104165 - Real functions

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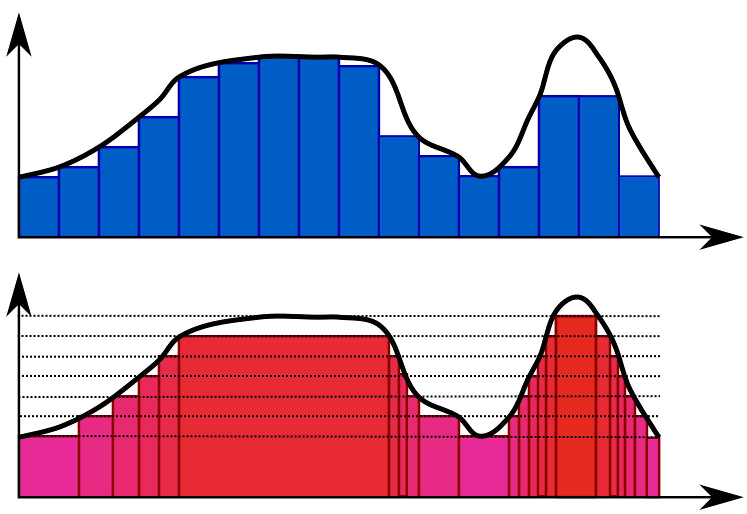
Abstract

1 Introduction

If $\forall x \quad f_n(x) \to f(x)$ (pointwise) does $\int_0^1 f_n(x)\,\mathrm{d}x \to \int_0^1 f(x)\,\mathrm{d}x$?
Define $f_n(x) = \chi_{r_1,r_2,\dots r_n}$, where $\{r_i\} = \mathbb{Q} \cap [0,1]$, i.e., first n rational numbers. Those functions are integrable since they are non-zero in finite number of points. However, $f(x) = \chi_{\mathbb{Q}\cap[0,1]}$ is not integrable.

Riemann integral: limit We defined Riemann integral as limit of Riemann sum:

$$\int_a^b f(x)\,\mathrm{d}x = \lim \sum f(x_i')(x_{i+1} - x_i)$$



By dividing on y, we bound the error by the size of each interval, ϵ:

$$g(x) = s\chi_{A_1} + (s+\epsilon)\chi_{A_2} + \dots$$

$$\forall x \quad |g(x) - f(x)| \leq \epsilon$$

2 Measure

For $A \subseteq \mathbb{R}$ we want to define size of A which we will denote $\lambda(A)$. What do we require from λ?

1. $\lambda([a,b]) = b - a$
2. $0 \leq \lambda(A) \leq \infty$
3. $\lambda(\emptyset) = 0$
4. If $A = \bigcup_{k=1}^{\infty} A_k$ and $\forall i,j \quad A_i \cap A_j = \emptyset$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. $\lambda(A + x) = \lambda(A)$, where $A + x = \{s + x : a \in A\}$.

From those properties we get additional properties:

- Additivity:
$$A = \dot{\bigcup_{i=1}^{n}} A_i \Rightarrow \lambda(A) = \sum_{i=1}^{n} \lambda(A_i)$$
- If $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$.

Theorem 2.1. Function λ fulfilling 1-5 and defined on every subset of $\mathbb{R}$ doesn't exist.

Proof. Suppose there exists such λ.
Define equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Define E choose from each equivalence class one representative from $\left[0, \frac{1}{2}\right]$. Note that if $q_1 \neq q_2$, then $q_1 + E \cap q_2 + E = \emptyset$, since else $e_1 - e_2 = q_1 - q_2$ and $e_1 \sim e_2$, in contradiction.
From definition $E \subset \left[0, \frac{1}{2}\right]$. Take a look at

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0, 1]$$

Thus

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \leq \lambda([0, 1]) = 1$$

On the other hand

$$\lambda\left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) = \sum_{k=2}^{\infty} \lambda\left(\frac{1}{k} + E\right)) = \lambda(E))$$

Thus $\lambda(E) = 0$. However,

$$\mathbb{R} = \dot{\bigcup_{r \in \mathbb{Q}}} r + E$$

From sigma-additivity

$$\lambda(\mathbb{E}) = \sum_{r \in \mathbb{Q}} \lambda(r + E) = 0$$

But $\lambda(\mathbb{R}) \geq \lambda([0, 1])$, in contradiction. □

Reqirements for measure in $\mathbb{R}$

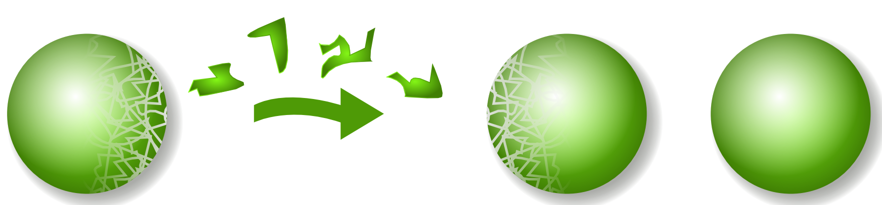
1. $0 \leq \lambda(E) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$
4. If $A = \dot{\bigcup}_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Note In $\mathbb{R}^3$ it is impossible to define measure that fulfills those requirements eve if we replace sigma-additivity with additivity.

Banach-Tarski paradox Denote B – unit ball in $\mathbb{R}^3$. We can write

$$B = \dot{\bigcup_{i=1}^{5}} A_i$$

and find C_i by rotation or translation of A_i such that $\dot{\bigcup}_{i=1}^{5} C_i$ is two unit balls.



2.1 Construction of λ

Definition 2.1 (Special boxes)**.** Let E box with edges parallel to axes:

$$E = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]$$

For E we define

$$\lambda(E) = \prod_{i=1}^{n} (b_i - a_i)$$

Definition 2.2 (Special polygons)**.** is a finite union of special boxes.

Note Each special polygon is a finite union of special boxes with disjoint interior.
Let P is special polygon written as $P = \bigcap_{i=1}^{k} A_i$ where A_i is special box and their interior is disjoint.

$$\lambda(P) = \sum_{i=1}^{k} \lambda(A_i)$$

Proposition 2.2. The definition is independent on choice of A_i.

Proof. Let $P = \bigcap A_i = \bigcap B_i$.
If we continue edges of both A_i and B_i we'll get net which divides P into C_i which refines both A_i and B_i and thus

$$\lambda(P) = \sum_i \lambda(A_i) = \sum_i \lambda(B_i) = \sum_i \lambda(C_i)$$

□

Proposition 2.3. If P_1, P_2 are special polygons and $P_1 \subseteq P_2$ then $\lambda(P_1) \leq \lambda(P_2)$.

Proof. Let $P_2 = \bigcap A_i$ and choose the refinement which divides P_1. □

Proposition 2.4. If P_1, P_2 are special polygons with disjoint interior then

$$\lambda(P_1 \cup P_2) = \lambda(P_1) + \lambda(P_2)$$

Proof. Find A_i which divides both P_1 and P_2. □

Proposition 2.5. For all $x \in \mathbb{R}^n$

$$\lambda(x + P) = \lambda(P)$$

Alternative proof. For special boxes

$$\lambda(E) = \lim_{N \to \infty} \frac{1}{N^n} \left| E \cap \frac{1}{N} \mathbb{Z}^n \right|$$

For $n = 1$, $I = [a, b] \subseteq \mathbb{R}$. We claim

$$b - a = \lim_{N \to \infty} \frac{1}{N} \left| E \cap \frac{1}{N} \mathbb{Z} \right|$$

First of all

$$b - a - 1 \leq |[a, b] \cap \mathbb{Z}| \leq b - a + 1$$

To find $\left|[a, b] \cap \frac{1}{2}\mathbb{Z}\right|$, we can use $|[2a, 2b] \cap \mathbb{Z}|$, which means

$$2b - 2a - 1 \leq \left| E \cap \frac{1}{2} \mathbb{Z} \right| \leq 2b - 2a + 1$$

And for any N:

$$Nb - Na - 1 \leq \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq Nb - Na + 1$$

$$b - a - \frac{1}{N} \leq \frac{1}{N} \left| [a, b] \cap \frac{1}{N} \mathbb{Z} \right| \leq b - a + \frac{1}{N}$$

By sandwich rule, we get the equality.
We can do the same for higher dimension and for open sets, and then we can easily proof the claim.
If P is special polygon and we take $\lim_{N \to \infty} \frac{1}{N^n} \left| P \cap \frac{1}{N} \mathbb{Z}^n \right| = \sum \lambda(A_i)$ when $P = \bigcap A_i$

□

Open sets

Definition 2.3. G is open if $\forall x \in G$ exists ball $B(x,r)$ such that $B \subset G$. Alternatively we can replace ball with special box. Thus for any open $G \neq \emptyset$

$$G = \bigcup \{P \text{ special polygon}\}$$

And we can define

$$\lambda(G) = \sup \{\lambda(P) | P \subseteq G\}$$

Lemma 2.1. Let $K \subseteq \mathbb{R}^n$ compact set and $\{G_i\}_{i \in I}$ open cover ($K \subseteq \bigcup G_i$). Then exists $\epsilon > 0$ such that $\forall x \in K$ exists $i \in I$ such that $B(x, \epsilon) \subseteq G_i$.

Lemma 2.2. For all polygon of dimension P

$$\lambda(P) = \inf \{\lambda(G) : P \subset G\}$$

Proof.

$$P \subseteq G \Rightarrow \lambda(P) \leq \lambda(G)$$

Infimum would give

$$\lambda(P) \leq \inf \{\lambda(G) : P \subset G\}$$

Write $P = \bigcup_{k=1}^N I_k$. Then

$$\lambda(P) = \sum_{k=1}^N \lambda(I_k)$$

For ϵ find I_k^ϵ such that

$$\begin{cases} \text{int } I_k^\epsilon \supseteq I_k \\ \lambda(I_k^\epsilon) \leq \lambda(I_k) + \frac{\epsilon}{N} \end{cases}$$

Denote $G = \bigcup_{k=1}^N \text{int}(I_k^\epsilon)$, then, from subadditivity

$$\lambda(G) \leq \sum_{k=1}^N \lambda(\text{int } I_k^\epsilon) = \sum_{k=1}^N \lambda(I_k^\epsilon) \leq \epsilon + \sum_{k=1}^N \lambda(I_k)$$

In addition,

$$\inf \lambda(G) \leq \lambda(P)$$

□

Proposition 2.6.

$$0 \leq \lambda(G) \leq \infty$$

Proof. Obvious □

Proposition 2.7.

$$\lambda(G) = 0 \iff G = \emptyset$$

Proof. If G is not empty, exists $x \in G$ and special box around x such that $P \subseteq G$and thus $\lambda(G) \leq \lambda(P) > 0$ □

Proposition 2.8.

$$\lambda(\mathbb{R}^n) = \infty$$

Proof. Any box is subset of $\mathbb{R}^n$ thus $\lambda(\mathbb{R}^n) = \infty$ □

Proposition 2.9.

$$G_1 \subseteq G_2 \Rightarrow \lambda(G_1) \leq \lambda(G_2)$$

Proof. Obvious □

Proposition 2.10.

$$\lambda\left(\bigcup_{k=1}^\infty G_k\right) \leq \sum \lambda(G_k)$$

Proof. Let P special polygon, $P \subseteq \bigcup_{k=1}^{\infty} G_k$. We'll show that it's possible to write

$$P = \bigcup_{j=1}^{N} I_j$$

finite union of special boxes with disjoint interior and for each j exists k such that $I_j \subset G_k$. Let ϵ from lemma for $K = P$. Write $P = \bigcup_{j=1}^{N} = I_j$ such that diameter of each $I_j < \epsilon$. If x_j is center of I_j, then $I_j \subseteq B(x_j, \epsilon) \subseteq G_k$.
If this is possible, for such P denote

$$P_k = \bigcup_{j=1}^{\infty} I_j | I_j \subset G_k, \forall i < k \quad I_j \not\subset G_i$$

Obviously $\bigcup P_k = P$ and union is finite since for some m, for every $k > m$ $P_m = \emptyset$, because there is finite number of I_j, and also internals of P_k are disjoint.
Thus $\lambda(P) = \sum \lambda(P_k) \leq \sum \lambda(G_k)$. This is right for any P, thus

$$\lambda\Big(\bigcup (G_k)\Big) = \sup \Big\{\lambda(P) | P \subseteq \bigcup (G_k)\Big\} \leq \sum_{k=1}^{\infty} \lambda(G_k)$$

□

Proposition 2.11.

$$\lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right) = \sum \lambda(G_k)$$

Proof. Since we have sigma-subadditivity, we need only one direction of inequality:

$$\lambda(G_k) = \sup \{\lambda(P) : P \subseteq G_k\}$$

For any N

$$\sum_{k=1}^{N} \lambda(G_k) = \sup \left\{\sum_{k=1}^{N} \lambda(P_k) : P_k \subseteq G_k\right\} = \sup \left\{\lambda\left(\dot{\bigcup_{k=1}^{N}} P_k\right) : P_k \subseteq G_k\right\} \leq \lambda\left(\dot{\bigcup_{k=1}^{N}} G_k\right) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

i.e.,

$$\sum_{k=1}^{\infty} \lambda(G_k) \leq \lambda\left(\dot{\bigcup_{k=1}^{\infty}} G_k\right)$$

□

Proposition 2.12.

$$\lambda(P) = \lambda(\operatorname{int} P) = \inf \{\lambda(G) : P \subseteq G\}$$

Proof. First, proof that $\lambda(P) = \lambda(\operatorname{int} P)$. If $I = P$ is non-empty special box $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. For any $\epsilon > 0$, $I_\epsilon = [a_1 + \epsilon, b_1 - \epsilon] \times [a_2 + \epsilon, b_2 - \epsilon] \times \cdots \times [a_n + \epsilon, b_n - \epsilon]$. $I_\epsilon \subseteq \operatorname{int} I$.
That means that $\lambda(I_\epsilon) \leq \lambda(\operatorname{int} I)$. Obviously, $\lambda(I_\epsilon) \to \lambda(I)$, i.e. $\lambda(I) \leq \lambda(\operatorname{int} I)$.
Generally, for $P = \bigcup_{k=1}^{N} I_k$,

$$\operatorname{int} P \geq \bigcup_{k=1}^{N} \operatorname{int} I_k$$

thus

$$\lambda(\operatorname{int} P) \geq \lambda\left(\bigcup_{k=1}^{N} \operatorname{int} I_k\right) = \sum_{k=1}^{N} \lambda(\operatorname{int} I_k) \geq \sum_{k=1}^{N} \lambda(I_k) = \lambda(P)$$

