theorem 2.1.** Function  $\lambda$  fulfilling 1-5 and defined on every subset of  $\mathbb{R}$  doesn't exist.

*Proof.* Suppose there exists such  $\lambda$ .

Define equivalence relation  $x \sim y$  iff  $x - y \in \mathbb{Q}$ . Define  $E$  choose from each equivalence class one representative from  $[0, \frac{1}{2}]$ . Note that if  $q_1 \neq q_2$ , then  $q_1 + E \cap q_2 + E = \emptyset$ , since else  $e_1 - e_2 = q_1 - q_2$  and  $e_1 \sim e_2$ , in contradiction.

From definition  $E \subset [0, \frac{1}{2}]$ . Take a look at

$$\bigcup_{k=2}^{\infty} \left( \frac{1}{k} + E \right) \subseteq [0, 1]$$

Thus

$$\lambda \left( \bigcup_{k=2}^{\infty} \left( \frac{1}{k} + E \right) \right) \leq \lambda([0, 1]) = 1$$

On the other hand

$$\lambda \left( \bigcup_{k=2}^{\infty} \left( \frac{1}{k} + E \right) \right) = \sum_{k=2}^{\infty} \lambda \left( \frac{1}{k} + E \right) = \lambda(E)$$

Thus  $\lambda(E) = 0$ . However,

$$\mathbb{R} = \bigcup_{r \in \mathbb{Q}} r + E$$

From sigma-additivity

$$\lambda(\mathbb{R}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$

But  $\lambda(\mathbb{R}) \geq \lambda([0, 1])$ , in contradiction. □

### Requirements for measure in $\mathbb{R}$

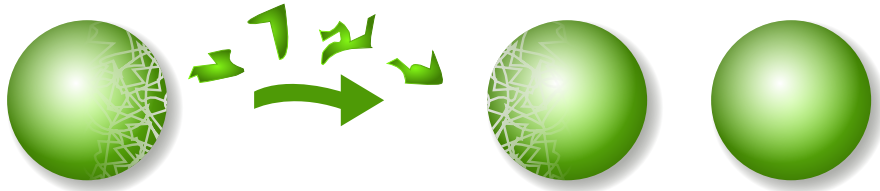
1.  $0 \leq \lambda(E) \leq \infty$
2.  $\lambda(\emptyset) = 0$
3.  $\lambda([a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i)$
4. If  $A = \bigcup_{k=1}^{\infty} A_k$ , then  $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$ .
5. If  $C$  is acquired from  $A$  by rotation or translation  $\lambda(C) = \lambda(A)$ .

**Note** In  $\mathbb{R}^3$  it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

**Banach–Tarski paradox** Denote  $B$  – unit ball in  $\mathbb{R}^3$ . We can write

$$B = \bigcup_{i=1}^5 A_i$$

and find  $C_i$  by rotation or translation of  $A_i$  such that  $\bigcup_{i=1}^5 C_i$  is two unit balls.



## 2.1 Construction of $\lambda$

**Definition 2.1** (Special boxes). Let  $E$  box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$$

For  $E$  we define

$$\lambda(E) = \prod_{i=1}^n (b_i - a_i)$$

**Definition 2.2** (Special polygons). is a finite union of special boxes.

**Note** Each special polygon is a finite union of special boxes with disjoint interior.

Let  $P$  is special polygon written as  $P = \bigcap_{i=1}^k A_i$  where  $A_i$  is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^k \lambda(A_i)$$

**Proposition 2.2.** The definition is independent on choice of  $A_i$ .

*Proof.* Let  $P = \bigcap A_i = \bigcap B_i$ .

If we continue edges of both  $A_i$  and  $B_i$  we'll get net which divides  $P$  into  $C_i$  which refines both  $A_i$  and  $B_i$  and thus

$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$

□

**Proposition 2.3.** If  $P_1, P_2$  are special polygons and  $P_1 \subseteq P_2$  then  $\lambda(P_1) \leq \lambda(P_2)$ .

*Proof.* Let  $P_2 = \bigcap A_i$  and choose the refinement which divides  $P_1$ .

□

**Proposition 2.4.** If  $P_1, P_2$  are special polygons with disjoint interior then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$

*Proof.* Find  $A_i$  which divides both  $P_1$  and  $P_2$ .

□

**Proposition 2.5.** For all  $x \in \mathbb{R}^n$

$$\lambda(x + P) = \lambda(P)$$

*Alternative proof.* For special boxes

$$\lambda(E) = \lim_{N \rightarrow \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For  $n = 1$ ,  $I = [a, b] \subseteq \mathbb{R}$ . We claim

$$b - a = \lim_{N \rightarrow \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find  $|[a, b] \cap \frac{1}{2} \mathbb{Z}|$ , we can use  $|[2a, 2b] \cap \mathbb{Z}|$ , which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any  $N$ :

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq b - a + \frac{1}{N}$$

By sandwich rule, we get the equality.

We can do the same for higher dimension and for open sets, and then we can easily proof the claim.

If  $P$  is special polygon and we take  $\lim_{N \rightarrow \infty} \frac{1}{N^n} |P \cap \frac{1}{N} \mathbb{Z}^n| = \sum \lambda(A_i)$  when  $P = \bigcap A_i$

□

### Open sets

**Definition 2.3.**  $G$  is open if  $\forall x \in G$  exists ball  $B(x, r)$  such that  $B \subset G$ . Alternatively we can replace ball with special box.

