104165 - Real functions

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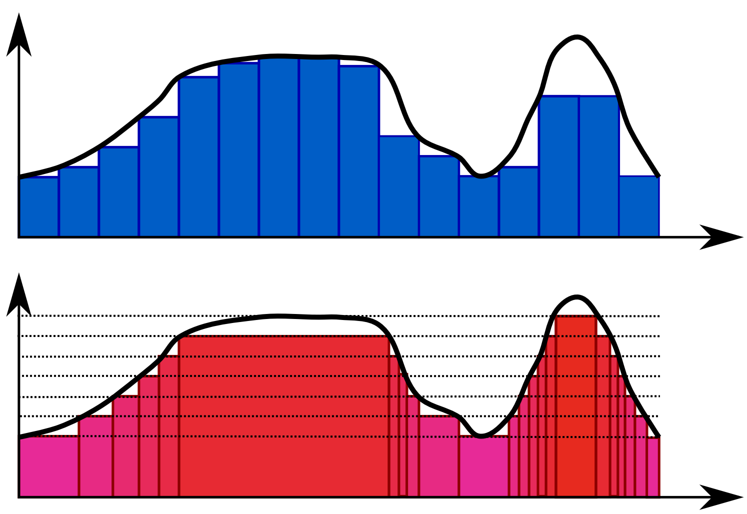
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1, r_2, \ldots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \ldots$$

$$\forall x \quad |g(x) - f(x)| \le \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \le \lambda(A) \le \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:
$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$
- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem 2.1. Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

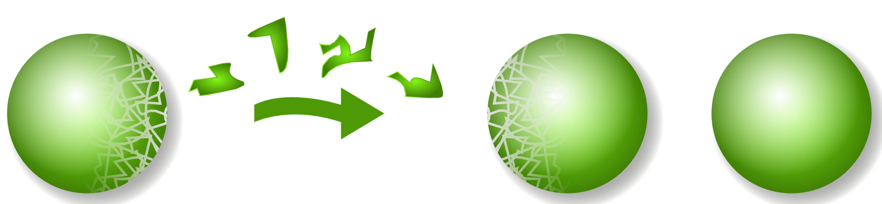
Proof. Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at
$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0, 1]$$
Thus
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0, 1]) = 1$$
On the other hand
$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$
Thus $\lambda(E) = 0$. However,
$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$
From sigma-additivity
$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$
But $\lambda(\mathbb{R}) \geq \lambda([0, 1])$, in contradiction. □

Reqirements for measure in $\mathbb{R}$

1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$
4. If $A = \dot\bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In $\mathbb{R}^3$ it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach–Tarski paradox Denote B – unit ball in $\mathbb{R}^3$. We can write
$$B = \dot{\bigcup_{i=1}^{5}} A_i$$
and find C_i by rotation or translation of A_i such that $\dot\bigcup_{i=1}^{5} C_i$ is two unit balls.



2.1 Construction of λ

Definition 2.1 (Special boxes)**.** Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^{n} (b_i - a_i)$$

Definition 2.2 (Special polygons)**.** is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.
Let P is special polygon written as $P = \bigcap_{i=1}^{k} A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^{k} \lambda(A_i)$$

Proposition 2.2. The definition is independent on choice of A_i.

Proof. Let $P = \bigcap A_i = \bigcap B_i$.
If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus

$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$

□

Proposition 2.3. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.

