104222 - Probability Theory

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Abstract

Contents

1 Intro

1.1 Probabilistic experiments

1. Throwing two dices. Possible results: $\{(i,j) : 1 \leq 1, 6 \leq 6\}$.
$$P(\{(i,j)\}) = \frac{1}{36}$$
2. Throwing of coin until we get heads (H). $\{1, 2, 3, \dots\}$. Probability is
$$P(\{n\}) \left(\frac{1}{2}\right)^n$$
3. Choosing a random number in $[0, 2]$. Random means that
$$P\Big((a,b)\Big) = \frac{b-a}{2}$$

State space is a space of all possible results of the experiment. Is denoted with Ω.

Event A is subset of state space. $A \subset \Omega$.

Function of probability (or measure of probability) is a function defined on particular set of events in Ω and has following properties:

1. $P(\emptyset) = 0$ and $P(\Omega) = 1$.
2. $0 \leq P(A) \leq 1$ if $P(A)$ is defined.
3. Sigma-additivity: if $\{A_n\}_{n=1}^{\infty}$ are disjoint then
$$P(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$
Sigma prefix means it works for countable infinite number of terms.

Example Since $P\Big((a,b)\Big) = \frac{b-a}{2}$ and $\{x\} \subset (x-\epsilon, x+\epsilon)$:
$$P(x) \leq P\Big((x-\epsilon, x+\epsilon)\Big) = \frac{2\epsilon}{2} = \epsilon$$
$$P(x) = 0$$
It turns out that it's impossible to define probability function on $[0, 2]$ such that it is invariant to shifts and, subsequently, $P\Big((a,b)\Big) = \frac{b-a}{2}$ and defined on every subset of $[0, 2]$.
For finite or infinite countable state spaces probability is defined for every event.
Notation:

1. $A^C = \Omega - A$. Since $\Omega = A + A^C$, $P(A^C) = 1 - P(A)$.
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Proof: $A = (A - B) \cup (A \cap B)$ and $B = (B - A) \cup (A \cap B)$. Also $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.
Then $P(A) = P(A - B) + P(A \cap B)$ and $P(B) = P(B - A) + P(A \cap B)$
$P(A - B) = P(A) - P(A \cap B)$ and $P(B - A) = P(B) - P(A \cap B)$
We get
$$P(A \cup B) = P(A-B) + P(B-A) + P(A \cap B) = P(A) - P(A \cap B) + P(B) - P(A \cap B) + P(A \cap B) = P(A) + P(B) - P(A \cap B)$$

3. Let $\{A_n\}_{n=1}^{\infty}$ is increasing sequence of events: $A_n \subset A_{n+1}$. Denote $A = \bigcup_{n=1}^{\infty} A_n$. We say that A_n goes to A and write

$$A = \lim_{n\to\infty} A_n$$

Similarly for decreasing sequence of $A_n \supset A_{n+1}$.

Theorem If $A = \lim_{n\to\infty} A_n$ then $P(A) = \lim_{n\to\infty} P(A_n)$.

Proof Via disjointing. Define $B_1 = A_1$ and $B_n = A_n - A_{n-1}$. From the construction $\{B_n\}_{n=1}^{\infty}$ are disjoint. Also

$$A_k = \bigcup_{n=1}^{k} B_n$$

then

$$A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

Then

$$P(A_k) = \sum_{n=1}^{k} B_n$$

$$P(A) = \sum_{n=1}^{\infty} B_n$$

Since $P(A_k)$ are partial sums of converging series, $P(A) = \lim_{n\to\infty} P(A_n)$.

1.2 Conditional probability

Conditional probability Let Ω is state space and $A, B \subset \Omega$ events. Suppose that $P(A) \neq 0$. Define

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

We say probability of B given A.

$$P(\Omega|A) = \frac{P(A\cap \Omega)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

$$P(\emptyset|A) = \frac{P(A\cap \emptyset)}{P(A)} = \frac{P(\emptyset)}{P(A)} = 0$$

It is easy to show

$$P\left(\bigcup_{j=1}^{\infty} B_j|A\right) = \sum_{j=1}^{\infty} P\left(B_j|A\right)$$

Also

$$P(B|A) = \frac{P(A\cap B)}{P(A)} \in [0,1]$$

Example

$$A = \begin{Bmatrix}\text{at least one} \\ \text{of dices is 6}\end{Bmatrix}$$

$$B = \begin{Bmatrix}\text{sum is} \\ 7\end{Bmatrix}$$

Then

$$P(A) = \frac{11}{36}$$

$$P(B) = \frac{1}{6}$$

$$P(A\cap B) = \frac{1}{18}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1}{3}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{2}{11}$$

Alternative form of conditional probability

$$P(A\cap B)=P(A)P(B|A)=P(B)P(A|B)$$

Definition A and B are independent if

$$P(A\cap B)=P(A)P(B)$$

Note If $P(A)=0$ or $P(B)=0$ then A and B are independent.

Theorem Suppose $P(A)\neq 0$, $P(B)\neq 0$. Then three following properties are equivalent:

1. A and B are independent
2. $P(A|B)=P(A)$
3. $P(B|A)=P(B)$.

Proof is trivial from definition of conditional probability.

Definition Independence of set of events $\{A_j\}_{j=1}^n$ is

$$\forall I\subseteq\{1,2,\dots n\}\quad P\left(\bigcap_{i\in I}A_i\right)=\prod_{j\in I}P(A_j)$$

Note Independence and disjointedness are different things:

$$P(A\cap B)=P(A)P(B)$$

$$P(A\cap B)=0$$

Example

$$A=\begin{Bmatrix}\text{at least one}\\ \text{of dices is 6}\end{Bmatrix}$$

$$B=\{\text{first is 6}\}$$

$$C=\{\text{second is 6}\}$$

$$B\cap C=A^C$$

$$P(A)=1-P\left(A^C\right)=1-\underbrace{P(B)P(C)}_{\text{independent}}=1-\frac{25}{36}=\frac{11}{36}$$

Coin experiment Flipping coin until getting H:

$$\Omega=\{1,2,\dots\}$$

$$P(\{n\})=\left(\frac{1}{2}\right)^n$$

$$A_n=\begin{Bmatrix}\text{it took}\\ \text{n flips}\end{Bmatrix}=\{n\}$$

$$B_j=\begin{Bmatrix}\text{heads}\\ \text{on } j^{th}\\ \text{flip}\end{Bmatrix}$$

$$C_j=\begin{Bmatrix}\text{tails on}\\ j^{th}\text{ flip}\end{Bmatrix}$$

$\{B_j\}_{j=1}^\infty$ and $\{C_j\}_{j=1}^\infty$ are independent.

$$A_n=C_1\cup C_2\cup\dots\cup C_{n-1}\cup B_n$$

$$P(A_n)=P\left(C_1\cup C_2\cup\dots\cup C_{n-1}\cup B_n\right)=P(C_1)\cdot P(C_2)\cdot\dots\cdot P(C_{n-1})\cdot P(B_n)=\left(\frac{1}{2}\right)^n$$

Domino experiment Taking domino out of box with 40 black and 30 white dominoes.

$$C = \left\{ \begin{array}{c} \text{black in} \\ \text{first} \\ \text{time} \end{array} \right\} = \{n\}$$

$$C = \left\{ \begin{array}{c} \text{black in} \\ \text{second} \\ \text{time} \end{array} \right\} = \{n\}$$

$$D = \{D \cap C\} \cup \{D \cap C^C\}$$

$$P(D) = P(C \cap D) + P(C^C \cup D)$$

$$P(C \cap D) = P(C)P(D|C) = \frac{4}{7} \cdot \frac{39}{69}$$

$$P(C^C \cap D) = P(C^C)P(D|C^C) = \frac{4}{7} \cdot \frac{40}{69}$$

$$P(D) = P(C \cap D) + P(C^C \cap D) = \frac{4}{7} \cdot \frac{39}{69} + \frac{4}{7} \cdot \frac{40}{69} = \frac{4}{7}$$

1.3 Total probability

Suppose we have

$$\bigcup_{j=1}^{n} A_j = \Omega$$

and

$$\forall j \neq k \; A_j \cap A_k = 0$$

We say that Ω is decomposed to

$$\{A_j\}_{j=1}^{n}$$

Now we can write any event $B \subset \Omega$ as

$$B = \bigcup_{j=1}^{n} (B \cap A_j)$$

Then

$$P(B) = \sum P(B \cap A_j)$$

Bayes' formula Suppose Ω is decomposed to

$$\{A_j\}_{j=1}^{n}$$

. Let $B \subset \Omega$. Then

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}$$

Proof

$$P(A_i|B) = \frac{P(A_i \cup B)}{P(B)} = \frac{P(A_i \cap B)}{\sum_{j=1}^{n} P(B \cap A_j)} = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{n} P(A_j)P(B|A_j)}$$

Exercise In college, 70% of people are students, and 30% are stuff. It's known that 80% of stuff and 20% of students come to college by car. We choose a random person and he came by car. What is probability he's a student.

Solution

$$S = \left\{ \begin{array}{c} \text{we} \\ \text{choose} \\ \text{a} \\ \text{student} \end{array} \right\} = \{n\}$$

$$W = \left\{ \begin{array}{c} \text{we} \\ \text{choose} \\ \text{a stuff} \end{array} \right\} = \{n\}$$

$$C = \left\{ \begin{array}{c} \text{came} \\ \text{by car} \end{array} \right\} = \{n\}$$

We need to find $P(S|C)$. $\Omega = S \cup W$. By Bayes' formula:

$$P(S|C) = \frac{P(S) \cdot P(C|S)}{P(S) \cdot P(C|S) + P(W) \cdot P(C|W)} = \frac{0.7 \cdot 0.2}{0.7 \cdot 0.2 + 0.3 \cdot 0.8} = \frac{0.14}{0.38} = \frac{7}{19}$$

2 Probability via combinatorics

Proof If in probability experiment there are n outcomes ($|\Omega| = n$) and they are equally probable, than

$$\forall A \subset \Omega \, P(A) = \frac{|A|}{|\Omega|}$$

Exercise What is probability to get exactly 2 aces in poker hand?

Solution

$$P(A) = \frac{\binom{4}{2} \cdot \binom{48}{3}}{\binom{52}{5}}$$

Exercise What is probability to get at least 3 aces in poker hand?

Solution

$$P(A) = \frac{\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}}{\binom{52}{5}}$$

Exercise What is probability to get a full house in poker hand?

Solution

$$P(A) = \frac{\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2}}{\binom{52}{5}}$$

Exercise What is probability to put n balls in n boxes such that exactly one box is empty?

Solution We choose one box which is empty, and one additional which contains to balls.

$$P(A) = \frac{n \cdot (n-1) \cdot \binom{n}{2}(n-2)!}{n^n}$$

Exercise What is probability to put n balls in n boxes such that at least one box is empty?

Solution

$$A = \left\{ \begin{matrix} \text{at least} \\ \text{one} \\ \text{empty} \end{matrix} \right\} = \{n\}$$

$$A^C = \left\{ \begin{matrix} \text{no} \\ \text{empty} \\ \text{boxes} \end{matrix} \right\} = \{n\}$$

$$P(A) = 1 - \frac{n!}{n^n}$$

Using Stirling approximation $n \approx n^n e^{-n} \sqrt{2\pi n}$

$$\frac{n!}{n^n} = \approx e^{-n} \cdot \sqrt{2\pi n}$$

2.1 Multinomial coefficients

Given n balls and k boxes and $\{r_j\}_{j=1}^{n}$, $\sum r_j = n$. How many ways there are to put r_j in j^{th} box.

$$\binom{n}{r_1}\binom{n-r_1}{r_2}\cdots\binom{r_{n-1}+r_n}{r_{n-1}}\binom{r_n}{r_n} = \frac{n!}{r_1! r_2! \ldots r_n!}$$

There k^n different ways to put n balls into k boxes, thus to get probability we need to divide first by second.

Example What is probability to get each result twice on 12 dices?

Solution We can see the analogy - dice a ball, and result is a box, so we have 12 balls and 6 dices.

3 Random variable

Definition (Ω, P) in called probability space.

Definition Random variable X on (Ω, P) is $X : \Omega \to \mathbb{R}$. We denote $\omega \in \Omega$.

$$\{X = a\} := \{\omega \in \Omega | X(\omega) = a\}$$
$$\{a \leq X \leq b\} := \{\omega \in \Omega | a \leq X(\omega) \leq b\}$$
$$P(X = a) := P\Big(\{X = a\}\Big) = P\Big(\{\omega \in \Omega : X(\omega) = a\}\Big)$$

Example Two dices.

$$P\Big((i,j)\Big) = \frac{1}{36}$$

We can define random variable

$$X\big((i,j)\big) = i + j$$

We can ask ourselves, what is probability that $X = 7$?

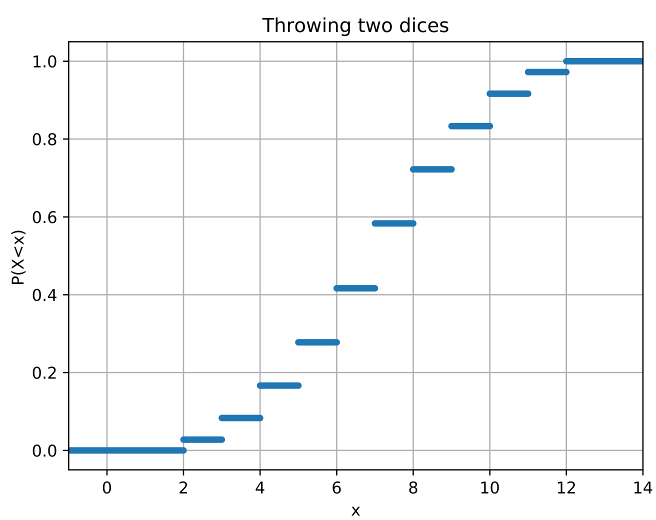
$$P(X = 7) = P\Big(\{(i,j) : X(i,j) = 7\}\Big) = P\Big(\{(i,j) : i + j = 7\}\Big) = \frac{1}{6}$$

Examples $\Omega = [0, 2]$. We can define $X(\omega) = \omega$.

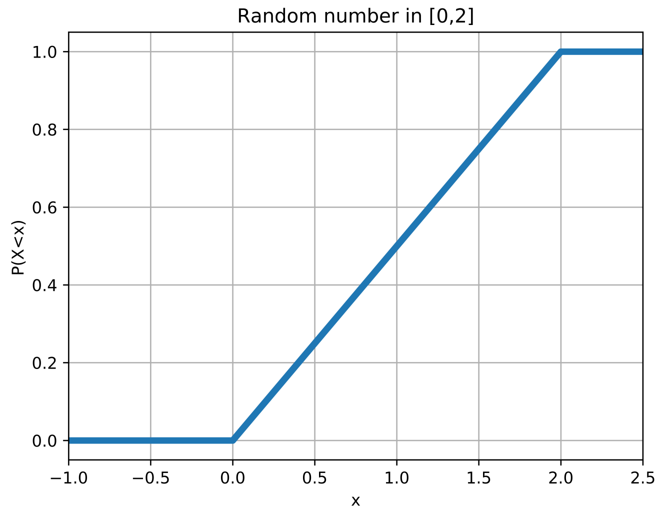
$$P(0 \leq X \leq 1) = P([0,1]) = \frac{1}{2}$$

Distribution function Let X random variable on (Ω, P). Define $F(x) = P(X \leq x)$.