Thus for any open  $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

**Lemma 2.1.** Let  $K \subseteq \mathbb{R}^n$  compact set and  $\{G_i\}_{i \in I}$  open cover ( $K \subseteq \bigcup G_i$ ). Then exists  $\epsilon > 0$  such that  $\forall x \in K$  exists  $i \in I$  such that  $B(x, \epsilon) \subseteq G_i$ .

**Lemma 2.2.** For all polygon of dimension  $P$

$$\lambda(P) = \inf \{\lambda(G) : P \subset G\}$$

*Proof.*

$$P \subseteq G \Rightarrow \lambda(P) \leq \lambda(G)$$

Infimum would give

$$\lambda(P) \leq \inf \{\lambda(G) : P \subset G\}$$

Write  $P = \bigcup_{k=1}^N I_k$ . Then

$$\lambda(P) = \sum_{k=1}^N \lambda(I_k)$$

For  $\epsilon$  find  $I_k^\epsilon$  such that

$$\begin{cases} \text{int } I_k^\epsilon \supseteq I_k \\ \lambda(I_k^\epsilon) \leq \lambda(I_k) + \frac{\epsilon}{N} \end{cases}$$

Denote  $G = \bigcup_{k=1}^N \text{int}(I_k^\epsilon)$ , then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^N \lambda(\text{int } I_k^\epsilon) = \sum_{k=1}^N \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^N \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

□

**Proposition 2.6.**

$$0 \leq \lambda(G) \leq \infty$$

*Proof.* Obvious

□

**Proposition 2.7.**

$$\lambda(G) = 0 \iff G = \emptyset$$

*Proof.* If  $G$  is not empty, exists  $x \in G$  and special box around  $x$  such that  $P \subseteq G$  and thus  $\lambda(G) \leq \lambda(P) > 0$

□

**Proposition 2.8.**

$$\lambda(\mathbb{R}^n) = \infty$$

*Proof.* Any box is subset of  $\mathbb{R}^n$  thus  $\lambda(\mathbb{R}^n) = \infty$

□

**Proposition 2.9.**

$$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$

*Proof.* Obvious

□

**Proposition 2.10.**

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum \lambda(G_k)$$

*Proof.* Let  $P$  special polygon,  $P \subseteq \bigcup_{k=1}^{\infty} G_k$ . We'll show that it's possible to write

$$P = \bigcup_{j=1}^N I_j$$

finite union of special boxes with disjoint interior and for each  $j$  exists  $k$  such that  $I_j \subset G_k$ . Let  $\epsilon$  from lemma for  $K = P$ . Write  $P = \bigcup_{j=1}^N I_j$  such that diameter of each  $I_j < \epsilon$ . If  $x_j$  is center of  $I_j$ , then  $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$ . If this is possible, for such  $P$  denote

$$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

Obviously  $\bigcup P_k = P$  and union is finite since for some  $m$ , for every  $k > m$   $P_m = \emptyset$ , because there is finite number of  $I_j$ , and also internals of  $P_k$  are disjoint.

Thus  $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$ . This is right for any  $P$ , thus

$$\lambda\left(\bigcup (G_k)\right) = \sup \left\{ \lambda(P) | P \subseteq \bigcup (G_k) \right\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

□

**Proposition 2.11.**

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) = \sum \lambda(G_k)$$

*Proof.* Since we have sigma-subadditivity, we need only one direction of inequality:

$$\lambda(G_k) = \sup \{ \lambda(P) : P \subseteq G_k \}$$

For any  $N$

$$\sum_{k=1}^N \lambda(G_k) = \sup \left\{ \sum_{k=1}^N \lambda(P_k) : P_k \subseteq G_k \right\} = \sup \left\{ \lambda\left(\bigcup_{k=1}^N P_k\right) : P_k \subseteq G_k \right\} \leq \lambda\left(\bigcup_{k=1}^N G_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$$

i.e.,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right)$$

□

**Proposition 2.12.**

$$\lambda(P) = \lambda(\text{int } P) = \inf \{ \lambda(G) : P \subseteq G \}$$

*Proof.* First, proof that  $\lambda(P) = \lambda(\text{int } P)$ . If  $I = P$  is non-empty special box  $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ . For any  $\epsilon > 0$ ,  $I_{\epsilon} = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$ .  $I_{\epsilon} \subseteq \text{int } I$ .

That means that  $\lambda(I_{\epsilon}) \leq \lambda(\text{int } I)$ . Obviously,  $\lambda(I_{\epsilon}) \rightarrow \lambda(I)$ , i.e.  $\lambda(I) \leq \lambda(\text{int } I)$ .

Generally, for  $P = \bigcup_{k=1}^N I_k$ ,

$$\text{int } P \supseteq \bigcup_{k=1}^N \text{int } I_k$$

thus

$$\lambda(\text{int } P) \geq \lambda\left(\bigcup_{k=1}^N \text{int } I_k\right) = \sum_{k=1}^N \lambda(\text{int } I_k) \geq \sum_{k=1}^N \lambda(I_k) = \lambda(P)$$

For any  $P$

$$\lambda(\text{int } P) \geq \lambda P$$

However

$$\lambda(\text{int } P) = \sum \{ \lambda(Q) : Q \subseteq \text{int } P \}$$

$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\text{int } P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2.