For any P

$$\lambda(\operatorname{int} P) \geq \lambda P$$

However

$$\lambda(\operatorname{int} P) = \sum \{\lambda(Q) : Q \subseteq \operatorname{int} P\}$$

$$Q \subseteq P \Rightarrow \lambda(Q) \leq \lambda(P) \Rightarrow \lambda(\operatorname{int} P) \leq \lambda(P)$$

Second part is obvious from Lemma 2.2. □

Proposition 2.13.

$$\lambda(x + G) = \lambda(G)$$

Proof. Obvious since it's right for polygons □

2.2 Compact sets

Definition 2.4. For compact $K \subseteq \mathbb{R}^n$

$$\lambda(K) = \inf \{\lambda(G) : K \subseteq G \quad G \text{ is open}\}$$

Proposition 2.14.

$$0 \leq \lambda(K) < \infty$$

Proof. Each K is subset of open box A and $\lambda(A) < \infty$ □

Proposition 2.15.

$$K_1 \subseteq K_2 \Rightarrow \lambda(K_1) \leq \lambda(K_2)$$

Proof. Obvious □

Proposition 2.16. Subadditivity

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

Proof.

$$K_i \subseteq G_i$$

$$K_1 \cup K_2 \subseteq G_1 \cup G_2$$

$$\lambda(K_1 \cup K_2) \leq \lambda(G_1 \cup G_2) \leq \lambda(G_1) + \lambda(G_2)$$

Thus

$$\lambda(K_1 \cup K_2) \leq \lambda(K_1) + \lambda(K_2)$$

□

Proposition 2.17.

$$K_1 \cup K_2 = \emptyset \Rightarrow \lambda(K_1 \cup K_2) = \lambda(K_1) + \lambda(K_2)$$

Proof. For K_1, K_2 exists $\epsilon > 0$ such that $\forall\, x \in K_1\; y \in K_2$, $d(x, y) \geq \epsilon$. Denote

$$U_i = \bigcup_{x \in K_i} B\Big(x, \frac{\epsilon}{2}\Big) \supset K_i$$

Let $K_1 \cup K_2 \subset G_i$, since $K_i \subset U_i$,

$$K_i \subset G \cap U_i$$

i.e.,

$$\forall i \quad \lambda(K_i) \leq \lambda(G \cap U_i)$$

Since $U_1 \cap U_2 = \emptyset$ (from construction)

$$(G \cap U_1) \cap (G \cap U_2) = \emptyset$$

$$\lambda(G \cap U_1) + \lambda(G \cap U_2) = \lambda\big((G \cap U_1) \cap (G \cap U_2)\big) \leq \lambda(G)$$

Thus

$$\lambda(G) \geq \lambda(G \cap U_1) + \lambda(G \cap U_2) \geq \lambda(K_1) + \lambda(K_2)$$

i.e.,

$$\lambda(K_1 \cup K_2) \geq \lambda(K_1) + \lambda(K_2)$$

□

2.3 General sets

Define outer and inner measure similar to Darboux sums:

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

$$\lambda_*(A) = \sup\{\lambda(K) : A \supset G, \text{ compact}\}$$

Proposition 2.18.

$$\lambda_*(A) \leq \lambda^*(A)$$

Proof. If G is open and K compact and $K \subset A \subset G$ then $K \subset G$, i.e. $\lambda(K) \leq \lambda(G)$. From that, taking supremum on K and infimum on G, we get the required result. □

Proposition 2.19.

$$A \subset B \Rightarrow \lambda^*(A) \leq \lambda^*(B) \quad \lambda_*(A) \leq \lambda_*(B)$$

Proof. Obvious. □

Proposition 2.20.

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof.

$$\lambda^*(A) = \inf\{\lambda(G) : A \subset G, \text{ open}\}$$

Thus exists G_k such that

$$\lambda(G_k) < \lambda^*(A_k) + \frac{\epsilon}{2^k}$$

$$\lambda^*\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \lambda\left(\bigcup_{k=1}^{\infty} G_k\right) \leq \sum_{k=1}^{\infty} \lambda(G_k) < \sum_{k=1}^{\infty} \left(\lambda^*(A_k) + \frac{\epsilon}{2^k}\right) = \lambda^*(A_k) + \epsilon$$

□

Proposition 2.21. For disjoint A_k

$$\lambda^*\left(\dot{\bigcup_{k=1}^{\infty}} A_k\right) \geq \sum_{k=1}^{\infty} \lambda^*(A_k)$$

Proof. For all i choose $K_i \subseteq A_i$. Choose some N, then

$$\bigcup_{k=1}^{N} K_k \subseteq \bigcup_{k=1}^{\infty} A_k$$

Since $\bigcup_{k=1}^{N} K_k$ is compact,

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \lambda\left(\bigcup_{k=1}^{N} K_k\right) = \sum_{k=1}^{N} \lambda(K_k)$$

By taking supremum on K_i, we get

$$\lambda_*\left(\bigcup_{k=1}^{\infty} A_n\right) \geq \sum_{k=1}^{N} \lambda_*(A_n)$$

□

Proposition 2.22. If A is open or compact then

$$\lambda(A) = \lambda^*(A) = \lambda_*(A)$$

Proof. If A is compact, obviously $\lambda_*(A) = \lambda(A)$, and $\lambda^*(A) = \lambda(A)$ by definition.
For open A, obviously $\lambda(A) = \lambda^*(A)$. In addition, for any special polygon $P \subset A$, $\lambda(P) \leq \lambda_*(A)$. However

$$\lambda^*(A) = \lambda(A) = \sup\{\lambda(P) : P \subseteq A\} \leq \lambda_*(A)$$

meaning

$$\lambda^*(A) = \lambda(A) = \lambda_*(A)$$

□

Denote

$$\mathcal{L}_0 = \{A \subset \mathbb{R}^n :: \lambda^*)A_{=}\lambda_*(A) < \infty\}$$

All compact sets and all open set with finite measure are in $\mathcal{L}_0$.

Proposition 2.23.

$$\lambda_*(A) = \lambda_*(A + x)$$
$$\lambda^*(A) = \lambda^*(A + x)$$

Definition 2.5. For set in $\mathcal{L}_0$, $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Lemma 2.3. If $A, B \in \mathcal{L}_0$ and $A \cap B = \emptyset$ then $A \cup B \in \mathcal{L}_0$ and

$$\lambda(A \cup B) = \lambda(A) + \lambda(B)$$

Proof.

$$\lambda^*(A \cup B) \le \lambda^*(A) + \lambda^*(B) = \lambda(A) + \lambda(B) == \lambda_*(A) + \lambda_*(B) \le \lambda_*(A \cup B) \le \lambda^*(A \cup B)$$

□

Theorem 2.24. $A \subseteq \mathbb{R}^n$ with $\lambda^*(A) < \infty$. $A \in \mathcal{L}_0$ iff for all $\epsilon > 0$ exists compact K and open G, $K \subseteq A \subseteq G$ and $\lambda(G \setminus K) < \epsilon$

Proof. $\Rightarrow$:
Let $A \in \mathcal{L}_0$. We can find compact K and open G, $K \subseteq A \subseteq G$ such that

$$\lambda(G) < \lambda^*(A) + \frac{\epsilon}{2}$$

$$\lambda(K) > \lambda_*(A) - \frac{\epsilon}{2}$$

Note that, by lemma

$$\lambda(G) = \lambda(K) + \lambda(G \setminus K)$$
$$\lambda(G \setminus K) = \lambda(G) - \lambda(K) < \epsilon$$

$\Leftarrow$:

$$\lambda^*(A) \le \lambda(G) = \lambda(K) + \lambda(G \setminus K) < \lambda(K) + \epsilon \le \lambda_*(A) + \epsilon$$

Thus $\lambda^*(A) = \lambda_*(A)$ and $A \in \mathcal{L}_0$. □

Collary 2.24.1. If $A, B \in \mathcal{L}_0$, then $A \cup B, A \cap B, A \setminus B \in \mathcal{L}_0$

Proof. First, show that $A \setminus B \in \mathcal{L}_0$. Take $K_1 \subseteq A \subseteq G_1$ and $K_2 \subseteq A \subseteq G_2$.

$$\lambda(G_1 \setminus K_1) < \frac{\epsilon}{2}$$

$$\lambda(G_2 \setminus K_2) < \frac{\epsilon}{2}$$

Denote $K = K_1 \setminus G_2$ and $G = G_1 \setminus K_2$.

$$K \subseteq A \setminus B \setminus G$$
$$G \setminus K = (G_1 \setminus K_1) \cup (G_2 \setminus K_2)$$
$$\lambda(G \setminus K) \le \lambda(G_1 \setminus K_1) + \lambda(G_2 \setminus K_2) < \epsilon$$

Now

$$A \cup B = (A \setminus B) \cup B \in \mathcal{L}_0$$
$$A \cap B = A \setminus (A \setminus B) \in \mathcal{L}_0$$

□

Theorem 2.25. Let $\{A_k\}$ set in $\mathcal{L}_0$ and $A \bigcup_{k=1}^{\infty} A_k$ such that $\lambda^*(A) < \infty$ then $A \in \mathcal{L}_0$ and

$$\lambda(A) \le \sum_{k=1}^{\infty} \lambda(A_k)$$

In addition, if $A_i \cap A_j = \emptyset$,

$$\lambda(A) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Suppose $\{A_k\}$ are disjoint.

$$\lambda^*(A) \leq \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda_*(A_k) \leq \lambda_*(A)$$

Thus $A \in \mathcal{L}_0$ and

$$\lambda(A) = \lambda^*(A) = \sum_{k=1}^{\infty} \lambda^*(A_k) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Now generally, define

$$B_1 = A_1 \in \mathcal{L}_0$$

$$B_2 = A_2 \setminus A_1$$

and so on:

$$B_k = A_k \setminus \bigcup_{i=1}^{k-1} A_i \in \mathcal{L}_0$$

Now $\{B_k\}$ are disjoint and $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{L}_0$. Thus

$$\lambda(A) = \lambda\left(\bigcup_{k=1}^{\infty} A_k\right) = \lambda\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} \lambda(B_k) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Note Any ball $B(0, R)$ is in $\mathcal{L}_\prime$, since it is inside special box large enough.

Definition 2.6. Let $A \subseteq \mathbb{R}^n$, we say A is Lebesgue measurable if $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$. It's measure equals

$$\lambda(A) = \sup\{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Denote a set of all such sets as $\mathcal{L}$.

Proposition 2.26. If $\lambda^*(A) < \infty$, $A \in \mathcal{L} \iff A \in \mathcal{L}_0$. For those sets λ definitions are equivalent.

Proof. If $A \in \mathcal{L}_0$ in, then $\forall M \in \mathcal{L}_0 \quad A \cap M \in \mathcal{L}_0$, thus $A \in \mathcal{L}$.
Now, if $A \in \mathcal{L}$ and $\lambda^*(A) < \infty$. For all $N \in \mathbb{N}$,

$$A \cap B(0, N) \in \mathcal{L}_0$$

However

$$A = \bigcup_{N=1}^{\infty} [A \cap B(0, N)]$$

And $\lambda^*(A) < \infty$, thus $A \in \mathcal{L}_0$.
Denote

$$\tilde{\lambda}(A) = \sup\{\lambda(A \cap M), M \in \mathcal{L}_0\}$$

Obviously, $\tilde{\lambda}(A) \geq \lambda(A)$ (take $M = A$). On the other side,

$$\forall M \in \mathcal{L}_0 \quad \lambda(A \cap M) \leq \lambda(A)$$

thus $\tilde{\lambda}(A) = \lambda(A)$

□

Proposition 2.27.

$$\emptyset \in \mathcal{L}$$

Proof.

$$\emptyset \in \mathcal{L}_0 \Rightarrow \emptyset \in \mathcal{L}$$

□

Proposition 2.28.

$$A \in \mathcal{L} \Rightarrow \mathbb{R}^n \setminus A \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$(\mathbb{R}^n \cap A) \cap M = M \setminus A = M \setminus (A \cap M) \in \mathcal{L}_0$$

□

Proposition 2.29.

$$\{A_i\}_{i=1}^{\infty} \in \mathcal{L} \Rightarrow A = \bigcup A_i \in \mathcal{L}$$

Proof. Take $M \in \mathcal{L}_0$.

$$A \cap M = \bigcup_{i=1}^{\infty} (A_k \cap M)$$

$$\lambda^*(A \cap M) \leq \lambda(M)$$

Thus

$$A \cap M \in\in \mathcal{L}_0$$

□

Proposition 2.30. If $\forall N \in \mathbb{N}$, $A \cap B(0, N) \in \mathcal{L}_0$, then $A \in \mathcal{L}$.

Definition 2.7. For some set X, set M of its subsets is called σ-algebra if

1. $\emptyset \in M$
2. $A \in M \Rightarrow X \setminus A \in M$
3. $\{A_i\}_{i=1}^{\infty} \in M \Rightarrow A = \bigcup A_i \in M$

Examples

1. 2^X for any X is σ-algebra
2. All subsets of $\mathbb{R}$ that are countable or their complement is countable.
3. All open sets in $\mathbb{R}$ is not σ-algebra.

Proposition 2.31. If M is σ-algebra and $\{A_k\}_{k=1}^{\infty} \subset M$, then

$$\bigcap_{k=1}^{\infty} A_k \in M$$

Proof.

$$X \setminus \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (X \setminus A_k) \in M$$

□

Proposition 2.32. All open and closed sets are in $\mathcal{L}$

Proof. Let A some open set. Then $A \cap B(0, N) \in \mathcal{L}_0$. Since $\mathcal{L}$ is closed for complementation, also closed sets are in $\mathcal{L}$. □

Proposition 2.33. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ then

$$\lambda\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. Denote $A = \bigcup_{k=1}^{\infty} A_k$. For $M \in \mathcal{L}_0$

$$\lambda(A \cap M) = \lambda\left(\bigcup_{k=1}^{\infty} (A_k \cap M)\right) \leq \sum_{k=1}^{\infty} \lambda(A_k \cap M) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

Since it right for any M,

$$\lambda(A) \leq \sum_{k=1}^{\infty} \lambda(A_k)$$

□

Proposition 2.34. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{L}$ and $A_i \cap A_j = 0$ then

$$\lambda\left(\bigcup_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} \lambda(A_k)$$

Proof. For some $N \in \mathbb{N}$, choose $\{M_p \in \mathcal{L}_0\}_{p=1}^{N}$. Define $\mathcal{L}_0 \ni M = \bigcup_{p=1}^{N} M_p$.

$$\lambda(A) \geq \lambda(A \cap M) = \sum_{k=1}^{\infty} \lambda(A_k \cap M) \geq \sum_{k=1}^{N} \lambda(A_k \cap M) \geq \sum_{k=1}^{N} \lambda(A_k \cap M_k)$$

Thus

$$\lambda_A \geq \sup\left\{\sum_{k=1}^{N} \lambda(A_k \cap M_k), M_k \in \mathcal{L}_0\right\} = \sum_{k=1}^{N} \sup\left\{\lambda(A_k \cap M_k), M_k \in \mathcal{L}_0\right\} = \sum_{k=1}^{N} \lambda(A_k)$$

Since it's right for any N,

$$\lambda_A \geq \sum_{k=1}^{\infty} \lambda(A_k)$$

$\square$

Theorem 2.35. The defined λ fulfills properties of measure.