Example Dices. $X = i + j$



Example $\Omega = [0, 2]$



Claim Let F distribution function.

1. F in monotonous non-decreasing
2. F is continuous to the right, i.e. $\lim_{y\to x^+} F(y) = F(x)$.
3. $F(x^-) = \lim_{y\to x^-} F(y) = P(X < x)$
4. $F(x) - F(x^-) = P(X = x)$
5. $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$

Proof

1. Obvious.
2. Denote $A = \{X \le x\}$ and $A_n = \left\{X \le x + \frac{1}{n}\right\}$. Then $P(A) = F(x)$ and $P(A_n) = F(x + \frac{1}{n})$. We need to show that
$$\lim_{n\to\infty} F(x + \frac{1}{n}) = F(x)$$
i.e.
$$\lim_{n\to\infty} P(A_n) = P(A)$$
Since $A = \bigcap A_n$, it's true.
3. Denote $B = \{X < x\}$ and $B_n = \left\{X - \frac{1}{n}\right\}$. That means
$$P(B_n) = F(x - \frac{1}{n})$$
Now we claim that
$$B = \bigcap_{n=1}^{\infty} B_n$$
Since $\omega \in B \iff X(\omega) < x$ and $\omega \in B_n \iff X(\omega) \le x - \frac{1}{n}$. Since
$$\lim_{n\to\infty} F(x - \frac{1}{n}) = P(X < x)$$
from monotonousness
$$F(x^-) = \lim_{y\to x^-} F(y) = \lim_{n\to\infty} F(x - \frac{1}{n})$$
4. Since
$$\{X \le x\} = \{X < x\} \cup \{X = x\}$$
$$P(X \le x) = P(X < x) + P(X = x)$$
Thus
$$F(x) - F(X^-) = P(X = x)$$

5. Denote $C_n = \{X \leq n\}$. Then
$$\bigcup_{n=1}^{\infty} = \Omega$$
Then
$$\lim_{n\to\infty} F(n) = \lim_{n\to\infty} P(C_n) = P(\Omega) = 1$$

Definition Random variable is called discrete if its distribution function is piecewise constant, i.e. is constant except countable set of points without accumulation points. Or, alternatively, if it can be written as a finite linear combination of indicator functions of intervals.
In discrete case exists sequence $\{x_j\}_{j=1}^N \subset \mathbb{R}$ and $\{p_j\}_{j=1}^N$ for $1 \leq N \leq \infty$ such that
$$P(X = x_j) = p_j$$
and
$$\sum_j p_j = 1$$

Definition Random variable is called continuous if its distribution function is continuous. Thus $P(X = x) = 0$
In this case we always assume that F is piecewise continuously differentiable, i.e. its derivative piecewise continuous.Thus,
$$F(b) - F(a) = \int_a^b F'(x)dx$$
Then we call $f(x) = F'(X)$ a density function. And
$$P(a \leq X \leq b) = \int_a^b f(x)dx$$
Then
$$F(x) = \int_{-\infty}^x f(t)dt$$

Expectation or expected value, denoted as $\mathbb{E}X$ and defined as
$$\mathbb{E}X = \sum_{i=1}^n p_j x_j \text{ if } \sum_{i=1}^n p_i |x_i| < \infty$$

Function of random variable Let $\Psi : \mathbb{R} \to \mathbb{R}$. Define $Y = \Psi(X)$. Then Y is random variable too. Also, then $\mathbb{E}Y = \mathbb{E}\Psi(X) = \sum_{i=1}^n p_j \Psi(x_j)$

Variance Lets look at
$$\sigma^2 = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}(X - \mu)^2 = \sum_{k=1}^{\infty} p_j (x_j - \mu)^2$$
Variance measures at closeness of random variable to the expectation.
Standard deviation σ is square root of variance. Also
$$\sigma^2 = \sum_{k=1}^{\infty} p_j (x_j - \mu)^2 = \sum_{k=1}^{\infty} p_j x_j^2 - 2\mu \sum_{k=1}^{\infty} p_j x_j + \mu^2 \sum_{k=1}^{\infty} p_j = \mathbb{E}X^2 - 2\mu \cdot mu + \mu^2 = \mathbb{E}X^2 - \mu^2$$
i.e.
$$\sigma^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}X^2 - \mu^2$$

3.1 Classical discrete distributions

3.1.1 Binomial distribution

Model:

1. There are n independent experiments (Bernoulli trials)

2. There are two possible results (success and fail)

3. In every experiment, probability of success is p.

Denote

$$X = \left\{ \begin{array}{c} \text{number of} \\ \text{successes} \end{array} \right\}$$

Then

$$P(X=k) = \binom{n}{k} \cdot p^k(1-p)^{n-k}$$

We say that $X \sim Bin(n,p)$ and $\mathbb{E}X = np$. Lets show that:

$$\mathbb{E}X = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=1}^{n} k\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} =$$

$$= \sum_{k=1}^{n} = \frac{n!}{(k-1)!(n-k)!}p^k(1-p)^{n-k} = np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}p^{k-1}(1-p)^{n-k}$$

Denote $j = k-1$

$$\mathbb{E}X = np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}p^{k-1}(1-p)^{n-k} = np\sum_{j=0}^{n-1}\underbrace{\binom{n-1}{j}p^j(1-p)^{(n-1)-j}}_{\sim Bin(n-1,p)\Rightarrow 1} = np$$

Note $\mathbb{E}X^n$ is n^{th} moment of X

Variance of X Since $\mathbb{E}X = np$ and $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \mathbb{E}X^2 - n^2p^2$ Lets calculate $\mathbb{E}X^2$:

$$\mathbb{E}X^2 = \sum_{k=0}^{n} k^2\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} + \underbrace{\sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k}}_{\mathbb{E}X}$$

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=2}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} =$$

$$= \sum_{k=2}^{n}\frac{n!}{(k-2)!(n-k)!}p^k(1-p)^{n-k} = n(n-1)p^2\underbrace{\sum_{k=2}^{n}\frac{(n-2)!}{(k-2)!(n-k)!}p^{k-2}(1-p)^{n-k}}_{\sim Bin(n-2,p)\Rightarrow 1} = n(n-1)p^2$$

Then

$$Var(X) = n(n-1)p^2 + np - n^2p^2 = n^2p^2 + np - np^2 - n^2p^2 = np(1-p)$$

3.1.2 Geometric distribution

Performing Bernoulli trials until success. Define random variable

$$X = \left\{ \begin{array}{c} \text{Number of} \\ \text{required} \\ \text{experimets} \end{array} \right\}$$

$$P(X=n) = (1-p)^{n-1}p$$

$$X \sim Geom(p)$$

Then

$$\mathbb{E}X = \sum_{n=1}^{\infty} n(1-p)^{n-1}p$$

Lets use generating functions. Define $g(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ Then $g'(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ Then $g'(1-p) = \sum_{n=1}^{\infty} n(1-p)^{n-1} = \frac{1}{p^2}$ i.e

$$\mathbb{E}X = \frac{p}{p^2} = \frac{1}{p}$$

Then

$$\mathbb{E}X^2 = \sum_{n=1}^{\infty} n^2(1-p)^{n-1}p = \sum_{n=2}^{\infty} n(n-1)(1-p)^{n-1}p + \frac{1}{p}$$

For same g, $g''(x) = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}$. Substituting

$$\mathbb{E}X^2 = \sum_{n=2}^{\infty} n(n-1)(1-p)^{n-1}p + \frac{1}{p} = p(1-p)\cdot\frac{2}{p^3} + \frac{1}{p} = \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2}{p^2} - \frac{1}{p}$$

Thus

$$Var(X) = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

Lemma Let X random variable with finite expectation. Then $\mathbb{E}(X+c) = \mathbb{E}X + c$ and $\mathbb{E}(cX) = c\mathbb{E}X$.

Proof Define $Y = X + c$ Then

$$\mathbb{E}Y = \sum p_i(x_i + c) = \sum p_i x_i + c\sum p_i = \mathbb{E}X + c$$

Similarly for $Z = cX$.

Lemma Let X random variable with finite expectation. Then $Var(X+c) = Var(X)$ and $Var(cX) = c^2 Var(x)$.

Proof Denote $Y = X + c$:

$$Var(Y) = \mathbb{E}(X + c - (\mathbb{E}X + c))^2 = \mathbb{E}(X - \mathbb{E}X)^2 = Var(X)$$

$Z = cX$:

$$Var(Z) == E(\mathbb{E}X - c\mathbb{E}X)^2 = \mathbb{E}c^2(X - \mathbb{E}X)^2 = c^2 Var(X)$$

3.1.3 Hypergeometric distribution

There are N balls in box, m are "good" and $N - m$ are "bad". We take out n balls out of box and count how much are good. There are n Bernoulli trials. Probability that it every of them would be successful is $\frac{m}{N}$. But experiments aren't independent, so it's not binomial distribution.
We know that $0 \le k \le m$ and $0 \le n - k \le N - m$. Thus

$$\max(0, n + m - N) \le k \le \min(m, n)$$

For k in this range

$$P(X = k) = \frac{\binom{m}{k}\binom{N-m}{n-k}}{\binom{N}{n}}$$

For hypergeometric distribution:

$$EX = \frac{nm}{N}$$

$$Var(X) = n\frac{m}{N}\left(1 - \frac{m}{N}\right)\frac{N-n}{N-1}$$

In limit of $N, m \to \infty$ and $\frac{m}{N} \to p$, then for fixed n, we get back the binomial distribution.

3.1.4 Poisson distribution

$$P(X = k) = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$X \sim Pois(\lambda)$$

Then $\mathbb{E}X = \sigma^2(X) = \lambda$.

Theorem (Poisson approximation of binomial distribution)

$$\lim_{n\to\infty\ p\to 0\ np\to\lambda} \binom{n}{k} p^k (1-p)^{n-k} = \frac{\lambda^k}{k!}e^{-\lambda}$$

If $X_{n,p} \sim Bin(n,p)$ and $X_\lambda \sim Pois(\lambda)$ then

$$\lim_{n\to\infty\ p\to 0\ np\to\lambda} P(X_{n,p} = k) = P(X_\lambda = k)$$

Proof We can rewrite $p_n = \frac{\lambda+\delta_n}{n}$, where $\delta_n \to 0$ Let $k \in \mathbb{N}$ and $n \geq k$

$$\binom{n}{k} p_n^k (1-p_n)^{n-k} = \frac{n(n-1)\dots(n-k+1)}{k!}\left(\frac{\lambda+\delta_n}{n}\right)^k\left(1-\frac{\lambda+\delta_n}{n}\right)^{n-k} = \frac{n(n-1)\dots(n-k+1)}{k!}\frac{(\lambda+\delta_n)^k}{n^k}\frac{\left(1-\frac{\lambda+\delta_n}{n}\right)^n}{\left(1-\frac{\lambda+\delta_n}{n}\right)^k} =$$

$$= \underbrace{\frac{n(n-1)\dots(n-k+1)}{n^k}}_{1}\frac{1}{k!}\underbrace{(\lambda+\delta_n)^k}_{\lambda^k}\underbrace{\left(1-\frac{\lambda+\delta_n}{n}\right)^n}_{e^{-\lambda}}\underbrace{\frac{1}{\left(1-\frac{\lambda+\delta_n}{n}\right)^k}}_{1} = \frac{\lambda^k}{k!}e^{-\lambda}$$

$$\mathbb{E}X = \sum_{k=1}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

$$\mathbb{E}X^2 = \sum_{k=1}^{\infty} k^2\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\left[\sum_{k=1}^{\infty}(k(k-1)+k)\frac{\lambda^k}{k!}\right] = \lambda^2+\lambda$$

$$\sigma^2(X) = \lambda$$

3.1.5 Uniform distribution on $[n] = \{1, 2, \dots, n\}$

Example of infinite expectation

$$P(X = 2^n) = 2^{-n}$$

The model is flipping coin until you get head and getting 2^n dollars where n is number of flips you made. Then expectancy of your profit

$$EX = \sum 2^n 2^{-n} = \infty$$

4 Continuous random variables

Since $f(x) = F'(x)$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

We define expectation of X as

$$\mu = \mathbb{E}X = \int_{-\infty}^{\infty} xf(x)dx$$

We can define expectation in broad sense. If one of $\int_{-\infty}^{0} xf(x)dx$ and $\int_{0}^{\infty} xf(x)dx$ is finite and second is infinite, we can define expectation as infinite.

4.1 Function of continuous of random variable

For $X : \Omega \to \mathbb{R}$ define

$$Y = \Psi(X)$$

Then

$$\mathbb{E}Y = \mathbb{E}\Psi(X) = \int_{-\infty}^{\infty} = \int_{-\infty}^{\infty}\Psi(x)f(x)dx$$

Now we can use it to define variance, by using $\psi(x) = (x-\mu)^2$:

$$\sigma^2 = \mathbb{E}(X-\mu)^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx$$

Claim

$$\sigma^2(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

Proof

$$\sigma^2(X) = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty}x^2 f(x)dx - 2\mu\int_{-\infty}^{\infty}\underbrace{xf(x)}_{\mu}dx + \mu^2\underbrace{\int_{-\infty}^{\infty}f(x)dx}_{1} = EX^2 - \mu^2$$

4.2 Classical continuous distributions

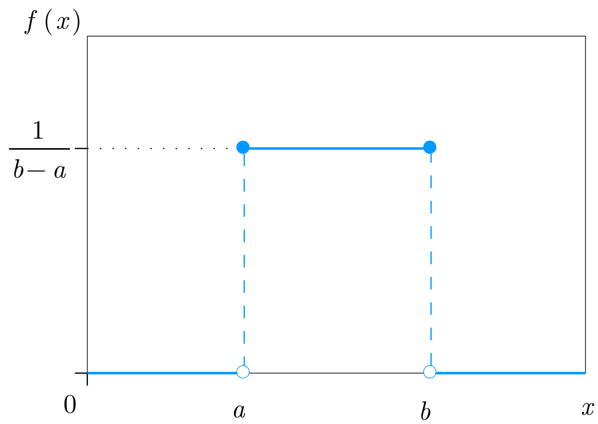
4.2.1 Uniform distribution

For $[a, b]$:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

$$EX = \frac{b+a}{2}$$

We write $X \sim U\big([a, b]\big)$.