□

**Proposition 2.13.**

$$\lambda(x + G) = \lambda(G)$$

*Proof.* Obvious since it's right for polygons

□

## 2.2 Compact sets

**Definition 2.4.** For compact  $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf \{ \lambda(G) : K \subseteq G \text{ } G \text{ is open} \}$$

**Proposition 2.14.**

$$0 \leq \lambda(K) < \infty$$

*Proof.* Each  $K$  is subset of open box  $A$  and  $\lambda(A) < \infty$  □

**Proposition 2.15.**

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

*Proof.* Obvious □

**Proposition 2.16.** Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

*Proof.*

$$K_i \subseteq G_i$$

$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$

$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

**Proposition 2.17.**

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

*Proof.* For  $K_1, K_2$  exists  $\epsilon > 0$  such that  $\forall x \in K_1 y \in K_2, d(x, y) \geq \epsilon$ . Denote

$$U_i = \bigcup_{x \in K_i} B\left(x, \frac{\epsilon}{2}\right) \supset K_i$$

Let  $K_1 \cup K_2 \subset G_i$ , since  $K_i \subset U_i$ ,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since  $U_1 \cap U_2 = \emptyset$  (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$

$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda((G \cap U_1) \cup (G \cap U_2)) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

## 2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subset G, \text{ open} \}$$

$$\lambda_*(A) = \sup \{ \lambda(K) : A \supset K, \text{ compact} \}$$

**Proposition 2.18.**

$$\lambda_*(A) \leq \lambda^*(A)$$

*Proof.* If  $G$  is open and  $K$  compact and  $K \subset A \subset G$  then  $K \subset G$ , i.e.  $\lambda(K) \leq \lambda(G)$ . From that, taking supremum on  $K$  and infimum on  $G$ , we get the required result.  $\square$

**Proposition 2.19.**

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

*Proof.* Obvious.  $\square$

**Proposition 2.20.**

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

*Proof.*

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subset G, \text{ open} \}$$

Thus exists  $G_k$  such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty} \left( \lambda^*(A_k) + \frac{\epsilon}{2^k} \right) = \lambda^*(A_k) + \epsilon$$

$\square$

**Proposition 2.21.** For disjoint  $A_k$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

*Proof.* For all  $i$  choose  $K_i \subseteq A_i$ . Choose some  $N$ , then

$$\bigcup_{k=1}^N K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since  $\bigcup_{k=1}^N K_k$  is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \lambda\left(\bigcup_{k=1}^N K_k\right) = \sum_{k=1}^N \lambda(K_k)$$

By taking supremum on  $K_i$ , we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_k\right) \geq \sum_{k=1}^N \lambda_*(A_k)$$

$\square$

**Proposition 2.22.** If  $A$  is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

*Proof.* If  $A$  is compact, obviously  $\lambda_*(A) = \lambda(A)$ , and  $\lambda^*(A) = \lambda(A)$  by definition.

For open  $A$ , obviously  $\lambda(A) = \lambda^*(A)$ . In addition, for any special polygon  $P \subset A$ ,  $\lambda(P) \leq \lambda_*(A)$ . However

$$\lambda^*(A) = \lambda(A) = \sup \{ \lambda(P) : P \subseteq A \} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

$\square$

Denote

$$\mathcal{L}_0 = \{A \subset \mathbb{R}^n :: \lambda^*(A) = \lambda_*(A) < \infty\}$$

All compact sets and all open set with finite measure are in  $\mathcal{L}_0$ .

**Proposition 2.23.**

$$\lambda_*(A) = \lambda_*(A + x)$$

$$\lambda^*(A) = \lambda^*(A + x)$$

**Definition 2.5.** For set in  $\mathcal{L}_0$ ,  $\lambda(A) = \lambda^*(A) = \lambda_*(A)$ .

**Lemma 2.3.** If  $A, B \in \mathcal{L}_0$  and  $A \cap B = \emptyset$  then  $A \cup B \in \mathcal{L}_0$  and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

*Proof.*

$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) = \lambda(A) + \lambda(B) = \lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) \leq \lambda^*(A \cup B)$$

□

**Theorem 2.24.**  $A \subseteq \mathbb{R}^n$  with  $\lambda^*(A) < \infty$ .  $A \in \mathcal{L}_0$  iff for all  $\epsilon > 0$  exists compact  $K$  and open  $G$ ,  $K \subseteq A \subseteq G$  and  $\lambda(G \setminus K) < \epsilon$

*Proof.*  $\Rightarrow$ :

Let  $A \in \mathcal{L}_0$ . We can find compact  $K$  and open  $G$ ,  $K \subseteq A \subseteq G$  such that

$$\lambda(G) < \lambda^*(A) + \frac{\epsilon}{2}$$

$$\lambda(K) > \lambda_*(A) - \frac{\epsilon}{2}$$

Note that, by lemma

$$\lambda(G) = \lambda(K) + \lambda(G \setminus K)$$

$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \epsilon$$

$\Leftarrow$ :

$$\lambda^*(A) \leq \lambda(G) = \lambda(K) + \lambda(G \setminus K) < \lambda(K) + \epsilon \leq \lambda_*(A) + \epsilon$$

Thus  $\lambda^*(A) = \lambda_*(A)$  and  $A \in \mathcal{L}_0$ .

