104165 - Real functions

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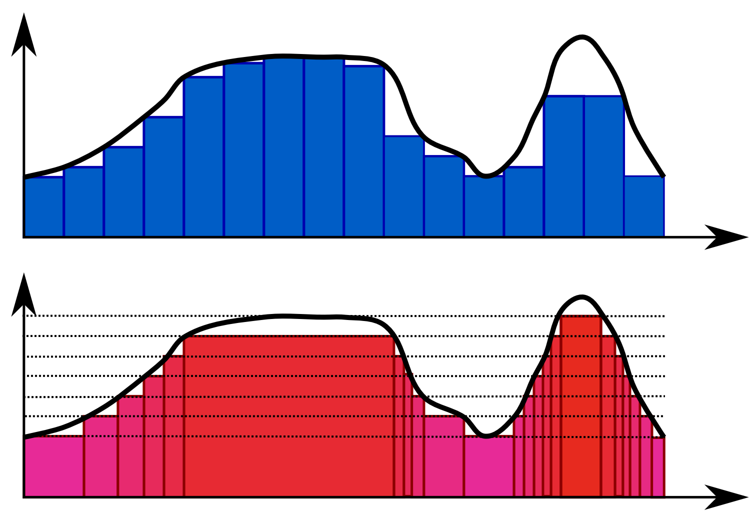
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1, r_2, \ldots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \ldots$$
$$\forall x \quad |g(x) - f(x)| \le \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \le \lambda(A) \le \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:
$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$
- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem 2.1. Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

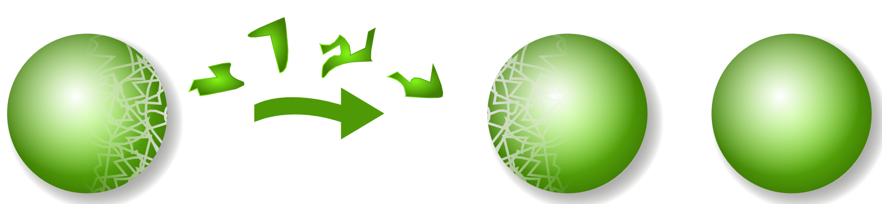
Proof. Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at
$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0, 1]$$
Thus
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0,1]) = 1$$
On the other hand
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$
Thus $\lambda(E) = 0$. However,
$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$
From sigma-additivity
$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$
But $\lambda(\mathbb{R}) \geq \lambda([0,1])$, in contradiction. □

Reqirements for measure in $\mathbb{R}$

1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$
4. If $A = \dot{\bigcup}_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In $\mathbb{R}^3$ it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach–Tarski paradox Denote B – unit ball in $\mathbb{R}^3$. We can write
$$B = \dot{\bigcup_{i=1}^{5}} A_i$$
and find C_i by rotation or translation of A_i such that $\dot{\bigcup}_{i=1}^{5} C_i$ is two unit balls.



2.1 Construction of λ

Definition 2.1 (Special boxes)**.** Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^{n} (b_i - a_i)$$

Definition 2.2 (Special polygons)**.** is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.
Let P is special polygon written as $P = \bigcap_{i=1}^{k} A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^{k} \lambda(A_i)$$

Proposition 2.2. The definition is independent on choice of A_i.

Proof. Let $P = \bigcap A_i = \bigcap B_i$.
If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus

$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$

□

Proposition 2.3. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.

Proof. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1. □

Proposition 2.4. If P_1, P_2 are special polygons with disjoint interior then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$

Proof. Find A_i which divides both P_1 and P_2. □

Proposition 2.5. For all $x \in \mathbb{R}^n$

$$\lambda(x + P) = \lambda(P)$$

Alternative proof. For special boxes

$$\lambda(E) = \lim_{N \to \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim

$$b - a = \lim_{N \to \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find $\left|[a, b] \cap \frac{1}{2}\mathbb{Z}\right|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any N:

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq b - a + \frac{1}{N}$$

By sandwich rule, we get the equality.
We can do the same for higher dimension and for open sets, and then we can easily proof the claim.
If P is special polygon and we take $\lim_{N \to \infty} \frac{1}{N^n} \left| P \cap \frac{1}{N} \mathbb{Z}^n \right| = \sum \lambda(A_i)$ when $P = \bigcap A_i$

□

Open sets

Definition 2.3. G is open if $\forall x \in G$ exists ball $B(x,r)$ such that $B \subset G$. Alternatively we can replace ball with special box. Thus for any open $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

Lemma 2.1. Let $K \subseteq \mathbb{R}^n$ compact set and $\{G_i\}_{i \in I}$ open cover ($K \subseteq \bigcup G_i$). Then exists $\epsilon > 0$ such that $\forall x \in K$ exists $i \in I$ such that $B(x,\epsilon) \subseteq G_i$.

Lemma 2.2. For all polygon of dimension P

$$\lambda(P) = \inf \{\lambda(G) : P \subset G\}$$

Proof.

$$P \subseteq G \Rightarrow \lambda(P) \leq \lambda(G)$$

Infimum would give

$$\lambda(P) \leq \inf \{\lambda(G) : P \subset G\}$$

Write $P = \bigcup_{k=1}^{N} I_k$. Then

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

For ϵ find I_k^ϵ such that

$$\begin{cases} \text{int } I_k^\epsilon \supseteq I_k \\ \lambda(I_k^\epsilon) \leq \lambda(I_k) + \frac{\epsilon}{N} \end{cases}$$

Denote $G = \bigcup_{k=1}^{N} \text{int}(I_k^\epsilon)$, then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^{N} \lambda(\text{int } I_k^\epsilon) = \sum_{k=1}^{N} \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^{N} \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

□

Proposition 2.6.

$$0 \leq \lambda(G) \leq \infty$$

Proof. Obvious □

Proposition 2.7.

$$\lambda(G) = 0 \iff G = \emptyset$$

Proof. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$and thus $\lambda(G) \leq \lambda(P) > 0$ □

Proposition 2.8.

$$\lambda(\mathbb{R}^n) = \infty$$

Proof. Any box is subset of $\mathbb{R}^n$ thus $\lambda(\mathbb{R}^n) = \infty$ □

Proposition 2.9.

$$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$

Proof. Obvious □

Proposition 2.10.