1. $0 \leq \lambda(A) \leq \infty$
2. $\lambda(\emptyset) = 0$
3. $\lambda([a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]) = \prod_{i=1}^{n}(b_i - a_i)$
4. If $A = \dot\bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.
5. If C is acquired from A by rotation or translation $\lambda(C) = \lambda(A)$.

Definition 2.8 (Measure)**.** For some set X, measure of X is function μ defined on σ-algebra M of subsets of X and fulfills

1. $0 \leq \mu(A) \leq \infty$
2. $\mu(\emptyset) = 0$
3. If $A = \dot\bigcup_{k=1}^{\infty} A_k$, then $\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_k)$.

We denote measure space as (X, μ, M).

Theorem 2.36. Let (X, μ, M) measure space.

1. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k\ A_k \subset A_{k+1}$, then
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$$
2. If $\{A_k\}_{k=1}^{\infty} \subset M$ and $\forall k\ A_k \supset A_{k+1}$ and $\mu(A_1) < \infty$, then
$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k)$$

Proof.

$$\bigcup_{k=1}^{\infty} A_k = A_1 \cup \left[\bigcup_{k=1}^{\infty} A_{k+1} \setminus A_k\right]$$

Since those sets are disjoint

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A_1) + \sum_{k=1}^{\infty} \mu(A_{k+1} \setminus A_k) = \lim_{N\to\infty} \mu(A_1) + \sum_{k=1}^{N} \mu(A_{k+1} \setminus A_k) = \lim_{N\to\infty} \mu\left(A_1 \cup \left[\bigcup_{k=1}^{N} A_{k+1} \setminus A_k\right]\right) = \lim_{N\to\infty} \mu(A_{N+1})$$

$\square$

Proposition 2.37. If $\lambda^*(A) = 0$, $A \in \mathcal{L}$ and for any $B \subset A$, $B \in \mathcal{L}$ and $\lambda(B) = 0$.

Proof.

$$\lambda_*(A) \le \lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}_0$$

Monotonity of upper measure ☐

Theorem 2.38. A is measurable iff $\forall \epsilon > 0$ exist open G and closed F such that

$$F \subseteq A \subseteq G$$

and

$$\lambda(G \setminus F) \le \epsilon$$

Proof. $\Leftarrow$:
Suppose exist such G and K. For all k choose G_k and F_k such that

$$\lambda(G_k \setminus F_k) < \frac{1}{k}$$

Denote

$$B = \bigcup_{k=1}^{\infty} F_k$$

$$\lambda^*(A \setminus B) = 0$$

and

$$A \setminus B \subseteq G_k \setminus B \subseteq G_k \setminus F_k$$

Thus

$$\lambda^*(A \setminus B) \le \lambda(G_k \setminus F_k) < \frac{1}{k}$$

Thus $\lambda^*(A \setminus B) = 0$ and $A \setminus B \in \mathcal{L}$.
However $B \in \mathcal{L}$ and $A = B \cup (A \setminus B)$, thus $A \in \mathcal{L}$.
$\Rightarrow$:
Suppose $A \in \mathcal{L}$. Denote $E_k = B(0, k) \setminus B(0, k-1)$. This is partition of $\mathbb{R}^n$. $E_k \in \mathcal{L}_0$ and so is $A \cap E_k \in \mathcal{L}$. Thus for all k there is

$$K_k \subseteq A \cap E_k \subseteq G_k$$

such that $\lambda(G_k \setminus K_k) < \frac{\epsilon}{2^k}$. Denote

$$F = \bigcup_{k=1}^{\infty} K_k$$

$$G = \bigcup_{k=1}^{\infty} G_k$$

$$\lambda(G \setminus F) = \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus F)\right) \le \lambda\left(\bigcup_{k=1}^{\infty} (G_k \setminus K_k)\right) \le \sum_{k=1}^{\infty} \lambda(G_k \setminus K_k) < \epsilon$$

Now, F is closed. Let $F \ni x_k \to x$. The sequence converges and thus bounded, and thus exists N such that $\{x_k\} \cup \{x\} \in B(0, N)$. Thus $\{x_k\} \subseteq \left(\bigcup_{i=1}^{N} E_i\right) \cap F$ and $\{x_k\} \subseteq \bigcup_{i=1}^{N} K_i$ and thus $\{x_k\} \cup \{x\} \in F$.

☐

Proposition 2.39. If A is measurable then $\lambda(A) = \lambda^*(A) = \lambda_*(A)$.

Proof. If $\lambda^*(A) < \infty$ we've already seen this.
Suppose $\lambda^(A) = \infty$. Thus $\inf \{\lambda(G) : A \subseteq G\} = \infty$. By previous theorem exists closed F such that $F \subseteq A \subseteq G$ and $\lambda(G \setminus A) \le 1$.

$$\infty = \lambda(G) = \lambda(G \setminus A) + \lambda(A) \le , bda(G \setminus F) + \lambda(A) \le 1 + \lambda(A)$$

Thus, $\lambda(A) = \infty$.
Now, take a look at $\{A \cap B(0, N)\}_N$.

$$\infty = \lambda(A) = \lambda\left(\bigcup_N (A \cap B(0, N))\right) = \lim_{N \to \infty} \lambda(A \cap B(0, N))$$

$$\infty \leftarrow \lambda(A \cap B(0, N)) = \lambda_*(A \cap B(0, N)) \le \lambda_*(A)$$

☐

Reminder We've built $E \subseteq \left[0, \frac{1}{2}\right]$ such that $q + E : q \in \mathbb{Q}$ is disjoint. And

$$\forall k \in \mathbb{N} \, \frac{1}{k} + E \subseteq [0,1]$$

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

Proposition 2.40. E is not measurable

Proof.

$$\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right) \subseteq [0,1]$$

$$1 = \lambda_* \left(\bigcup_{k=2}^{\infty} \left(\frac{1}{k} + E\right)\right) \geq \sum_{k=2}^{\infty} \lambda_* \left(\frac{1}{k} + E\right)$$

i.e., $\lambda_*(E) = 0$. On the other hand

$$\infty = \lambda^*(\mathbb{R}) = \lambda^* \left(\bigcup_{q \in \mathbb{Q}} q + E\right) \leq \sum_q \lambda^*(q + E) = \sum_q \lambda^*(E)$$

Thus $\lambda^*(E) > 0$, i.e., E is not measurable. □

Proposition 2.41. For any measurable $A \subseteq \mathbb{R}$ such that $\lambda(A) > 0$, exists non-measurable $B \subseteq A$.

Proof. We've seen that

$$\bigcup_{q \in \mathbb{Q}} q + E = \mathbb{R}$$

thus

$$A = \bigcup_{q \in \mathbb{Q}} A \cap (q + E)$$

$$0 \leq \lambda^*(A) = \lambda^* \left(\bigcup_{q \in \mathbb{Q}} A \cap (q + E)\right) \leq \sum_q \lambda^*(A \cap (q + E))$$

Thus exists q_0 such that $0 < \lambda^*(A \cap (q + E))$, denote

$$B = A \cap (q_0 + E)$$

$$\lambda_*(B) \leq \lambda_*(q_0 + E) = \lambda_*(E) = 0$$

i.e. $B \notin \mathcal{L}$.

□

Proposition 2.42. B measurable, $A \subseteq B$, then

$$\lambda^*(A) + \lambda_*(B \setminus A) = \lambda(B)$$

Proof.

$$\lambda^*(A) = \inf \{\lambda(G) : A \cap G\}$$

$$\lambda(G) + \lambda_*(B \setminus A) \geq \lambda(G) + \lambda_*(B \setminus B) = \lambda(G) + \lambda(B \setminus G) \geq \lambda(B)$$

On the other hand, for any $K \subseteq B \setminus A$

$$\lambda^*(A) + \lambda(K) \leq \lambda(B \setminus K) + \lambda(K) = \lambda(B)$$

By taking supremum on K, we get

$$\lambda^*(A) + \lambda(B \setminus A) \leq \lambda(B)$$

□

Proposition 2.43 (Carathéodory's condition)**.**

$$A \subseteq \mathbb{R}^n \text{ measurable} \iff \forall E \subseteq \mathbb{R}^n \quad \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Proof. $\Rightarrow$:
Let A measurable set. Choose general E. For open $G \supset E$,

$$\lambda(G) = \lambda(G \cap A) + \lambda(G \setminus A) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

Since it's right for any G, by taking infimum:

$$\lambda^*(E) \geq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

And by subadditivity

$$\lambda^*(E) \leq \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

i.e.,

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

$\Leftarrow$:
Suppose the condition is right for A. Let $M \in \mathcal{L}_0$, then

$$\lambda(M) = \lambda^*(M \cap A) + \lambda^*(M \setminus A)$$

From previous proposition

$$\lambda(M) = \lambda^*(M \cap A) + \lambda_*(M \setminus A)$$

Thus

$$\lambda_*(M \setminus A) = \lambda^*(M \setminus A)$$

and thus $M \setminus A \in \mathcal{L}_0$, i.e. $A \in \mathcal{L}$. $\square$

Lemma 2.4. Let $A \subseteq \mathbb{R}$ with positive measure, and let $\epsilon > 0$ then there exists an interval $J \subseteq \mathbb{R}$ $\frac{\lambda(A \cap J)}{\lambda(J)} = 1 - \epsilon$

Proof. Denote $C = \lambda(A) > 0$.

$$\lambda(A) = \lambda^*(A) = C$$

Thus exists open $G \supseteq A$ such that $\lambda(G) < \big(1 + \frac{\epsilon}{2}\big)C$.
Since G is open, it is disjoint union of open intervals:

$$G = \bigcup_{i=1}^{\infty} J_i$$

$$\Big(1 + \frac{\epsilon}{2}\Big)C > \lambda(G) = \sum \lambda(J_i)$$

Assume that $\forall i$ $\lambda(A \cap J) leq (1 - \epsilon)\lambda(J)$. Then

$$C = \lambda(A) = \lambda\left(A \cap \left(\bigcup_{i=1}^{\infty} J_i\right)\right) = \sum_{i=1}^{\infty} \lambda(A \cap J_i) \leq (1 - \epsilon)\sum \lambda_J = (1 - \epsilon)\lambda(G) = (1 - \epsilon)\Big(1 + \frac{\epsilon}{2}\Big)C < C$$

$\square$

Theorem 2.44. Let $A \subset \mathbb{R}$ measurable set with positive measure. $A - A = \{x - y | x, y \in A\}$.

Proof. If A has non-empty interior, the theorem is obvious. since there exists $a \in A$, $(a - \delta, a + \delta) \subset A$ and thus $(-\delta, \delta) \subset A - A$.

$$t \in A - A \iff A + t \cap A \neq \emptyset$$

Let $J = (a, b)$ from previous lemma with $\epsilon = \frac{1}{3}$. Assume $t \notin A - A$, i.e. $A \cap (A + t) = \emptyset$. And thus

$$(A \cap J) \cap [(A + t) \cap (J + t)] = \emptyset$$

$$\lambda(A \cap J) \geq \frac{2}{3}\lambda(J)$$

$$\frac{2}{3}\lambda(J) + \frac{2}{3}\lambda(J) \leq \lambda(A \cap J) + \lambda\big((A + t) \cap (J + t)\big) = \lambda\big((A \cap J) \cup \big[(A + t) \cap (J + t)\big]\big) \leq \lambda(J \cup (J + t))$$

Now, if $t \geq 0$, $J \cup (J+t) \subseteq (a, b+t)$, and if $t < 0$, $J \cup (J+t) \subseteq (a+t, b)$. Anyway

$$\frac{4}{3}\lambda(J) \leq \lambda(J \cup (J+t)) \leq \lambda(J) + |t|$$

i.e.,

$$|t| \geq \frac{1}{3}\lambda(J)$$

Thus $\forall\, 0 < t < \frac{1}{3}\lambda(J)$, $(-t, t) \subseteq A - A$. ☐

Let a set of subsets in $\mathbb{R}^n$. Exists σ-algebra that is superset of a, and also

$$\bigcup \{m : \quad a \subset m|; \sigma\text{-algebra}\}$$

is σ-algebra and is called σ-algebra generated by a.
Denote $\mathcal{B}$ σ-algebra generated by all open sets in $\mathbb{R}^n$. $\mathcal{B}$ is Borel σ-algebra. Since all open sets are in $\mathcal{L}$, $\mathcal{B} \subseteq \mathcal{L}$.

Theorem 2.45. Let measurable $A \subseteq \mathbb{R}^n$, we can write $A = E \cup N$, such that

1. $E \cap N = 0$
2. $E \in \mathcal{B}$
3. $\lambda(N) = 0$

Proof. For all $k \in \mathbb{N}$, find

$$F_k \subseteq A \subseteq G_k$$

G_k open and F_k closed, and

$$\lambda(G_k \setminus F_k) \leq \frac{1}{k}$$

Denote $E = \bigcup_{k=1}^{\infty} F_k \in \mathcal{E}$. $N = A \setminus E \in \mathcal{L}$.

$$\lambda(N) = \lambda(A \setminus E) \leq \lambda(G_k \setminus F_k) < \frac{1}{k}$$

i.e., $\lambda(N) = 0$. ☐

Reminder $f : E \to \mathbb{R}^n$ is continuous iff $\forall \quad G \subseteq \mathbb{R}^n$, $f^{-1}(G)$ is open in E.

Theorem 2.46. Let $f : E \to \mathbb{R}^n$ be continuous for Borel set $E \subseteq \mathbb{R}^n$. Then $f^{-1}(\mathcal{B}) \subseteq \mathcal{B}$.