□

**Collary 2.1.** If  $A, B \in \mathcal{L}_0$ , then  $A \cup B, A \cap B, A \setminus B \in \mathcal{L}_0$

*Proof.* First, show that  $A \setminus B \in \mathcal{L}_0$ . Take  $K_1 \subseteq A \subseteq G_1$  and  $K_2 \subseteq B \subseteq G_2$ .

$$\lambda(G_1 \setminus K_1) < \frac{\epsilon}{2}$$

$$\lambda(G_2 \setminus K_2) < \frac{\epsilon}{2}$$

Denote  $K = K_1 \setminus G_2$  and  $G = G_1 \setminus K_2$ .

$$K \subseteq A \setminus B \subseteq G$$

$$G \setminus K = (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$$

$$\lambda(G \setminus K) \leq \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) < \epsilon$$

Now

$$A \cup B = (A \setminus B) \cup B \in \mathcal{L}_0$$

$$A \cap B = A \setminus (A \setminus B) \in \mathcal{L}_0$$

□

**Theorem 2.25.** Let  $\{A_k\}$  set in  $\mathcal{L}_0$  and  $A = \bigcup_{k=1}^{\infty} A_k$  such that  $\lambda^*(A) < \infty$  then  $A \in \mathcal{L}_0$  and

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

In addition, if  $A_i \cap A_j = \emptyset$ ,

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$



*Proof.* Suppose  $\{A_k\}$  are disjoint.

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda_*(A_k) \leq \lambda_*(A)$$

Thus  $A \in \mathcal{L}_0$  and

$$\lambda(A) = \lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Now generally, define

$$B_1 = A_1 \in \mathcal{L}_0$$

$$B_2 = A_2 \setminus A_1$$

and so on:

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \in \mathcal{L}_0$$

Now  $\{B_k\}$  are disjoint and  $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{L}_0$ . Thus

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

**Note** Any ball  $B(0, R)$  is in  $\mathcal{L}$ , since it is inside special box large enough.

**Definition 2.6.** Let  $A \subseteq \mathbb{R}^n$ , we say  $A$  is Lebesgue measurable if  $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$ . It's measure equals

$$\lambda(A) = \sup \{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Denote a set of all such sets as  $\mathcal{L}$ .

**Proposition 2.26.** If  $\lambda^*(A) < \infty$ ,  $A \in \mathcal{L} \iff A \in \mathcal{L}_0$ . For those sets  $\lambda$  definitions are equivalent.

*Proof.* If  $A \in \mathcal{L}_0$  in, then  $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$ , thus  $A \in \mathcal{L}$ .

Now, if  $A \in \mathcal{L}$  and  $\lambda^*(A) < \infty$ . For all  $N \in \mathbb{N}$ ,

$$A \cap B(0, N) \in \mathcal{L}_0$$

However

$$A = \bigcup_{N=1}^{\infty} [A \cap B(0, N)]$$

And  $\lambda^*(A) < \infty$ , thus  $A \in \mathcal{L}_0$ .

Denote

$$\tilde{\lambda}(A) = \sup \{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Obviously,  $\tilde{\lambda}(A) \geq \lambda(A)$  (take  $M = A$ ). On the other side,

$$\forall M \in \mathcal{L}_0 \quad \lambda(A \cap M) \leq \lambda(A)$$

thus  $\tilde{\lambda}(A) = \lambda(A)$

□

**Proposition 2.27.**

$$\emptyset \in \mathcal{L}$$

*Proof.*

$$\emptyset \in \mathcal{L}_0 \Rightarrow \emptyset \in \mathcal{L}$$

□

**Proposition 2.28.**

$$A \in \mathcal{L} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{L}$$

*Proof.* Take  $M \in \mathcal{L}_0$ .

$$(\mathbb{R}^n \cap A) \cap M = M \setminus A = M \setminus (A \cap M) \in \mathcal{L}_0$$

□

**Proposition 2.29.**

$$\{A_i\}_{i=1}^{\infty} \in \mathcal{L} \Rightarrow A = \bigcup A_i \in \mathcal{L}$$

*Proof.* Take  $M \in \mathcal{L}_0$ .

$$A \cap M = \bigcup_{i=1}^{\infty} (A_i \cap M)$$

$$\lambda^*(A \cap M) \leq \lambda(M)$$

Thus

$$A \cap M \in \mathcal{L}_0$$

□

**Proposition 2.30.** If  $\forall N \in \mathbb{N}$ ,  $A \cap B(0, N) \in \mathcal{L}_0$ , then  $A \in \mathcal{L}$ .

**Definition 2.7.** For some set  $X$ , set  $M$  of its subsets is called  $\sigma$ -algebra if

1.  $\emptyset \in M$
2.  $A \in M \Rightarrow X \setminus A \in M$
3.  $\{A_i\}_{i=1}^{\infty} \in M \Rightarrow A = \bigcup A_i \in M$

**Examples**

1.  $2^X$  for any  $X$  is  $\sigma$ -algebra
2. All subsets of  $\mathbb{R}$  that are countable or their complement is countable.
3. All open sets in  $\mathbb{R}$  is not  $\sigma$ -algebra.

**Proposition 2.31.** If  $M$  is  $\sigma$ -algebra and  $\{A_k\}_{k=1}^{\infty} \subset M$ , then

$$\bigcap_{k=1}^{\infty} A_k \in M$$

*Proof.*

$$X \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \setminus A_k) \in M$$

□

**Proposition 2.32.** All open and closed sets are in  $\mathcal{L}$

*Proof.* Let  $A$  some open set. Then  $A \cap B(0, N) \in \mathcal{L}_0$ . Since  $\mathcal{L}$  is closed for complementation, also closed sets are in  $\mathcal{L}$ .