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum \lambda(G_k)$$

Proof. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

$$P = \bigcup_{j=1}^{N} I_j$$

finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^{N} = I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j, then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$.
If this is possible, for such P denote

$$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

Obviously $\bigcup P_k = P$ and union is finite since for some m, for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j, and also internals of P_k are disjoint.
Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P, thus

$$\lambda\Big(\bigcup(G_k)\Big) = \sup\Big\{\lambda(P) | P \subseteq \bigcup(G_k)\Big\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

□

Proposition 2.11.

$$\lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right) = \sum \lambda(G_k)$$

Proof. Since we have sigma-subadditivity, we need only one direction of inequality:

$$\lambda(G_k) = \sup\{\lambda(P) : P \subseteq G_k\}$$

For any N

$$\sum_{k=1}^{N} \lambda(G_k) = \sup\left\{\sum_{k=1}^{N} \lambda(P_k) : P_k \subseteq G_k\right\} = \sup\left\{\lambda\left(\dot{\bigcup_{k=1}^{N}} P_k\right) : P_k \subseteq G_k\right\} \leq \lambda\left(\dot{\bigcup_{k=1}^{N}} G_k\right) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

i.e.,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

□

Proposition 2.12.

$$\lambda(P) = \lambda(\operatorname{int} P) = \inf\{\lambda(G) : P \subseteq G\}$$

Proof. First, proof that $\lambda(P) = \lambda(\operatorname{int} P)$. If $I = P$ is non-empty special box $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. For any $\epsilon > 0$, $I_\epsilon = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$. $I_\epsilon \subseteq \operatorname{int} I$.
That means that $\lambda(I_\epsilon) \leq \lambda(\operatorname{int} I)$. Obviously, $\lambda(I_\epsilon) \to \lambda(I)$, i.e. $\lambda(I) \leq \lambda(\operatorname{int} I)$.
Generally, for $P = \bigcup_{k=1}^{N} I_k$,

$$\operatorname{int} P \geq \bigcup_{k=1}^{N} \operatorname{int} I_k$$

thus

$$\lambda(\operatorname{int} P) \geq \lambda\left(\bigcup_{k=1}^{N} \operatorname{int} I_k\right) = \sum_{k=1}^{N} \lambda(\operatorname{int} I_k) \geq \sum_{k=1}^{N} \lambda(I_k) = \lambda(P)$$

For any P

$$\lambda(\operatorname{int} P) \geq \lambda P$$

However

$$\lambda(\operatorname{int} P) = \sum\{\lambda(Q) : Q \subseteq \operatorname{int} P\}$$

$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\operatorname{int} P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2. □

Proposition 2.13.

$$\lambda(x + G) = \lambda(G)$$

Proof. Obvious since it's right for polygons □

2.2 Compact sets

Definition 2.4. For compact $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf \{\lambda(G) : K \subseteq G \quad G \text{ is open}\}$$

Proposition 2.14.

$$0 \leq \lambda(K) < \infty$$

Proof. Each K is subset of open box A and $\lambda(A) < \infty$ □

Proposition 2.15.

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

Proof. Obvious □

Proposition 2.16. Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

Proof.

$$K_i \subseteq G_i$$

$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$

$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

Proposition 2.17.

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

Proof. For K_1, K_2 exists $\epsilon > 0$ such that $\forall\, x \in K_1\ y \in K_2$, $d(x, y) \geq \epsilon$. Denote

$$U_i = \bigcup_{x \in K_i} B\Big(x, \frac{\epsilon}{2}\Big) \supset K_i$$

Let $K_1 \cup K_2 \subset G_i$, since $K_i \subset U_i$,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since $U_1 \cap U_2 = \emptyset$ (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$

$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda\big((G \cap U_1) \cap (G \cap U_2)\big) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

$$\lambda_*(A) = \sup\{\lambda(K) : A \supset G, \text{ compact}\}$$

Proposition 2.18.

$$\lambda_*(A) \leq \lambda^*(A)$$

Proof. If G is open and K compact and $K \subset A \subset G$ then $K \subset G$, i.e. $\lambda(K) \leq \lambda(G)$. From that, taking supremum on K and infimum on G, we get the required result. □

Proposition 2.19.

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

Proof. Obvious. □

Proposition 2.20.

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof.

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

Thus exists G_k such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty}\left(\lambda^*(A_k) + \frac{\epsilon}{2^k}\right) = \lambda^*(A_k) + \epsilon$$

□

Proposition 2.21. For disjoint A_k

$$\lambda^*\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. For all i choose $K_i \subseteq A_i$. Choose some N, then

$$\bigcup_{k=1}^{N} K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since $\bigcup_{k=1}^{N} K_k$ is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \lambda\left(\bigcup_{k=1}^{N} K_k\right) = \sum_{k=1}^{N} \lambda(K_k)$$

By taking supremum on K_i, we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \sum_{k=1}^{N} \lambda_*(A_n)$$

□

Proposition 2.22. If A is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Proof. If A is compact, obviously $\lambda_*(A) = \lambda(A)$, and $\lambda^*(A) = \lambda(A)$ by definition.
For open A, obviously $\lambda(A) = \lambda^*(A)$. In addition, for any special polygon $P \subset A$, $\lambda(P) \leq \lambda_*(A)$. However

$$\lambda^*(A) = \lambda(A) = \sup\{\lambda(P) : P \subseteq A\} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

□

Denote
$$\mathcal{L}_0 = \{A \subset \mathbb{R}^n :: \lambda^*)A_= \lambda_*(A) < \infty\}$$
All compact sets and all open set with finite measure are in $\mathcal{L}_0$.