Proof. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1. □

Proposition 2.4. If P_1, P_2 are special polygons with disjoint interior then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$

Proof. Find A_i which divides both P_1 and P_2. □

Proposition 2.5. For all $x \in \mathbb{R}^n$

$$\lambda(x + P) = \lambda(P)$$

Alternative proof. For special boxes

$$\lambda(E) = \lim_{N \to \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim

$$b - a = \lim_{N \to \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find $\left|[a, b] \cap \frac{1}{2}\mathbb{Z}\right|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any N:

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq b - a + \frac{1}{N}$$

By sandwich rule, we get the equality.
We can do the same for higher dimension and for open sets, and then we can easily proof the claim.
If P is special polygon and we take $\lim_{N \to \infty} \frac{1}{N^n} \left| P \cap \frac{1}{N} \mathbb{Z}^n \right| = \sum \lambda(A_i)$ when $P = \bigcap A_i$

□

Open sets

Definition 2.3. G is open if $\forall x \in G$ exists ball $B(x,r)$ such that $B \subset G$. Alternatively we can replace ball with special box. Thus for any open $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

Lemma 2.1. Let $K \subseteq \mathbb{R}^n$ compact set and $\{G_i\}_{i \in I}$ open cover ($K \subseteq \bigcup G_i$). Then exists $\epsilon > 0$ such that $\forall x \in K$ exists $i \in I$ such that $B(x, \epsilon) \subseteq G_i$.

Lemma 2.2. For all polygon of dimension P

$$\lambda(P) = \inf \{\lambda(G) : P \subset G\}$$

Proof.

$$P \subseteq G \Rightarrow \lambda(P) \leq \lambda(G)$$

Infimum would give

$$\lambda(P) \leq \inf \{\lambda(G) : P \subset G\}$$

Write $P = \bigcup_{k=1}^{N} I_k$. Then

$$\lambda(P) = \sum_{k=1}^{N} \lambda(I_k)$$

For ϵ find I_k^ϵ such that

$$\begin{cases} \text{int } I_k^\epsilon \supseteq I_k \\ \lambda(I_k^\epsilon) \leq \lambda(I_k) + \frac{\epsilon}{N} \end{cases}$$

Denote $G = \bigcup_{k=1}^{N} \text{int}(I_k^\epsilon)$, then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^{N} \lambda(\text{int } I_k^\epsilon) = \sum_{k=1}^{N} \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^{N} \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

□

Proposition 2.6.

$$0 \leq \lambda(G) \leq \infty$$

Proof. Obvious □

Proposition 2.7.

$$\lambda(G) = 0 \iff G = \emptyset$$

Proof. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$and thus $\lambda(G) \leq \lambda(P) > 0$ □

Proposition 2.8.

$$\lambda(\mathbb{R}^n) = \infty$$

Proof. Any box is subset of $\mathbb{R}^n$ thus $\lambda(\mathbb{R}^n) = \infty$ □

Proposition 2.9.

$$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$

Proof. Obvious □

Proposition 2.10.

$$\lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum \lambda(G_k)$$

Proof. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

$$P = \bigcup_{j=1}^{N} I_j$$

finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^{N} = I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j, then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$.
If this is possible, for such P denote

$$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

Obviously $\bigcup P_k = P$ and union is finite since for some m, for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j, and also internals of P_k are disjoint.
Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P, thus

$$\lambda\Big(\bigcup (G_k)\Big) = \sup \Big\{\lambda(P) | P \subseteq \bigcup (G_k)\Big\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

□

Proposition 2.11.

$$\lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right) = \sum \lambda(G_k)$$

Proof. Since we have sigma-subadditivity, we need only one direction of inequality:

$$\lambda(G_k) = \sup\{\lambda(P) : P \subseteq G_k\}$$

For any N

$$\sum_{k=1}^{N} \lambda(G_k) = \sup\left\{\sum_{k=1}^{N} \lambda(P_k) : P_k \subseteq G_k\right\} = \sup\left\{\lambda\left(\dot{\bigcup_{k=1}^{N}} P_k\right) : P_k \subseteq G_k\right\} \leq \lambda\left(\dot{\bigcup_{k=1}^{N}} G_k\right) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

i.e.,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

□

Proposition 2.12.

$$\lambda(P) = \lambda(\operatorname{int} P) = \inf\{\lambda(G) : P \subseteq G\}$$