Proof. Let

$$m = \left\{A \subseteq \mathbb{R}^n : f^{-1}(A) \in \mathcal{B}\right\}$$

We need to show that $\mathcal{B} \subseteq m$, i.e., that m is σ-algebra containing all open sets.
$\emptyset \in m$, since $\emptyset = f^{-1}(\emptyset)$.
If $\{A_k\} \subseteq m$, then $f^{-1}(A_k) \in \mathcal{B}$ and

$$f^{-1}\left(\bigcup_k A_k\right) = \bigcup_k f^{-1}(A_k) \in \mathcal{B}$$

If $A \in m$, then

$$f^{-1}(\mathbb{R}^n \setminus A) = E \setminus f^{-1}(A) \in \mathcal{B}$$

Now lets show that all open sets are in m. If G is open,

$$f^{-1}(G) = E \cap U_G \in \mathcal{B}$$

☐

Theorem 2.47. There exists measurable set in $\mathbb{R}$ which is not Borel.

Proof. Define $f : [0,1] : \mathbb{R}$. Let x in ternary basis $0.a_1a_2\dots$. Then

$$f(x) = \frac{1}{2^N} + \sum_{1}^{N-1} \frac{1}{2^n}\frac{a_n}{2}$$

where N is first index such that $a_N = 1$.
Note that f is constant on $I \subset [0,1]$ such that $I \not\subset C$ (Cantor set).
f is monotonous and onto, and thus continuous.
Define also $g(x) = x + f(x)$, which is one-to-one and onto, thus it is homeomorphism. □

Denote $\mathcal{C}$ set of intervals in $[0,1] \setminus C$. Any interval $J \in \mathcal{C}$ exists r such that

$$g(x) = x + r$$

(f is constant on J). That means $\lambda(g(J)) = \lambda(J)$.
We see that

$$\lambda(G) - \lambda\left([0,2] \setminus \bigcup_{J\in\mathcal{C}} g(J)\right) = 2 - \sum_{J\in\mathcal{C}} \lambda(g(J)) = 2 - \sum_{J\in\mathcal{C}} \lambda(J) = 2 - 1 = 1$$

Let $B \subseteq g(C)$ which is not measurable. Denote

$$A = g^{-1}(B)$$

It is obvious that $A \subseteq C$, and since $\lambda(C) = 0$, $\lambda(A) = 0$.
If A was Borel, then, since $B = g(A)$ and g is homeomorphism, we get that B is Borel. However, this is impossible, since B is non-measurable.

3 Measurable functions and integrals

We want to define integral as the sum of possible values of function times the size of set for which function gets this values:

$$\int f \sim \sum f(t_i \in A_i) \times \lambda(A_i)$$

where

$$A_i = \{x : f(x) \in [a, a+\epsilon]\}$$

Let X space with σ-algebra M. We work with functions

$$f : X \to [-\infty, \infty]$$

Definition 3.1. We say f is M-measurable if for all $-\infty \leq t \leq \infty$

$$f^{-1}(-\infty, t) \in M$$

Proposition 3.1. The following conditions are equivalent:

1. f is M-measurable:
$$\forall -\infty < t \leq \infty \quad f^{-1}([-\infty, t]) \in M$$
2. $$\forall -\infty < t \leq \infty \quad f^{-1}([-\infty, t)) \in M$$
3. $$\forall -\infty \leq t \leq \infty \quad f^{-1}([t, \infty]) \in M$$
4. $$\forall -\infty \leq t < \infty \quad f^{-1}((t, \infty]) \in M$$
5. $f^{-1}(\infty) \in M$, $f^{-1}(-\infty) \in M$, and $\forall E \in \mathcal{B}(\mathbb{R})$ $f^{-1}(E) \in M$
6. $f^{-1}(\infty) \in M$, $f^{-1}(-\infty) \in M$, and $\forall a, b \in \mathbb{R}$ $f^{-1}([a, b]) \in M$

Proof. $1 \Rightarrow 2$:

$$f^{-1}([-\infty, t)) = \bigcup_{\mathbb{Q} \ni r < t} f^{-1}([-\infty, r])$$

thus $f^{-1}([-\infty, t)) \in M$.
$2 \Rightarrow 3$:
If $t = -\infty$, $f^{-1}([-\infty, \infty]) = X \in M$. Otherwise

$$f^{-1}([t, \infty]) = X \setminus \left(f^{-1}([-\infty, t))\right)$$

thus $f^{-1}([t, \infty]) \in M$.
$3 \Rightarrow 4$: just like $1 \Rightarrow 2$
$4 \Rightarrow 1$: just like $2 \Rightarrow 3$
$1 - 4 \Rightarrow 5$:
Taking $t = \pm\infty$, we get $f^{-1}(\infty)$ and $f^{-1}(-\infty)$.
Let

$$S = \left\{E \subset \mathbb{R} | f^{-1}(E) \in M\right\}$$

S is σ-algebra.

$$f^{-1}\Big((a, b)\Big) = f^{-1}((a, \infty]) \cap f^{-1}([-\infty, b)) \in M$$

Thus open intervals are in $\mathbb{R}$, and thus open sets and thus $\mathcal{B} \subset S$.
$5 \Rightarrow 6$: Obvious, since 5 is stronger
$6 \Rightarrow 1$: Left as an exercise $\square$

Collary 3.1.1. If $f : E \to [-\infty, \infty]$ and $E \in M$, then the definition is conserved.

Collary 3.1.2. $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ is measurable iff $A \in M$.

Proof. $A = \chi_A^{-1}(\{1\})$, thus one direction is obvious.
Else,

$$\chi_A^{-1}(A) = \begin{cases} X & 0, 1 \in E \\ A & 1 \in E, 0 \notin E \\ X \setminus A & 1 \notin E, 0 \in E \\ \emptyset & 0, 1 \notin E \end{cases}$$

$\square$

Collary 3.1.3. $f : E \to \mathbb{R}$, for Borel set $E \subset \mathbb{R}$. If f is continuous then f Borel-measurable and Lebesgue-measurable.

Theorem 3.2. Let $f : X \to \mathbb{R}$ M-measurable functions.
If $\phi : B \to \mathbb{R}$ for Borel set $B \subseteq \mathbb{R}$ and $f(x) \subseteq B$ and ϕ Borel-measurable, then $\phi \circ f$ are M-measurable.

Proof. We need to show

$$f^{-1}(\phi^{-1}(E)) = (\phi \circ f)^{-1}(E) \in M$$

Now, $\phi^1(E) \in \mathcal{B}$, since ϕ is Borel-measurable. Then $f^{-1}(\phi^{-1}(E)) \in M$. $\square$

Collary 3.2.1. If f is non-zero, $\frac{1}{f}$ is measurable.

Collary 3.2.2. If $0 < p < \infty$, $|f|^p$ is measurable.

Proposition 3.3. If f is weaker (for example, Lebesgue-measurable), the theorem is not true, even if ϕ is homeomorphism. For example, we've seen g and non measurable $g(A)$ for measurable A. Then

$$\chi_A \circ \phi = \chi_{g(A)}$$

which is non-measurable.

Theorem 3.4. Let $f, g : X \to \mathbb{R}$ M-measurable functions. Then $f + g$, cf, $f \cdot g$ are M-measurable.

Proof.
$$(f+g)^{-1}(-\infty,t)\bigcup_{r\in\mathbb{Q}}\left[f^{-1}(-\infty,r)\cap g^{-1}(-\infty,t-r)\right]$$
That means measurable functions are vector space.
$$f\cdot g=\frac{1}{4}(f+g)^2-\frac{1}{4}(f-g)^2$$
□

Theorem 3.5. Let $\{f_k\}_{k=1}^{\infty}:X\to[-\infty,\infty]$ sequence of M-measurable functions. Then also $\liminf f_k$, $\limsup f_k$, $\sup f_k$, $\inf f_k$ and so is $\lim f_k$ if exists.

Proof.
$$(\sup f_k)^{-1}([-\infty,t])=\{x:\sup f_k(x)\le t\}=\bigcap\{x:f_k(x)\le t\}=\bigcap f_k^{-1}([-\infty,t])\in M$$
$$\limsup f_k(x)=\inf_n\left(\sup_n j\le n f_j(x)\right)$$
□

Definition 3.2 (Simple function). $f:X\to[-\infty,\infty]$ is called simple function if it acquires only finite number of values. If we denote those values as $\{a_i\}_{i=1}^n$ and $A_k=\{x:f(x)=a_k\}$. Then we can rewrite function as
$$f(x)=\sum_{k=1}^n a_k\chi_{A_k}(x)$$
In fact, all functions that can be written as
$$f(x)=\sum_{k=1}^m b_K\chi_{B_k}(x)$$
is simple. If $\{B_k\}$ ar disjoint and b_k are not equal, this is called canonical representation.

Proposition 3.6. f is measurable iff $\forall k\ A_k\in M$

Proof. χ_A measurable $\Rightarrow$ f is measurable.
$A_k=\{x:f(x)=a_k\}$ is measurable. □

Theorem 3.7. $f:X\to[-\infty,\infty]$. f m-measurable if there is sequence $\{s_k\}$ of measurable simple functions such that $\forall x\, s_k(x)\to f(x)$. We can choose s_k such that $|s_{k-1}|\le|s_k|$.

Proof. $\Leftarrow$ obvious.
$\Rightarrow$:
Suppose $f\ge 0$. Define
$$s_k(x)=\begin{cases}k & f(x)\ge k\\ \frac{i-1}{2^k} & \frac{i-1}{2^k}\le f(x)<\frac{i}{2^k}\end{cases}$$
We can rewrite as
$$s_k(x)=k\chi_{A_k f^{-1}(k,\infty)}+\sum_{i=1}^{k\cdot 2^k}\frac{i-1}{2^k}\chi_{f^{-1}\left[\frac{i-1}{2^k},\frac{i}{2^k}\right]}$$
which is canonical form, and we conclude s_k are measurable.
Obviously, $s_k\le s_{k+1}$.
If $f(x)=\infty$, $s_k=k\to\infty=f(x)$.
Else, $\exists k_0>f(x)$, and then
$$s_k(x)\le f(x)\le s_k(x)+\frac{1}{2^k}$$
i.e., $s_k(x)\to f(x)$.
In general case we define $f_+=\max\{f(x),0\}$ and $f_-=\max\{-f(x),0\}$. Note that $f=f_+-f_-$ and $f_-\cdot f_+=0$. Both f_-,f_+ are measurable. For $f_\pm$ exist sequences $\{s_k'\}$, $\{s_k''\}$, we can define $s_k=s_k'-s_k''$.
For any x either $s_k'(x)$ or $s_k''(x)$ is 0, thus in any point $s_k=s_k'$ or $s_k=-s_k''$, i.e., $|s_{k-1}|\le|s_k|$. □

Definition 3.3. If some property is fulfilled for all x except, maybe, some set A which is subset of set of measure 0, we say that property is fulfilled almost everywhere (a.e.). In probability we say the property fulfilled almost surely (a.s.).

Theorem 3.8. Let $f : \mathbb{R}^n \to [-\infty, \infty]$ be Lebesgue-measurable function. Then $\exists g(x)$, Borel-measurable function, such that $\lambda(\{x : f(x) \neq g(x)\}) = 0$, i.e. $f(x) = g(x)$ a.e.

Proof. Suppose $f \geq 0$. Let $\{s_k\}$ as in previous theorem and thus $f = \sup s_k$.

$$s_k = \sum_{j=1}^{m} a_j \chi_{A_j}$$

Since $A_j \in \mathcal{L}$ we can rewrite it as $A_j = E_j \cup N_j$.
Define

$$h_k = \sum_{j=1}^{m} a_j \chi_{E_j} \leq s_k$$

Since $h_k = s_k$ except for $\bigcup N_j$, which is of measure 0, $h_k = s_k$ a.e.
Denote $N = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} N_j$, obviously $\lambda(N) = 0$. Also define $g = \sup\limits_k h_k$.
$g(x) = f(x)$ if $x \notin N$, i.e., a.e. and $g(x)$ is Borel-measurable as supremum of Borel-measurable functions.
For general f, we do same with $f_{\pm}$ and acquire $g_{\pm}$. □

Lemma 3.1. If f is Lebesgue measurable, then if $g : \mathbb{R}^n \to [-\infty, \infty]$ fulfilling

$$\lambda^* \{x : f(x) \neq g(x)\} = 0$$

then g is measurable.

Proof. Let $-\infty \leq t \leq \infty$, we need to show that $B = g^{-1}([-\infty, t])$ is Lebesgue-measurable.
We now that $A = f^{-1}[-\infty, t]$ is Lebesgue-measurable.

$$B \setminus A \subseteq \{x : f(x) \neq g(x)\}$$

Thus $B \setminus A$ is measurable with measure 0.

$$B = (A \cup B) \setminus (A \setminus B) \in \mathcal{L}$$

□

Theorem 3.9 (Tietze extension theorem)**.** Let $Y \subseteq \mathbb{R}^n$ and $f : Y \to \mathbb{R}$ continuous and bounded ($|f| \leq M$). Exists continuous function $F : \mathbb{R}^n \to \mathbb{R}$ such that $F = f$ on Y and $|F| \leq M$.

Theorem 3.10 (Lusin's theorem)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$ which vanishes outside of measurable set A. The for all $\epsilon > 0$ exists closed set $E \subseteq A$ and continuous function $g : \mathbb{R}^n \to \mathbb{R}$ such that $f = g$ on E and $\lambda(A \setminus E) < \epsilon$.

Proof. Let f be a simple function in canonical form:

$$f(x) = \sum_{j=1}^{m} a_j \chi_{A_j}(x)$$

and $A = \bigcup_{j=1}^{m} A_i$. A_i are measurable, thus exists closed set $F_i \subseteq A_i \subseteq A$ such that $\lambda(A_i \setminus F_i) < \frac{\epsilon}{m}$.
Denote $E = \bigcup_{i=1}^{m} F_i$.

$$\lambda(A \setminus E) = \sum_i \lambda(A_i \setminus F_i) <\leq \epsilon$$

Define f_0 on E such that $f_0\big|_{F_i} = a_i$. f is continuous and thus by 3.9 exists g as required.
Now, let f be measurable and bounded. Let $\epsilon > 0$. We know there exists $\{s_k\}_{k=1}^{\infty}$ such that $s_k \to f$ uniformly.
For all k exists continuous g_k and L_k such that $\lambda(A \setminus L_k) < \frac{\epsilon}{2^k}$ and $g_k = s_k$ on L_k. Denote $E = \bigcap L_k$.

$$\lambda(E \setminus E) = \lambda\Big(A \setminus \bigcap L_k\Big) = \lambda\Big(\bigcup (A \setminus L_k)\Big) \leq \sum \lambda(A \setminus L_k) < \epsilon$$

On E, g_k converges uniformly to f, thus f is continuous of E and from 3.9 we get what we wanted.
Let f measurable function which vanishes outside of measurable set A such that $\lambda(A) < \infty$.

$$\bigcap_N \underbrace{\{x \in A : |f(x)| \geq N\}}_{A_N} = \emptyset$$

$A_N \subset A_{N+1}$, measurable and $\lambda(A) < \infty$, then $0 = \lambda(\bigcap_N A_N) = \lim_{N \to \infty} \lambda(A_N)$.