Proposition 2.23.
$$\lambda_*(A) = \lambda_*(A + x)$$
$$\lambda^*(A) = \lambda^*(A + x)$$

Definition 2.5. For set in $\mathcal{L}_0$, $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Lemma 2.3. If $A, B \in \mathcal{L}_0$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{L}_0$ and
$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

Proof.
$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) = \lambda(A) + \lambda(B) == \lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) \leq \lambda^*(A \cup B)$$

□

Theorem 2.24. $A \subseteq \mathbb{R}^n$ with $\lambda^*(A) < \infty$. $A \in \mathcal{L}_0$ iff for all $\epsilon > 0$ exists compact K and open G, $K \subseteq A \subseteq G$ and $\lambda(G \setminus K) < \epsilon$

Proof. $\Rightarrow$:
Let $A \in \mathcal{L}_0$. We can find compact K and open G, $K \subseteq A \subseteq G$ such that
$$\lambda(G) < \lambda^*(A) + \frac{\epsilon}{2}$$
$$\lambda(K) > \lambda_*(A) - \frac{\epsilon}{2}$$
Note that, by lemma
$$\lambda(G) = \lambda(K) + \lambda(G \setminus K)$$
$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \epsilon$$

$\Leftarrow$:
$$\lambda^*(A) \leq \lambda(G) = \lambda(K) + \lambda(G \setminus K) < \lambda(K) + \epsilon \leq \lambda_*(A) + \epsilon$$
Thus $\lambda^*(A) = \lambda_*(A)$ and $A \in \mathcal{L}_0$. □

Collary 2.1. If $A, B \in \mathcal{L}_0$, then $A \cup B, A \cap B, A \setminus B \in \mathcal{L}_0$

Proof. First, show that $A \setminus B \in \mathcal{L}_0$. Take $K_1 \subseteq A \subseteq G_1$ and $K_2 \subseteq A \subseteq G_2$.
$$\lambda(G_1 \setminus K_1) < \frac{\epsilon}{2}$$
$$\lambda(G_2 \setminus K_2) < \frac{\epsilon}{2}$$
Denote $K = K_1 \setminus G_2$ and $G = G_1 \setminus K_2$.
$$K \subseteq A \setminus B \setminus G$$
$$G \setminus K = (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$$
$$\lambda(G \setminus K) \leq \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) < \epsilon$$
Now
$$A \cup B = (A \setminus B) \cup B \in \mathcal{L}_0$$
$$A \cap B = A \setminus (A \setminus B) \in \mathcal{L}_0$$

□

Theorem 2.25. Let $\{A_k\}$ set in $\mathcal{L}_0$ and $A \bigcup_{k=1}^{\infty} A_k$ such that $\lambda^*(A) < \infty$ then $A \in \mathcal{L}_0$ and
$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$
In addition, if $A_i \cap A_j = \emptyset$,
$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Suppose $\{A_k\}$ are disjoint.

$$\lambda^*(A) \le \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda_*(A_k) \le \lambda_*(A)$$

Thus $A \in \mathcal{L}_0$ and

$$\lambda(A) = \lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Now generally, define

$$B_1 = A_1 \in \mathcal{L}_0$$

$$B_2 = A_2 \setminus A_1$$

and so on:

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \in \mathcal{L}_0$$

Now $\{B_k\}$ are disjoint and $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{L}_0$. Thus

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \le \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Note Any ball $B(0,R)$ is in $\mathcal{L}_{\prime}$, since it is inside special box large enough.

Definition 2.6. Let $A \subseteq \mathbb{R}^n$, we say A is Lebesgue measurable if $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$. It's measure equals

$$\lambda(A) = \sup\{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Denote a set of all such sets as $\mathcal{L}$.

Proposition 2.26. If $\lambda^*(A) < \infty$, $A \in \mathcal{L} \iff A \in \mathcal{L}_0$. For those sets λ definitions are equivalent.

Proof. If $A \in \mathcal{L}_0$ in, then $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$, thus $A \in \mathcal{L}$.
Now, if $A \in \mathcal{L}$ and $\lambda^*(A) < \infty$. For all $N \in \mathbb{N}$,

$$A \cap B(0,N) \in \mathcal{L}_0$$

However

$$A = \bigcup_{N=1}^{\infty} [A \cap B(0,N)]$$

And $\lambda^*(A) < \infty$, thus $A \in \mathcal{L}_0$.
Denote

$$\tilde{\lambda}(A) = \sup\{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Obviously, $\tilde{\lambda}(A) \ge \lambda(A)$ (take $M = A$). On the other side,

$$\forall M \in \mathcal{L}_0 \quad \lambda(A \cap M) \le \lambda(A)$$

thus $\tilde{\lambda}(A) = \lambda(A)$

□

Proposition 2.27.

$$\emptyset \in \mathcal{L}$$

Proof.

$$\emptyset \in \mathcal{L}_0 \Rightarrow \emptyset \in \mathcal{L}$$

□

Proposition 2.28.

$$A \in \mathcal{L} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$(\mathbb{R}^n \cap A) \cap M = M \setminus A = M \setminus (A \cap M) \in \mathcal{L}_0$$

□

Proposition 2.29.

$$\{A_i\}_{i=1}^{\infty} \in \mathcal{L} \Rightarrow A = \bigcup A_i \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$A \cap M = \bigcup_{i=1}^{\infty} (A_k \cap M)$$

$$\lambda^*(A \cap M) \leq \lambda(M)$$

Thus

$$A \cap M \in\in \mathcal{L}_0$$

□

Proposition 2.30. If $\forall N \in \mathbb{N}$, $A \cap B(0, N) \in \mathcal{L}_0$, then $A \in \mathcal{L}$.

Definition 2.7. For some set X, set M of its subsets is called σ-algebra if

1. $\emptyset \in M$
2. $A \in M \Rightarrow X \setminus A \in M$
3. $\{A_i\}_{i=1}^{\infty} \in M \Rightarrow A = \bigcup A_i \in M$

Examples

1. 2^X for any X is σ-algebra
2. All subsets of $\mathbb{R}$ that are countable or their complement is countable.
3. All open sets in $\mathbb{R}$ is not σ-algebra.

