106349 - Advanced probability

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Abstract

1 Introduction. Summary of course through an example. Branching process

We have an individual that gives a birth to a random number of offsprings – random variable X. X define a distribution, i.e., $P : \mathbb{Z}^+ \to [0,1]$, i.e., $P(X=k) \in [0,1]$, and $\sum_{k=0}^{\infty} P(X=k) = 1$.

Definition 1.1. $f_X(\theta) = \sum_{k=0}^{\infty} \theta^k P(X=k)$ – moment-generating function.

The series is absolutely convergent for $\theta \in [-1,1]$ since k sums to 1. For $\theta \in (-1,1)$, f_x is analytic, thus we can differentiate it term-by-term:

$$f'_X(\theta) = \sum_{k\geq 1} \theta^{k-1} P(X=k)$$

Since, f_X is analytic, knowing it means knowing $P(X=k)$ and vice versa.
Note that $f_X(0) = P(X=0)$ and $f_X(1) = 1$. Also

$$f'_X(1) = \sum_{k\geq 0}^{\infty} kP(X=k) = \mathbb{E}X = \mu$$

$$\lim_{\theta\to 1} \frac{f_X(1) - f_X(\theta)}{1-\theta} = \lim_{\theta\to 1} \frac{1 - f_X(\theta)}{1-\theta}$$

Note also that f_X is convex, since second derivative is positive.

Size of n^{th} generation Let $\left(X_r^{(n)}\right)_{n,r=1\infty}$, where n is generation and r is offspring number (index) in n^{th} generation.
Assume $X_r^{(n)}$ are i.i.d. (independent, identically distributed) random variables.
Identically distributed means

$$P(X_n^r = k) = P(X = k)$$

.
Independence means

$$P\big(\forall i < J\ X_{r_i}^{n_i} = k\big) = \prod_{i=1}^{J} P\big(X_{r_i}^{n_i} = k\big)$$

.
Define $z_1 = X_1^1$. $z_2 = \sum_{r=1}^{z_1} X_r^2$ an so on:

$$z_{n+1} = \sum_{r=1}^{z_n} X_r^n$$

We want to study asymptotics of z_n.
Given U and V taking values in $\mathbb{Z}^+$,

$$\mathbb{E}][U|V=k] = \sum_{j=0}^{\infty} jP(U=j|V=k)$$

, where

$$P(U=j|V=k) = \frac{P(U=j, V=k)}{P(V=k)}$$

If U, V are independent, $P(U=j|V=k) = P(U=j)$ and thus $\mathbb{E}[U|V=k] = \mathbb{E}U$.

Definition 1.2. Define random variable $\mathbb{E}[U|V]$ such that

$$\mathbb{E}[U|V] = \mathbb{E}[U|V=k]$$

if $V = k$.

Definition 1.3 (Tower property)**.**

$$\mathbb{E}\big[\mathbb{E}[U|V]\big] = \mathbb{E}U$$

Define

$$f_n = \sum_{k=0}^{\infty}\sum_{k=0}^{\infty} \theta^k P(z_n = k) = \mathbb{E}\theta^{z_n}$$

.

Theorem 1.1.

$$f_{n+1}(\theta) = f_n(f_X(\theta))$$

or

$$f_n(\theta) = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(\theta))$$

Proof. Use tower property with $U^{z_{n+1}}$ and $V = \theta^{z_n}$. By tower property

$$\mathbb{E}[\theta^{z_{n+1}}] = \mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big]$$

$$\mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big] = \sum_{k=0}^{\infty} P(z_n = k)\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k]$$

What is $\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k]$?

$$\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k] = \mathbb{E}\Big[\theta^{\sum_{j=1}^{k} X_j^{n+1}}|\theta^{z_n} = k\Big] \overset{\text{independence}}{=} \mathbb{E}\Big[\theta^{\sum_{j=1}^{z_n} X_j^{n+1}}\Big] \overset{\text{independence}}{=} \prod_{j=1}^{k} \mathbb{E}\Big[\theta^{X_j^{n+1}}\Big] \overset{\text{i.d.}}{=} (f_X(\theta))^k$$

Thus

$$\mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big] = \sum_{k=0}^{\infty} P(z_n = k)(f_X(\theta))^k = f_n(f(\theta))$$

Also we can say

$$\mathbb{E}[\theta^{z_{n+1}}|z_n] = (f_X(\theta))^{z_n}$$

□

Study of z_n What is $\pi_n = P(z_n = 0) = f_n(0) = f(\pi_{n-1})$, probability that population is extinguished. Since $z_{n-1} = 0 \Rightarrow z_n = 0$, i.e. π_n is non-decreasing.
Let $P(z_n = 0 \text{ for some n}) = \pi$.
We hope that $\{z_n = 0\}$ such that

$$\bigcup_n \{z_n = 0\} = \{z_n = 0 \text{ for some n}\}$$

i.e., $\pi = \lim_{n\to\infty} \pi_n$. We call π the extinction probability.