4.2.2 Exponential distribution

For $\lambda > 0$

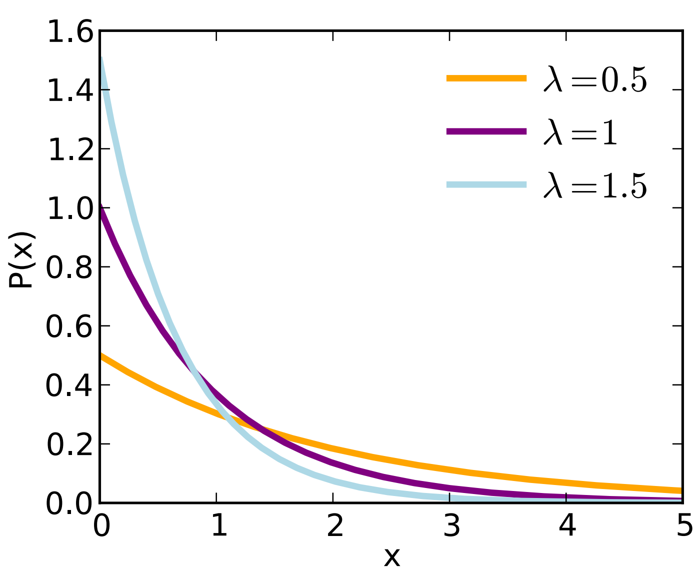
$$f(x) = \lambda e^{-\lambda x},\ x \geq 0$$

Then

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} f(x)dt = 1 - e^{-\lambda x}$$

$$EX = \int_{0}^{\infty} x\lambda e^{-\lambda x}dx = \left[-xe^{-\lambda x}\right]_0^\infty + \int_0^\infty e^{-\lambda x}dx = \frac{1}{\lambda}$$

$$\sigma^2 = EX^2 - (EX)^2 = EX^2 - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$



If $X \sim Exp(\lambda)$ then
$$P(X \geq a+b | X \geq a) = P(X \geq b)$$
This property is called memorylessness.

Proof that exponential distribution is memoryless

$$P(X \geq a+b|X \geq a) = \frac{P(X \geq a+b,\ X \geq a)}{P(X \geq a)} = \frac{P(X \geq a+b)}{P(X \geq a)} = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}$$

Notation $P(A, B) = P(A \cap B)$

Model For example, the time until a radioactive particle decays or time until you get cellphone call.

4.2.3 Normal (Gaussian) distribution

Proof of integral Denote
$$l = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$
Since l is even;
$$\frac{l}{2} = \int_0^{\infty} e^{-\frac{x^2}{2}} dx$$
Now
$$\frac{l^2}{4} = \int_0^{\infty} e^{-\frac{x^2}{2}} dx \int_0^{\infty} e^{-\frac{y^2}{2}} dy = \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy$$
The integral is in first quarter. Switching to polar:
$$dxdy = rdrd\theta$$
$$\frac{l^2}{4} = \int_0^{\infty}\int_0^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_0^{\infty} drre^{-\frac{r^2}{2}} = \int_0^{\frac{\pi}{2}} d\theta \left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = \int_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$
$l = \sqrt{2\pi}$

Normal distribution

$$f(x) = ce^{-\frac{(x-a)^2}{2b}}$$
$$\int_{-\infty}^{\infty} f(x) = c\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2b}} = 1$$
Substitute $z = \frac{x-a}{\sqrt{b}}$ and $dx = \sqrt{b}dz$.
$$c\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2b}} = c\sqrt{b}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dzc\sqrt{2\pi b}$$
Then
$$c = \frac{1}{\sqrt{2\pi b}}$$
and
$$f(x) = \frac{1}{\sqrt{2\pi b}} e^{-\frac{(x-a)^2}{2b}}$$
which is density function.

Expectation of normal distribution

$$\int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} xf(x)\frac{e^{-\frac{(x-a)^2}{2b}}}{\sqrt{2\pi b}}dx = \underbrace{\int_{-\infty}^{\infty}(x-a)f(x)\frac{e^{-\frac{(x-a)^2}{2b}}}{\sqrt{2\pi b}}dx}_{\text{odd function}\Rightarrow=0} + \int_{-\infty}^{\infty} af(x)\frac{e^{-\frac{(x-a)^2}{2b}}}{\sqrt{2\pi b}}dx = a\int_{-\infty}^{\infty} f(x)\frac{e^{-\frac{(x-a)^2}{2b}}}{\sqrt{2\pi b}}dx = a$$
Since a is mean, we'll replace it with μ:
$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2b}}}{\sqrt{2\pi b}}$$

Variance of normal distribution

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

By parts: $u = z$ and $v' = ze^{-\frac{z^2}{2}}$. Then

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{z^2}{2}} dz = \left[-ze^{-\frac{z^2}{2}}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = \sqrt{2\pi}$$

lets calculate variance:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{e^{-\frac{(x-\mu)^2}{2b}}}{\sqrt{2\pi b}} dx$$

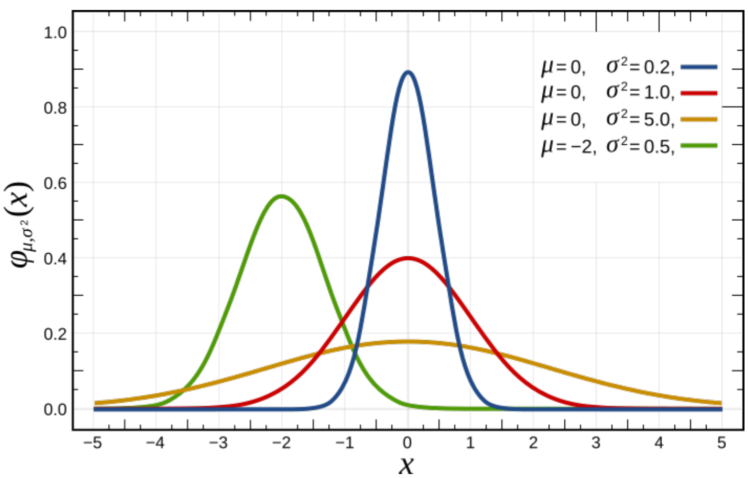
With same substitution $z = \frac{x-\mu}{\sqrt{b}}$:

$$\sigma^2 = \int_{-\infty}^{\infty} bz^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz = b \underbrace{\int_{-\infty}^{\infty} z^2 \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dz}_{1} = b$$

Thus $\sqrt{b} = \sigma$.
Which leads us to final form of $f(x)$:

$$f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}$$

We write $X \sim N(\mu, \sigma^2)$.



It can be shown that $x = \mu \pm \sigma$ are inflection point.

$$F(x) = \int_{-\infty}^{x} \frac{e^{-\frac{(y-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dy$$

Lets denote $Z \sim N(0,1)$ is standard distribution. Canonical substitution $z = \frac{x-\mu}{\sigma}$: For $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, we can show $P(\mu + a\sigma \leq X \leq \mu + b\sigma)$ is independent on a and b:

$$P(\mu + a\sigma \leq X \leq \mu + b\sigma) = \int_{\mu+a\sigma}^{\mu+b\sigma} \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx = \int_{a}^{b} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} dx = P(a \leq Z \leq b)$$

Conclusion

$$\frac{X_{\mu,\sigma^2} - \mu}{\sigma} = Z$$

Claim Suppose X random variable such that $\exists a, b \quad -\infty \leq a < b \leq \infty \; \forall x \notin [a,b] \quad f_X(x) = 0$.
Let $\Psi : [a,b] \to \mathbb{R}$ continuous, monotonous, piecewise continuously differentiable. Define $Y = \Psi(X)$, then Y is continuous random variable and

$$f_Y(y) = \frac{f_X(x)}{|\Psi'(x)|} = \frac{f_X(\Psi^{-1}(y))}{|\Psi'(\Psi^{-1}(y))|}$$

Proof Without loss of generality, Ψ is decreasing.

$$F_Y(y) = P(Y \leq y) = P(\Psi(X) \leq y) = P(X \geq \Psi^{-1}(y)) = 1 - P(X \leq \Psi^{-1}(y)) = 1 - F_X(\Psi^{-1}(y)) = 1 - F_X(\Psi^{-1}(y))$$

Thus

$$f_Y(y) = F_Y'(y) = -(F_X(\Psi^{-1}(y))) = -F_X'(\Psi^{-1}(y)) \cdot \left(\Psi^{-1}(y)\right)'$$

Since $\left(\Psi^{-1}\right)'(y) = \frac{1}{\Psi'(\Psi^{-1}(y))}$:

$$f_Y(y) = -f_X(\Psi^{-1}(y)) \cdot \frac{1}{\Psi'(\Psi^{-1}(y))} = \frac{f_X(x)}{|\Psi'(x)|}$$

Claim Let ψ piecewise monotonous, i.e. consists of finite or countable intervals in each of which its monotonous, then

$$f_Y(y) = \sum_{x:\psi(x)=y} \frac{f_X(x)}{|\psi'(x)|}$$

Simulation

$$U \sim U\left([0,1]\right)$$

$$F_U(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$f_U(x) = 1,\ 0 \leq x \leq 1$$

To acquire a number which is distributed (approximately) by U:
Choose $n \in N$, toss a coin n times and write down a number as binary: $0.a_1a_2a_3\ldots a_n$.

Claim Let X continuous random variable with distribution function F_X. Suppose $\exists -\infty \leq a < b \leq -\infty$ such that $F_x(a) = 0$ and $F_x(b) = 1$, such that F_X is completely monotonic on (a, b). Suppose $U \sim U\big([0,1]\big)$ and define $Y = F_X^{-1}(U)$ then $Y \sim X$.

Proof

$$F_Y(y) = P(Y \leq y) = P(F_X^{-1}(U) \leq y) = P(U \leq F_X(y)) = F_U(F_X(y)) = F_X(y)$$

Example

$$X \sim Exp(\lambda)$$

$$f_X(x) = \lambda e^{-\lambda x}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

Lets find $F_X^{-1}(y)$

$$y = 1 - e^{-\lambda x}$$

$$-\lambda x = \log(1-y)$$

$$x = -\frac{\log(1-y)}{\lambda}$$

Thus

$$F_X^{-1}(y) = -\frac{1}{\lambda}\log(1-y)$$

Thus we define

$$Y = -\frac{1}{\lambda}\log(1-U)$$

and $Y \sim X$.

Claim Suppose F_X is completely monotonic and piecewise continuously differentiable. Then $Y = F_X(X) \sim U([0,1])$.

Proof

$$F_Y(y) = P(Y \leq y) = P(F_X(X) \leq y) = P(X \leq F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$$

Note

$$X = X^+ + X^-$$

$$\mathbb{E}X^+ = \begin{cases} \sum_{j:x_j>0} p_j e^{tx_j} & \text{discrete} \\ \int_0^\infty e^{tx} f(x)dx & \text{continuous} \end{cases}$$

$$\mathbb{E}X^+ = \begin{cases} -\sum_{j:x_j<0} p_j e^{tx_j} & \text{discrete} \\ -\int_{-\infty}^0 e^{tx} f(x)dx & \text{continuous} \end{cases}$$

Moment-generating function

$$M_X(t) = \mathbb{E}e^{tX}$$

Obviously $M(0) = 1$. Generally:

$$M_X(t) = \begin{cases} \sum_{j=1}^N p_j e^{tx_j} & \text{discrete} \\ \int_{-\infty}^\infty e^{tx} f(x)dx & \text{continuous} \end{cases}$$

Now, if $\mathbb{E}(X^+)^n = \infty$, then $\forall t > 0 \quad M_X(t) = \infty$, and if $\mathbb{E}(X^-)^n = \infty$, then $\forall t < 0 \quad M_X(t) = \infty$. That means that necessary condition for existence of M_X in neighborhood of 0 is $\mathbb{E}|X|^n < \infty$.

Note If $M_X(t) < \infty$ for t in neighborhood of t_0, then M_X if differentible infinite times and

$$\frac{d^n}{dt^n} M_X(t) = \mathbb{E}X^n e^{tX}$$

In particular,

$$\frac{d^n}{dt^n} M_X(0) = \mathbb{E}X^n$$

By Taylor

$$M_X(t) = \sum_{n=0}^\infty \frac{M_X^{(n)}(0)}{n!} t^n$$

Example

$$X \sim N(0,1)$$

$$f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

$$M_X(t) = \int_{-\infty}^\infty e^{tx} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx = e^{\frac{t^2}{2}} \int_{-\infty}^\infty \frac{e^{-\frac{(x-t)^2}{2}}}{\sqrt{2\pi}} dx = e^{\frac{t^2}{2}}$$

Thus

$$M_X(t) = \sum_{n=0}^\infty \frac{t^{2n}}{2^n n!} = \sum_{n=0}^\infty \frac{(2n)!}{2^n n!} \frac{t^{2n}}{(2n!)}$$

That means

$$\mathbb{E}X^{2n+1} = 0$$

(that can be seen from direct calculation, acquiring integral of odd function on whole $\mathbb{R}$)

$$\mathbb{E}X^{2n} = \frac{(2n)!}{2^n} = \frac{2n(2n-1)(2n-2)(2n-3)\dots 1}{2^n n(n-1)(n-2)\dots 1} = \frac{2^n \cdot n(2n-1)(n-1)(2n-3)(n-2)\dots 1}{2^n n(n-1)(n-2)\dots 1} = (2n-1)(2n-3)(2n-5)\cdot\dots\cdot 5\cdot 3\cdot 1$$

5 Markov's and Chebyshev's inequalities

Markov's inequality Let $Y \geq 0$ and $\mathbb{E}Y < \infty$. Then,

$$\forall \lambda > 0 \quad P(Y \geq \lambda) \leq \frac{\mathbb{E}Y}{\lambda}$$

Proof Discrete:

$$P(Y \geq \lambda) = \sum_{j:y_j>\lambda} p_j \leq \sum_{j:y_j>\lambda} \frac{y_j}{\lambda} p_j \leq \frac{1}{\lambda} \sum_j p_j y_j = \frac{\mathbb{E}Y}{\lambda}$$

Continuous:

$$P(Y \geq \lambda) = \int_\lambda^\infty f(y)dy \leq \int_\lambda^\infty \frac{y}{\lambda} f(y)dy \leq \frac{1}{\lambda} \int_0^\infty y f(y)dy = \frac{\mathbb{E}Y}{\lambda}$$

Chebyshev's inequality Let X random variable for which second moment exists. Denote $\mu = \mathbb{E}X$ and $\sigma^2 = Var(x)$ then

$$P(|X-\mu| \geq \lambda) \leq \frac{\sigma^2}{\lambda^2}$$

Proof Define $Y = (X-\mu)^2$ By Markov inequality:

$$P(|X-\mu| \geq \lambda) = P\left((X-\mu)^2 \geq \lambda^2\right) = P\left(Y^2 \geq \lambda^2\right) \leq \frac{1}{\lambda^2}\mathbb{E}Y = \frac{\mathbb{E}(X-\mu)^2}{\lambda^2} = \frac{\sigma^2}{\lambda^2}$$

6 Multiple random variable

Random vector Let X and Y two random variables on same probability space. Then (X,Y) is random vector. Define

$$F(x,y) = P(X \leq x, Y \leq y)$$

which is called joint distribution function.