Proposition 2.31. If M is σ-algebra and $\{A_k\}_{k=1}^{\infty} \subset M$, then

$$\bigcap_{k=1}^{\infty} A_k \in M$$

Proof.

$$X \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \setminus A_k) \in M$$

□

Proposition 2.32. All open and closed sets are in $\mathcal{L}$

Proof. Let A some open set. Then $A \cap B(0, N) \in \mathcal{L}_0$. Since $\mathcal{L}$ is closed for complementation, also closed sets are in $\mathcal{L}$. □

Proposition 2.33. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Denote $A = \bigcup_{k=1}^{\infty} A_k$. For $M \in \mathcal{L}_0$

$$\lambda(A \cap M) = \lambda\left(\bigcup_{k=1}^{\infty} (A_k \cap M)\right) \leq \sum_{k=1}^{\infty} \lambda(A_k \cap M) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Since it right for any M,

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Proposition 2.34. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ and $A_i \cap A_j = 0$ then

$$\lambda\left(\bigcup_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. For some $N \in \mathbb{N}$, choose $\{M_p \in \mathcal{L}_0\}_{p=1}^{N}$. Define $\mathcal{L}_0 \ni M = \bigcup_{p=1}^{N} M_p$.

$$\lambda(A) \geq \lambda(A \cap M) = \sum_{k=1}^{\infty} \lambda(A_k \cap M) \geq \sum_{k=1}^{N} \lambda(A_k \cap M) \geq \sum_{k=1}^{N} \lambda(A_k \cap M_k)$$

Thus

$$\lambda_A \geq \sup\left\{\sum_{k=1}^{N} \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0\right\} = \sum_{k=1}^{N} \sup\left\{\lambda(A_k \cap M_k), M_k \in \mathcal{L}_0\right\} = \sum_{k=1}^{N} \lambda(A_k)$$

Since it's right for any N,

$$\lambda_A \geq \sum_{k=1}^{\infty} \lambda(A_k)$$

$\square$

Theorem 2.35. The defined λ fulfills properties of measure.

1. $0 \leq \lambda(A) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$
4. If $A = \dot{\bigcup}_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Definition 2.8 (Measure). For some set X, measure of X is function μ defined on σ-algebra M of subsets of X and fulfills

1. $0 \leq \mu(A) \leq \infty$
2. $\mu(\emptyset) = 0$
3. If $A = \dot{\bigcup}_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.

We denote measure space as (X, μ, M).

Theorem 2.36. Let (X, μ, M) measure space.

1. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k\ A_k \subset A_{k+1}$, then
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$$
2. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k\ A_k \supset A_{k+1}$ and $\mu(A_1) < \infty$, then
$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$$

Proof.

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup \left[\bigcup_{k=1}^{\infty} A_{k+1} \setminus A_k\right]$$

Since those sets are disjoint

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) = \lim_{N\to\infty} \mu(A_1) + \sum_{k=1}^{N} \mu(A_{k+1} \setminus A_k) = \lim_{N\to\infty} \mu\left(A_1 \cup \left[\bigcup_{k=1}^{N} A_{k+1} \setminus A_k\right]\right) = \lim_{N\to\infty} \mu(A_{N+1})$$

$\square$

Proposition 2.37. If $\lambda^*(A) = 0$, $A \in \mathcal{L}$ and for any $B \subset A$, $B \in \mathcal{L}$ and $\lambda(B) = 0$.

Proof.

$$\lambda_*(A) \leq \lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}_0$$

Monotonity of upper measure □

Theorem 2.38. A is measurable iff $\forall \epsilon > 0$ exist open G and closed F such that

$$F \subseteq A \subseteq G$$

and

$$\lambda(G \setminus F) \leq \epsilon$$

Proof. $\Leftarrow$:
Suppose exist such G and K. For all k choose G_k and F_k such that

$$\lambda(G_k \setminus F_k) < \frac{1}{k}$$

Denote

$$B = \bigcup_{k=1}^{\infty} F_k$$

$$\lambda^*(A \setminus B) = 0$$

and

$$A \setminus B \subseteq G_k \setminus B \subseteq G_k \setminus F_k$$

Thus

$$\lambda^*(A \setminus B) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

Thus $\lambda^*(A \setminus B) = 0$ and $A \setminus B \in \mathcal{L}$.
However $B \in \mathcal{L}$ and $A = B \cup (A \setminus B)$, thus $A \in \mathcal{L}$.
$\Rightarrow$:
Suppose $A \in \mathcal{L}$. Denote $E_k = B(0,k) \setminus B(0,k-1)$. This is partition of $\mathbb{R}^n$. $E_k \in \mathcal{L}_0$ and so is $A \cap E_k \in \mathcal{L}$. Thus for all k there is

$$K_k \subseteq A \cap E_k \subseteq G_k$$

such that $\lambda(G_k \setminus K_k) < \frac{\epsilon}{2^k}$. Denote

$$F = \bigcup_{k=1}^{\infty} K_k$$

$$G = \bigcup_{k=1}^{\infty} G_k$$

$$\lambda(G \setminus F) = \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus F)\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus K_k)\right) \leq \sum_{k=1}^{\infty} \lambda(G_k \setminus K_k) < \epsilon$$

Now, F is closed. Let $F \ni x_k \to x$. The sequence converges and thus bounded, and thus exists N such that $\{x_k\} \cup \{x\} \in B(0,N)$. Thus $\{x_k\} \subseteq \left(\bigcup_{i=1}^{N} E_i\right) \cap F$ and $\{x_k\} \subseteq \bigcup_{i=1}^{N} K_i$ and thus $\{x_k\} \cup \{x\} \in F$.

□

Proposition 2.39. If A is measurable then $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Proof. If $\lambda^*(A) < \infty$ we've already seen this.
Suppose $\lambda^(A) = \infty$. Thus $\inf \{\lambda(G) : A \subseteq G\} = \infty$. By previous theorem exists closed F such that $F \subseteq A \subseteq G$ and $\lambda(G \setminus A) \leq 1$.

$$\infty = \lambda(G) = \lambda(G \setminus A) + \lambda(A) \leq å, bda(G \setminus F) + \lambda(A) \leq 1 + \lambda(A)$$

Thus, $\lambda(A) = \infty$.
Now, take a look at $\{A \cap B(0,N)\}_N$.

$$\infty = \lambda(A) = \lambda\left(\bigcup_N (A \cap B(0,N))\right) = \lim_{N \to \infty} \lambda(A \cap B(0,N))$$

$$\infty \leftarrow \lambda(A \cap B(0,N)) = \lambda_*(A \cap B(0,N)) \leq \lambda_*(A)$$

□

Reminder We've built $E \subseteq \left[0, \frac{1}{2}\right]$ such that $q + E : q \in \mathbb{Q}$ is disjoint. And

$$\forall k \in \mathbb{N}\ \frac{1}{k} + E \subseteq [0, 1]$$

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

Proposition 2.40. E is not measurable

Proof.