Theorem 1.2. If $\mu = \mathbb{E} > 1$ then π is a unique root of $\pi = f(\pi)$ and $\pi \in [0,1)$. If $\mu \leq 1$, $\pi = 1$.
If we look at $f(\pi)$ and π, they intersect in 1, and they can intersect in two points since $f(x)$ is convex. There is second intersection iff $f'(1) = \mu > 1$.

Construction of X_n^r Construct set Ω, $f_{n,r} : \Omega \to \mathbb{Z}^+$ and $\mathcal{F}$ a collection of subsets of Ω with $P : \mathcal{F} \to [0,1]$.
Let $\Omega = \mathbb{Z}^+ \times \mathbb{Z}^+$, $\mathcal{F} = \{0,1\}^{\Omega}$.
The problem is when we have infinitely number of variables.

Example Example of not well-behaved triple $(\Omega, \mathcal{F}, P)$. $\Omega = \mathbb{N}$. Now $\mathcal{F} = \{C \subset \mathbb{N} : C \text{ has density}\}$.
C has density means

$$\frac{|C \cap \mathbb{N}|}{n} \overset{n\to\infty}{\to} \rho(C)$$

However, for $C(m) = \{1, 2, \ldots, m\}$, $\forall m \quad \rho(c_m)$, and

$$\rho\Big(\bigcup C_m\Big) = 1$$

Thus $(\mathbb{N}, \mathcal{F}, \rho)$ is not a good probability space, since it doesn't fulfills this $\pi_n \to \pi$ property. Note we can define other probabilities on naturals, for example

$$P(\{i\}) = 2^{-i}$$

Asymptotics of z_i Assuming $\pi \in (0, 1)$, what is behavior of z_n?

Definition 1.4. z_n is a Markov chain if

$$P(z_{n+1} = j | z_i = k_i \quad \forall i \leq n) = P(z_{n+1} = j | z_n = k_n)$$

We can use to compute expectation:

$$\mathbb{E}[z_{n+1} | z_i = k_i \quad \forall i < n] = E[z_{n+1} | z_n = k_n]$$

Then, since $E\Big[\sum_{i=1}^{J} X_i^n\Big] = J\mu$

$$E[z_{n+1} | z_n] = \mu z_n$$

Let $M_n = \frac{z_n}{\mu^n}$ then $\mathbb{E}[M_n] = 1$. Also

$$\mathbb{E}[M_{n+1} | z_0, \ldots, z_n] = M_n$$

This is a definition of martingale with respect to $z_0, \ldots, z_n$.
Let $(\Omega, \mathcal{F}, P)$ we say S happens almost surely (a.s.) if

$$P(\{w \in \Omega : S \text{ is true for w}\}) = 1$$

Theorem 1.3 (Martingale convergence theorem). If M_n is a positive martingale then $\lim_{n\to\infty} M_N = M_\infty$ exists a.s. and

- $\mu \leq 1$. $M_\infty = 0$ a.s. That means $\mathbb{E}M_\infty = 0$ but $\mathbb{E}M_=1$, i.e.,

$$\mathbb{E}\Big[\liminf_{n\to\infty} M_n\Big] < [\liminf_{n\to\infty} \mathbb{E}[M_n]$$

- $\mu > 1$. If $M_\infty > 0$ with positive probability then $z_n \sim \mu^n M_\infty$.

Lemma 1.1 (Fatou's lemma).

$$\mathbb{E}\Big[\liminf_{n\to\infty} M_n\Big] \leq \liminf_{n\to\infty} \mathbb{E}[M_n]$$

Theorem 1.4.

$$\mathbb{E}[M_\infty] = 1 \iff \mu > 1 \quad \text{ and } \mathbb{E}[X \log(X)] < \infty$$

2 Overview of measure theory

Notation

- S is a set.
- $\mathcal{A}$ is algebra of subsets of S
  1. $S \in \mathcal{A}$
  2. $$E \in \mathcal{A} \Rightarrow E^C \in \mathcal{A}$$
     , where $E^C = S \setminus E$
  3. $$E_1, E_2 \in \mathcal{A} \Rightarrow E_1 \cup E_2 \in \mathcal{A}$$
     meaning
     $$E_1, E_2 \in \mathcal{A} \Rightarrow E_1 \cap E_2 \in \mathcal{A}$$

- $\mathcal{F}$ is a σ-algebra if the last item works for countable union.
- $E \Delta F = E \setminus F \cup F \setminus E$

Definition 2.1. A measurable space is a pair $\{S, \mathcal{F}\}$.

