104165 - Real functions

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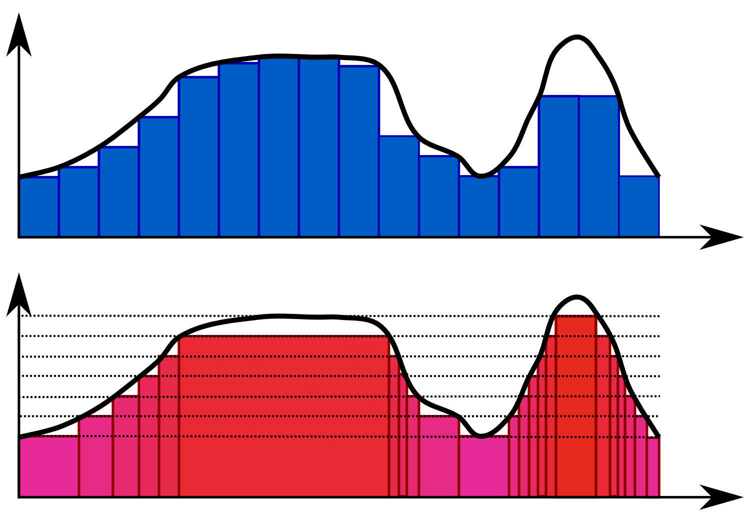
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1,r_2,\ldots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \ldots$$

$$\forall x \quad |g(x) - f(x)| \le \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \le \lambda(A) \le \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:

$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$

- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

Proof Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0,1]$$

Thus

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0,1]) = 1$$

On the other hand

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$

Thus $\lambda(E) = 0$. However,

$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$

From sigma-additivity

$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$

But $\lambda(\mathbb{R}) \geq \lambda([0,1])$, in contradiction.