□

**Proposition 2.33.** If  $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$  then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

*Proof.* Denote  $A = \bigcup_{k=1}^{\infty} A_k$ . For  $M \in \mathcal{L}_0$

$$\lambda(A \cap M) = \lambda\left(\bigcup_{k=1}^{\infty} (A_k \cap M)\right) \leq \sum_{k=1}^{\infty} \lambda(A_k \cap M) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Since it right for any  $M$ ,

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

**Proposition 2.34.** If  $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$  and  $A_i \cap A_j = \emptyset$  then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

*Proof.* For some  $N \in \mathbb{N}$ , choose  $\{M_p \in \mathcal{L}_0\}_{p=1}^N$ . Define  $\mathcal{L}_0 \ni M = \bigcup_{p=1}^N M_p$ .

$$\lambda(A) \geq \lambda(A \cap M) = \sum_{k=1}^{\infty} \lambda(A_k \cap M) \geq \sum_{k=1}^N \lambda(A_k \cap M) \geq \sum_{k=1}^N \lambda(A_k \cap M_k)$$

Thus

$$\lambda_A \geq \sup \left\{ \sum_{k=1}^N \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0 \right\} = \sum_{k=1}^N \sup \{ \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0 \} = \sum_{k=1}^N \lambda(A_k)$$

Since it's right for any  $N$ ,

$$\lambda_A \geq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

**Theorem 2.35.** The defined  $\lambda$  fulfills properties of measure.

1.  $0 \leq \lambda(A) \leq \infty$
2.  $\lambda(\emptyset) = 0$
3.  $\lambda([a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]) = \prod_{i=1}^n (b_i - a_i)$
4. If  $A = \bigcup_{k=1}^{\infty} A_k$ , then  $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$ .
5. If  $C$  is acquired from  $A$  by rotation or translation  $\lambda(C) = \lambda(A)$ .

**Definition 2.8 (Measure).** For some set  $X$ , measure of  $X$  is function  $\mu$  defined on  $\sigma$ -algebra  $M$  of subsets of  $X$  and fulfills

1.  $0 \leq \mu(A) \leq \infty$
2.  $\mu(\emptyset) = 0$
3. If  $A = \bigcup_{k=1}^{\infty} A_k$ , then  $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$ .

We denote measure space as  $(X, \mu, M)$ .

**Theorem 2.36.** Let  $(X, \mu, M)$  measure space.

1. If  $\{A_k\}_{k=1}^{\infty} \subset M$  and  $\forall k A_k \subset A_{k+1}$ , then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

2. If  $\{A_k\}_{k=1}^{\infty} \subset M$  and  $\forall k A_k \supset A_{k+1}$  and  $\mu(A_1) < \infty$ , then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

*Proof.*

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup \left[ \bigcup_{k=1}^{\infty} A_{k+1} \setminus A_k \right]$$

Since those sets are disjoint

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) = \lim_{N \rightarrow \infty} \mu(A_1) + \sum_{k=1}^N \mu(A_{k+1} \setminus A_k) = \lim_{N \rightarrow \infty} \mu\left(A_1 \cup \left[ \bigcup_{k=1}^N A_{k+1} \setminus A_k \right]\right) = \lim_{N \rightarrow \infty} \mu(A_{N+1})$$

□

**Proposition 2.37.** If  $\lambda^*(A) = 0$ ,  $A \in \mathcal{L}$  and for any  $B \subset A$ ,  $B \in \mathcal{L}$  and  $\lambda(B) = 0$ .

*Proof.*

$$\lambda_*(A) \leq \lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}_0$$

Monotonicity of upper measure □

**Theorem 2.38.**  $A$  is measurable iff  $\forall \epsilon > 0$  exist open  $G$  and closed  $F$  such that

$$F \subseteq A \subseteq G$$

and

$$\lambda(G \setminus F) \leq \epsilon$$

*Proof.*  $\Leftarrow$ :

Suppose exist such  $G$  and  $K$ . For all  $k$  choose  $G_k$  and  $F_k$  such that

$$\lambda(G_k \setminus F_k) < \frac{1}{k}$$

Denote

$$B = \bigcup_{k=1}^{\infty} F_k$$

$$\lambda^*(A \setminus B) = 0$$

and

$$A \setminus B \subseteq G_k \setminus B \subseteq G_k \setminus F_k$$

Thus

$$\lambda^*(A \setminus B) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

Thus  $\lambda^*(A \setminus B) = 0$  and  $A \setminus B \in \mathcal{L}$ .

However  $B \in \mathcal{L}$  and  $A = B \cup (A \setminus B)$ , thus  $A \in \mathcal{L}$ .

$\Rightarrow$ :

Suppose  $A \in \mathcal{L}$ . Denote  $E_k = B(0, k) \setminus B(0, k-1)$ . This is partition of  $\mathbb{R}^n$ .  $E_k \in \mathcal{L}_0$  and so is  $A \cap E_k \in \mathcal{L}$ . Thus for all  $k$  there is

$$K_k \subseteq A \cap E_k \subseteq G_k$$

such that  $\lambda(G_k \setminus K_k) < \frac{\epsilon}{2^k}$ . Denote

$$F = \bigcup_{k=1}^{\infty} K_k$$

$$G = \bigcup_{k=1}^{\infty} G_k$$

$$\lambda(G \setminus F) = \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus F)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus K_k)\right) \leq \sum_{k=1}^{\infty} \lambda(G_k \setminus K_k) < \epsilon$$

Now,  $F$  is closed. Let  $F \ni x_k \rightarrow x$ . The sequence converges and thus bounded, and thus exists  $N$  such that  $\{x_k\} \cup \{x\} \in B(0, N)$ .