Claim

1. F is monotonic in both x and y.
2. $\lim_{x\to-\infty} F(x,y) = \lim_{y\to-\infty} F(x,y) = 0$.
3. $\lim_{x\to\infty} F(x,y) = F_Y(y)$
   $\lim_{y\to\infty} F(x,y) = F_X(x)$.
   F_X and F_Y are called marginal distribution function.
4. $\lim_{x\to\infty,y\to\infty} F(x,y) = 1$
5. F is continuous from the right for both of x and y.

Claim

$$-\infty \leq a < b \leq +\infty$$
$$-\infty \leq c < d \leq +\infty$$

Then

$$P(a < X \leq b, c < Y \leq d) = F(b,d) - F(b,c) - F(a,b) + F(a,c)$$

Discrete vector $F(x,y)$ is piecewise constant. In other words, X,Y are discrete random variables:

$$P(X = x_i) = p_i^X : p_i^X > 0$$
$$P(Y = y_j) = p_j^Y : p_j^Y > 0$$

and

$$P\Big((X,Y) = (x_i,y_j)\Big) = P(X = x_i, Y = y_j) = p_{ij} : p_{ij} \geq 0$$

Then we can write

$$p_i^X = P(X = x_i) = \sum_{j=1}^{m} p_{ij}$$
$$p_j^Y = P(Y = y_i) = \sum_{i=1}^{n} p_{ij}$$

Now if we take the previous equation:

$$P(a < X \leq b, c < Y \leq d) = F(b,d) - F(a,d) - F(b,c) - F(a,c)$$

Then if $a \to b$ and $c \to d$:

$$\begin{aligned} P\Big((X,Y) = (b,d)\Big) = P(a < X \leq b, c < Y \leq d) &= F(b,d) - F(b^-,d) - F(b,d^-) + F(b^-,d^-) = \\ &= \big(F(b,d) - F(b^-,d)\big) - \big(F(b,d^-) - F(b^-,d^-)\big) \end{aligned}$$

Continuous variable $F(x, y)$ continuous in both x and y. We always suppose that F is piecewise twice differentiable and define

$$f(x,y) = \frac{\partial^2 F}{\partial x \partial y}$$

Then

$$\int_a^b \int_c^d f(x,y)dxdy = \int_a^b dx \int_c^d dy \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)(x,y) =$$

$$= \int_a^b dx \left(\frac{\partial F}{\partial x}(x,d) - \frac{\partial F}{\partial x}(x,b)\right) = F(b,d) - F(a,d) - F(b,c) + F(a,c) = P(a < X \leq b, c < Y \leq d)$$

Lemma Suppose h, g integrable on (c, d), $-\infty \leq c < d \leq \infty$. Suppose

$$\forall c \leq a < b \leq d \int_a^b g(x)dx = \int_a^b h(x)dx$$

Then if g and h is continuous, $g(x) = h(x)$.

Proof Suppose $x \in (c, d)$ such that g and h continuous.

$$\forall d - x > \delta > 0 \qquad \int_x^{x+\delta} g(y)dy = \int_x^{x+\delta} h(y)dy$$

$$\frac{1}{\delta}\int_x^{x+\delta} g(y)dy = \frac{1}{\delta}\int_x^{x+\delta} h(y)dy$$

In limit $\delta \to 0$, for continuous functions $g(x) = h(x)$.

Claim

$$\int_{-\infty}^{\infty} f(x,y)dy = f_X(x)$$

and

$$\int_{-\infty}^{\infty} f(x,y)dx = f_Y(y)$$

Proof For $-\infty \leq a < b \leq \infty$ and $-\infty \leq c < d \leq \infty$:

$$P(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f(x,y)dxdy$$

In limit $c \to -\infty$, $d \to \infty$:

$$P(a \leq X \leq b) = \int_{-\infty}^{\infty} dy \int_a^b f(x,y)dx = \int_a^b \int_{-\infty}^{\infty} f(x,y)dydx$$

But

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

Then

$$\int_a^b \int_{-\infty}^{\infty} f(x,y)dydx = \int_a^b f_X(x)dx$$

From lemma

$$\int_{-\infty}^{\infty} f(x,y)dy = f_X(x)$$

Generalization

$$P\big((X,Y) \in A\big) = \int_a^b dx \int_c^d dy f(x,y) = \iint\limits_A f(x,y)dxdy$$

Dimension reduction

$$\Psi : \mathbb{R}^2 \to \mathbb{R}$$

Discrete

$$\mathbb{E}\Psi(X,Y) = \sum_{i=1}^{n}\sum_{j=1}^{m}\Psi(x_i,y_j)p_{ij}$$

Continuous

$$\mathbb{E}\Psi(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(x,y)f(x,y)dxdy$$

Example $\Psi(x,y) = \mathbb{1}_A$.
Then

$$\mathbb{E}\Psi(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(x,y)dxdy = \iint\limits_A f(x,y)dxdy = P\big((X,Y)\in A\big)$$

Example

$$\Psi(x,y) = x$$

Obviously

$$\mathbb{E}\Psi(X,Y) = \mathbb{E}X$$

Formally:

$$\mathbb{E}\Psi(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf(x,y)dxdy = \int_{-\infty}^{\infty}x\int_{-\infty}^{\infty}f(x,y)dydx = \int_{-\infty}^{\infty}xf_X(x)dx = \mathbb{E}X$$

Note Everything said is naturally generalized to N dimensions.

6.1 Independence

Definition X and Y are independent, if

$$P\big(a < X \le b, c < Y \le d\big) = P\big(a < X \le b\big)P\big(c < Y \le d\big)$$

Which is also generalized to any pair of (measurable) set:

$$P(X\in A, Y\in B) = P(X\in A)P(Y\in B)$$

Claim The following 3 are equivalent:

1. X and Y are independent
2. $$F(x,y) = F_X(x)F_Y(y)$$
3. $$p_{ij} = p_i^X p_j^Y$$
   or
   $$f(x,y) = f_X(x)f_Y(y)$$
   for discrete and continuous respectively.

Proof

1. $1 \Rightarrow 2$ Immediate. For $a = c = -\infty$ and $b = x$, $d = y$:
   $$F(x,y) = P\left(X\le x, Y\le y\right) = P(X\le x)P(Y\le y) = F_X(x)F_Y(y)$$
2. $1 \Leftarrow 2$
   $$\begin{aligned} P(a < X \le b, c < Y \le d) &= F(b,d) - F(a,d) - F(b,c) + F(a,c) = \\ &= F_X(b)F_Y(d) - F_X(a)F_Y(d) - F_X(b)F_Y(c) + F_X(a)F_Y(c) = \\ &= F_Y(d)\left(F_X(b) - F_X(a)\right) - F_Y(c)\left(F_X(b)-F_X(a)\right) = \left(F_Y(d)-F_Y(c)\right)\left(F_X(b)-F_X(a)\right) = P(a<X\le b)P(c<Y\le d)\end{aligned}$$

3. $2 \Rightarrow 3$ (continuous)

$$F(x,y) = F_X(x)F_Y(y)$$
$$\frac{\partial^2 F(x,y)}{\partial xy} = \frac{\partial^2}{\partial xy} F_X(x)F_Y(y)$$
$$f(x,y) = F'_X(x)F'_Y(y) = f_X(x)f_Y(y)$$

4. $2 \Leftarrow 3$ (continuous)

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(x,y)dxdy = \int_{-\infty}^{x}\int_{-\infty}^{y} f_X(x)f_Y(y)dxdy = \int_{-\infty}^{x} f_X(x)dx \int_{-\infty}^{y} f_Y(y)dy = F_X(x)F_Y(y)$$

7 Conditional distribution

7.1 Discrete case

$$p_{Y|X}(y|x) = P(Y=y|X=x) = \frac{P(X=x,Y=y)}{P(X=x)} = \frac{p(x,y)}{p_X(x)}$$

We'll call it distribution function. The definition is consistent:

$$\sum_y p_{Y|X}(y|x) = P(Y=y|X=x) = \sum_y \frac{p(x,y)}{p_X(x)} = 1$$

$$F_{Y|X=x} = P\left(Y \le y|X=x\right) = \sum_{y' \le y} P\left(Y = y'|X=x\right) = \sum_{y'\le y} p_{Y|X=x}\left(y'|x\right) = \frac{1}{p(x)}\sum_{y'\le y} p\left(x,y'\right)$$

$$\mathbb{E}(Y|X=x) = \sum_y y p_{Y|X}(y|x) = \sum_y y\frac{p(x,y)}{p_X(x)} = \frac{1}{p_X(x)}\sum_y yp(x,y)$$

$$\mathbb{E}\big(\Psi(Y)|X=x\big) = \sum_y \Psi(y)p_{Y|X}(y|x)$$

Example

$$X \sim Pois(\lambda)$$
$$Y|X=x \sim Bin(x,p)$$

Find distribution of Y

Solution

$$P(Y=y) = \sum_x P(X=x,Y=y) = \sum_x P(X=x)P(Y=y|X=x)$$

or

$$P(Y=y) = \sum_x p(x,y) = \sum_x p_X(x)p_{Y|X}(y|x)$$

Now

$$P(X=x) = \frac{e^{-\lambda x}}{x!}$$
$$= P(Y=y|X=x) = \binom{x}{y}p^y(1-p)^{x-y}$$

Thus

$$P(Y=y) = \sum_{x=y}^{\infty}\frac{e^{-\lambda}\lambda^x}{x!}\binom{x}{y}p^y(1-p)^{x-y} = e^{-\lambda}p^y\sum_{x=y}^{\infty}\frac{1}{x!}\frac{x!\lambda^x}{y!(x-y)!}(1-p)^{x-y} = \frac{e^{-\lambda}p^y}{y!}\lambda^y\sum_{x=y}^{\infty}\frac{\lambda^{x-y}}{(x-y)!}(1-p)^{x-y}$$

Denote $x-y=m$

$$P(Y=y) = \frac{e^{-\lambda}p^y}{y!}\lambda^y\sum_{m=0}^{\infty}\frac{\lambda^m}{(m)!}(1-p)^m = \frac{e^{-\lambda}p^y}{y!}\lambda^y e^{\lambda(1-p)} = e^{-\lambda p}\frac{(\lambda p)^y}{y!}$$

Note We write $Y|X \sim Bin(X,p)$ for $Y|X=x \sim Bin(x,p)$.

7.2 Continuous case

We can't use $P(X=x)$ in continuous case, since it equals 0. Thus we modify the equation as follows:

$$P\big(Y\le y|X\in[x,x+\epsilon]\big)=\frac{P\big(x\le X\le x+\epsilon, Y\le y\big)}{P\big(x\le X\le x+\epsilon\big)}=\frac{\int_{-\infty}^{y}dy'\int_{x}^{x+\epsilon}dx' f(x',y')}{\int_{x}^{x+\epsilon}f_X(x')dx'}=$$

$$=\frac{\frac{1}{\epsilon}\int_{x}^{x+\epsilon}\left(\int_{-\infty}^{y}f(x',y')dy'\right)dx'}{\frac{1}{\epsilon}\int_{x}^{x+\epsilon}f_X(x')dx'}\xrightarrow{\epsilon\to 0}\frac{\int_{-\infty}^{y}f(x,y')dy'}{f_X(x)}$$

Thus

$$F_{Y|X}(y|x)=\frac{\int_{-\infty}^{y}f(x,y')dy'}{f_X(x)}$$

$$f_{Y|X}(y|x)=\frac{d}{dy}F_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}$$

"Full density" formula

$$f_Y(y)=\int_{-\infty}^{\infty}f_x(x)f_{Y|X}(y|x)dx$$

Example

$$X\sim U([0,1])\quad Y|X\sim U([0,x])$$

Solution $f_X(x)=1\quad 0\le x\le 1$ and $f_{Y|X}f_X(x)=\frac{1}{x}\quad 0\le y\le x$. Thus

$$f(x,y)=f_X(x)f_{Y|X}(y|x)=\frac{1}{x}\quad 0\le y\le x\le 1$$

And

$$f_Y(y)=\int_{-\infty}^{\infty}f(x,y)dx=\int_{y}^{1}\frac{1}{x}=-\log y\quad 0\le y\le 1$$

Exercise Find $\mathbb{E}Y$

Solution

$$\mathbb{E}Y=\int_{0}^{1}y(-\log y)dy=-\int_{0}^{1}y\log ydy=\frac{1}{4}$$

Linearity of expectation Define $\Psi(X,Y)=X+Y$.
Then

$$\mathbb{E}(X+Y)=\mathbb{E}\Psi(X,Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Psi(x,y)f(x,y)dxdy=\int_{-\infty}^{\infty}dxx\int_{-\infty}^{\infty}dyf(x,y)+\int_{-\infty}^{\infty}dyy\int_{-\infty}^{\infty}dxf(x,y)=$$

$$=\int_{-\infty}^{\infty}xf_X(x)dx+\int_{-\infty}^{\infty}yf_Y(y)dy=\mathbb{E}X+\mathbb{E}Y$$

Sum of independent variables Suppose X,Y independent.

$$f(x,y)=f_X(x)f_Y(y)$$

$$p(x,y)=p_X(x)p_Y(y)$$

We denote $Z=X+Y$ and want to find $f_Z(z)=f_{X+Y}(z)=?$

Discrete

$$p_Z(z)=p_{X+Y}(z)=P(X+Y=z)=\sum_x P(X=x,Y=z-x)\overset{ind}{=}\sum_x P(X=x)P(Y=z-x)=\sum_x p_x(x)p_y(z-x)$$

Continuous Denote $A_z = \{(x,y) : x+y \leq z\}$. Then

$$F_{X+Y}(z) = P(X+Y \leq z) = P((X,Y) \in A_z) = \iint\limits_{A_z} f(x,y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} dy f(x,y) \overset{ind}{=}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{z-x} dy f_X(x) f_Y(y) = \int_{-\infty}^{\infty} dx f_X(x) \int_{-\infty}^{z-x} dy f_Y(y)$$

Then

$$f_{X+Y}(z) = \frac{d}{dz} \int_{-\infty}^{\infty} dx f_X(x) \int_{-\infty}^{z-x} dy f_Y(y) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx = (f_X * f_Y)(z)$$

7.3 Law of total expectation (smoothing theorem)

Denote $h(x) = \mathbb{E}(Y|X=x)$. Then what is $\mathbb{E}h(X)$?

$$\mathbb{E}h(X) = \int_{-\infty}^{\infty} h(x) f_X(x)dx = \int_{-\infty}^{\infty} f_X(x) \left[\int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy\right] dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x) y f_{Y|X}(y|x)dydx =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x) y \frac{f(x,y)}{f_X(x)} dydx = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)dxdy = \int_{-\infty}^{\infty} y f_y(y)dy = \mathbb{E}Y$$

We denote

$$h(X) = E(Y|X)$$

Example

$$X \sim [0,1]$$
$$Y|X \sim [0,x]$$
$$h(x) = \mathbb{E}(Y|X=x) = \frac{x}{2}$$
$$\mathbb{E}(Y) = \mathbb{E}h(X) = \frac{1}{4}$$

Example Suppose we choose random point (X,Y) on unitary disk. Denote $R = \sqrt{X^2+Y^2}$ and $f_{R,X}$ density of vector (R,X). Find $f_{R,X}$.