Proposition 2.1. If we have $(\mathcal{F}_i)_{i \in I}$, then $\bigcap_{i \in I} \mathcal{F}$ is also a σ-algebra.

Definition 2.2. Let C be a collection of subsets of S. $\sigma(C)$ is a smallest σ-algebra containing C (σ-algebra generated by C). It is easy to construct one

$$I = \{\mathcal{F} : \mathcal{F} \supset C\}$$

and then

$$\sigma(C) = \bigcap_{\mathcal{F} \in I} \mathcal{F}$$

Definition 2.3. Let $\{S, \mathcal{F}\}$ be a topological space. $\mathcal{B}(X)$ (Borel σ-algebra) is defined as σ-algebra generated by open sets. We denote $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

Exercise

$$\pi(\mathbb{R}) = \{(-\infty, x],\ x \in \mathbb{R}\}$$

Show that $\sigma(\pi(\mathbb{R})) = B$

Definition 2.4. Additive set function on a collection of sets $\mathcal{F}$ is

$$\mu : \mathcal{F} \to [0, \infty)$$

$$\forall E, F \in \mathcal{F}\ E \cap F = \emptyset \quad \mu(E \cup F) = \mu(E) + \mu(F)$$

We say μ is σ-additive if same holds of countable infinite sets

$$\forall \{E_i\}_{i=1}^{\infty}\ E_i \cap E_j = \emptyset \quad \mu(E \cup F) = \sum_{i=1}^{\infty} \mu(E_i)$$

Definition 2.5. A triple $(S, \mathcal{F}, \mu)$ is a measure space if $\mathcal{F}$ is a σ-algebra on S and μ is σ-additive on $\mathcal{F}$.

Definition 2.6. $(S, \mathcal{F}, \mu)$ is finite if $\mu(S) < \infty$
$(S, \mathcal{F}, \mu)$ is σ-finite if

$$\exists \{E_i,\ \mu(E_i) < \infty\}_{i=1}^{\infty} \quad S = \bigcup_{i=1}^{\infty} E_i$$

Definition 2.7. If $\mu(S) = 1$, $(S, \mathcal{F}, \mu)$ is probability space.

Definition 2.8. E is null if $\mu(E) = 0$.

Definition 2.9. ϕ is said to be true almost everywhere with respect to μ if

$$\mu(\{X : \phi(X) = \text{False}\}) = 0$$

2.1 Results from measure theory

Definition 2.10. A collection of sets $\mathcal{D}$ is called a π-system if $E, F \in D \Rightarrow E \cap F \in \mathcal{D}$

Theorem 2.2 (Uniqness)**.** Let $\mathcal{D}$ be a π-system generating a σ-algebra $\mathcal{F}$. Let μ_1 and μ_2 be two finite measures on $\mathcal{F}$ which agree on $\mathcal{D}$. Then $\mu_1 = \mu_2$.

Collary 2.1. $(S, \mathcal{F}, P_1)$, $(S, \mathcal{F}, P_2)$ probability spaces, $P1 = P2$ on π-system $\mathcal{D}$, then $P_1 = P_2$.

Theorem 2.3 (Carathéodory's extension theorem)**.** Let $\mathcal{A}$ be an algebra of sets. $\mu_0 : \mathcal{A} \to \mathbb{R}^+$ σ-additive set function on $\mathcal{A}$. Then exists unique extension $\bar{\mu} : \sigma(\mathcal{A}) \to \mathbb{R}^+$ such that $\bar{\mu} = \mu_0$.

Homework Lebesgue on $\mathbb{R}$. $\mathcal{A} = \{\text{open set}\}$. If we have

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

then

$$\mu_0(O) = \sum_{i=1}^{\infty} b_i - a_i$$

Check that μ_0 is well defined and σ-additive.