Proof. First, proof that $\lambda(P) = \lambda(\operatorname{int} P)$. If $I = P$ is non-empty special box $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. For any $\epsilon > 0$, $I_\epsilon = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$. $I_\epsilon \subseteq \operatorname{int} I$.
That means that $\lambda(I_\epsilon) \leq \lambda(\operatorname{int} I)$. Obviously, $\lambda(I_\epsilon) \to \lambda(I)$, i.e. $\lambda(I) \leq \lambda(\operatorname{int} I)$.
Generally, for $P = \bigcup_{k=1}^{N} I_k$,

$$\operatorname{int} P \geq \bigcup_{k=1}^{N} \operatorname{int} I_k$$

thus

$$\lambda(\operatorname{int} P) \geq \lambda\left(\bigcup_{k=1}^{N} \operatorname{int} I_k\right) = \sum_{k=1}^{N} \lambda(\operatorname{int} I_k) \geq \sum_{k=1}^{N} \lambda(I_k) = \lambda(P)$$

For any P

$$\lambda(\operatorname{int} P) \geq \lambda P$$

However

$$\lambda(\operatorname{int} P) = \sum \{\lambda(Q) : Q \subseteq \operatorname{int} P\}$$

$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\operatorname{int} P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2. □

Proposition 2.13.

$$\lambda(x + G) = \lambda(G)$$

Proof. Obvious since it's right for polygons □

2.2 Compact sets

Definition 2.4. For compact $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf\{\lambda(G) : K \subseteq G \quad G \text{ is open}\}$$

Proposition 2.14.

$$0 \leq \lambda(K) < \infty$$

Proof. Each K is subset of open box A and $\lambda(A) < \infty$ □

Proposition 2.15.

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

Proof. Obvious □

Proposition 2.16. Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

Proof.

$$K_i \subseteq G_i$$

$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$

$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

Proposition 2.17.

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

Proof. For K_1, K_2 exists $\epsilon > 0$ such that $\forall\, x \in K_1\; y \in K_2$, $d(x, y) \geq \epsilon$. Denote

$$U_i = \bigcup_{x \in K_i} B\Big(x, \frac{\epsilon}{2}\Big) \supset K_i$$

Let $K_1 \cup K_2 \subset G_i$, since $K_i \subset U_i$,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since $U_1 \cap U_2 = \emptyset$ (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$

$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda\big((G \cap U_1) \cap (G \cap U_2)\big) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

$$\lambda_*(A) = \sup\{\lambda(K) : A \supset G, \text{ compact}\}$$

Proposition 2.18.

$$\lambda_*(A) \leq \lambda^*(A)$$

Proof. If G is open and K compact and $K \subset A \subset G$ then $K \subset G$, i.e. $\lambda(K) \leq \lambda(G)$. From that, taking supremum on K and infimum on G, we get the required result. □

Proposition 2.19.

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

Proof. Obvious. □

Proposition 2.20.

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof.

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

Thus exists G_k such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty}\left(\lambda^*(A_k) + \frac{\epsilon}{2^k}\right) = \lambda^*(A_k) + \epsilon$$

□

Proposition 2.21. For disjoint A_k

$$\lambda^*\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. For all i choose $K_i \subseteq A_i$. Choose some N, then

$$\bigcup_{k=1}^{N} K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since $\bigcup_{k=1}^{N} K_k$ is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \lambda\left(\bigcup_{k=1}^{N} K_k\right) = \sum_{k=1}^{N} \lambda(K_k)$$

By taking supremum on K_i, we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \sum_{k=1}^{N} \lambda_*(A_n)$$

□

Proposition 2.22. If A is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Proof. If A is compact, obviously $\lambda_*(A) = \lambda(A)$, and $\lambda^*(A) = \lambda(A)$ by definition.
For open A, obviously $\lambda(A) = \lambda^*(A)$. In addition, for any special polygon $P \subset A$, $\lambda(P) \leq \lambda_*(A)$. However

$$\lambda^*(A) = \lambda(A) = \sup\{\lambda(P) : P \subseteq A\} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

□

Denote

$$\mathcal{L}_0 = \{A \subset \mathbb{R}^n :: \lambda^*)A_{=}\lambda_*(A) < \infty\}$$

All compact sets and all open set with finite measure are in $\mathcal{L}_0$.