Thus exists N_0 such that $\lambda(A_{N_0}) < \frac{\epsilon}{2}$. Denote

$$G = \{x \in A : |f(x)| < N_0\}$$

$$A_{N_0} = \{x \in A : |f(x)| \geq N_0\}$$

Then $\lambda(A \setminus G) = \lambda(A_{N_0}) < \frac{\epsilon}{2}$.
Then $\chi_G f : G \to \mathbb{R}$ is bounded and measurable, i.e., exists closed $E \subseteq G$ such that $\chi_G f$ is continuous on E and $\lambda(G \setminus E) < \frac{\epsilon}{2}$.
Since $\lambda(A \setminus E) < \epsilon$, once again we use 3.9 and get what we wanted.
Denote $A_k = A \cap (B(0,k) \setminus B(0,k-1))$. Define $f_k = f\big|_{A_k}$ then exists closed $E_k \subseteq A_k$ such that $f_k\big|_{E_k}$ such that $\lambda(A_k \setminus E_k) < \frac{\epsilon}{2^k}$.
$E = \bigcup E_k$ is closed and $f\big|_E$ is continuous and $\lambda(A \setminus E) < \epsilon$. ☐

Collary 3.10.1. Let f measurable and vanishing outside measurable A. Then there exists sequence of continuous functions $\{g_k\}$ such that $g_k \to f$ a.e. on A.

Proof. For all natural K exists closed $E_k \subseteq A$ and continuous g_k suxh that $g_k = f$ on E_k and $\lambda(A \setminus E_k) < \frac{1}{2^k}$. Denote $E \bigcup_m \left(\bigcap_{k \geq m} E_k\right) \subseteq A$.
$\forall x \in E$ exists m such that $\forall k \geq m$ $x \in E_k$. That means $\forall k \geq m$ $g_k(x) = f(x)$, which means $g_k(x) \to f(x)$ for every such x, i.e. for all $x \in E$.

$$A \setminus E = \bigcap_m \bigcup_{k \geq m} (A \setminus E_k)$$

$$\lambda(A \setminus E) \leq \lambda\left(\bigcup_{k \geq m} A \setminus E_k\right) \leq \sum_{k=m}^{\infty} \lambda(A \setminus E_k) \leq \sum_{k=m}^{\infty} \frac{1}{2^k} = \frac{1}{2^{m-1}}$$

☐

Definition 3.4 (Almost uniform convergence)**.** Let $\{f_n\}$, f be real functions measurable on A. We say that $f_n \to f$ almost uniformly, if for all $\epsilon > 0$ exists measurable E such that $\lambda(A \setminus E) \leq \epsilon$ and on E $f_n \to f$ uniformly.

Theorem 3.11 (Egorov's theorem)**.** Let $A \subseteq \mathbb{R}^n$ be a measurable set with finite measure. Let $\{f_n\}$, f real functions measurable on A and $f_n \to f$ a.e. on A. Then for all $\epsilon > 0$ exists E such that $\lambda(A \setminus E) \leq \epsilon$ and on E $f_n \to f$ uniformly a.e.

Proof. $f_n \to f$, then exists n such that for $j \geq n$ $|f_j - f| < \frac{1}{k}$.
Denote

$$E_n^k = \left\{c : |f_j(x) - f(x)| < \frac{1}{k} \quad \forall k \geq n\right\}$$

Then for all k $x \in \bigcup_{n=1}^{\infty} E_n^k$.
From the assumption we get $\lambda\big(A \setminus \bigcup_{n=1}^{\infty} E_n^k\big) = 0$. Note that $E_n^k \subseteq E_{n+1}^k$ thus

$$\forall k \; A \setminus E_n^k \supseteq A \setminus E_{n+1}^k$$

$$A \setminus \bigcup_{n=1}^{\infty} E_n^k = \bigcap_{n=1}^{\infty} A \setminus E_n^k$$

$$0 = \lambda\left(A \setminus \bigcup_{n=1}^{\infty} E_n^k\right) = \lambda\left(\bigcap_{n=1}^{\infty} A \setminus E_n^k\right) = \lim_{n \to \infty} \lambda(A \setminus A_n^k)$$

Thus for all k $\lambda(A \setminus A_n^k) \to 0$. For all $\epsilon > 0$ exists n_k such that

$$n \geq n_k \Rightarrow \lambda\big(A \setminus E_n^k\big) < \frac{\epsilon}{2^k}$$

Denote $E = \bigcup_{k=1}^{\infty}$ (it is measurable).

$$\lambda(A \setminus E) = \lambda\left(\bigcup_k A \setminus E_{n_k}^k\right) \leq \sum \lambda(A \setminus E_{n_k}^k) < \sum \frac{\epsilon}{2^k} = \epsilon$$

For all $x \in E \subseteq E_{n_k}^k \subseteq E_n^k$, thus for all k exists n_k such that

$$|f_k(x) - f(x)| \leq \frac{1}{k}$$

Thus $f_n \to f$ uniformly on E. ☐

Example The condition $\lambda(A) < \infty$ is necessary. Take $A = \mathbb{R}$ and

$$f_n = \chi_{[n,\infty]}$$

$f_n \to 0$ for all x.

4 Lebesgue integral

4.1 Stage 1

Let s simple measurable function we can write

$$s(x) = \sum_{i=1}^{m} a_i \chi_{A_i}(x)$$

Such that $\{A_i\}$ are measurable and disjoint and $\mathbb{R}^n = \bigcup_{i=1}^{n} A_i$ and $0 \le a_i \in \mathbb{R}$. We define

$$\int s \, \mathrm{d}\lambda = \sum_{i=1}^{m} a_i \lambda(A_i)$$

where $0 \cdot \infty = 0$

Proposition 4.1. $\int s \, \mathrm{d}\lambda$ is well-defined

Proof. Directly from 4.5 □

Proposition 4.2.

$$0 \le \int s \, \mathrm{d}\lambda \le \infty$$

Proposition 4.3. For all $0 \le c \in \mathbb{R}$, $\int cs \, \mathrm{d}\lambda = c \int s \, \mathrm{d}\lambda$

Proposition 4.4.

$$\int (s+t) \, \mathrm{d}\lambda = \int s \, \mathrm{d}\lambda + \int t \, \mathrm{d}\lambda$$

Proposition 4.5. $s \le t$ a.e. $\Rightarrow \int s \, \mathrm{d}\lambda \le \int t \, \mathrm{d}\lambda$

Proof. Denote $N = \{x : s(x) > t(t)\}$

$$s(x) = \sum_{i=1}^{m} a_i \chi_{A_i}(x)$$

$$t(x) = \sum_{i=1}^{k} b_i \chi_{B_i}(x)$$

For all i

$$A_i = \bigcup_{j=1}^{k} (A_i \cap B_j)$$

Then

$$\int s \, \mathrm{d}\lambda = \sum a_i \lambda(A_i)$$

$$\lambda(A_i) = \lambda\left(\bigcup_{j=1}^{k} A_i \cap B_j\right) = \sum_{j=1}^{k} \lambda(A_i \cap B_j)$$

$$\int s \, \mathrm{d}\lambda = \sum_{i,j} a_i \lambda(A_i \cap B_j)$$

$$\int t \, \mathrm{d}\lambda = \sum_{i,j} b_j \lambda(A_i \cap B_j)$$

For all i, j,

$$a_i \lambda(A_i \cap B_j) \le b_j \lambda(A_i \cap B_j)$$

If $\lambda(A_i \cap B_j) = 0$ it's obvious. Else $\exists x \in A_i \cap B_j \setminus N$, and for that x $s(x) \le t(x)$ and thus $a_i \le b_j$ and thus $\int s \, \mathrm{d}\lambda \le \int t \, \mathrm{d}\lambda$. □

Proposition 4.6. $s = t$ a.e. $\Rightarrow \int s \, \mathrm{d}\lambda = \int t \, \mathrm{d}\lambda$

Proof. Directly from 4.5 □

Proposition 4.7. If $\alpha \in \mathbb{R}$ and $s'(x) = s(x + \alpha)$, then s' is simple and

$$\int s' \, \mathrm{d}\lambda = \int s \, \mathrm{d}\lambda$$

4.2 Stage 2

Let $f : \mathbb{R}^n \to [0, \infty]$ be a measurable function. We define

$$\int f \, \mathrm{d}\lambda = \sup \left\{ \int s \, \mathrm{d}\lambda : s < \infty, s \leq f, \text{ simple masurable} \right\}$$

Proposition 4.8. $\int f \, \mathrm{d}\lambda$ is well-defined

Proposition 4.9.

$$0 \leq \int f \, \mathrm{d}\lambda \leq \infty$$

Proposition 4.10. For all $0 \leq c \in \mathbb{R}$, $\int cf \, \mathrm{d}\lambda = c \int f \, \mathrm{d}\lambda$

Proposition 4.11. If $f \leq g$ a.e.

$$\int f \, \mathrm{d}\lambda \leq \int g \, \mathrm{d}\lambda$$

Proof. Denote $N = \{x; f(x) > g(x)\}$ and $I = \int f \, \mathrm{d}\lambda$.
For all $\epsilon > 0$ exists s such that

$$\int s \, \mathrm{d}\lambda \geq I - \epsilon$$

$$\tilde{s} = \begin{cases} s(x) & x \notin N \\ 0 & x \in N \end{cases}$$

Then $\tilde{s} \leq g$.

$$I - \epsilon = \int s \, \mathrm{d}\lambda = \int \tilde{s} \, \mathrm{d}\lambda \leq \int g \, \mathrm{d}\lambda$$

Thus

$$\int \tilde{s} \, \mathrm{d}\lambda \geq I$$

□

Proposition 4.12. If $f = g$ a.e.

$$\int f \, \mathrm{d}\lambda = \int g \, \mathrm{d}\lambda$$

Proposition 4.13. If $f'(x) = f(x + \alpha)$ a.e.

$$\int f \, \mathrm{d}\lambda = \int f' \, \mathrm{d}\lambda$$

Theorem 4.14 (Monotone convergence theorem)**.** Let $\{f_n\}$ sequence of measurable functions such that $0 \leq f_1 \leq \ldots$ Let $f(x) = \lim_{n\to\infty} f_n(x)$ then $\int f \, \mathrm{d}\lambda = \lim_{n\to\infty} \int f_n \, \mathrm{d}\lambda$.

Proof. For all n

$$\int f_n \, \mathrm{d}\lambda \leq \int f \, \mathrm{d}\lambda$$

Denote $I = \lim_{n\to\infty} \infty f_n$.
Assume $I < \int f \, \mathrm{d}\lambda$. Then $\exists c \in \mathbb{R}$ such that

$$I < c < \int f \, \mathrm{d}\lambda$$

From definition of $\int f \, \mathrm{d}\lambda$ exists simple t such that $0 \leq t \leq f$ such that $\int t \, \mathrm{d}\lambda > c$. Exists $0 < q < 1$ such that $\int t \, \mathrm{d}\lambda > \frac{c}{q}$
Define $s = qt$, then $c < \int s \, \mathrm{d}\lambda$ and $f(x).s(x)$.

Define

$$E_k = \{x : f_k(x) \geq s(x)\}$$

, then

$$E_1 \subseteq E_2 \subseteq \dots$$

$$\mathbb{R}^n = \bigcup_{k=1}^{\infty} E_k$$

For all k

$$f_k \geq f_k \chi_{E_k} \geq s(x)\chi_{E_k}$$

By writing $s(x) = \sum_{i=1}^{m} a_i \chi_{A_i}$ we get

$$s(x)\chi_{E_k} = \sum a_i \chi_{A_i \cap E_k}$$

Then

$$\int f_k \, \mathrm{d}\lambda \geq \int s(x)\chi_{E_k} \, \mathrm{d}\lambda = \sum a_i \lambda(A_i \cap E_k)$$

But

$$\lambda(A_i) = \lim_{k\to\infty} \lambda(A_i \cap E_k)$$

and

$$I = \lim_{k\to\infty} \int f_k \, \mathrm{d}x = \lim_{k\to\infty} \sum_{i=1}^{m} a_i \lambda(A_i \cap E_k) = \sum_{i=1}^{m} a_i \lambda(A_i) = \int s \, \mathrm{d}x > c$$

□

Collary 4.14.1.

$$\int (f+g) \, \mathrm{d}\lambda = \int f \, \mathrm{d}\lambda + \int g \, \mathrm{d}\lambda$$

Proof. Find $\{s_k\} \to f$, $\{t_k\} \to g$, then

$$\int f + g \, \mathrm{d}\lambda = \lim_{k\to\infty} \int s_k + t_k \, \mathrm{d}\lambda = \lim_{k\to\infty} \int s_k \, \mathrm{d}\lambda + \lim_{k\to\infty} \int t_k \, \mathrm{d}\lambda = \int s \, \mathrm{d}\lambda + \int t \, \mathrm{d}\lambda$$

□

Collary 4.14.2. Let $\{f_k\}$ sequence of measurable functions, $f_k \geq 0$ on $\mathbb{R}^n$ then

$$\int \left[\sum_{k=1}^{\infty} f_k\right] \mathrm{d}\lambda = \sum_{k=1}^{\infty} \int f_k \, \mathrm{d}\lambda$$

Theorem 4.15 (Fatou's lemma)**.** Let $\{f_k\}_{k=1}^{\infty}$ a sequence of nonegative measurable functions.

$$\int \liminf f_k \, \mathrm{d}\lambda \leq \liminf \int f_k \, \mathrm{d}\lambda$$

Proof.

$$\liminf f_k(x) = \sup_m \inf_{j\geq m} f_j(x)$$

$$g_m(x) = \inf_{j \geq m} f_j(x)$$

Since $0 \leq g_m \leq g_{m+!}$ and $g_m \leq f_m$,

$$\liminf f_k(x) = \sup_m g_m(x) = \lim g_m(x)$$

$$\int \liminf f_k \, \mathrm{d}\lambda = \int \lim_{m\to\infty} g_m \, \mathrm{d}\lambda = \int \lim_{m\to\infty} g_m \, \mathrm{d}\lambda \leq \liminf \int f_m \, \mathrm{d}\lambda$$

□

4.3 Stage 3

Definition 4.1. Let $f : \mathbb{R}^n \to [-\infty, \infty]$ measurable. f is called integrable if $\int f^+(x)\, d\lambda < \infty$ and $\int f^-(x)\, d\lambda < \infty$ and in this case

$$\int f\, d\lambda = \int f^+(x)\, d\lambda - \int f^-(x)\, d\lambda$$

Set of all integrable functions is denoted $\mathcal{L}^!(\mathbb{R}^n)$.