Lemma 2.1. $(S, \mathcal{F}, \mu)$ measure space. $A, B \in \mathcal{F}$, then

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mu(F_i)$$

If $\mu(S) < \infty$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

From that we get inclusion-exclusion:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A)_i - \sum_{i \neq j} \mu(A_i \cap A_j) + \cdots + (-1)^{n-1} \mu\left(\bigcap_{i=1}^{n} A_i\right)$$

Exercise Proof the lemma

Lemma 2.2. If $F_n \subseteq F_{n+1}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n \to \infty} \mu(F_n)$$

If $\mu(S) < \infty$ and $F_n \supseteq F_{n+1}$ then

$$\mu\left(\bigcap_{i=1}^{\infty} F_i\right) = \lim_{n \to \infty} \mu(F_n)$$

Proof. Assume $\mu(S) < \infty$. Define $F_\infty = \bigcup_{i=1}^{\infty} F_i$. Let $G_n = F_n \setminus F_{n+1}$. Then

$$F_\infty = \bigcup_{i=1}^{\infty} G_i$$

Meaning

$$\mu(F_\infty) = \sum_{i=1}^{\infty} G_i$$

$$\mu(F_n) = \sum_{k=1}^{n} G_k$$

Since measure is finite, the tail of series tends to 0, thus

$$\mu(F_\infty) - \mu(F_n) = \sum_{k=n}^{\infty} G_k \to 0$$

Then we can take complements and get the second statement. □

Exercise Proof unconditionally

3 Recasting measure theory as probability

Definition 3.1. A probability space is a $(\Omega, \mathcal{F}, P)$ is a measure space such that $P(\Omega) = 1$. We call $\omega \in \Omega$ an outcome. $E \in \mathcal{F}$ is an event. $P(E)$ is probability of the event.

Example Tossing finite or infinite sequence of coins.

Tossing 4 coins

$$\Omega = \{HHHH, HHHT, HHTH, \ldots, TTTH, TTTT\}$$
$$\mathcal{F} = 2^{\Omega}$$
$$P(\omega \in \Omega) = \frac{1}{|\Omega|}$$

Tossing infinite number coins

$$\Omega = \{0,1\}^{\mathbb{N}}$$

Ω has a natural topology which is called a product topology. It is coarsest topology such that $\pi_i : \Omega \to \{0,1\}$ $\pi_i(\omega) = \omega_i$ is continuous.
Let $\mathcal{F} = \mathcal{B}(\Omega)$.
Smallest σ-algebra such that

$$\pi_i^{-1}(0) \subset \Omega \in \mathcal{F}$$
$$\pi_i^{-1}(1) \subset \Omega \in \mathcal{F}$$

$$\pi_i(\Omega, \mathcal{F}) \to \left(\{0,1\}, \{0,1\}^{\{0,1\}}\right)$$

Natural π-system $\mathcal{F}_n$ smallest σ-algebra making $\pi_1, \ldots, \pi_n$ measurable.
Note that

Proposition 3.1.

$$\bigcup_n \mathcal{F}_n \neq \mathcal{F}$$

Proof. Define $S_n(\omega) = \sum_{i=1}^n \omega_n$.

$$X_n = \frac{S_n(\omega)}{n}$$

Define

$$Y(\omega) = \limsup X_n(\omega)$$
$$E = \left\{\omega : Y(\omega) \geq \frac{1}{3}\right\}$$
$$E \in \mathcal{F} \setminus \bigcup_n \mathcal{F}_n$$

□

What $\mathcal{F}_n$ looks like? For example, $\mathcal{F}_2$ has 4 outcomes, deciding only first two tosses.

Note If we take $(\Omega_4, \mathcal{F}^{(4)}, P_4)$, restricting to $(\Omega_3, \mathcal{F}^{(3)}, P_3)$

$$P_4(\{(0,0,0,\omega_4)\}) = P_3(\{(0,0,0)\})$$

Thus we want P_{fair} defined on Ω to fulfill same property:

$$P_{fair}(E) = P_n(\tilde{E})$$

where $E \in \mathcal{F}_n$ and $\tilde{E} \in F^{(n)}$.

Definition 3.2. $E \subset \mathcal{F}$ occures almost surely (a.s.) if $P(E) = 1$.