Proposition 2.23.

$$\lambda_*(A) = \lambda_*(A + x)$$
$$\lambda^*(A) = \lambda^*(A + x)$$

Definition 2.5. For set in $\mathcal{L}_0$, $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Lemma 2.3. If $A, B \in \mathcal{L}_0$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{L}_0$ and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

Proof.

$$\lambda^*(A \cup B) \leq \lambda^*(A) + \lambda^*(B) = \lambda(A) + \lambda(B) == \lambda_*(A) + \lambda_*(B) \leq \lambda_*(A \cup B) \leq \lambda^*(A \cup B)$$

□

Theorem 2.24. $A \subseteq \mathbb{R}^n$ with $\lambda^*(A) < \infty$. $A \in \mathcal{L}_0$ iff for all $\epsilon > 0$ exists compact K and open G, $K \subseteq A \subseteq G$ and $\lambda(G \setminus K) < \epsilon$

Proof. $\Rightarrow$:
Let $A \in \mathcal{L}_0$. We can find compact K and open G, $K \subseteq A \subseteq G$ such that

$$\lambda(G) < \lambda^*(A) + \frac{\epsilon}{2}$$

$$\lambda(K) > \lambda_*(A) - \frac{\epsilon}{2}$$

Note that, by lemma

$$\lambda(G) = \lambda(K) + \lambda(G \setminus K)$$
$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \epsilon$$

$\Leftarrow$:

$$\lambda^*(A) \leq \lambda(G) = \lambda(K) + \lambda(G \setminus K) < \lambda(K) + \epsilon \leq \lambda_*(A) + \epsilon$$

Thus $\lambda^*(A) = \lambda_*(A)$ and $A \in \mathcal{L}_0$. □

Collary 2.1. If $A, B \in \mathcal{L}_0$, then $A \cup B, A \cap B, A \setminus B \in \mathcal{L}_0$

Proof. First, show that $A \setminus B \in \mathcal{L}_0$. Take $K_1 \subseteq A \subseteq G_1$ and $K_2 \subseteq A \subseteq G_2$.

$$\lambda(G_1 \setminus K_1) < \frac{\epsilon}{2}$$

$$\lambda(G_2 \setminus K_2) < \frac{\epsilon}{2}$$

Denote $K = K_1 \setminus G_2$ and $G = G_1 \setminus K_2$.

$$K \subseteq A \setminus B \setminus G$$
$$G \setminus K = (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$$
$$\lambda(G \setminus K) \leq \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) < \epsilon$$

Now

$$A \cup B = (A \setminus B) \cup B \in \mathcal{L}_0$$
$$A \cap B = A \setminus (A \setminus B) \in \mathcal{L}_0$$

□

Theorem 2.25. Let $\{A_k\}$ set in $\mathcal{L}_0$ and $A \bigcup_{k=1}^{\infty} A_k$ such that $\lambda^*(A) < \infty$ then $A \in \mathcal{L}_0$ and

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

In addition, if $A_i \cap A_j = \emptyset$,

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Suppose $\{A_k\}$ are disjoint.

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda_*(A_k) \leq \lambda_*(A)$$

Thus $A \in \mathcal{L}_0$ and

$$\lambda(A) = \lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Now generally, define

$$B_1 = A_1 \in \mathcal{L}_0$$

$$B_2 = A_2 \setminus A_1$$

and so on:

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \in \mathcal{L}_0$$

Now $\{B_k\}$ are disjoint and $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{L}_0$. Thus

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□