Proposition 4.16.

$$\int |f|\, d\lambda \leq \infty \iff f \in \mathcal{L}^1$$

Proof. $f \in \mathcal{L}^1$, then

$$\int |f|\, d\lambda = \int (f_+ + f_-)\, d\lambda = \int f_+\, d\lambda + \int f_-\, d\lambda < \infty$$

$f_+, f_- \leq |f|$, then $\int f_+\, d\lambda, \int f_-\, d\lambda \leq \int |f|\, d\lambda < \infty$. □

Proposition 4.17.

$$\int |f|\, d\lambda \geq \left| \int f\, d\lambda \right|$$

Proof.

$$\left| \int f\, d\lambda \right| = \left| \int f_+\, d\lambda + \int f_-\, d\lambda \right| \leq \int f_+\, d\lambda + \int f_-\, d\lambda = \int |f|$$

□

Proposition 4.18. If $f \in \mathcal{L}^1$ then

$$\lambda(f^{-1}(-\infty)) = \lambda(f^{-1}(\infty)) = 0$$

Proof. Let $A = \lambda(f^{-1}(\infty))$ and $\lambda(A) > 0$. $f_+ \geq \infty \cdot \chi_A$.
Then $\int f_+\, d\lambda \geq \int \infty \cdot \chi_A\, d\lambda = \infty$. □

Proposition 4.19. If $f = g$ a.e. and $f \in \mathcal{L}_1$, then $g \in \mathcal{L}_1$ and $\int f\, d\lambda = \int g\, d\lambda$.

Proposition 4.20. If $f \in \mathcal{L}_1$ and $\int |f|\, d\lambda = 0$, then $f = 0$ a.e.

Proof. Define

$$A_k = \left\{ x : |f(x)| > \frac{1}{k} \right\}$$

$$|f| \geq |f|\chi_{A_k} \geq \frac{1}{k}\chi_{A_k}$$

$$0 = \int |f|\, d\lambda \geq \frac{1}{k} \int \chi_{A_k}$$

Thus $\lambda(A_k) = 0$ and thus

$$\lambda(\bigcup A_k) = 0$$

□

Proposition 4.21. $\mathcal{L}^1$ is vector space.

Proof. For $a \geq 0$,

$$af = af_+ + af_-$$

$$\int af\, d\lambda = \int af_+ - \int af_- = a \int f_+ - a \int f_- = a \int f$$

For $a < 0$, we note that $(af)_\pm = -af_\mp$.
Let $h = f + g$

$$h_+ - h_- = h = (f_+ - f_-) + (g_+ - g_-)$$

$$h_+ + f_- + g_- = h_- + f_+ + g_+$$

$$\int h_+ + \int f_- + \int g_- = \int h_- + \int f_+ + \int g_+$$

Then $h_+ \leq f_+ + g_+$ and $\int h_+ < \infty$, i.e. $h \in \mathcal{L}^1$. □

Theorem 4.22 (Monotonic convergence theorem)**.** Let $\{f_n\}$ sequence of integrable functions such that $f_1 \leq f_2 \leq \ldots$ and $\sup_n \int f_n \,\mathrm{d}\lambda < \infty$ Let $f(x) = \lim_{n\to\infty} f_n(x)$ then $\int f \,\mathrm{d}\lambda = \lim_{n\to\infty} \int f_n \,\mathrm{d}\lambda$.

Proof. Define $g_n = f_n - f_1$ then g_n are non-negative and thus converge as we've shown. □

Theorem 4.23 (Dominated convergence theorem)**.** Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}^n$ such that exists $0 \leq g \in \mathcal{L}^1$ such that $|f_k(x)| \leq g_x$ a.e.
Let

$$f(x) = \lim_{k\to\infty} f_k(x)$$

then

$$\int f \,\mathrm{d}\lambda = \lim_{k\to\infty} \int f_k \,\mathrm{d}\lambda$$

Proof. □

4.4 Riemann integral

Reminder: Riemann integral in rectangle is limit of Darboux sums: for I_j subrectangle

$$U(P) = \sum M_j \lambda(I_j)$$

$$D(P) = \sum m_j \lambda(I_j)$$

where m_j and M_j are supremum and infimum on subrectangle. f is integrable if sums converge to same number.
Define

$$\tau_P(x) = \begin{cases} M_j & x \in \operatorname{int} I_j \\ M & \text{otherwise} \end{cases}$$

$$\sigma_P(x) = \begin{cases} m_j & x \in \operatorname{int} I_j \\ m & \text{otherwise} \end{cases}$$

σ, τ are simple, Lebesgue integrable, in particular measurable, and Riemann integrable and continuous a.e.

$$\int \sigma_p \,\mathrm{d}\lambda = L(P) = \int \sigma_p \,\mathrm{d}x$$

$$\int \tau_p \,\mathrm{d}\lambda = U(P) = \int \sigma_p \,\mathrm{d}x$$

and $\sigma_p \leq f \leq \tau_p$.

Theorem 4.24. Let f bounded ($|f| \leq M$) on special box I then

1. f is Riemann integrable iff f continuous a.e.
2. If f is Riemann integrable it is measurable and

$$\int f \,\mathrm{d}\lambda = \int f \,\mathrm{d}x$$

Proof. Suppose f is Riemann integrable, for each k exists partition P_k such that $U(P_k) - L(P_k) < \frac{1}{k}$ and $P_{k-1} \subset P_k$. Denote σ_k, τ_k. $\int \tau_k \,\mathrm{d}\lambda = U(P_k)$ and $\tau_k \geq f$.
In every point of continuity of σ_p, $\sigma_k \leq f$ thus

$$\sigma_k \leq \underline{\sigma_k} \leq \underline{f}$$

Thus

$$\sigma_k \leq \underline{f} \leq \bar{f} \leq \tau_k$$

a.e. Obviously $\sigma_{k+1} \geq \sigma_k$ and $\tau_{k+1} \leq \tau_k$.
Define $g = \sup \sigma_k$ and $h = \inf \tau_k$. By definition

$$g \leq \underline{f} \leq \bar{f} \leq h$$

a.e., and $h - g \leq \tau_k - \sigma_k$.

$$\int (h-g) \,\mathrm{d}\lambda \leq \in t\tau_k - \sigma_k \,\mathrm{d}\lambda = U(P_k) - L(P_k) < \frac{1}{k}$$

$$\int (h-g)\,\mathrm{d}\lambda = 0$$

Thus $h = g$ a.e., and $\underline{f} = \bar{f}$ a.e., thus f continuous a.e.

$$\int_I f\,\mathrm{d}x \leq U(P_k) < L(P_k) + \frac{1}{k} = \int \sigma_k\,\mathrm{d}\lambda + \frac{1}{k} \leq \int f\,\mathrm{d}\lambda + \frac{1}{k}$$

$$\int_I f\,\mathrm{d}x \geq L(P_k) > U(P_k) - +\frac{1}{k} = \int \tau_k\,\mathrm{d}\lambda - \frac{1}{k} \geq \int f\,\mathrm{d}\lambda - \frac{1}{k}$$

Thus

$$\int f\,\mathrm{d}\lambda - \frac{1}{k} \leq \int f\,\mathrm{d}x \leq \int f\,\mathrm{d}\lambda + \frac{1}{k}$$

i.e., $\int f\,\mathrm{d}\lambda = \int f\,\mathrm{d}x$

Suppose f continuous a.e. denote by P_k partition of each rectangle side to 2^k equal parts. We know that

$$\sigma_1 \leq \sigma_2 \leq \cdots \leq f$$

For x which is not on the boundary,

$$\underline{f}(x) \leq \lim_{k\to\infty} \sigma_k(x)$$

else there is t such that

$$\lim_{k\to\infty} \sigma_k(x) < t < \underline{f}(x)$$

Thus exists $\delta > 0$ such that $\inf\{f(y) : y \in B(x,\delta)\} > t$ and

$$\lim_{k\to\infty} \sigma_k(x) < t < f(y)$$

However we can increase k such that the partition will include the whole δ-neighborhood, and thus there is contradiction with the definition of the σ. Thus

$$\underline{f}(x) \leq \lim_{k\to\infty} \sigma_k(x)$$

and similarly

$$\bar{f}(x) \geq \lim_{k\to\infty} \tau_k(x)$$

a.e.

Since f is continuous a.e., $f = \underline{f} = \bar{f}$ a.e. Thus

$$\lim_{k\to\infty} \sigma_k(x) \geq f \geq \lim_{k\to\infty} \tau_k(x)$$

a.e. However $\sigma_k \leq \tau_k$, i.e.

$$\lim_{k\to\infty} \sigma_k(x) = f = \lim_{k\to\infty} \tau_k(x)$$

a.e.

$$f = \lim_{k\to\infty} \int \sigma_k(x)\,\mathrm{d}\lambda = \lim_{k\to\infty} \int \tau_k(x)\,\mathrm{d}\lambda$$

where we switch limit and integral by DCT with $M\chi$. We got that

$$\lim U(P_k) = \lim L(P_k)$$

and thus f is Riemann integrable. □

Proposition 4.25. The Riemann and Lebesgue integrals are equivalent only for bounded functions on finite intervals.

Theorem 4.26. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $A \subseteq \mathbb{R}^n$. Then

$$\lambda^*(TA) = |\det\{T\}|\lambda^*(A)$$

$$\lambda_*(TA) = |\det\{T\}|\lambda_*(A)$$

and if A is measurable, TA is measurable and

$$\lambda(TA) = |\det\{T\}|\lambda(A)$$

Proof. If T is not invertible, then dimension of the image is less than dimension of the space, and thus

$$\lambda^*(TA) = \lambda_*(TA) = 0$$

and since $\det T = 0$, we are done.
If T is not invertible, denote matrix corresponding to T as B. We can rewrite B as

$$B = E_1 E_2 \dots E_n$$

. Assuming A is special rectangle, without loss of generality it is

$$A = [0, b_1] \times [0, b_2] \times \dots \times [0, b_n] \times$$

If E_i multiplies row by c then

$$EA = [0, b_1] \times [0, b_2] \times \dots \times [0, cb_i] \times \dots \times [0, b_n] \times$$

$$\lambda(EA) = |c| \lambda(A) = |\det E| \lambda(A)$$

If E_i swaps two rows the area doesn't change. If E_i adds row to another row, we get a parallelogram whose area still equals to original.
We continue step-by-step for steps of building of Lebesgue measure. □

Theorem 4.27. Let T invertible linear transformation on $\mathbb{R}^n$ and f function on $\mathbb{R}^n$.

1. If f is measurable $f \circ T$ is measurable
2. If $f \geq 0$ measurable then $\int f \, d\lambda = |\det T| \int f(Tx) \, d\lambda$.
3. If f is measurable then $\int f \, d\lambda = |\det T| \int f(Tx) \, d\lambda$.

Theorem 4.28. Let T linear transformation and f function defined on $\mathbb{R}^n$ then

1. If f measurable then so is $f \circ T$
2. If f is measurable and $f \geq 0$ then $\int f \, d\lambda = |\det(T)| \int f(Tx) \, d\lambda$.
3. If $f \in \mathcal{L}^1$ then $f \circ T \in L^1$.

5 $L^1(\mathbb{R}^n)$ space

Definition 5.1. Vector space X over $\mathbb{R}$ is called norm space if exists function $\|\cdot\| : X \to \mathbb{R}$ such that

1. $\|x\| \geq 0$
2. $\|x\| = 0 \iff x = 0$
3. For $c \in \mathbb{R}$ $\|cx\| = |c| \|x\|$
4. $\|x + y\| \leq \|x\| + \|y\|$

Norm space is metric space with metric $d(x, y) = \|x - y\|$.
Definition 5.2 (Convergence)**.** $x_n \to x$ if $d(x_n, x) \to 0 \iff \|x_n - x\| \to 0$.
If $x_n \to x$ then $\|x_n\| \to \|x\|$.
Definition 5.3 (Open and closed balls)**.** Open ball: $B(x, r) = \{y \in X | \|y - x\| < r\}$ Closed ball: $B(x, r) = \{y \in X | \|y - x\| \leq r\}$
Definition 5.4 (Open set)**.** $A \subset X$ is open if $\forall a \in A \exists r > 0$ such that $B(a, r) \subseteq A$.
Definition 5.5 (Closed set)**.** $A \subset X$ is open if $A \ni x_n \to x \Rightarrow x \in A$.
Definition 5.6 (Closure)**.** $\bar{A}$, the closure of A:

$$\bar{A} = \{x : \exists x_n \in A, x_n \to x\}$$

Definition 5.7 (Dense set)**.** A is dense if $\bar{A} = X$.

For $f \in \mathcal{L}^1(\mathbb{R}^n)$ denote

$$\|f\|_1 = \int |f| \, \mathrm{d}\lambda$$

Note that this is almost norm, except that there exists non-zero functions with norm 0. To fix it, we define

$$N = \left\{ f \in \mathcal{L}^1 : f = 0 \text{ a.e.} \right\}$$

and

$$L_1(\mathbb{R}^n) = \mathcal{L}^1(\mathbb{R}^n)/N$$

i.e., we define equivalence relation, $f \sim g$ if $f = g$ a.e.
Usually we'll look at members of L_1 as functions and not as equvivalence classes. However, it's impossible to talk about value or limit of function in point in this case. Now we can define

$$\|f\|_1 = \int |f| \, \mathrm{d}\lambda$$

norm on $L_1(\mathbb{R}^n)$.

Definition 5.8. $C_c(\mathbb{R}^n)$ are continuous functions with compact support, i.e., $f = 0$ outside of some ball.

Definition 5.9. $[f] \in L_1(\mathbb{R}^n)$ is continuous if there is continuous $f \in [f]$.

6 Question 1

$L^1_C(\mathbb{R}^n)$ - functions in L^1 with compact support a.e.
$\mathcal{C}^m$ - function with continuous partial derivatives up to order m.
$\mathcal{C}^m_c$ - function in $\mathcal{C}^m$ with constant support.
$\mathcal{C}^\infty = \bigcap_{k=1}^\infty \mathcal{C}^k$.
Convergence in L^1:

$$f_n \overset{L^1}{\to} f \Rightarrow \int |f_n - f| \, \mathrm{d}\lambda \to 0$$

If $Y_1 \subset Y_2$ then $\bar{Y}_1 \subset \bar{Y}_2$. Also $\bar{\bar{Y}} = \bar{Y}$.

Lemma 6.1. If $f_n \to f$ a.e. and $|f_n| < |f|$ and $\{f_n, f\} \subseteq L^1$ then $f_n \overset{L^1}{\to} f$.