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0, 1]$$

$$1 = \lambda_* \left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \geq \sum_{k=2}^{\infty} \lambda_* \left(\frac{1}{k} + E\right)$$

i.e., $\lambda_*(E) = 0$. On the other hand

$$\infty = \lambda^*(\mathbb{R}) = \lambda^* \left(\bigcup_{q \in \mathbb{Q}} q + E\right) \leq \sum_q \lambda^*(q + E) = \sum_q \lambda^*(E)$$

Thus $\lambda^*(E) > 0$, i.e., E is not measurable. □

Proposition 2.41. For any measurable $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$, exists non-measurable $B \subseteq A$.

Proof. We've seen that

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

thus

$$A = \bigcup_{q \in \mathbb{Q}} A \cap (q + E)$$

$$0 \leq \lambda^*(A) = \lambda^* \left(\bigcup_{q \in \mathbb{Q}} A \cap (q + E)\right) \leq \sum_q \lambda^*(A \cap (q + E))$$

Thus exists q_0 such that $0 < \lambda^*(A \cap (q + E))$, denote

$$B = A \cap (q_0 + E)$$

$$\lambda_*(B) \leq \lambda_*(q_0 + E) = \lambda_*(E) = 0$$

i.e. $B \notin \mathcal{L}$.

□

Proposition 2.42. B measurable, $A \subseteq B$, then

$$\lambda^*(A) + \lambda_*(B \setminus A) = \lambda(B)$$

Proof.

$$\lambda^*(A) = \inf \{\lambda(G) : A \cap G\}$$

$$\lambda(G) + \lambda_*(B \setminus A) \geq \lambda(G) + \lambda_*(B \setminus B) = \lambda(G) + \lambda(B \setminus G) \geq \lambda(B)$$

On the other hand, for any $K \subseteq B \setminus A$

$$\lambda^*(A) + \lambda(K) \leq \lambda(B \setminus K) + \lambda(K) = \lambda(B)$$

By taking supremum on K, we get

$$\lambda^*(A) + \lambda(B \setminus A) \leq \lambda(B)$$

□

Proposition 2.43 (Carathéodory's condition)**.**

$$A \subseteq \mathbb{R}^n \text{ measurable} \iff \forall E \subseteq \mathbb{R}^n \quad \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Proof. $\Rightarrow$:
Let A measurable set. Choose general E. For open $G \supset E$,

$$\lambda(G) = \lambda(G \cap A) + \lambda(G \setminus A) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Since it's right for any G, by taking infimum:

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

And by subadditivity

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

i.e.,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

$\Leftarrow$:
Suppose the condition is right for A. Let $M \in \mathcal{L}_0$, then

$$\lambda(M) = \lambda^*(M \cap A) + \lambda^*(M \setminus A)$$

From previous proposition

$$\lambda(M) = \lambda^*(M \cap A) + \lambda_*(M \setminus A)$$

Thus

$$\lambda_*(M \setminus A) = \lambda^*(M \setminus A)$$

and thus $M \setminus A \in \mathcal{L}_0$, i.e. $A \in \mathcal{L}$. $\square$

Lemma 2.4. Let $A \subseteq \mathbb{R}$ with positive measure, and let $\epsilon > 0$ then there exists an interval $J \subseteq \mathbb{R}$ $\frac{\lambda(A \cap J)}{\lambda(J)} = 1 - \epsilon$

Proof. Denote $C = \lambda(A) > 0$.

$$\lambda(A) = \lambda^*(A) = C$$

Thus exists open $G \supseteq A$ such that $\lambda(G) < \left(1 + \frac{\epsilon}{2}\right)C$.
Since G is open, it is disjoint union of open intervals:

$$G = \bigcup_{i=1}^{\infty} J_i$$

$$\left(1 + \frac{\epsilon}{2}\right)C > \lambda(G) = \sum \lambda(J_i)$$

Assume that $\forall i \; \lambda(A \cap J) leq (1 - \epsilon)\lambda(J)$. Then

$$C = \lambda(A) = \lambda\left(A \cap \left(\bigcup_{i=1}^{\infty} J_i\right)\right) = \sum_{i=1}^{\infty} \lambda(A \cap J_i) \leq (1 - \epsilon) \sum \lambda_J = (1 - \epsilon)\lambda(G) = (1 - \epsilon)\left(1 + \frac{\epsilon}{2}\right)C < C$$

$\square$

Theorem 2.44. Let $A \subset \mathbb{R}$ measurable set with positive measure. $A - A = \{x - y | x, y \in A\}$.

Proof. If A has non-empty interior, the theorem is obvious. since there exists $a \in A$, $(a - \delta, a + \delta) \subset A$ and thus $(-\delta, \delta) \subset A - A$.

$$t \in A - A \iff A + t \cap A \neq \emptyset$$

Let $J = (a, b)$ from previous lemma with $\epsilon = \frac{1}{3}$. Assume $t \notin A - A$, i.e. $A \cap (A + t) = \emptyset$. And thus

$$(A \cap J) \cap [(A + t) \cap (J + t)] = \emptyset$$

$$\lambda(A \cap J) \geq \frac{2}{3}\lambda(J)$$

$$\frac{2}{3}\lambda(J) + \frac{2}{3}\lambda(J) \leq \lambda(A \cap J) + \lambda\big((A + t) \cap (J + t)\big) = \lambda\big((A \cap J) \cup \big[(A + t) \cap (J + t)\big]\big) \leq \lambda(J \cup (J + t))$$

Now, if $t \geq 0$, $J \cup (J+t) \subseteq (a, b+t)$, and if $t < 0$, $J \cup (J+t) \subseteq (a+t, b)$. Anyway

$$\frac{4}{3}\lambda(J) \leq \lambda(J \cup (J+t)) \leq \lambda(J) + |t|$$

i.e.,

$$|t| \geq \frac{1}{3}\lambda(J)$$

Thus $\forall\, 0 < t < \frac{1}{3}\lambda(J)$, $(-t, t) \subseteq A - A$. □

Let a set of subsets in $\mathbb{R}^n$. Exists σ-algebra that is superset of a, and also

$$\bigcup \{m : \quad a \subset m|; \sigma\text{-algebra}\}$$

is σ-algebra and is called σ-algebra generated by a.
Denote $\mathcal{B}$ σ-algebra generated by all open sets in $\mathbb{R}^n$. $\mathcal{B}$ is Borel σ-algebra. Since all open sets are in $\mathcal{L}$, $\mathcal{B} \subseteq \mathcal{L}$.