Solution We have

$$f_{X,Y} = \frac{1}{\pi} \quad x^2+y^2 < 1$$

Now, we want first to find

$$F_{R,X}(r,x) = P(R \leq r, X \leq x)$$

we can limit $0 \leq r \leq 1$ and $0 \leq x \leq 1$. Define

$$A_{r,x} = \left\{(x',y') : (x')^2 + (y')^2 \leq r^2, x' \leq x\right\}$$

$$P\left(X^2+Y^2 \leq r^2, X \leq x\right) = P((X,Y) \in A_{r,x}) = \iint\limits_{A_{r,x}} f_{X,Y}(x,y)dxdy = \frac{1}{\pi}\iint\limits_{A_{r,x}} dxdy$$

Denote

$$A_{r,x}^{(1)} = \{(x',y') \in A_{r,x} : x' \leq 0\}$$

and

$$A_{r,x}^{(2)} = A_{r,x} - A_{r,x}^{(1)}$$

$$P\left(X^2+Y^2 \leq r^2, X \leq x\right) = \frac{1}{\pi}\iint\limits_{A_{r,x}} dxdy = \frac{1}{\pi}\iint\limits_{A_{r,x}^{(1)}} dxdy + \frac{1}{\pi}\iint\limits_{A_{r,x}^{(2)}} dxdy = \frac{r^2}{2} + \frac{1}{\pi}\iint\limits_{A_{r,x}^{(2)}} dxdy$$

Now

$$\iint\limits_{A_{r,x}^{(2)}} dxdy = 2\int_0^x (r^2 - (x')^2)^{\frac{1}{2}} dx'$$

Thus

$$F_{R,X}(r,x) = \frac{r^2}{2} + \frac{2}{\pi}\int_0^x (r^2-(x')^2)^{\frac{1}{2}}dx'$$

$$\frac{\partial F_{R,X}}{\partial x} = \frac{2}{\pi}(r^2-x^2)^{\frac{1}{2}}$$

$$f_{R,X}(r,x) = \frac{\partial^2 F_{R,X}}{\partial xr} = \frac{\partial}{\partial r}\frac{2}{\pi}(r^2-x^2)^{\frac{1}{2}} = \frac{2r}{\pi}(r^2-x^2)^{-\frac{1}{2}} = \frac{2r}{\pi(r^2-x^2)^{\frac{1}{2}}}$$

Claim For independent X and Y

$$\mathbb{E}\Phi(X)\Psi(Y) = \mathbb{E}\Phi(X)\mathbb{E}\Psi(Y)$$

if $\mathbb{E}\Phi(X)\Psi(Y) < \infty$.

Proof

$$\mathbb{E}\Phi(X)\Psi(Y) = \sum_{x,y}\Phi(x)\Psi(y)p(x,y) = \sum_{x,y}\Phi(x)\Psi(y)p_X(x)p_Y(y) = \sum_x \Phi(x)p_X(x)\sum_y \Psi(y)p_Y(y) = \mathbb{E}\Phi(X)\mathbb{E}\Psi(Y)$$

Claim For independent X and Y

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

Proof

$$M_{X+Y}(t) = \mathbb{E}e^{t(X+Y)} = \mathbb{E}e^{tX}e^{tY} = \mathbb{E}e^{tX}\mathbb{E}e^{tY} = M_X(t)M_Y(t)$$

Claim For independent X and Y

$$\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)$$

Proof

$$\sigma^2(X+Y) = \mathbb{E}(X+Y-\mathbb{E}(X+Y))^2 = \mathbb{E}\left(X+Y-\mathbb{E}X-\mathbb{E}Y\right)^2 = \mathbb{E}\left(X-\mathbb{E}X+Y-\mathbb{E}Y\right)^2 =$$
$$= \mathbb{E}(X-\mathbb{E}X)^2 + (Y-\mathbb{E}Y)^2 + 2\mathbb{E}(X-\mathbb{E}X)(Y-\mathbb{E}Y) = \mathbb{E}(X-\mathbb{E}X)^2 + (Y-\mathbb{E}Y)^2 = \sigma^2(X) + \sigma^2(Y) + 2\mathbb{E}(X-\mathbb{E}X)\mathbb{E}(Y-\mathbb{E}Y)$$

7.4 Indicator random variable

Let $\{Y_j\}_{j=1}^n$ independent identically distributed (IID) random variables such that $P(Y_j=1)=p$ and $P(Y_i=0)=1-p$ (i.e. $Y_j \sim Ber(p)$). Denote $B(n,p) \sim X = \sum_{j=1}^n Y_j$.
Lets find $\mathbb{E}X$:

$$\mathbb{E}X = \mathbb{E}\sum_{j=1}^n Y_j = n\mathbb{E}Y = np$$

$$\sigma^2(Y) = \mathbb{E}\left(Y-\mathbb{E}Y\right)^2 = \mathbb{E}\left(Y-p\right)^2 = (0-p)^2(1-p) + (1-p)^2p = p^2(1-p) + (1-p)^2p = p(1-p)$$

$$\sigma^2(X) = \mathbb{E}X = \sigma^2\left(\sum_{j=1}^n Y_j\right) = n\sigma^2\left(Y\right) = np(1-P)$$

Example Suppose there are n pairs of shoes and m of shoes are bad. What is expectation of number of pairs of good shoes?

Example Denote $Y_i = \begin{cases} 1, & \text{good pair} \\ 0, & \text{bad pair} \end{cases}$ Then $\mathbb{E}X = \mathbb{E}\sum_j Y_j$.

$$P(Y_j=1) = \frac{\binom{2n-2}{m}}{\binom{2n}{m}} = \frac{(2n-m)(2n-m-1)}{2n(2n-1)}$$

$$\mathbb{E}X = \mathbb{E}\sum_j Y_j = n\mathbb{E}Y_j = n\frac{(2n-m)(2n-m-1)}{2n(2n-1)} = \frac{(2n-m)(2n-m-1)}{2(2n-1)}$$

Example $X_i \sim Exp(\lambda)$ IID. X_1 is time until first call. X_i is time between $(i-1)^{th}$ and i^{th} calls. Then time until call is $S_n = \sum_{i=1}^{n} X_i$.
Choose a time window T. Then number of calls in interval $[0, T]$ is denoted N.
Denote density of S_n as f_n. Since $S_1 = X_1$, $f_1(t) = \lambda e^{-\lambda t}$

Theorem $f_n(t) = \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t}$.

Proof By induction. Basis is trivial.
Step.

$$S_{n+1} = S_n + X_{n+1}$$

and S_n and X_{n+1} are independent. Then f_{n+1}:

$$f_{n+1}(t) = (f_n * f_1)(t) = \int_{-\infty}^{\infty} f_n(s) f_1(t-s) ds = \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!} e^{-\lambda s} \lambda e^{-\lambda(t-s)} ds =$$
$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \int_0^t s^{n-1} ds = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \frac{t^n}{n} = \frac{\lambda^{n+1} t^n}{n!} e^{-\lambda t}$$

Theorem $N \sim Pos(\lambda T)$.

Proof

$$N = \max\{n \geq 0 : S_n \leq T\}$$

Also

$$\{N = 0\} = \{S_1 > T\}$$
$$P(N=0) = P(S_1 > T) = \int_T^{\infty} \lambda e^{-\lambda T} dt = e^{-\lambda T}$$

$$\{N \geq n\} = \{S_n \leq T\}$$

$$P(N \geq n) = P(S_n \leq T) = \int_0^T f_n(t) dt = \int_0^T \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t} dt = \frac{\lambda^n}{(n-1)!} \int_0^T t^{n-1} e^{-\lambda t} dt =$$
$$= -\frac{\lambda^n}{(n-1)!} e^{-\lambda T} \sum_{i=1}^{n-1} \frac{T^{n-i}}{\lambda^i} = - - \frac{\lambda^n}{(n-1)!} e^{-\lambda T} + P(N \geq n-1)$$

I.e.

$$P(N = n-1) = P(N \geq n-1) - P(N \geq n) = \frac{(\lambda T)^{n-1}}{(n-1)!} e^{-\lambda T}$$

8 Weak law of large numbers (WLLN)

Suppose $\{X_n\}_{n=1}^{\infty}$ IID random variables. Denote

$$S_n = \sum_{j=1}^{n} X_j$$

Suppose also that first two moments exist, denote $\mu = \mathbb{E}X_n$ and $\sigma^2 = Var(X_n)$ exist.

Weak law of large numbers

$$\lim_{n \to \infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0$$

or

$$\frac{S_n}{n} \overset{\text{prob}}{\to} \mu$$

Proof Chebyshev inequality:

$$P\left(|Y-\mathbb{E}Y|>\lambda\right)\leq\frac{\sigma^2}{\lambda^2}$$

With $Y=\frac{S_n}{n}$.

$$\mathbb{E}\frac{S_n}{n}=\frac{1}{n}\mathbb{E}S_n=\frac{1}{n}\mathbb{E}\sum_{j=1}^{n}X_j=\frac{1}{n}\mathbb{E}nX_1=\mu$$

$$\sigma^2\left(\frac{S_n}{n}\right)=\frac{1}{n^2}\sigma^2(S_n)=\frac{1}{n^2}\sigma^2\left(\sum_{j=1}^{n}X_j\right)=\frac{1}{n^2}\sum_{j=1}^{n}\sigma^2\left(X_j\right)=\frac{1}{n^2}\cdot n\sigma^2=\frac{\sigma^2}{n}$$

$$P\left(\left|\frac{S_n}{n}-\mu\right|>\lambda\right)\leq\frac{\sigma^2}{n\epsilon^2}\to 0$$

Same can be proofed without second moment, though the proof is more complicated.

Example If $X_j\sim Ber(p)$ $X=S_n\sim Bin(n,p)$. Thus

$$\lim_{n\to\infty}=P\left(\left|\frac{X}{n}-p\right|\geq\epsilon\right)=0$$

Example Note that $P(S_n=\mu)\not\to 1$.
If $X_j\sim Ber(p)$, then

$$P\left(\frac{S_n}{n}=p\right)=P\left(S_n=pn\right)$$

Let $p=\frac{1}{2}$ and substitute $2n$:

$$P\left(\frac{S_n}{n}=p\right)=P\left(S_n=n\right)=\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}=\frac{(2n)!}{(n!)^2}\left(\frac{1}{2}\right)^{2n}\approx\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}(2\pi n)2^{2n}}=\frac{1}{\sqrt{\pi n}}\to 0$$

Notation For $A\cap B=\Omega$ and $A\cup B=\emptyset$. then we can write

$$\mathbb{E}X=\mathbb{E}\left(XI_A+XI_B\right)$$

When

$$\mathbb{E}XI_A=E(X,A)=\begin{cases}\sum_{x\in A}xp(x)\\ \int\limits_A xf(x)dx\end{cases}$$

Weierstrass approximation theorem Suppose f is a continuous real-valued function defined on the real interval $[a,b]$. For every $\epsilon>0$, there exists a polynomial p such that for all $x\in[a,b]$, we have

$$|f(x)-p(x)|<\epsilon$$

or equivalently, the supremum norm

$$||f(x)-p(x)||<\epsilon$$

.

Proof $\forall p\in[0,1]$ let $\{X_{n,p}\}_{n=1}^{\infty}$ IID random variables such that $X_i\sim Ber(p)$. Denote $S_{n,p}=\sum_{j=1}^{n}X_{j,p}$. Then

$$\mathbb{E}\frac{S_{n,p}}{n}=p$$

and

$$\sigma^2\left(\frac{S_{n,p}}{n}\right)=\frac{1}{n^2}\sigma^2\left(S_{n,p}\right)=\frac{1}{n^2}\sigma^2\left(\sum_{j=1}^{n}X_{j,p}\right)=\frac{1}{n}\sigma^2(X_1)=\frac{1}{n}p(1-p)$$

Thus from Chebyshev

$$P\left(\left|\frac{S_{n,p}}{n}-p\right|\geq\epsilon\right)\leq\frac{p(1-p)}{\epsilon^2 n}\leq\frac{1}{4\epsilon^2 n}$$

Let $f \in \mathcal{C}([0,1])$ and $\epsilon > 0$. Lets find polynomial $Q(x)$ such that

$$|f(x) - Q(x)| < \epsilon$$

Denote

$$M = \max_{x\in[a,b]} |f(x)|$$

Also f is uniformly continuous, we can choose $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$.
Since $\frac{S_n}{n} \in [0,1]$, we can look at $f(S)$. Then

$$\mathbb{E}f\left(\frac{S_n}{n}\right) = \sum_{j=0}^{\infty} f\left(\frac{j}{n}\right)\binom{n}{j} p^j (1-p)^{n-j} = Q_n(p)$$

$$|Q_n(p) - f(p)| = \left|\mathbb{E}f\left(\frac{S_n}{n}\right) - f(p)\right| = \left|\mathbb{E}\left[f\left(\frac{S_n}{n}\right) - f(p)\right]\right| \leq \mathbb{E}\left|f\left(\frac{S_n}{n}\right) - f(p)\right|$$

Now divide domain of expectation into two:

$$\mathbb{E}\left|f\left(\frac{S_n}{n}\right) - f(p)\right| = \mathbb{E}\left(\left|f\left(\frac{S_n}{n}\right) - f(p)\right|, \frac{S_n}{n} - p \leq \delta\right) + \mathbb{E}\left(\left|f\left(\frac{S_n}{n}\right) - f(p)\right|, \frac{S_n}{n} - p > \delta\right)$$

From uniform continuousness:

$$\mathbb{E}\left(\left|f\left(\frac{S_n}{n}\right) - f(p)\right|, \frac{S_n}{n} - p \leq \delta\right) \leq \frac{\epsilon}{2}\cdot P\left(\frac{S_n}{n} - p \leq \delta\right) \leq \frac{\epsilon}{2}$$

And from Chebyshev, as show earlier

$$\mathbb{E}\left(\left|f\left(\frac{S_n}{n}\right) - f(p)\right|, \frac{S_n}{n} - p > \delta\right) < 2M\cdot P\left(\frac{S_n}{n} - p > \delta\right) \leq \frac{2M}{4\delta^2 n}$$

$$\exists n_0 > 0\ \forall n > n_0 \quad \frac{2M}{4\delta^2 n} < \frac{\epsilon}{2}$$

Thus

$$|Q_{n_0}(p) - f(p)| < \epsilon$$

and required polynomial is

$$Q_{n_0}(p) = \sum_{j=0}^{n_0} f\left(\frac{j}{n}\right)\binom{n}{j} p^j (1-p)^{n-j}$$

We can obviously map from $[a,b]$ to $[0,1]$ and back to acquire polynomial for any function on compact interval.