Proof. Since

$$\int |f_n - f| \, \mathrm{d}\lambda \overset{a.e.}{\to} 0$$

$$|f_n - f| \leq |f_n| + |f| \leq 2|f| \in L^1$$

From dominated convergence,

$$\int |f_n - f| \, \mathrm{d}\lambda = 0$$

□

Lemma 6.2 (Urysohn's lemma)**.** Let B,C closed disjoint sets, then exists continuous $f : \mathbb{R}^n \to [0,1]$ such that $f\big|_C = 1$ and $f\big|_B = 0$.

Lemma 6.3. Simple function are dense in L^1.

Proof. First lets show that L^1_c is dense in L^1. Let $f \in L^1$ and denote

$$f_n = f\chi_{B_0(n)}$$

Since $f_n \overset{a.e.}{\to} f$ and $|f_n| \leq |f|$ we get

$$L^1_c \ni f_n \overset{L^1}{\to} f$$

Thus L^1_c is dense in L^1.
Now lets show that simple functions are dense in L^1_c. It's enough to show that for nonnegative functions. For each function $f \in L^1_c$ exists sequence $s_k \geq 0$ such that $s_k \overset{a.e.}{\to} f$ and $|s_k| \leq |f|$ we get

$$s_k \overset{L^1}{\to} f$$

□

Lemma 6.4. C_c is dense in L^1.

Proof. We want to show that any nonnegative simple function with compact support can be approximated with functions from C_c. It's enough to show that for χ_A, where A is closed and measurable. Suppose $A \subseteq B_0(n)$. $\forall \epsilon > 0$ exists open G and compact K such that

$$K \subseteq A \subseteq G$$

$G \subseteq B_0(n)$.
From 6.2, exists $g : \mathbb{R}^n \to [0,1]$ such that $g\big|_K = 1$ and $g\big|_{\mathbb{R}^n \setminus G} = 0$. $g \in C_c$.

$$g(x) - \chi_A(x) = \begin{cases} 0 & x \in K \\ 0 & x \in \mathbb{R}^n \setminus G \\ \leq 1 & x \in G \setminus K \end{cases}$$

$$\int |g(x) - \chi_A(x)| \, \mathrm{d}\lambda = \int\limits_{G \setminus K} |g(x) - \chi_A(x)| \, \mathrm{d}\lambda \leq \lambda(G \setminus K) < \epsilon$$

□

Theorem 6.1. Let $f \in L_1$ and $f_y(x) = f(x+y)$

$$\|f_y - f\| \overset{y \to 0}{\to} 0$$

Proof. Let $f \in L^1$ and $\epsilon > 0$. Exists $g \in C_c$ such that $\|f - g\| < \frac{\epsilon}{3}$. Suppose $g = 0$ outside of $B_0(r)$.

$$\int |f - g| \, \mathrm{d}\lambda < \frac{\epsilon}{3}$$

But

$$\int |f_y - g_y| \, \mathrm{d}\lambda < \frac{\epsilon}{3}$$

i.e., $\|f_y - g_y\| < \frac{\epsilon}{3}$. g is uniformly continuous. Thus exists $1 > \delta > 0$ such that if $\|y\| < \delta$ then

$$|g(x+y) - g(x)| < \frac{\epsilon}{3\lambda(B_0(r+1)}$$

For $\|x\| > r+1$

$$|g(x+1) - g(x)| = 0$$

Thus for $\|y\| < \delta$

$$\int |g_y(x) - g(x)| \, \mathrm{d}\lambda = \int\limits_{B_0(r+1)} |g(x+y) - g(x)| \leq \lambda(B_0(r+1)) \frac{\epsilon}{3\lambda(B_0(r+1)} = \frac{\epsilon}{3}$$

Thus

$$\|f_y - f\| \leq \|f_y - g_y\| + \|g_y - g\| + \|g_y - f\| < \epsilon$$

□

Definition 6.1. We say that $f_n \to f$ in measure if

$$\lambda(\{x : |f_n(x) - f(x)| \geq \epsilon\}) = 0$$

Note that $f_n \overset{L^1}{\to} f \Rightarrow f_n \overset{\text{measure}}{\to} f$.
Definition 6.2. $\{x_n\}$ in metric space X is called Cauchy sequence if for all $\epsilon > 0$ exists N such that for all $n, m > N$

$$\|x_n - x_m\| < \epsilon$$

Definition 6.3. Metric space is called complete space if each Cauchy sequence converges.
Definition 6.4. Complete normed space is called Banach space.
Theorem 6.2. L^1 is Banach space.

Proof. Let $\{f_n\}$ Cauchy sequence in L^1. For all $\epsilon > 0$ exists $N(\epsilon)$ such that $\|f_n - f_m\| < \epsilon$ for $n, m > N$. We can require that $\left\{N\left(\frac{1}{2^k}\right)\right\}$ is non-decreasing sequence. Denote $n_k = N\left(\frac{1}{2^k}\right)$ and $g_k = f_{n_k}$.
For all k:

$$\int |g_k - g_{k+1}| = \|g_k - g_{k+1}\| = \left|f_{n_k} - f_{n_{k+1}}\right| < \frac{1}{2^k}$$

Thus

$$\sum_{k=1}^{\infty} \int |g_k - g_{k+1}| < \infty$$

We've shown that from that we can conclude that

$$h(x) = \sum_{k=1}^{\infty} (g_{k+1} - g_k)$$

exists and $h \in L^1$.

$$\int h \, \mathrm{d}\lambda = \sum_{k=1}^{\infty} g_{k+1} - g_k \, \mathrm{d}\lambda$$

$$h(x) = \lim_{N \to \infty} \sum_{k=1}^{N} g_{k+1} - g_k = \lim_{N \to \infty} g_N(x) - g_1(x)$$

Thus

$$\lim_{N \to \infty} g_N(x) = h(x) + g_1(x)$$

On the other hand

$$\sum_{k=m}^{\infty} g_{k+1}(x) - g_k(x) = \lim_{N \to \infty} g_N(x) - g_m(x) = g_1(x) + h(x) - g_m(x)$$

$$\|g_1 + h - g_m\| = \left\| \sum_{k=m}^{\infty} g_{k+1} - g_k \right\| \le \sum \|g_k - g_{k+1}\| \le \sum_{k=m}^{\infty} \frac{1}{2^k} \overset{m \to \infty}{\to} 0$$

Thus $g_m \overset{L_1}{\to} g_1 + h$ meaning partial sequence of $\{f_n\}$ converges and thus $\{f_n\}$ converges. $\square$

Collary 6.2.1. If $f_n \overset{L_1}{\to} f$ then exists subsequence $\{f_{n_k}\}$ such that $f_n \overset{\text{a.e.}}{\to} f$

Proof. $g_k = f_{n_k} \to g_1 + h$ a.e. and $g_k = f_{n_k} \overset{L_1}{\to} g_1 + h$ but $g_k \overset{L_1}{\to} f_1$ thus $f = g_1 + h$ a.e. and thus $g_k \to f$ a.e. $\square$

Proposition 6.3. In normed space if exists $\sum x_n$ then

$$\left\| \sum x \right\| \le \sum \|x\|$$

Note that since there is unique completion of metric space we could define L^1 differently. Start with $X = C_c(\mathbb{R}^n)$ with Riemann integral as a metric. Its completion is L^1 and we can define integral as a limit of Riemann integrals.

6.1 Parameter-dependent integrals

Let $J \subseteq \mathbb{R}$ and $f : \mathbb{R}^n \times \mathbb{R} \to [-\infty, \infty]$ and

$$f_t : x \mapsto f(x, t) \in \mathbb{L}^1(\mathbb{R}^n)$$

Denote

$$F(t) = \int f_t(x) \, \mathrm{d}x$$

Theorem 6.4. Let f, F. Suppose $h \in L^1(\mathbb{R}^n)$ such that

$$\forall x, t \quad |f(x, t)| \le g(x)$$

Let $t_0 \in J$. Suppose for almost every x $t \mapsto f(x, t)$ if continuous in t_0. Then F is continuous in t_0.

Proof. We want to show that

$$\lim F(t_n) = F(t_0)$$

for $t_n \to t_0$.

$$F(t_n) = \int f_{t_n} \,\mathrm{d}\lambda$$

$$F(t_0) = \int f_{t_0} \,\mathrm{d}\lambda$$

also, $f_{t_n} \to f_{t_0}$ a.e., from DCT we get the required. $\square$

Theorem 6.5. Let J open and assume that for almost all x $(x \notin N)$ $t \mapsto f_t(x)$ differentiable for all $t \in J$. Also assume that exists $h \in L^1$ such that for all x and t

$$\frac{\partial f}{\partial t}(x,t) \leq (x)$$

Then F is differentiable in J and

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t) = \int \frac{\partial f}{\partial t}(x,t) \,\mathrm{d}\lambda(x)$$

Proof. For all $t \in J$ $\frac{\partial f}{\partial t}(x,t)$ is measurable as a limit of measurable functions. Choose $t \in J$ and $\delta > 0$ such that $t + s \in J$ for all $|s| < \delta$. Define

$$g(x,s) = \begin{cases} \frac{\partial f}{\partial t}(x,t) & x \notin N, s = 0 \\ \frac{f(x,t+s)-f(x,t)}{s} & x \notin N, s \neq 0 \\ 0 & x \in N \end{cases}$$

Now

$$\int g(x,0) \,\mathrm{d}\lambda(x) = \int \frac{\partial f}{\partial t}(x,t) \,\mathrm{d}\lambda(x)$$

$$\int g(x,s) \,\mathrm{d}\lambda(x) = \frac{F(t+s)+F(t)}{s}$$

Thus we need

$$\lim_{s \to 0} \int g(x,s) \,\mathrm{d}\lambda(x) = \int g(x,0) \,\mathrm{d}\lambda(x)$$

Note that $s \mapsto g(x,s)$ is continuous in $s = 0$ and $x \mapsto g(x,s)$ is integrable for all s. Also $|g(x,s)| \leq h(x)$ (for $s > 0$ from Lagrange) and thus from previous theorem we get the required. $\square$

Functions in C_c^∞

$$h(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

h is differentiable infinite times

$$h^{(n)} = P_{2n}\left(\frac{1}{t}\right)e^{-\frac{1}{t}}$$

Define

$$\tilde{\phi}(x) = h(1 - x_1^2 - x_2^2 - \ldots x_n^2) \in C^\infty(\mathbb{R}^n)$$

Note that $\tilde{\phi}(x) = 0$ if $\|x\| \geq 1$, thus $\tilde{\phi}(x) \in C_c^\infty(\mathbb{R}^n)$ denote $C = \int \tilde{\phi} \,\mathrm{d}\lambda$ and define

$$\phi(x) = \frac{1}{C}\tilde{\phi}(x)$$

We get $\phi(x) \in C_c^\infty(\mathbb{R}^n)$, $\phi \geq 0$, $\int \phi \,\mathrm{d}\lambda = 1$ and $\phi(x) > 0$ iff $\|x\| < 1$. Let $f : \mathbb{R}^n = \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$. Denote for $y \in \mathbb{R}^m$ and $x \in \mathbb{R}^l$ $f_y(x) = f(x,y)$

Lemma 6.5. If the theorem is right for a sequence of functions $0 \leq f_1 \leq f_2 \ldots$ and $f = \lim_{j\to\infty} f_j$ then the theorem is right for f.

Proof. For all $y \in \mathbb{R}^m$ $(f_j)_y \uparrow f_y$, thus f is measurable for almost all y:

$$F_j(y) = \int (f_j)_y \, \mathrm{d}x \uparrow \int f_y \, \mathrm{d}x = F(y)$$

$$\int\limits_{\mathbb{R}^m} F \, \mathrm{d}y = \lim_{j \to \infty} \int F_j \, \mathrm{d}y = \lim_{j \to \infty} \int f_j \, \mathrm{d}\lambda = \int f \, \mathrm{d}\lambda$$

□

Theorem 6.6 (Fubini theorem)**.** Let $f : \mathbb{R}^n \to [-, \infty]$ be a measurable function. THen for almost all $y \in \mathbb{R}^m$ a function f_y is measurable and thus $F(y) \int\limits_{\mathbb{R}^l} f_y(x) \, \mathrm{d}\lambda(x)$ and F is measurable on $\mathbb{R}^m$ and

$$\int\limits_{\mathbb{R}^m} F(y) \, \mathrm{d}\lambda(y) = \int\limits_{\mathbb{R}^n} f \, \mathrm{d}\lambda$$

7 Differentiation and integration in $\mathbb{R}$

Definition 7.1. Let $E \subseteq \mathbb{R}$ and M set of closed intervals of positive lengths. M is called Vitali cover if for all $\epsilon > 0$ and $\forall x \in E$ $\exists I \in M$ such that $x \in I$ and $\lambda(I) < \epsilon$.

Theorem 7.1 (Vitaly)**.** Let $E \subset \mathbb{R}$, $\lambda^*(E) < \infty$ and M Vitaly cover of E. Then either E is contained in finite disjoint union $\bigcup_{k=1}^N I_K$ for $I_k \in M$ or exists $\{I_k\}_{k=0}^\infty \subset M$ such that $I_k \cap I_j = \emptyset$ and for all $\epsilon > 0$ exists $N = N(\epsilon)$ such that $\lambda^*(E \setminus \bigcup^N I_k) < \epsilon$

Definition 7.2. Let g defined in neighbourhood of x_0. Define

$$\liminf_{x \to x_0} g(x) = \sup_{\delta > 0} \left(\inf \{ g(y) : 0 < |y - x_0| < \delta \} \right)$$

$$\limsup_{x \to x_0} g(x) = \inf_{\delta > 0} \left(\sup \{ g(y) : 0 < |y - x_0| < \delta \} \right)$$

Lemma 7.1.