Thus  $\{x_k\} \subseteq \left(\bigcup_{i=1}^N E_i\right) \cap F$  and  $\{x_k\} \subseteq \bigcup_{i=1}^N K_i$  and thus  $\{x_k\} \cup \{x\} \in F$ . □

**Proposition 2.39.** If  $A$  is measurable then  $\lambda(A) = \lambda^*(A) = \lambda_*(A)$ .

*Proof.* If  $\lambda^*(A) < \infty$  we've already seen this.

Suppose  $\lambda^*(A) = \infty$ . Thus  $\inf \{\lambda(G) : A \subseteq G\} = \infty$ . By previous theorem exists closed  $F$  such that  $F \subseteq A \subseteq G$  and  $\lambda(G \setminus A) \leq 1$ .

$$\infty = \lambda(G) = \lambda(G \setminus A) + \lambda(A) \leq \lambda(G \setminus A) + \lambda(A) \leq 1 + \lambda(A)$$

Thus,  $\lambda(A) = \infty$ .

Now, take a look at  $\{A \cap B(0, N)\}_N$ .

$$\infty = \lambda(A) = \lambda\left(\bigcup_N (A \cap B(0, N))\right) = \lim_{N \rightarrow \infty} \lambda(A \cap B(0, N))$$

$$\infty \leftarrow \lambda(A \cap B(0, N)) = \lambda_*(A \cap B(0, N)) \leq \lambda_*(A)$$

□

**Reminder** We've built  $E \subseteq [0, \frac{1}{2}]$  such that  $q + E : q \in \mathbb{Q}$  is disjoint. And

$$\forall k \in \mathbb{N} \quad \frac{1}{k} + E \subseteq [0, 1]$$

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

**Proposition 2.40.**  $E$  is not measurable

*Proof.*

$$\begin{aligned} \bigcup_{k=2}^{\infty} \left( \frac{1}{k} + E \right) &\subseteq [0, 1] \\ 1 = \lambda_* \left( \bigcup_{k=2}^{\infty} \left( \frac{1}{k} + E \right) \right) &\geq \sum_{k=2}^{\infty} \lambda_* \left( \frac{1}{k} + E \right) \end{aligned}$$

i.e.,  $\lambda_*(E) = 0$ . On the other hand

$$\infty = \lambda^*(\mathbb{R}) = \lambda^* \left( \bigcup_{q \in \mathbb{Q}} q + E \right) \leq \sum_q \lambda^*(q + E) = \sum_q \lambda^*(E)$$

Thus  $\lambda^*(E) > 0$ , i.e.,  $E$  is not measurable. □

**Proposition 2.41.** For any measurable  $A \subseteq \mathbb{R}$  such that  $\lambda(A) > 0$ , exists non-measurable  $B \subseteq A$ .

*Proof.* We've seen that

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

thus

$$\begin{aligned} A &= \bigcup_{q \in \mathbb{Q}} A \cap (q + E) \\ 0 \leq \lambda^*(A) &= \lambda^* \left( \bigcup_{q \in \mathbb{Q}} A \cap (q + E) \right) \leq \sum_q \lambda^*(A \cap (q + E)) \end{aligned}$$

Thus exists  $q_0$  such that  $0 < \lambda^*(A \cap (q_0 + E))$ , denote

$$B = A \cap (q_0 + E)$$

$$\lambda_*(B) \leq \lambda_*(q_0 + E) = \lambda_*(E) = 0$$

i.e.  $B \notin \mathcal{L}$ . □

**Proposition 2.42.**  $B$  measurable,  $A \subseteq B$ , then

$$\lambda^*(A) + \lambda_*(B \setminus A) = \lambda(B)$$

*Proof.*

$$\lambda^*(A) = \inf \{ \lambda(G) : A \subseteq G \}$$

$$\lambda(G) + \lambda_*(B \setminus A) \geq \lambda(G) + \lambda_*(B \setminus G) = \lambda(G) + \lambda(B \setminus G) \geq \lambda(B)$$

On the other hand, for any  $K \subseteq B \setminus A$

$$\lambda^*(A) + \lambda(K) \leq \lambda(B \setminus K) + \lambda(K) = \lambda(B)$$

By taking supremum on  $K$ , we get

$$\lambda^*(A) + \lambda(B \setminus A) \leq \lambda(B)$$

□

**Proposition 2.43** (Carathéodory's condition).

$$A \subseteq \mathbb{R}^n \text{ measurable} \iff \forall E \subseteq \mathbb{R}^n \quad \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

*Proof.*  $\Rightarrow$ :

Let  $A$  measurable set. Choose general  $E$ . For open  $G \supset E$ ,

$$\lambda(G) = \lambda(G \cap A) + \lambda(G \setminus A) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Since it's right for any  $G$ , by taking infimum:

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

And by subadditivity

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

i.e.,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

$\Leftarrow$ :

Suppose the condition is right for  $A$ . Let  $M \in \mathcal{L}_0$ , then

$$\lambda(M) = \lambda^*(M \cap A) + \lambda^*(M \setminus A)$$

From previous proposition

$$\lambda(M) = \lambda^*(M \cap A) + \lambda_*(M \setminus A)$$

Thus

$$\lambda_*(M \setminus A) = \lambda^*(M \setminus A)$$

and thus  $M \setminus A \in \mathcal{L}_0$ , i.e.  $A \in \mathcal{L}$ . □

**Lemma 2.4.** Let  $A \subseteq \mathbb{R}$  with positive measure, and let  $\epsilon > 0$  then there exists an interval  $J \subseteq \mathbb{R}$   $\frac{\lambda(A \cap J)}{\lambda(J)} = 1 - \epsilon$

*Proof.* Denote  $C = \lambda(A) > 0$ .

$$\lambda(A) = \lambda^*(A) = C$$

Thus exists open  $G \supseteq A$  such that  $\lambda(G) < (1 + \frac{\epsilon}{2})C$ .