Theorem 2.45. Let measurable $A \subseteq \mathbb{R}^n$, we can write $A = E \cup N$, such that

1. $E \cap N = 0$
2. $E \in \mathcal{B}$
3. $\lambda(N) = 0$

Proof. For all $k \in \mathbb{N}$, find

$$F_k \subseteq A \subseteq G_k$$

G_k open and F_k closed, and

$$\lambda(G_k \setminus F_k) \leq \frac{1}{k}$$

Denote $E = \bigcup_{k=1}^{\infty} F_k \in \mathcal{E}$. $N = A \setminus E \in \mathcal{L}$.

$$\lambda(N) = \lambda(A \setminus E) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

i.e., $\lambda(N) = 0$. □

Reminder $\quad f : E \to \mathbb{R}^n$ is continuous iff $\forall \quad G \subseteq \mathbb{R}^n$, $f^{-1}(G)$ is open in E.

Theorem 2.46. Let $f : E \to \mathbb{R}^n$ be continuous for Borel set $E \subseteq \mathbb{R}^n$. Then $f^{-1}(\mathcal{B}) \subseteq \mathcal{B}$.

Proof. Let

$$m = \left\{A \subseteq \mathbb{R}^n : f^{-1}(A) \in \mathcal{B}\right\}$$

We need to show that $\mathcal{B} \subseteq m$, i.e., that m is σ-algebra containing all open sets.
$\emptyset \in m$, since $\emptyset = f^{-1}(\emptyset)$.
If $\{A_k\} \subseteq m$, then $f^{-1}(A_k) \in \mathcal{B}$ and

$$f^{-1}\left(\bigcup_k A_k\right) = \bigcup_k f^{-1}(A_k) \in \mathcal{B}$$

If $A \in m$, then

$$f^{-1}(\mathbb{R}^n \setminus A) = E \setminus f^{-1}(A) \in \mathcal{B}$$

Now lets show that all open sets are in m. If G is open,

$$f^{-1}(G) = E \cap U_G \in \mathcal{B}$$

□

Theorem 2.47. There exists measurable set in $\mathbb{R}$ which is not Borel.

Proof. Define $f : [0,1] : \mathbb{R}$. Let x in ternary basis $0.a_1a_2\dots$. Then

$$f(x) = \frac{1}{2^N} + \sum_{1}^{N-1} \frac{1}{2^n}\frac{a_n}{2}$$

where N is first index such that $a_N = 1$.
Note that f is constant on $I \subset [0,1]$ such that $I \not\subset C$ (Cantor set).
f is monotonous and onto, and thus continuous.
Define also $g(x) = x + f(x)$, which is one-to-one and onto, thus it is homeomorphism. □

Denote $\mathcal{C}$ set of intervals in $[0,1] \setminus C$. Any interval $J \in \mathcal{C}$ exists r such that

$$g(x) = x + r$$

(f is constant on J). That means $\lambda(g(J)) = \lambda(J)$.
We see that

$$\lambda(G) - \lambda\left([0,2] \setminus \bigcup_{J\in\mathcal{C}} g(J)\right) = 2 - \sum_{J\in\mathcal{C}} \lambda(g(J)) = 2 - \sum_{J\in\mathcal{C}} \lambda(J) = 2 - 1 = 1$$

Let $B \subseteq g(C)$ which is not measurable. Denote

$$A = g^{-1}(B)$$

It is obvious that $A \subseteq C$, and since $\lambda(C) = 0$, $\lambda(A) = 0$.
If A was Borel, then, since $B = g(A)$ and g is homeomorphism, we get that B is Borel. However, this is impossible, since B is non-measurable.

3 Measurable functions and integrals

We want to define integral as the sum of possible values of function times the size of set for which function gets this values:

$$\int f \sim \sum f(t_i \in A_i) \times \lambda(A_i)$$

where

$$A_i = \{x : f(x) \in [a, a+\epsilon]\}$$

Let X space with σ-algebra M. We work with functions

$$f : X \to [-\infty, \infty]$$

Definition 3.1. We say f is M-measurable if for all $-\infty \le t \le \infty$

$$f^{-1}(-\infty, t) \in M$$

Proposition 3.1. The following conditions are equivalent:

1. f is M-measurable:
$$\forall\ -\infty < t \le \infty \quad f^{-1}([-\infty, t]) \in M$$
2. $$\forall\ -\infty < t \le \infty \quad f^{-1}([-\infty, t)) \in M$$
3. $$\forall\ -\infty \le t \le \infty \quad f^{-1}([t, \infty]) \in M$$
4. $$\forall\ -\infty \le t < \infty \quad f^{-1}((t, \infty]) \in M$$
5. $f^{-1}(\infty) \in M$, $f^{-1}(-\infty) \in M$, and $\forall E \in \mathcal{B}(\mathbb{R})\ f^{-1}(E) \in M$
6. $f^{-1}(\infty) \in M$, $f^{-1}(-\infty) \in M$, and $\forall a, b \in \mathbb{R}\ f^{-1}([a, b]) \in M$