8.1 First and second moment methods

Let $\{Y_n\}_{n=0}^{\infty}$ non-negative random variables (non necessarily on same probability space).

First moment method If $\lim_{n\to\infty}\mathbb{E}Y_n = 0$ then $\forall\epsilon > 0 \quad \lim_{n\to\infty} P(Y_n > \epsilon) = 0$. We also write $Y \overset{\text{prob}}{\to} 0$.

Proof From Markov:

$$P(Y_n > \epsilon) \leq \frac{1}{\epsilon}\mathbb{E}Y_n \to 0$$

Second moment method If $\lim_{n\to\infty}\mathbb{E}Y_n = \infty$ and $\sigma^2(Y_n) = o\left((\mathbb{E}Y)^2\right)$ then $\forall\epsilon > 0 \quad \lim_{n\to\infty} P\left(\left|\frac{Y_n}{\mathbb{E}Y_n} - 1\right| > \epsilon\right) = 0$, i.e. $\frac{Y_n}{\mathbb{E}Y_n} \overset{\text{prob}}{\to} 1$.

Proof From Chebyshev, since $\mathbb{E}\frac{Y_n}{\mathbb{E}Y_n} = \frac{1}{\mathbb{E}Y_n}\mathbb{E}Y_n = 1$:

$$P\left(\left|\frac{Y_n}{\mathbb{E}Y_n} - 1\right| > \epsilon\right) \leq \frac{1}{\epsilon^2} Var\left(\frac{Y_n}{\mathbb{E}Y_n}\right) = \frac{1}{\epsilon^2}\frac{\sigma^2(Y_n)}{(\mathbb{E}Y_n)^2} \to 0$$

Note If Y_n is integer, then from first moment method $\lim_{n\to\infty} P(Y_n = 0) = 1$.

Conclusion If second moment method conditions are fulfilled,

$$\forall M \quad \lim_{n\to\infty} P(Y_n > M) = 1$$

Proof Let $\epsilon = 0.2018$.

$$1 \leftarrow P\left(\left|\frac{Y_n}{\mathbb{E}Y_n} - 1\right| \leq 0.2018\right) = P\left(0.7982 \leq \frac{Y_n}{\mathbb{E}Y_n} \leq 1.2018\right) \leq P\left(0.7982 \leq \frac{Y_n}{\mathbb{E}Y_n}\right) = P\left(0.7982\mathbb{E}Y_n \leq Y_n\right)$$

Since $\mathbb{E}Y_n \to \infty$, for n sufficiently large, for all M $\mathbb{E}Y_n > M$ and thus

$$1 \leftarrow P\left(0.7982\mathbb{E}Y_n \leq Y_n\right) \leq P\left(M \leq Y_n\right)$$

9 Random Graphs

Reminder Graph is pair of set of vertices and set of edges: $G = (V, \mathcal{E})$.

Full graph K_n is full graph, if $K = (G, \mathcal{E})$ and

$$V = [n] = \{1, 2, 3, \dots\}$$

$$\mathcal{E} = \{\{i, j\} : 1 \leq i < j \leq n\}$$

Then $|\mathcal{E}| = \binom{n}{2}$

Take random graph of order n. Vertices are deterministic: $V = [n]$. Choose $0 \leq p \leq 1$. Then for every potential edge $\{i, j\}$ we throw a coin with probability p of heads. If we get heads, $\{i, j\} \in \mathcal{E}$, else $\{i, j\} \notin \mathcal{E}$. The tosses are independent for each pair of potential edges.
We denote

$$G_n(p) = \{[n], \mathcal{E}_n(p)\}$$

These graphs are called Erdös-Reny graphs.

Notation For Erdös-Reny graphs we write $P_{n,p}$ for probability and $\mathbb{E}_{n,p}$ for expectation.

Exercise

$$\mathbb{E}_{n,p} \left|\mathcal{E}_n(p)\right| = p\binom{n}{2}$$

Definition Vertex $j \in [n]$ is called disconnected if its $\deg(j) = 0$.

Exercise What is probability that there is disconnected vertex in $G_n(p)$.

Solution Denote $A_{j,n}$ event that j^{th} node is disconnected. Then we search a probability of $\bigcup_{j=1}^n A_{j,n}$.

$$P_{n,p}\left(\bigcup_{j=1}^n A_{j,n}\right) \leq \sum_{j=1}^n P_{n,p}(A_{j,n}) = nP_{n,p}\left(A_{j,n}\right) = n(1-p)^{n-1}$$

Thus, in limit

$$\lim_{n\to\infty} P_{n,p}\left(\bigcup_{j=1}^n A_{j,n}\right) = 0$$

Thus we'll need to make p depend on n such that $\lim_{n\to\infty} p_n = 0$. We define sequence of sparse graphs $G_n(p_n)$ such that $p_n \to 0$.

Example Suppose $p_n = \frac{c}{n}$. Then

$$P_{n,\frac{c}{n}}(A_{j,n}) = (1-p_n)^{n-1} = \left(1 - \frac{c}{n}\right)^n \to e^{-c}$$

Then obviously

$$\liminf_{n\to\infty} P_{n,\frac{c}{n}}\left(\bigcup_{j=1}^n A_{j,n}\right) > 0$$

Theorem Denote number of disconnected vertices in G_{n,p_n} as D_n. Then $\{D_n \geq 1\} = \bigcup_{j=1}^{n} A_{j,n}$. Let $p_n = \frac{\log n + c_n}{n}$.

1. If $\lim_{n\to\infty} c_n = \infty$, then $\lim_{n\to\infty} P_{n,p_n}(D_n = 0) = 1$.
2. If $\lim_{n\to\infty} c_n = -\infty$, then
$$\lim \mathbb{E}_{n,p_n} D_n = \infty$$
and for all $\epsilon > 0$,
$$\lim_{n\to\infty} P_{n,p_n}\left(\left|\frac{D_n}{\mathbb{E}_{n,p_n} D_n} - 1\right| > \epsilon\right) = 0$$
. In particular, for all M
$$\lim_{n\to\infty} P_{n,p_n}(D_n > M) = 1$$
and
$$\lim_{n\to\infty} P_{n,p_n}(D_n = 0) = 0$$

We say that $\frac{\log n}{n}$ is threshold for the property of "having at least one vertex disconnected". It can be shown that the same threshold applies to connectedness of the graph.

Proof Without loss of generality, $c_n = o\left(n^{\frac{1}{2}}\right)$ (if c_n is larger, it obviously true). By first moment method, it's enough to show $\lim_{n\to\infty} \mathbb{E}_{n,p_n} D_n = 0$. Define $D_{j,n} = \mathbb{1}_{A_{j,n}}$, then $D_n = \sum_{j=1}^{n} D_{j,n}$. Then

$$\mathbb{E}_{n,p_n} D_n = \mathbb{E}_{n,p_n} \sum_{j=1}^{n} D_{j,n} = n\mathbb{E}_{n,p_n} D_{1,n} = nP_{n,p_n}(A_{1,n}) = n(1-p_n)^{n-1} = n\left(1 - \frac{\log n + c_n}{n}\right)^{n-1}$$

Take a look at $\mathbb{E}_{n,p_n} D_{1,n}$:

$$\mathbb{E}_{n,p_n} D_{1,n} = \left(1 - \frac{\log n + c_n}{n}\right)^{n-1}$$
$$\log \mathbb{E}_{n,p_n} D_{1,n} = (n-1)\log\left(1 - \frac{\log n + c_n}{n}\right)$$

By Taylor:

$$\exists 0 < x^\star < x \quad f(x) = f(0) + f'(0)x + f''(x^\star)\frac{x^2}{2}$$

For $f(x) = \log(1-x)$:

$$\exists 0 < x^\star < x \quad \log(1-x) = -x - \frac{x^2}{(1-x^\star)^2}$$

Thus

$$\log \mathbb{E}_{n,p_n} D_{1,n} = (n-1)\left[-\frac{\log n + c_n}{n} - \frac{1}{(1-x^\star)^2}\left(\frac{\log n + c_n}{n}\right)^2\right] = -\log n - c_n + o(1)$$

That means

$$\mathbb{E}_{n,p_n} D_{1,n} = e^{-\log n - c_n + o(1)} = \frac{1}{n}e^{-c_n}(1 + o(1))$$

Back to total expectation

$$\mathbb{E}_{n,p_n} D_n = n\mathbb{E}_{n,p_n} D_{1,n} = e^{-c_n}(1 + o(1))$$

Note, that this equality is right for any $c_n = o\left(n^{\frac{1}{2}}\right)$.

1. Now suppose $\lim_{n\to\infty} c_n = \infty$, then $\mathbb{E}_{n,p_n} D_n \to 0$.
2. If $\lim_{n\to\infty} c_n = -\infty$, then $\mathbb{E}_{n,p_n} D_n = e^{-c_n}(1 + o(1)) \to \infty$.

   Now we want to use second moment method and for that we need to show that $\sigma^2(D_n) = o\left((\mathbb{E}_{n,p_n} D_n)^2\right)$:
$$\sigma^2(D_n) = \mathbb{E}_{n,p_n} D_n^2 - (\mathbb{E} D_n)^2$$
   or
$$\mathbb{E}_{n,p_n} D_n^2 = \sigma^2(D_n) + (\mathbb{E} D_n)^2$$
   Thus we want to show
$$\mathbb{E}_{n,p_n} D_n^2 = (1 + o(1))(\mathbb{E} D_n)^2$$

Since

$$D_n = \sum_{j=1}^{n} D_{j,n}$$

$$D_n^2 = \sum_{j=1}^{n} D_{j,n} \sum_{k=1}^{n} D_{k,n} = \sum_{j=1}^{n} D_{j,n}^2 + \sum_{\substack{1\le j,k\le n \\ j\ne k}} D_{j,n}D_{k,n} = \sum_{j=1}^{n} D_{j,n} + \sum_{\substack{1\le j,k\le n \\ j\ne k}} D_{j,n}D_{k,n}$$

Then

$$\mathbb{E}_{n,p_n} D_n^2 = \mathbb{E}_{n,p_n} D_n + n(n-1)\mathbb{E}_{n,p_n} D_{1,n}D_{2,n}$$

$$\mathbb{E}_{n,p_n} D_{1,n}D_{2,n} = P_{n,p_n}\left(D_{1,n} \cap D_{2,n}\right) = (1-p_n)^{n-1} \cdot (1-p_n)^{n-2} = (1-p_n)^{2n-3} = \left(1 - \frac{\log n + c_n}{n}\right)^{2n-3}$$

Taking logarithm:

$$\log \mathbb{E}_{n,p_n} D_{1,n}D_{2,n} = (2n-3)\log\left(1 - \frac{\log n + c_n}{n}\right)$$

With similar mehtod we get

$$\log \mathbb{E}_{n,p_n} D_{1,n}D_{2,n} = (2n-3)\left[-\frac{\log n + c_n}{n} - \frac{1}{(1-x^\star)^2}\left(\frac{\log n + c_n}{n}\right)^2\right] = -2\log n - 2c_n + o(1)$$

$$\mathbb{E}_{n,p_n} D_{1,n}D_{2,n} = e^{-2\log n}e^{-2c_n}e^{o(1)} = \frac{1}{n^2}e^{-2c_n}(1+o(1))$$

back to expectation of D_n^2:

$$\mathbb{E}_{n,p_n} D_n^2 = \mathbb{E}_{n,p_n} D_n + \frac{n(n-1)}{n^2}e^{-2c_n}(1+o(1)) = \mathbb{E}_{n,p_n} D_n + e^{-2c_n}(1+o(1))$$

$$\left(\mathbb{E}_{n,p_n} D_n\right)^2 = \left(e^{-c_n}\left(1+o(1)\right)\right)^2 = e^{-2c_n}\left(1+o(1)\right)$$

$$\mathbb{E}_{n,p_n} D_n^2 = e^{-c_n}\left(1+o(1)\right) + e^{-2c_n}\left(1+o(1)\right) = e^{-2c_n}\left(e^{c_n}\left(1+o(1)\right) + 1 + o(1)\right) = e^{-2c_n}\left(1+o(1)\right) = \left(\mathbb{E}_{n,p_n} D_n\right)^2\left(1+o(1)\right)$$

10 Center limit theorem

Given IID variables $\{X_n\}_{n=1]^\infty}$. Denote

$$\mu = \mathbb{E}X_n$$

$$\sigma^2 = Var(X_n)$$

Since

$$\mathbb{E}S_n = n\mu$$

$$\mathbb{E}S_n - n\mu = 0$$

$$Var(S_n - n\mu) = Var(S_n) = n\sigma^2$$

Thus

$$\begin{cases} E\frac{S_n - n\mu}{\sqrt{n}\sigma} = 0 \\ Var\left(\frac{S_n - n\mu}{\sqrt{n}\sigma}\right) = \frac{1}{n\sigma^2}Var(S_n) = 1 \end{cases}$$

Center limit theorem Let $\{X_n\}_{n=1}^\infty$ IID random variables with two moments. Denote $\mu = \mathbb{E}X_n$ and $\sigma^2 = Var(X_n)$. Define $S_n = \sum_{i=1}^n X_i$ and $Z \sim N(0,1)$ then for all $-\infty \le a < b \le \infty$

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - n\mu}{\sqrt{n}\sigma} \le b\right) = \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx = P(a \le Z \le b)$$

Depending on distribution, the exact value of n such that we can claim that

$$P\left(n\mu + a\sigma\sqrt{n} \le S_n \le n\mu + b\sigma\sqrt{n}\right) \approx \int_a^b \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}dx$$

For fairly good distribution $n > 30$.

Two words on proof We can define characteristic function of random variable

$$\Phi_X(t) = \mathbb{E}e^{itX} = \int_{-\infty}^{\infty} e^{itx} f(x)dx = \mathcal{F}[f(x)]$$

Characteristic function is continuous and unique for a distribution. Then, from $M_Z(t) = e^{\frac{t^2}{2}}$ we can acquire $\Phi_Z(t) = e^{-\frac{t^2}{2}}$
The continuity theorem claims that for $\{X_n\}_{n=1}^{\infty}$, if $\lim_{n\to\infty} \Phi_{X_n}(t) = \Psi(t)$, then

$$\exists X \quad \Psi(t) = \mathbb{E}e^{itX}$$

and

$$\lim_{n\to\infty} P(a \leq X_n \leq b) = P(a \leq X \leq b)$$

We can show that

$$\lim_{n\to\infty} \Phi_{\frac{S_n - n\mu}{\sqrt{n}\sigma}}(t) = e^{-\frac{t^2}{2}}$$

and thus acquire that $\frac{S_n - n\mu}{\sqrt{n}\sigma} \sim Z$.