Since  $G$  is open, it is disjoint union of open intervals:

$$G = \bigcup_{i=1}^{\infty} J_i$$

$$\left(1 + \frac{\epsilon}{2}\right)C > \lambda(G) = \sum \lambda(J_i)$$

Assume that  $\forall i \lambda(A \cap J_i) \leq (1 - \epsilon)\lambda(J_i)$ . Then

$$C = \lambda(A) = \lambda\left(A \cap \left(\bigcup_{i=1}^{\infty} J_i\right)\right) = \sum_{i=1}^{\infty} \lambda(A \cap J_i) \leq (1 - \epsilon) \sum \lambda(J_i) = (1 - \epsilon)\lambda(G) = (1 - \epsilon)\left(1 + \frac{\epsilon}{2}\right)C < C$$

□

**Theorem 2.44.** Let  $A \subset \mathbb{R}$  measurable set with positive measure.  $A - A = \{x - y | x, y \in A\}$ .

*Proof.* If  $A$  has non-empty interior, the theorem is obvious. since there exists  $a \in A$ ,  $(a - \delta, a + \delta) \subset A$  and thus  $(-\delta, \delta) \subset A - A$ .

$$t \in A - A \iff A + t \cap A \neq \emptyset$$

Let  $J = (a, b)$  from previous lemma with  $\epsilon = \frac{1}{3}$ . Assume  $t \notin A - A$ , i.e.  $A \cap (A + t) = \emptyset$ . And thus

$$(A \cap J) \cap [(A + t) \cap (J + t)] = \emptyset$$

$$\lambda(A \cap J) \geq \frac{2}{3}\lambda(J)$$

$$\frac{2}{3}\lambda(J) + \frac{2}{3}\lambda(J) \leq \lambda(A \cap J) + \lambda((A + t) \cap (J + t)) = \lambda((A \cap J) \cup [(A + t) \cap (J + t)]) \leq \lambda(J \cup (J + t))$$

Now, if  $t \geq 0$ ,  $J \cup (J + t) \subseteq (a, b + t)$ , and if  $t < 0$ ,  $J \cup (J + t) \subseteq (a + t, b)$ . Anyway

$$\frac{4}{3}\lambda(J) \leq \lambda(J \cup (J + t)) \leq \lambda(J) + |t|$$

i.e.,

$$|t| \geq \frac{1}{3}\lambda(J)$$

Thus  $\forall 0 < t < \frac{1}{3}\lambda(J)$ ,  $(-t, t) \subseteq A - A$ . □

Let  $a$  set of subsets in  $\mathbb{R}^n$ . Exists  $\sigma$ -algebra that is superset of  $a$ , and also

$$\bigcup \{m : a \subset m; \sigma\text{-algebra}\}$$

is  $\sigma$ -algebra and is called  $\sigma$ -algebra generated by  $a$ .

Denote  $\mathcal{B}$   $\sigma$ -algebra generated by all open sets in  $\mathbb{R}^n$ .  $\mathcal{B}$  is Borel  $\sigma$ -algebra. Since all open sets are in  $\mathcal{L}$ ,  $\mathcal{B} \subseteq \mathcal{L}$ .

**Theorem 2.45.** Let measurable  $A \subseteq \mathbb{R}^n$ , we can write  $A = E \cup N$ , such that

1.  $E \cap N = \emptyset$
2.  $E \in \mathcal{B}$
3.  $\lambda(N) = 0$

*Proof.* For all  $k \in \mathbb{N}$ , find

$$F_k \subseteq A \subseteq G_k$$

$G_k$  open and  $F_k$  closed, and

$$\lambda(G_k \setminus F_k) \leq \frac{1}{k}$$

Denote  $E = \bigcup_{k=1}^{\infty} F_k \in \mathcal{E}$ .  $N = A \setminus E \in \mathcal{L}$ .

$$\lambda(N) = \lambda(A \setminus E) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

i.e.,  $\lambda(N) = 0$ . □

**Reminder**  $f : E \rightarrow \mathbb{R}^n$  is continuous iff  $\forall G \subseteq \mathbb{R}^n$ ,  $f^{-1}(G)$  is open in  $E$ .

**Theorem 2.46.** Let  $f : E \rightarrow \mathbb{R}^n$  be continuous for Borel set  $E \subseteq \mathbb{R}^n$ . Then  $f^{-1}(\mathcal{B}) \subseteq \mathcal{B}$ .

*Proof.* Let

$$m = \{A \subseteq \mathbb{R}^n : f^{-1}(A) \in \mathcal{B}\}$$

We need to show that  $\mathcal{B} \subseteq m$ , i.e., that  $m$  is  $\sigma$ -algebra containing all open sets.