1. $$\liminf g \leq \limsup g$$
2. If $g(x) \geq M$
$$\liminf g \geq M$$
3. If $g(x) \leq M$
$$\limsup g \leq M$$
4. $\lim g$ exists if $\liminf g = \limsup g$

Definition 7.3.

$$UDf(x) = \limsup_{h \to 0} [f(x+h) - f(x)]$$

$$LDf(x) = \liminf_{h \to 0} [f(x+h) - f(x)]$$

Lemma 7.2. Let $A \subseteq [a, b]$ such that for all $x \in A$, $0 \leq LDf(x) \leq r$ then

$$\lambda^*(f(A)) \leq r \lambda^*(A)$$

Proof. Let $\epsilon > 0$ and G open such that $A \subseteq G$ and $\lambda(G) \leq \lambda^*(A) + \epsilon$. Thus

$$r. \sup_{\delta > 0} \left(\inf \left\{ \frac{1}{h} [f(x+h) - f(x)] : 0 < |h| < \delta \right\} \right)$$

For all $x \in A$ choose $\delta_0 > 0$ such that $(x - \delta_0, x + \delta_0) \subseteq G$. For all $\delta < \delta_0$ exists $0 < |h| < \delta$ such that $\frac{1}{h}[f(x+h) - f(x)] < r$. Denote

$$M = \left\{ [f(x), f(x+h)] : [x, x+h] \subseteq G, \; \frac{1}{h}[f(x+h) - f(x)] < r \right\}$$

Then M is Vitali cover of $f(A)$, since

$$\lambda([f(x), f(x+h)]) < |h| r < \delta r$$

By theorem, exists

$$\{[f(x+h)-f(x)]\} \subseteq M$$

of disjoint intervals such that

$$\lambda^*\Big(f(A) \setminus \bigcup [f(x_n), f(x_n+h_n)]\Big) = 0$$

Thus

$$\lambda^*(f(A)) \leq \lambda^*\Big(f(A) \setminus \bigcup [f(x_n), f(x_n+h_n)]\Big) + \lambda^*\Big(\bigcup [f(x_n), f(x_n+h_n)]\Big) \leq \sum \lambda[f(x_n), f(x_n+h_n)] = \sum |[f(x_n) - f(x_n+h_n)| \leq \sum$$

□

Lemma 7.3. Let $A \subseteq [a,b]$ such that for all $x \in A$

$$UDf(x) > s > 0$$

then $\lambda^*(f(A)) \geq s\lambda^*(A)$

Proof. Let G open. $f(A) \subseteq G$ and $\lambda(G) \leq \lambda^*(A) + \epsilon$. Since f has only countable number of discontinuity points, we can remove a null set of such points and assume f is continuous on A.
Thus, for all $x \in A$ exists $\delta_0(x) > 0$ such that

$$[f(x-\delta), f(x+\delta)] \subseteq G$$

for all $0 < \delta < \delta_0$ and for all δ_0 exists $|h|$ such that $0 < |h| < \delta$ and

$$s < \frac{1}{h}[f(x+h) - f(x)]$$

and $[f(x), f(x+h)] \subseteq G$.
Let

$$M = \{[x, x+h]\}$$

which is Vitali cover.

$$\lambda^*(A) \leq \lambda^*\Big(A \setminus \bigcup [x_n, x_n+h_n]\Big) + \lambda^*\Big(\bigcup [x_n, x_n+h_n]\Big) = \leq \lambda \bigcup [x_n, x_n+h_n] = \sum |h_n| \leq \frac{1}{s}\sum |f(x_n+h_n) - f(x_n)| < \frac{1}{s}\lambda(G) \leq \frac{1}{s}\lambda^*($$

□

Collary 7.1.1. If f is monotonic on $[a,b]$ and $A = \{x \in [a,b] : UDf(x) = \infty\}$ then $\lambda^*(A) = 0$.

Proof. For all $s > 0$

$$\infty > f(b) - f(a) = \lambda([f(a), f(b)]) = \lambda^*(f([a,b])) \geq \lambda^*(f(A)) \geq s\lambda^*(A)$$

Thus $\lambda^*(A) = 0$. □

Theorem 7.2 (Lebesgue)**.** If f is monotoneous on $[a,b]$ it is differentiable a.e. there.

Proof. Assume f is non-decreasing and if we replace f with $f(x) + x$ it is increasing. From corollary, $UDf(x) < \infty$ a.e. and it has to be $UDf(x) = LDf(x)$ a.e.
For all $r < s$ in $\mathbb{Q}$ define

$$A_{r,s} = \{x : LDf(x) < r < s < UDf(x)\}$$

$$\bigcup A_{r,s} = \{x : LDf(x) < UDf(x)\}$$

It's enough to show that for any r, s $\lambda^*(A_{r,s}) = 0$.
From lemmas we get

$$s\lambda^*(A_{r,s}) \leq \lambda^*(fA_{r,s}) \leq r\lambda^*(A_{r,s})$$

Bur $r < s$ and thus $\lambda^*(A_{r,s}) = 0$. □

Collary 7.2.1. For all $f \in L^1[a,b]$ the function $F(x) = \int_a^x f \,\mathrm{d}\lambda$ os differentiable a.e.

Proof. Use f_+ and f_-. □

Proposition 7.3. Let f non-decreasing on $[a,b]$ then f' is measurable and integrable and

$$\int_a^b f' \,\mathrm{d}\lambda \leq f(b) - f(a)$$

Proof. Define f to be $f(b)$ on $[b, b+1]$ from Lebesgue theorem f' exists a.e. If f' exists,

$$f'(x) = \lim_{n\to\infty} \underbrace{n\left[f\left(x+\frac{1}{n}\right) - f(x)\right]}_{g_n(x)\geq 0}$$

g_n is measurable and so is f'. By Fatou lemma

$$\int_a^b f' \,\mathrm{d}\lambda \leq \liminf \int g_n \,\mathrm{d}\lambda = \liminf \left(n\int_a^b \left[f\left(x+\frac{1}{n}\right) - f(x)\right] \mathrm{d}\lambda\right) = \liminf n \int_a^b f\left(x+\frac{1}{n}\right) \mathrm{d}\lambda - n\int_a^b f(x) \,\mathrm{d}\lambda = \liminf n \int_{a+\frac{1}{n}}^{b+\frac{1}{n}} f(x$$

$\square$

7.1 Functions of bounded variation

Definition 7.4. Let f real on $[a,b]$ and $T = \{T_i\}_{i=0}^n$ partition of $[a,b]$. Denote

$$V_f(T) = \sum_{i=1}^n |f(t_i) - f(t_{i-1})|$$

And

$$V_a^b f = \sup_T \{V_F(T)\}$$

variation of f

Definition 7.5. We say f has bounded variation if $f \in BV[a,b]$. Note that $V_a^b f + V_b^c f = V_a^c f$.

Proposition 7.4. Let $f \in BV[a,b]$ and define for all $x \in [a,b]$

$$g(x) = V_a^x f$$

Then g is monotonous non decreasing, and if f is continuous from left/right, so is g.

Proof.

$$V_a^y f = V_a^x f + V_x^y f$$

Now suppose f is continuous from the left in x_0. Choose partition of $[a, x_0]$ such that $V_f(T) > V_a^{x_0} f - \frac{\epsilon}{2}$. From continuity from the left exists $t_{n-1} < y < x_0$ such that $|f(x_0) - f(y)| < \frac{\epsilon}{2}$ Now

$$g(y) = V_a^y f \geq \sum_{i=1}^{n-1} |f(t_i) - f(t_{i-1})| + |f(y) - f(t_{n-1})| > \sum_{i=1}^{n-1} |f(t_i) - f(t_{i-1})| + |f(y) - f(t_{n-1})| + |f(x_0) - f(y)| - \frac{\epsilon}{2} \geq V_f(T) - \frac{\epsilon}{2} > g(x_0) -$$

Now let $y \leq z \leq x_0$ then $|g(x_0) - g(z)| = g(x_0) - g(z) \leq g(x_0) - g(y) < \epsilon$ $\square$

Theorem 7.5. $f \in BV[a,b]$ iff f is difference of two non-decreasing monotonic functions. If f is continuous we get difference of two continuous functions then $f \in BV[a,b]$ and f differentiable a.e. and Reimann integrable.

Proof. $f(x) = V_a^x(f) - [V_a^x f - f(x)]$, we need $v_a^x f - f(x) = h(x)$ monotonous non-decreasing. If $x < y$

$$f(y) - f(x) \leq |f(x) - f(y)| \leq V_x^y f$$

$$h(y) - h(x) = V_a^y - f(y) - V_a^x + f(x) = V - x^y - f(y) + f(x) \geq 0$$

$\square$

Proposition 7.6. Let f integrable on $[a,b]$, define $F(x) = \int_a^x f \,\mathrm{d}\lambda$ then F is continuous and $F \in BV[a,b]$.

Proof. Continuity of integral relative to the boundaries gives us continuity of F. $F \in BV[a,b]$ since $F = \int_a^x f_+ - \int_a^x f$, i.e., F is a difference of two non-decreasing monotonic functions. $\square$

Theorem 7.7. Let f integrable on $[a,b]$. Define
$$F(x) = \int_a^x f \,\mathrm{d}\lambda$$
Then F is differentiable a.e. and $F' = f$ a.e.

Proof. We'll proof that $\int_a^x F' \,\mathrm{d}\lambda = \int_a^x f \,\mathrm{d}\lambda$.
Denote $M = \sup |x|$.
$$f_n(x) = n\left[F\left(x + \frac{1}{n}\right) - F(x)\right] \to F'(x)$$
Define $f(x) = 0$ for $x > b$, i.e. $F(x) = F(b)$ for those x.
$$|f_n(x)| = \left|n\left[F\left(x + \frac{1}{n}\right) - F(x)\right]\right| = n\left|\int_x^{x+\frac{1}{n}} f \,\mathrm{d}\lambda\right| \leq M$$
Since $|f_n| \leq M\chi_{[a,b]}$ and $f_n \to F'$ a.e., we get
$$\int_a^c F' \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_a^c f_n \,\mathrm{d}\lambda = \lim_{n\to\infty} n \int_a^c F\left(x + \frac{1}{n}\right) - F(x) \,\mathrm{d}\lambda = \lim_{n\to\infty} n \int_a^c F\left(x + \frac{1}{n}\right) \mathrm{d}\lambda - n \int_a^c F(x) \,\mathrm{d}\lambda =$$
$$= \lim_{n\to\infty} n \int_{a+\frac{1}{n}}^{c+\frac{1}{n}} F(x) \,\mathrm{d}\lambda - n \int_a^c F(x) \,\mathrm{d}\lambda = \lim_{n\to\infty} n \int_a^{a+\frac{1}{n}} F(x) \,\mathrm{d}\lambda - \lim_{n\to\infty} n \int_c^{c+\frac{1}{n}} F(x) \,\mathrm{d}\lambda =$$
$$= F(c) - F(a) = \int_a^c f \,\mathrm{d}\lambda$$
□

Definition 7.6. f is uniformly continuous on $[a,b]$ if for all $\epsilon > 0$ exists $\delta > 0$ such that
$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Definition 7.7. f is absolutely continuous on $[a,b]$ ($f \in AC[a,b]$) if for $\{(x_i, y_i)\}_{i=1}^m$ are disjoint open intervals in $[a,b]$ such that $\sum(y_i - x_i) < \delta$ then
$$|f(x)_i - f(y)_i| < \epsilon$$

Proposition 7.8. AC is subspace.

Proposition 7.9. $f \in AC[a,b] \Rightarrow f \in BV[a,b]$

Proof. □

Proposition 7.10. $f \in AC[a,b]$ and $f' = 0$ a.e., then f is constant.

Proof. Choose $a < c \leq b$ and show that $f(c) = f(a)$. Denote $E = \{x \in (a,c) : f'(x)\}$ and $\lambda((a,c) \setminus E) = 0$. Let $s > 0$. For all $x \in E$
$$\lim_{h\to 0} \frac{|f(x+h) - f(x)|}{|h|} = 0$$
Thus for $h > 0$ small enough
$$|f(x+h) - f(x)| < \frac{sh}{n - a}$$
$$[x, x+h] \subseteq (a,c)$$
Denote
$$M = \{[x, x+h] : x \in E, h > 0\}$$
Thus is Vitaly cover (it obviously covers $[a,c]$ an the size of interval depends on h).
Let $\epsilon > 0$ and corresponding $\delta > 0$ by absolute continuity. By Vitaly theorem exists finite set of disjoint intervals $\{[x_k, x_k + h_k]\} \subseteq M$ such that
$$\lambda\Big(E \setminus \bigcup [x_k, x_k + h_k]\Big) < \delta$$
and then
$$\lambda\Big((a,c) \setminus \bigcup [x_k, x_k + h_k]\Big) < \delta$$

Then

$$\sum x_{i+1} - y_i < \delta$$

and thus

$$\sum |f(x_{i+1}) - f(y_i)| < \epsilon$$

$$\sum |f(y_i) - f(y_i)| < \sum \frac{s}{b-a}(y_k - x_k) \leq s$$

$$|f(c) - f(a)| = \left|\sum f(x_{k+1}) - f(y_k) + \sum f(y_k) - f(x_k)\right| \leq \sum |f(x_{k+1}) - f(y_k)| + \sum |f(y_k) - f(x_k)| \leq \epsilon + s$$

□

Theorem 7.11. f on $[a,b]$ is absolutely continuous iff exists $h \in \mathcal{L}^[a,b]$ and $c \in \mathbb{R}$ such that

$$f(x) = c + \int_a^x h \, \mathrm{d}\lambda$$

In this case $h = f'$ a.e.

Proof. $\Rightarrow$: $h \in \mathcal{L}^1$, $f(x) = c + \int_a^x h \, \mathrm{d}\lambda$ thus we've seen that for $\lambda(E) < \delta$

$$\int_E |h| \, \mathrm{d}\lambda < \epsilon$$

We'll use it to show that $\int_a^x h \, \mathrm{d}\lambda$ is absolutely continuous. Let $\{x_i, y_i\}$ set of disjoint intervals such that $\lambda(\bigcup^m (x_i, y_i)) < \delta$ denote $E = \bigcup^m (x_i, y_i)$, thus

$$\int_E |h| \, \mathrm{d}\lambda < \epsilon$$

$$\sum^m |f(y_i) - f(x_i)| = \sum^m \left| \int_{x_i}^{y_i} h \, \mathrm{d}\lambda \right| \leq \sum^m \int_{x_i}^{y_i} |h| \, \mathrm{d}\lambda = \int_E |h| \, \mathrm{d}\lambda < \epsilon$$

$\Leftarrow$:
Suppose $f \in AC[a,b]$ thus $f \in BV[a,b]$ and thus f' exists a.e. and $f' \in \mathcal{L}^1$. Define $g(x) = \int_a^x f' \, \mathrm{d}\lambda$, by previous direction $g \in AC[a,b]$ and thus $f - g \in AC[a,b]$. We know that g' exists a.e. and $g' = f'$ a.e. Thus $f' - g' = 0$ a.e., i.e., $f - g = 0$. □

Collary 7.11.1. If $f \in AC[a,b]$ then for all $a \leq x \leq b$

$$\int_a^x f' \, \mathrm{d}\lambda = f(x) - f(a)$$

Proposition 7.12. $f \in AC[a,b]$ and $E \subseteq [a,b]$ such that $\lambda(E) = 0$ then $f(E)$ measurable and $\lambda(f(E)) = 0$.