Example Denote R_{100} number of heads in 100 tosses of fair coin. Find for which a

$$P(50 - a \leq R_{100} \leq 50 + a) \approx 0.95$$

Solution Denote $\{X_n\}_{n=1}^{100}$ such that $X_i \sim Ber\left(\frac{1}{2}\right)$. Denote $S_{100} = \sum_{j=1}^{100} X_i$, then $S_{100} = R_{100}$.

$$P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$$

$$\mu = \mathbb{E}X_n = \frac{1}{2}$$

$$\sigma^2 = \mathbb{E}\left(X_n - \mathbb{E}X_n\right)^2 = \mathbb{E}\left(X_n - \frac{1}{2}\right)^2 = \frac{1}{2}\left(1 - \frac{1}{2}\right)^2 + \frac{1}{2}\left(0 - \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$Z \approx \frac{S_{100} - 100 \cdot \frac{1}{2}}{\sqrt{100} \cdot \frac{1}{2}} = \frac{S_{100} - 50}{5}$$

$$P(50 - a \leq R_{100} \leq 50 + a) = P(50 - a \leq S_{100} \leq 50 + a) = P\left(-\frac{a}{5} \leq \frac{S_{100} - 50}{5} \leq \frac{a}{5}\right) \approx P\left(-\frac{a}{5} \leq Z \leq \frac{a}{5}\right) =$$

$$= 2P\left(-\frac{a}{5} \leq Z \leq 0\right) = 2\left(\frac{1}{2} - P\left(Z - \frac{a}{5}\right)\right) = 1 - 2P\left(Z - \frac{a}{5}\right) = 0.95$$

$$P\left(Z - \frac{a}{5}\right) = 0.025$$

$$\frac{a}{5} = 1.96 \Rightarrow a = 9.8 \approx 10$$

What if $n = 10000$?

$$Z \approx \frac{S_{10000} - 10000 \cdot \frac{1}{2}}{\sqrt{10000} \cdot \frac{1}{2}} = \frac{S_{100} - 50000}{50}$$

$$P(500 - a \leq R_{10000} \leq 500 + a) = P\left(-\frac{a}{50} \leq \frac{S_{10000} - 500}{50} \leq \frac{a}{50}\right) \approx \cdots = 1 - 2P\left(Z - \frac{a}{50}\right) = 0.95$$

$$P\left(Z - \frac{a}{50}\right) = 0.025$$

$$\frac{a}{50} = 1.96 \Rightarrow a = 98$$

As we can see, with 10000 we are with high probability withing 1% around expectation, while for 100 tosses we were only withing 10%.

11 The Simple Symmetric Random Walk

$\{X_n\}_{n=1}^{\infty}$ IID random vectors distributed as

- $d = 1$: $X_n^1 \sim Ber\left(\frac{1}{2}\right)$
$$P\left(X_n^1 = 1\right) = P\left(X_n^1 = -1\right) = \frac{1}{2}$$
- $d \geq 2$
$$P(X_n^d = -e_j) = P(X_n^d = e_j) = \frac{1}{2d}$$

Denote
$$S_0^d = \vec{0} \in \mathbb{Z}_d$$
and
$$S_n^d = \sum_{j=1}^{n} X_j^d$$
Then $\left\{S_n^d\right\}_{n=0}^{\infty}$ is d-dimensional SSRW.

Denote
$$\hat{R}_d = \left\{\exists n \geq 1 \quad S_n^{(d)} = 0\right\}$$
i.e. if we ever return to 0
$$p_d = P(\hat{R}_d)$$
Denote also
$$T_d = \left\{\lim_{n\to\infty} S_n^{(d)} = \infty\right\}$$
We claim that $S_n \in T_d \cup S_d$. If $S_d \notin T_d$ in comes infinite time into ball of radius N, and thus will some tome return to 0. Now we ask if $p_d = 1$.

Reminder Stirling approximation
$$n! \approx n^n e^{-n}\sqrt{2\pi n}$$
$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

Definition If $p_d = 1$ then walk is called recurrent. If $p_d < 1$ walk is called transient.

$$N^d = \left\{ \begin{smallmatrix}\text{Number of} \\ \text{returns to 0}\end{smallmatrix} \right\}$$
$$N^d = \sum_{n=1}^{\infty} \mathbb{1}_{\{0\}}\left(S_n^d\right)$$
Now, if $p_d < 1$.
$$P(N^d = 0) = 1 - p_d$$
$$P(N^d = 1) = p_d(1 - p_d)$$
$$P(N^d = m) = p_d^m(1 - p_d)$$
So, $N^d \sim Geom(1 - p_d)$.
If $p_d = 1$, then $P(N^D = \infty) = 1$ and thus
$$\mathbb{E}N^d = \begin{cases} \frac{1}{1-p_d} & p_d < 1 \\ \infty & p_d = 1 \end{cases}$$
Now, $\mathbb{E}N_d < \infty$ iff $p_d < 1$, and thus we can check if $p_d = 1$ by checking if $\mathbb{E}N_d = \infty$
Lets use indicator method:
$$\mathbb{E}N^d = \mathbb{E}\sum_{n=1}^{\infty} \mathbb{1}_{\{0\}}\left(S_n^d\right) = \sum_{n=1}^{\infty} \mathbb{E}\mathbb{1}_{\{0\}}\left(S_n^d\right) = \sum_{n=1}^{\infty} \mathbb{E}P\left(S_n^d = 0\right) = \sum_{n=1}^{\infty} \mathbb{E}P\left(S_{2n}^d = 0\right)$$
Thus $p_d = 1$ iff $\sum_{n=1}^{\infty} \mathbb{E}P\left(S_{2n}^d = 0\right) = \infty$.

For $d=1$:

$$P(S_{2n}^1=0)=\frac{\binom{2n}{n}}{2^{2n}}=\frac{(2n)!}{(n!)^2}\frac{1}{2^{2n}}\approx\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{\left(n^ne^{-n}\sqrt{2\pi n}\right)^n2^{2n}}=\frac{1}{(\pi n)^{\frac{1}{2}}}$$

Since $\sum_{n=1}^{\infty}\frac{1}{(\pi n)^{\frac{1}{2}}}=\infty$, $\sum_{n=1}^{\infty}\mathbb{E}P\left(S_{2n}^1=0\right)=\infty$ and thus $p_1=1$.
For $d=2$:

$$P(S_{2n}^2=0)=\frac{\sum_{j=0}^{n}\binom{2n}{j}\binom{2n-j}{j}\binom{2n-2j}{n-j}}{4^{2n}}=\sum_{j=0}^{n}\frac{(2n)!}{j!j!(n-j)!(n-j)!4^{2n}}=$$

$$=\frac{(2n)!}{(n!)^24^{2n}}\sum_{j=0}^{n}\frac{n!}{j!j!(n-j)!(n-j)!}=\frac{(2n)!}{(n!)^24^{2n}}\sum_{j=0}^{n}\binom{n}{j}^2=\frac{1}{4^{2n}}\binom{2n}{n}\binom{2n}{n}=\left[\frac{1}{2^{2n}}\binom{2n}{n}\right]^2=P^2(S_{2n}^1=0)\approx\frac{1}{\pi n}$$

Thus the series diverge and $p_2=1$
For $d=3$

$$P(S_{2n}^3=0)=\frac{\sum_{0\le j+k\le n}\binom{2n}{j}\binom{2n-j}{j}\binom{2n-2j}{k}\binom{2n-2j-k}{n-j-k}}{4^{2n}}=\sum_{0\le j+k\le n}\frac{(2n)!}{(j!)^2(k!)^2((n-j-k)!)^26^{2n}}=$$

$$=\frac{(2n)!}{(n!)^26^{2n}}\sum_{0\le j+k\le n}\frac{(n!)^2}{(j!)^2(k!)^2((n-j-k)!)^2}=\frac{(2n)!}{(n!)^26^{2n}}\sum_{0\le j+k\le n}\left(\frac{n!}{j!k!(n-j-k)!}\right)^2\le$$

$$\le\frac{(2n)!}{(n!)^26^{2n}}\frac{n!}{\left(\frac{n}{3}\right)!\left(\frac{n}{3}\right)!\left(\frac{n}{3}\right)!}\sum_{0\le j+k\le n}\frac{n!}{j!k!(n-j-k)!}=$$

$$=\frac{(2n)!}{(n!)^26^{2n}}\frac{n!}{\left(\left(\frac{n}{3}\right)!\right)^3}\sum_{0\le j+k\le n}3^n\approx\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}3^n}{n^ne^{-n}\sqrt{2\pi n}6^{2n}\left(\left(\frac{n}{3}\right)^{\frac{n}{3}}e^{-\frac{n}{3}}\sqrt{2\pi\frac{n}{3}}\right)^3}=$$

$$=\frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}3^n}{n^ne^{-n}\sqrt{2\pi n}6^{2n}\left(\frac{n}{3}\right)^ne^{-n}\left(2\pi\frac{n}{3}\right)^{\frac{3}{2}}}=\frac{\sqrt{2}}{\left(2\pi\frac{n}{3}\right)^{\frac{3}{2}}}=\frac{\sqrt{\frac{16\pi^3}{27}}}{n^{\frac{3}{2}}}$$

Thus $p_3<1$, i.e. for $d=3$ the walk is diverging. This is counter-intuitive since each pair of coordinates returns to 0, but all three go to infinity.
Denote R_d event that walk returns to 0 infinite times. Then $P(R_d)+P(T_d)=0$. Note that $P(\hat{R}_d)\ge\frac{1}{2d}$.

11.1 1D Random Walk

$\{X_n\}_{n=1}^{\infty}$ IID, $X_n\sim Ber\left(\frac{1}{2}\right)$ such that $P\left(X_n=1\right)=P\left(X_n=-1\right)=\frac{1}{2}$. Denote

$$S_n=j+\sum_{i=1}^{n}X_i$$

Denote as P_j and $\mathbb{E}_j$ probability and expectation for specific value of j. Now define

$$\tau_k=\inf\{n\ge0:S_n=k\}$$

Theorem For $0\le j\le L$

$$h_j=P_j(\tau_0<\tau_L)=\frac{L-j}{L}$$

Proof Lets make first step analysis:

$$h_j=\frac{1}{2}h_{j+1}+\frac{1}{2}h_{j-1}$$

Denote events "went right" and "went left" as $\mathcal{R}$ and $\mathcal{L}$ correspondingly. Then

$$\{\tau_0<\tau_L\}=\{\{\tau_0<\tau_L\}\cap\mathcal{R}\}\{\{\tau_0<\tau_L\}\cap\mathcal{L}\}$$

$$h_j=P_j(\tau_0<\tau_L)=P_j\left(\{\tau_0<\tau_L\}\cap\mathcal{R}\right)+P_j\left(\{\tau_0<\tau_L\}\cap\mathcal{L}\right)=$$

$$=P_j(\mathcal{R})P_j\left(\tau_0<\tau_L|\mathcal{R}\right)+P_j(\mathcal{L})P_j\left(\tau_0<\tau_L|\mathcal{L}\right)=\frac{1}{2}P_{j+1}\left(\tau_0<\tau_L\right)+\frac{1}{2}P_{j-1}\left(\tau_0<\tau_L\right)$$

Now we got difference equation with bound conditions

$$2h_j = h_{j+1} + h_{j-1}$$

$$\forall 1 \le j \le L-1 \quad h_j - h_{j+1} = h_{j-1} - h_j$$

And bound conditions are $h_0 = 1$ and $h_L = 0$. Denote $\gamma = h_j - h_{j-1}$. Then

$$1 - 0 = h_0 - h_L = \sum_{j=1}^{L} h_{j-1} - h_j = \gamma L$$

$$\gamma = \frac{1}{L}$$

Also

$$h_j = h_j - 0 = h_j - h_L = \sum_{k=j+1}^{L} h_{k-1} - h_K = (L-j)\gamma = \frac{L_j}{L}$$

Notation

$$\tau_{0,L} = \inf\{n \ge 0 : \ S_n = 0 \text{ or } S_n = L\} = \tau_0 \vee \tau_L$$

Theorem

$$\mathbb{E}_j \tau_{0,L} = j(L-j)$$

Proof

$$h_j = \mathbb{E}_j \tau_{0,L}$$

Bound conditions are $h_0 = h_L = 0$.

$$h_j = \frac{1}{2}(1 + h_{j+1}) + \frac{1}{2}(1 + h_{j+1})$$

$$2h_j = 2 + h_{j+1} + h_{j+1}$$

$$h_j - h_{j-1} - 2 = h_{j+1} - h_j$$

Denote $\gamma = h_1 - h_0$

$$h_2 - h_1 = h_1 - h_0 - 2 = \gamma - 2$$

$$h_3 - h_2 = h_2 - h_1 - 2 = \gamma - 4$$

And generally

$$h_j - h_{j-1} = \gamma - 2(j-1)$$

$$0 - 0 = h_L - h_0 = \sum_{1}^{L} h_j - h_{j-1} = \sum_{j=1}^{L} \gamma - 2(j-1) = \gamma L - 2\frac{L(L-1)}{2} = L(\gamma - (L-1))$$

$$\gamma = L - 1$$

$$h_j = h_j - 0 = h_j - h_0 = \sum_{k=1}^{j} h_k - h_{k-1} = \sum_{k=1}^{j} \gamma - 2(k-1) = \gamma j - 2\frac{j(j-1)}{2} = j(\gamma - (j-1)) = j(L-1-j+1) = j(L-j)$$

Notation

$$\hat{\tau}_j = \inf\{n \ge : S_n = j\}$$

Conclusion

$$\mathbb{E}_0 \hat{\tau}_0 = \infty$$

Proof

$$\mathbb{E}_0 \hat{\tau}_0 = \frac{1}{2}(1 + \mathbb{E}_1 \tau_0) + \frac{1}{2}(1 + \mathbb{E}_{-1} \tau_0) = 1 + \mathbb{E}_1 \tau_0 \overset{\forall L \ge 2}{\ge} 1 + \mathbb{E}_1 \tau_{0,L} = L = \infty$$

11.2 Asymmetrical random walk

$\{X_n\}_{n=1}^{\infty}$ IID, $X_n \sim \mathit{Ber}(p)$ such that $P(X_n = 1) = 1 - P(X_n = -1) = p$. WLG $p > 0.5$.