Proof. $1 \Rightarrow 2$:

$$f^{-1}([-\infty, t)) = \bigcup_{\mathbb{Q} \ni r < t} f^{-1}([-\infty, r])$$

thus $f^{-1}([-\infty, t)) \in M$.
$2 \Rightarrow 3$:
If $t = -\infty$, $f^{-1}([-\infty, \infty]) = X \in M$. Otherwise

$$f^{-1}([t, \infty]) = X \setminus \left(f^{-1}([-\infty, t))\right)$$

thus $f^{-1}([t, \infty]) \in M$.
$3 \Rightarrow 4$: just like $1 \Rightarrow 2$
$4 \Rightarrow 1$: just like $2 \Rightarrow 3$
$1 - 4 \Rightarrow 5$:
Taking $t = \pm\infty$, we get $f^{-1}(\infty)$ and $f^{-1}(-\infty)$.
Let

$$S = \left\{E \subset \mathbb{R} | f^{-1}(E) \in M\right\}$$

S is σ-algebra.

$$f^{-1}\Big((a, b)\Big) = f^{-1}((a, \infty]) \cap f^{-1}([-\infty, b)) \in M$$

Thus open intervals are in $\mathbb{R}$, and thus open sets and thus $\mathcal{B} \subset S$.
$5 \Rightarrow 6$: Obvious, since 5 is stronger
$6 \Rightarrow 1$: Left as an exercise ◻

Collary 3.1. If $f : E \to [-\infty, \infty]$ and $E \in M$, then the definition is conserved.

Collary 3.2. $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ is measurable iff $A \in M$.

Proof. $A = \chi_A^{-1}(\{1\})$, thus one direction is obvious.
Else,

$$\chi_A^{-1}(A) = \begin{cases} X & 0, 1 \in E \\ A & 1 \in E, 0 \notin E \\ X \setminus A & 1 \notin E, 0 \in E \\ \emptyset & 0, 1 \notin E \end{cases}$$

◻

Collary 3.3. $f : E \to \mathbb{R}$, for Borel set $E \subset \mathbb{R}$. If f is continuous then f Borel-measurable and Lebesgue-measurable.

Theorem 3.2. Let $f : X \to \mathbb{R}$ M-measurable functions.
If $\phi : B \to \mathbb{R}$ for Borel set $B \subseteq \mathbb{R}$ and $f(x) \subseteq B$ and ϕ Borel-measurable, then $\phi \circ f$ are M-measurable.

Proof. We need to show

$$f^{-1}(\phi^{-1}(E)) = (\phi \circ f)^{-1}(E) \in M$$

Now, $\phi^1(E) \in \mathcal{B}$, since ϕ is Borel-measurable. Then $f^{-1}(\phi^{-1}(E)) \in M$. ◻

Collary 3.4. If f is non-zero, $\frac{1}{f}$ is measurable.

Collary 3.5. If $0 < p < \infty$, $|f|^p$ is measurable.

Proposition 3.3. If f is weaker (for example, Lebesgue-measurable), the theorem is not true, even if ϕ is homeomorphism. For example, we've seen g and non measurable $g(A)$ for measurable A. Then

$$\chi_A \circ \phi = \chi_{g(A)}$$

which is non-measurable.

Theorem 3.4. Let $f, g : X \to \mathbb{R}$ M-measurable functions. Then $f + g$, cf, $f \cdot g$ are M-measurable.

Proof.
$$(f+g)^{-1}(-\infty,t)\bigcup_{r\in\mathbb{Q}}\left[f^{-1}(-\infty,r)\cap g^{-1}(-\infty,t-r)\right]$$
That means measurable functions are vector space.
$$f\cdot g=\frac{1}{4}(f+g)^2-\frac{1}{4}(f-g)^2$$
□

Theorem 3.5. Let $\{f_k\}_{k=1}^{\infty}:X\to[-\infty,\infty]$ sequence of M-measurable functions. Then also $\liminf f_k$, $\limsup f_k$, $\sup f_k$, $\inf f_k$ and so is $\lim f_k$ if exists.

Proof.
$$(\sup f_k)^{-1}([-\infty,t])=\{x:\sup f_k(x)\le t\}=\bigcap\{x:f_k(x)\le t\}=\bigcap f_k^{-1}([-\infty,t])\in M$$
$$\limsup f_k(x)=\inf_n\left(\sup_n j\le nf_j(x)\right)$$
□

Definition 3.2 (Simple function)**.** $f:X\to[-\infty,\infty]$ is called simple function if it acquires only finite number of values. If we denote those values as $\{a_i\}_{i=1}^{n}$ and $A_k=\{x:f(x)=a_k\}$. Then we can rewrite function as
$$f(x)=\sum_{k=1}^{n}a_k\chi_{A_k}(x)$$
In fact, all functions that can be written as
$$f(x)=\sum_{k=1}^{m}b_K\chi_{B_k}(x)$$
is simple. If $\{B_k\}$ ar disjoint and b_k are not equal, this is called canonical representation.

Proposition 3.6. f is measurable iff $\forall k\ A_k\in M$

Proof. χ_A measurable $\Rightarrow f$ is measurable.
$A_k=\{x:f(x)=a_k\}$ is measurable. □

Theorem 3.7. $f:X\to[-\infty,\infty]$. f m-measurable if there is sequence $\{s_k\}$ of measurable simple functions such that $\forall x\, s_k(x)\to f(x)$. We can choose s_k such that $|s_{k-1}|\le|s_k|$.

Proof. $\Leftarrow$ obvious.
$\Rightarrow$:
Suppose $f\ge 0$. Define
$$s_k(x)=\begin{cases}k & f(x)\ge k\\ \frac{i-1}{2^k} & \frac{i-1}{2^k}\le f(x)<\frac{i}{2^k}\end{cases}$$
We can rewrite as
$$s_k(x)=k\chi_{A_kf^{-1}(k,\infty)}+\sum_{i=1}^{k\cdot 2^k}\frac{i-1}{2^k}\chi_{f^{-1}\left[\frac{i-1}{2^k},\frac{i}{2^k}\right]}$$
which is canonical form, and we conclude s_k are measurable.
Obviously, $s_k\le s_{k+1}$.
If $f(x)=\infty$, $s_k=k\to\infty=f(x)$.
Else, $\exists k_0>f(x)$, and then
$$s_k(x)\le f(x)\le s_k(x)+\frac{1}{2^k}$$
i.e., $s_k(x)\to f(x)$.
In general case we define $f_+=\max\{f(x),0\}$ and $f_-=\max\{-f(x),0\}$. Note that $f=f_+-f_-$ and $f_-\cdot f_+=0$. Both f_-,f_+ are measurable. For $f_\pm$ exist sequences $\{s_k'\}$, $\{s_k''\}$, we can define $s_k=s_k'-s_k''$.
For any x either $s_k'(x)$ or $s_k''(x)$ is 0, thus in any point $s_k=s_k'$ or $s_k=-s_k''$, i.e., $|s_{k-1}|\le|s_k|$. □

Definition 3.3. If some property is fulfilled for all x except, maybe, some set A which is subset of set of measure 0, we say that property is fulfilled almost everywhere (a.e.). In probability we say the property fulfilled almost surely (a.s.).