$\emptyset \in m$ , since  $\emptyset = f^{-1}(\emptyset)$ .

If  $\{A_k\} \subseteq m$ , then  $f^{-1}(A_k) \in \mathcal{B}$  and

$$f^{-1}\left(\bigcup_k A_k\right) = \bigcup_k f^{-1}(A_k) \in \mathcal{B}$$

If  $A \in m$ , then

$$f^{-1}(\mathbb{R}^n \setminus A) = E \setminus f^{-1}(A) \in \mathcal{B}$$

Now let's show that all open sets are in  $m$ . If  $G$  is open,

$$f^{-1}(G) = E \cap U_G \in \mathcal{B}$$

□

**Theorem 2.47.** There exists measurable set in  $\mathbb{R}$  which is not Borel.

*Proof.* Define  $f : [0, 1] : \mathbb{R}$ . Let  $x$  in ternary basis  $0.a_1a_2\dots$ . Then

$$f(x) = \frac{1}{2^N} + \sum_1^{N-1} \frac{1}{2^n} \frac{a_n}{2}$$

where  $N$  is first index such that  $a_N = 1$ .

Note that  $f$  is constant on  $I \subset [0, 1]$  such that  $I \not\subset C$  (Cantor set).

$f$  is monotonous and onto, and thus continuous.

Define also  $g(x) = x + f(x)$ , which is one-to-one and onto, thus it is homeomorphism. □

Denote  $\mathcal{C}$  set of intervals in  $[0, 1] \setminus C$ . Any interval  $J \in \mathcal{C}$  exists  $r$  such that

$$g(x) = x + r$$

( $f$  is constant on  $J$ ). That means  $\lambda(g(J)) = \lambda(J)$ .

We see that

$$\lambda(G) - \lambda\left([0, 2] \setminus \bigcup_{J \in \mathcal{C}} g(J)\right) = 2 - \sum_{J \in \mathcal{C}} \lambda(g(J)) = 2 - \sum_{J \in \mathcal{C}} \lambda(J) = 2 - 1 = 1$$

Let  $B \subseteq g(C)$  which is not measurable. Denote

$$A = g^{-1}(B)$$

It is obvious that  $A \subseteq C$ , and since  $\lambda(C) = 0$ ,  $\lambda(A) = 0$ .

If  $A$  was Borel, then, since  $B = g(A)$  and  $g$  is homeomorphism, we get that  $B$  is Borel. However, this is impossible, since  $B$  is non-measurable.

### 3 Measurable functions and integrals

We want to define integral as the sum of possible values of function times the size of set for which function gets this values:

$$\int f \sim \sum f(t_i \in A_i) \times \lambda(A_i)$$

where

$$A_i = \{x : f(x) \in [a, a + \epsilon]\}$$

Let  $X$  space with  $\sigma$ -algebra  $M$ . We work with functions

$$f : X \rightarrow [-\infty, \infty]$$

**Definition 3.1.** We say  $f$  is  $M$ -measurable if for all  $-\infty \leq t \leq \infty$

$$f^{-1}(-\infty, t) \in M$$

**Proposition 3.1.** The following conditions are equivalent:

1.  $f$  is  $M$ -measurable:

$$\forall -\infty < t \leq \infty \quad f^{-1}([-\infty, t]) \in M$$

2.

$$\forall -\infty < t \leq \infty \quad f^{-1}([-\infty, t)) \in M$$

3.

$$\forall -\infty \leq t \leq \infty \quad f^{-1}([t, \infty]) \in M$$

4.

$$\forall -\infty \leq t < \infty \quad f^{-1}((t, \infty]) \in M$$

5.  $f^{-1}(\infty) \in M$ ,  $f^{-1}(-\infty) \in M$ , and  $\forall E \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(E) \in M$

6.  $f^{-1}(\infty) \in M$ ,  $f^{-1}(-\infty) \in M$ , and  $\forall a, b \in \mathbb{R} \quad f^{-1}([a, b]) \in M$



*Proof.*  $1 \Rightarrow 2$ :

$$f^{-1}([-\infty, t)) = \bigcup_{\mathbb{Q} \ni r < t} f^{-1}([-\infty, r])$$

thus  $f^{-1}([-\infty, t)) \in M$ .

$2 \Rightarrow 3$ :

If  $t = -\infty$ ,  $f^{-1}([-\infty, \infty]) = X \in M$ . Otherwise

$$f^{-1}([t, \infty]) = X \setminus (f^{-1}([-\infty, t)))$$

thus  $f^{-1}([t, \infty]) \in M$ .

$3 \Rightarrow 4$ : just like  $1 \Rightarrow 2$

$4 \Rightarrow 1$ : just like  $2 \Rightarrow 3$

$1 - 4 \Rightarrow 5$ :

Taking  $t = \pm\infty$ , we get  $f^{-1}(\infty)$  and  $f^{-1}(-\infty)$ .

Let

$$S = \{E \subset \mathbb{R} \mid f^{-1}(E) \in M\}$$

$S$  is  $\sigma$ -algebra.

$$f^{-1}\left((a, b)\right) = f^{-1}((a, \infty]) \cap f^{-1}([-\infty, b)) \in M$$

Thus open intervals are in  $\mathbb{R}$ , and thus open sets and thus  $\mathcal{B} \subset S$ .

$5 \Rightarrow 6$ : Obvious, since 5 is stronger

$6 \Rightarrow 1$ : Lest as an exercise

□