Theorem For $j \geq 0$:

$$P_J\left(\tau_0 < \tau_L\right) = \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^L}{1 - \left(\frac{1-p}{p}\right)^L}$$

$$P_j(\tau_0 < \infty) = \left(\frac{1-p}{p}\right)^j$$

Proof Denote

$$h_j = P_j(\tau_0 < \tau_L)$$

And bound conditions are $h_0 = 1$ and $h_L = 0$. Then

$$h_j = ph_{j+1} + (1-p)h_{j-1}$$

$$p\left(h_j - h_{j+1}\right) = (1-p)\left(h_{j-1} - h_j\right)$$

$$h_j - h_{j+1} = \frac{1-p}{p}\left(h_{j-1} - h_j\right)$$

Denote

$$\gamma = h_0 - h_1$$

$$h_1 - h_2 = \frac{1-p}{p}\gamma$$

And generally

$$h_j - h_{j+1} = \left(\frac{1-p}{p}\right)^j \gamma$$

$$1 - 0 = h_0 - h_L = \sum_{j=0}^{L-1} h_j - h_{j+1} = \gamma \sum_{j=0}^{L-1}\left(\frac{1-p}{p}\right)^j = \gamma \frac{1 - \left(\frac{1-p}{p}\right)^L}{1 - \frac{1-p}{p}}$$

Thus

$$\gamma = \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^L}$$

Now

$$h_j = h_j - 0 = h_j - h_L = \sum_{k=j}^{L-1} h - K - h_{k+1} = \gamma \sum_{k=j}\left(\frac{1-p}{p}\right)^k = \gamma \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^L}{1 - \frac{1-p}{p}} =$$

$$= \frac{1 - \frac{1-p}{p}}{1 - \left(\frac{1-p}{p}\right)^L} \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^L}{1 - \frac{1-p}{p}} = \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^L}{1 - \left(\frac{1-p}{p}\right)^L}$$

For second part, for $0 \leq j \leq L$

$$\{\tau_0 < \infty\} = \bigcup_{L=1}^{\infty} \{\tau_0 < \tau_L\}$$

We can look at sequence of events

$$\{\tau_0 < \tau_L\} \nearrow \{\tau_0 < \infty\}$$

$$P_J\left(\tau_0 < \infty\right) = \lim_{L\to\infty} P_J\left(\tau_0 < \tau_L\right) = \lim_{L\to\infty} \frac{\left(\frac{1-p}{p}\right)^j - \left(\frac{1-p}{p}\right)^L}{1 - \left(\frac{1-p}{p}\right)^L} = \left(\frac{1-p}{p}\right)^j$$

12 Branching processes

Let W random variable distributed on $\mathbb{N}$ and denote $p_n = P(W = n)$. W is offspring distribution.
Lets start with population of one particle at $t = 0$, written as $X_0 = 1$. At $t = 1$ particle generates random number of offsprings and dies. Denote X_1 number of its offsprings.
On $t = 2$ each of particles independently generates offsprings and dies. Denote X_2 total number of offsprings.
$\{X_n\}_{n=0]^\infty}$ is called Galton–Watson process. If $X_n = 0$ then $X_m = 0$ for all $m \geq n$. Denote

$$S_n = \{X_n \geq 1\}$$

i.e. probability that process survives until $t = n$. And

$$S = \bigcap_{n=0}^{\infty} S_n$$

i.e. probability that process is alive forever. Define also complementary events, extinctions

$$S_n^c = \{X_n = 0\}$$

$$S^c = \bigcup_{n=0}^{\infty} S_n^c$$

Trivial cases If $p_0 = 0$, then $P(S) = 1$.
If $p_0 > 0$ then $P(s) < P(S_1) = 1 - p_0 < 1$ and

$$P(S_{n+1} = 0 | S_n = m) = p_0^m$$

Question $P(S) = 0$ or $P(S) > 0$? Denote $\mu = \mathbb{E}W$

Claim

$$\mathbb{E}X_n = \mu_n$$

Proof For $n = 1$:

$$\mathbb{E}X_1 = \mathbb{E}W = \mu$$

For $n >\geq 2$:

$$\mathbb{E}\left(X_n | X_{n-1} = m\right) = \mathbb{E}\sum_{i=1}^{m} W_i = \mu m$$

From smoothing theorem

$$\mathbb{E}\left(X_n | X_{n-1}\right) = \mu X_{n-1}$$

$$\mathbb{E}X_n = \mathbb{E}\mathbb{E}\left(X_n | X_{n-1}\right) = \mathbb{E}\mu X_{n-1} = \mu \mathbb{E}X_{n-1}$$

Thus $\mathbb{E}X_n = \mu^n$.

Conclusion

$$\mu < 1 \Rightarrow P(S) = 0$$

Proof From Markov inequality

$$P(S_n) = P(X_n \geq 1) \leq \frac{1}{1}\mathbb{E}X_n = \mu^n \to 0$$

Claim

Proof Define

$$\Phi(t) = \sum_{n=0}^{\infty} p_n t^n$$

, probability generating function. Note that $\Phi(1) = \sum p_n = 1$. By Taylor,

$$p_n = \frac{\Phi^{(n)}(0)}{n!}$$

Since radius of convergence is at least 1, for $|t| < 1$ we get

$$\Phi'(t) = \sum_{n=1}^{\infty} n p_n t^{n-1}$$

$$\Phi''(t) = \sum_{n=2}^{\infty} n(n-1) p_n t^{n-2}$$

Then $\Phi'(t) \geq 0$ and $\Phi''(t) \geq 0$. If $p_0 + p_1 < 1$ then $\Phi'(t) \geq 0$ and $\Phi''(t) > 0$.
Thus $\Phi'(t)$ is monotonous increasing, and thus has limit.

Claim

$$\lim_{t \to 1} \Phi'(t) = \mu$$

Proof

$$\Phi'(t) = \sum_{n=1}^{\infty} n p_n t^{n-1} \leq \sum_{n=1}^{\infty} n p_n = \mu$$

On the other hand

$$\Phi'(t) \geq \sum_{n=1}^{N} n p_n t^{n-1}$$

$$\lim_{t \to 1} \Phi'(t) \geq \lim_{t \to 1} \sum_{n=1}^{N} n p_n t^{n-1} = \sum_{n=1}^{N} \lim_{t \to 1} n p_n t^{n-1} = \sum_{n=1}^{N} n p_n$$

Claim Suppose $p_0 > 0$ and $p_0 + p_1 < 1$, then a smallest root $\alpha \in [0, 1]$ of equation $\phi(t) = t$ fulfills:

$$\begin{cases} \alpha = 1 & \mu \leq 1 \\ \alpha \in (0, 1) & \mu > 1 \end{cases}$$

Also if $\mu > 1$ then $\Phi(t) > t$ for $t \in (0, \alpha)$ and $\Phi(t) < t$ for $t \in (\alpha, 1)$.

Proof Denote $\psi(t) = \Phi(t) - t$. Then $\psi(0) = p_0$ and $\psi(1) = 0$. Since ψ' is monotonically increasing, ψ' is monotonically increasing too. By previous claim,

$$\lim_{t \to 1} \psi'(t) = \mu - 1$$

If $\mu \leq 1$, then

$$\lim_{t \to 1} \psi'(t) \leq 0$$

Since ψ' is monotonically increasing, it's negative for $t \in [0, 1)$.Since $\psi(1) = 0$ and derivative is negative, then $\psi(t) > 0$ for $t \in [0, 1)$, and thus $\psi(t) > t$.
Now if $\mu > 1$, then $\mu - 1 > 0$, i.e.

$$\lim_{t \to 1} \psi'(t) > 0$$

and exists neighborhood of 1 such that $\psi'(t) > 0$. Since $\psi(0) > 0$ and $\psi(1) > 1$, and also Φ is convex, exists unique $\alpha \in (0, 1)$ such that $\psi(\alpha) = 0$. Moreover,

$$\begin{cases} \psi(s) > 0 & s \in (0, \alpha) \\ \psi(s) < 0 & s \in (\alpha, 1) \end{cases}$$

i.e.

$$\begin{cases} \Phi(t) > t & t \in (0, \alpha) \\ \Phi(t) < t & t \in (\alpha, 1) \end{cases}$$

Theorem Suppose $p_0 > 0$. If $\mu \le 1$ then $P(S) = 0$. If $\mu > 1$ then $P(S) = \alpha > 0$, where α is smallest root of equation $\Phi(t) = t$.

Proof If $p_0 + p_1 = 1$, then $\mu < 1$, and then we already proofed $P(S) = 0$. Thus suppose $p_0 + p_1 < 1$. Denote

$$q = P(S^c)$$

and

$$q_n = P(S_n^c) = P(X_n = 0)$$

We know that

$$q = \lim_{n\to\infty} q_n$$

Thus we want to show that if $\mu \le 1$, then $q = 1$ and if $\mu > 1$, $q = \alpha$.

Claim

$$q_n = \Phi(q_{n-1})$$

Proof For $n = 1$, $q_1 = p_0$, and $\Phi(q_0) = \Phi(0) = p_0$.
Now, for $n \ge 2$:

$$P(X_n = 0 | X_1 = m) = q_{n-1}^m$$

Thus

$$q_n = P(X_n = 0) = \sum_{m=0}^{\infty} P(X_n = 0|X_1 = m)P(X_1 = m) = \sum_{m=0}^{\infty} p_m q_{n-1}^m = \Phi(q_{n-1})$$

Proof of theorem (continued) Thus we get, in limit, that

$$\Phi(q) = q$$

From claim on roots of this equation we get that if $\mu \le 1$, $q = 1$. If $\mu > 1$ then $q \in \{\alpha, 1\}$.
If $q = 1$ then $q_n > \alpha$ for some big n. Then, from claim on roots of $\Phi(t) = t$, we get

$$q_{n+1} = \Phi(q_n) < q_n$$

which is contradiction, since q is monotonically increasing. That means that $q = \alpha$.

13 The Probabilistic method

13.1 Ramsey theory

Suppose we have full graph $K_n = ([n], E)$, where $|E| = \binom{n}{2}$.

2-coloring Is coloring of graph's edges into two colors: red and blue.

$R(j) = R(j,j)$ is smallest number R such that any 2-coloring of K_R exists monochromatic clique K_j. Obviously $R(j) \le \binom{2j-2}{j-1} \le 4^{j-1}$

Theorem (Erdös)

$$R(j) \ge \frac{1}{e}(1 + o(1))\, j \cdot 2^{\frac{j}{2}}$$

when $j \to \infty$.

Proof Let $j \geq 3$. Choose random coloring of K_n, when each edge is colored independently from others with probability $\frac{1}{2}$ for each of colors.
Let $W \subset K_n$ clique of size j. Define indicator

$$\mathbb{1}_W = \begin{cases} 1 & W \text{ is monochromatic} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$P\left(\mathbb{1}_W = 1\right) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{j}{2}} = 2^{1-\binom{j}{2}}$$

Define

$$X_j = \sum_{W \subset K_n : |W| = j} \mathbb{1}_W$$

which is number of monochromatic cliques of size j. The expectation of X_j:

$$\mathbb{E}X_j = \binom{n}{j} 2^{1-\binom{j}{2}}$$

Since average number of monochromatic cliques of size j is $\binom{n}{j}2^{1-\binom{j}{2}}$, exists coloring such that it has exactly M monochromatic cliques of size j for some M such that $M \leq \binom{n}{j}2^{1-\binom{j}{2}}$.
For such coloring, remove one vertex for each of M cliques. Denote by M' number of removed vertices.

$$M' \leq M \leq \binom{n}{j} 2^{1-\binom{j}{2}}$$

Take a look at full graph of $n - M'$ vertices. By construction, there is no monochromatic cliques of size j in this graph. Thus

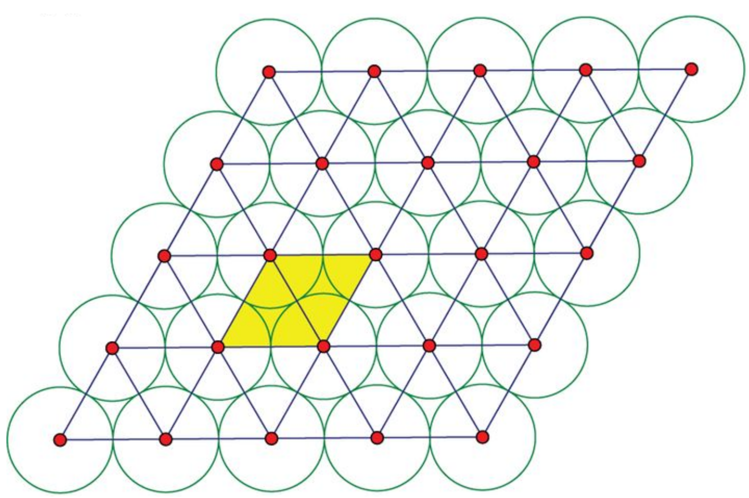
$$R(j) > n - m \geq n - \binom{n}{j} 2^{1-\binom{j}{2}}$$

In can be shown that

$$\max_{n : j \leq n < \infty} n - \binom{n}{j} 2^{1-\binom{j}{2}} = \frac{1}{e}\left(1 + o(1)\right) j \cdot 2^{\frac{j}{2}}$$

13.2 Disk Cover Problem

Theorem For each choice of 10 points on plane $\{x_i\}_{i=1}^{10}$ and each choice of $R > 0$ it is always possible to cover points by non-overlapping disks of radius R.
Lets start from dense (hexagonal) packing of disks on plane.



Area of each disk is

$$A(D) = \pi R^2$$

and area of "triangle" is

$$A(T) = \left(\sqrt{3} - \frac{\pi}{2}\right) R^2$$

Total covered area is

$$\frac{A(D)}{A(D)+2A(T)} = \frac{\pi}{\sqrt{3}-\frac{\pi}{2}} \approx 0.907$$

Thus choosing random packing (two coordinates of center of one disk), probability of point to be not covered is

$$P(x_i) \approx 0.093$$

Then probability that at least one point is uncovered

$$P\left(\{x_n\}\right) \leq 10 \cdot P(x_i) = 10 \cdot 0.093 = 0.93 < 1$$

i.e. exist packing for which all 10 are covered.

13.3 Secretary problem

In this game Alice, the informed player, writes secretly distinct numbers on n cards. Bob, the stopping player, observes the actual values and can stop turning cards whenever he wants, winning if the last card turned has the overall maximal number. Call strategy j (for $j > 0$) - to skip first j numbers and then choose the first number bigger then all previous numbers. Denote by W event of winning. What is $P_j(W)$ - probability to win using strategy j. Denote by A_k event that k^{th} number is biggest.

$$P_j(A_k) = \frac{1}{n}$$

Find probability of winning via full probability

$$P_j(W) = \sum_{k=1}^{n} P(A_k)P(W|A_k) = \frac{1}{n}\sum_{k=1}^{n} P_j(W|A_k)$$

Now, if $1 \leq k \leq j$,

$$P_j(W|A_k) = 0$$

In other case, if maximum of first $k-1$ numbers is in first j numbers, we win, else we lose, i.e.

$$P_j(W|A_k) = \frac{j}{k-1}$$

Back to probability

$$P_j(W) = \frac{1}{n}\sum_{k=j+1}^{n} \frac{j}{k-1}$$

Now lets perform asymptotic analysis for $j = [xn]$, $0 < x < 1$:

$$\sum_{k=[xn]+1}^{n} \frac{1}{k-1} = \sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{[xn]} \frac{1}{k-1} \approx \log n - \log xn = -\log x$$

Thus

$$P_j(W) \approx -\frac{[xn]}{n}\log x = -x\log x$$

To maximize probability:

$$P_j'(W) = \log x + 1$$

$$\log x = -1$$

$$x = \frac{1}{e} \approx 0.368$$

And probability to win is

$$P_j(W) = -e^{-1}\log e^{-1} = e^{-1} \approx 0.368$$