Theorem 3.8. Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be Lebesgue-measurable function. Then $\exists g(x)$, Borel-measurable function, such that $\lambda(\{x : f(x) \neq g(x)\}) = 0$, i.e. $f(x) = g(x)$ a.e.

Proof. Suppose $f \geq 0$. Let $\{s_k\}$ as in previous theorem and thus $f = \sup s_k$.

$$s_k = \sum_{j=1}^{m} a_j \chi_{A_j}$$

Since $A_j \in \mathcal{L}$ we can rewrite it as $A_j = E_j \cup N_j$.
Define

$$h_k = \sum_{j=1}^{m} a_j \chi_{E_j} \leq s_k$$

Since $h_k = s_k$ except for $\bigcup N_j$, which is of measure 0, $h_k = s_k$ a.e.
Denote $N = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} N_j$, obviously $\lambda(N) = 0$. Also define $g = \sup\limits_k h_k$.
$g(x) = f(x)$ if $x \notin N$, i.e., a.e. and $g(x)$ is Borel-measurable as supremum of Borel-measurable functions.
For general f, we do same with $f_\pm$ and acquire $g_\pm$. □

Lemma 3.1. If f is Lebesgue measurable, then if $g : \mathbb{R}^n \to [-\infty, \infty]$ fulfilling

$$\lambda^* \{x : f(x) \neq g(x)\} = 0$$

then g is measurable.

Proof. Let $-\infty \leq t \leq \infty$, we need to show that $B = g^{-1}([-\infty, t])$ is Lebesgue-measurable.
We now that $A = f^{-1}[-\infty, t]$ is Lebesgue-measurable.

$$B \setminus A \subseteq \{x : f(x) \neq g(x)\}$$

Thus $B \setminus A$ is measurable with measure 0.

$$B = (A \cup B) \setminus (A \setminus B) \in \mathcal{L}$$

□

Theorem 3.9 (Tietze extension theorem)**.** Let $Y \subseteq \mathbb{R}^n$ and $f : Y \to \mathbb{R}$ continuous and bounded ($|f| \leq M$). Exists continuous function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on Y and $|F| \leq M$.

Theorem 3.10 (Lusin's theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ which vanishes outside of measurable set A. The for all $\epsilon > 0$ exists closed set $E \subseteq A$ and continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f = g$ on E and $\lambda(A \setminus E) < \epsilon$.

Proof. Let f be a simple function in canonical form:

$$f(x) = \sum_{j=1}^{m} a_j \chi_{A_j}(x)$$

and $A = \bigcup_{j=1}^{m} A_i$. A_i are measurable, thus exists closed set $F_i \subseteq A_i \subseteq A$ such that $\lambda(A_i \setminus F_i) < \frac{\epsilon}{m}$.
Denote $E = \bigcup_{i=1}^{m} F_i$.

$$\lambda(A \setminus E) = \sum_i \lambda(A_i \setminus F_i) <\leq \epsilon$$

Define f_0 on E such that $f_0\big|_{F_i} = a_i$. f is continuous and thus by 3.9 exists g as required.
Now, let f be measurable and bounded. Let $\epsilon > 0$. We know there exists $\{s_k\}_{k=1}^{\infty}$ such that $s_k \to f$ uniformly.
For all k exists continuous g_k and L_k such that $\lambda(A \setminus L_k) < \frac{\epsilon}{2^k}$ and $g_k = s_k$ on L_k. Denote $E = \bigcap L_k$.

$$\lambda(E \setminus E) = \lambda\Big(A \setminus \bigcap L_k\Big) = \lambda\Big(\bigcup (A \setminus L_k)\Big) \leq \sum \lambda(A \setminus L_k) < \epsilon$$

On E, g_k converges uniformly to f, thus f is continuous of E and from 3.9 we get what we wanted.
Let f measurable function which vanishes outside of measurable set A such that $\lambda(A) < \infty$.

$$\bigcap_N \underbrace{\{x \in A : |f(x)| \geq N\}}_{A_N} = \emptyset$$

$A_N \subset A_{N+1}$, measurable and $\lambda(A) < \infty$, then $0 = \lambda(\bigcap_N A_N) = \lim_{N\to\infty} \lambda(A_N)$.

Thus exists N_0 such that $\lambda(A_{N_0}) < \frac{\epsilon}{2}$. Denote

$$G = \{x \in A : |f(x)| < N_0\}$$

$$A_{N_0} = \{x \in A : |f(x)| \geq N_0\}$$

Then $\lambda(A \setminus G) = \lambda(A_{N_0}) < \frac{\epsilon}{2}$.
Then $\chi_G f : G \to \mathbb{R}$ is bounded and measurable, i.e., exists closed $E \subseteq G$ such that $\chi_G f$ is continuous on E and $\lambda(G \setminus E) < \frac{\epsilon}{2}$.
Since $\lambda(A \setminus E) < \epsilon$, once again we use 3.9 and get what we wanted.
Denote $A_k = A \cap (B(0,k) \setminus B(0,k-1))$. Define $f_k = f\big|_{A_k}$ then exists closed $E_k \subseteq A_k$ such that $f_k\big|_{E_k}$ such that $\lambda(A_k \setminus E_k) < \frac{\epsilon}{2^k}$.
$E = \bigcup E_k$ is closed and $f\big|_E$ is continuous and $\lambda(A \setminus E) < \epsilon$. □