Definition 3.3 (lim sup and lim inf)**.** Let $\{E_n\}$ be a sequence of events.

$$\limsup E_n = \bigcap_m \bigcup_{n \geq m} E_n = \{E_n \text{ occurs infenetely often (i.o.)}\} = \{\omega \in \Omega : \forall m \, \exists n(\omega) > m \quad \omega \in E_n(\omega)\}$$

Alternatively, $(\Omega, \mathcal{F})$ and $\{E_n\}$ there is a natural map

$$I : \Omega \to \{0,1\}^N$$

$$\omega \mapsto \{1_{E_n}(\omega)\}$$

where
$$1_E(\omega) = \begin{cases} 0 & \omega \notin E \\ 1 & \omega \in E \end{cases}$$

Now
$$\liminf E_n = \bigcup_m \bigcap_{n \geq m} E_n = \{E_n \text{ occurs eventually}\} = \{\omega \in \Omega : \exists m(\omega)\, \forall n \geq m(\omega) \quad \omega \in E_n(\omega)\}$$

Remark Since everything is countable, if $E_n \in \mathcal{F}$, then $\limsup E_n, \liminf E_n \in \mathcal{F}$
We can write
$$\left\{\frac{S_n}{n} \to \frac{1}{2}\right\} = \left\{\limsup \frac{S_n}{n} \leq \frac{1}{2}\right\} \cap \left\{\liminf \frac{S_n}{n} \geq \frac{1}{2}\right\}$$
.
Choose $q \in \mathbb{Q}^+$ and take a look at
$$\left\{\liminf \frac{S_n}{n} > q\right\} = \liminf E_n(q)$$
where $E_n = \left\{\omega : \frac{S_n}{n} > q\right\}$.
In addition
$$\left\{\limsup \frac{S_n}{n} < q\right\} = \liminf F_n(q)$$
where $F_n = \left\{\omega : \frac{S_n}{n} < q\right\}$.
Therefore $\left\{\liminf \frac{S_n}{n} > q\right\} \in \mathcal{F}$.
Finally,
$$\left\{\liminf \frac{S_n}{n} \geq \alpha\right\} = \bigcap_{q<\alpha} \left\{\liminf \frac{S_n}{n} > q\right\}$$

Lemma 3.1 (Fatou's lemma)**.**
$$P\left[\liminf_{n\to\infty} E_n\right] \leq \liminf_{n\to\infty} p(E_n)$$

Proof.
$$\liminf_{n\to\infty} E_n = \bigcup_m \bigcap_{n\geq m} E_n$$

Sets $F_m = \bigcap_{n\geq m} E_n$ are increasing and $F_n \subseteq E_n$, thus
$$P\left[\liminf_{n\to\infty} E_n\right] = \lim_{n\to\infty} P(F_n) \leq \liminf_{n\to\infty} P(E_n)$$

□

Lemma 3.2 (Fatou's lemma)**.**
$$P\left[\limsup_{n\to\infty} E_n\right] \geq \limsup_{n\to\infty} p(E_n)$$

Proof. Note that $(\limsup E_n)^C = \liminf E_n^C$, thus this is straightforward form previous lemma. □

Lemma 3.3 (First Borel-Cantelli lemma)**.** Let $\{E_n\} \subseteq \mathcal{F}$ be a sequence of events s.t. $\sum_n P(E_n) < \infty$, then
$$P(E_n \text{ happens i.o.}) = 0$$

Proof.
$$P(E_n \text{ i.o.}) = P\left(\bigcap_m \bigcup_{n\geq m} E_n\right) \leq P\left(\bigcup_{n\geq m} E_n\right) \leq \sum_{n=m}^{\infty} P(E_n) \overset{m\to\infty}{\to} 0$$

Since $P(E_n \text{ i.o.})$ is independent on m, it got to be 0. □

Example Fix $\epsilon > 0$. Look at $P\left(\left|\frac{S_n(\omega)}{n} - \frac{1}{2}\right| > \epsilon\right)$.

Claim

$$P\left(\left|\frac{S_n(\omega)}{n}-\frac{1}{2}\right|>\epsilon\right)\leq\frac{12}{\epsilon^4}\frac{1}{n^2}$$

By 3.3 $P\left(\left|\frac{S_n(\omega)}{n}-\frac{1}{2}\right|>\epsilon \text{ i.o.}\right)=0$ thus

$$\left\{\frac{S_n}{n}\to\frac{1}{2}\right\}=\bigcap_{\epsilon>0}\left\{\left|\frac{S_n(\omega)}{n}-\frac{1}{2}\right|<\epsilon \text{ eventually}\right\}=1$$

Definition 3.4. Let $(S,\mathcal{F})$, $(\Omega,\mathcal{B})$ be measurable spaces.

$$\phi:S\to\Omega$$

ϕ is $\big((\mathcal{F},\mathcal{B})\big)$-measurable if $\forall B\in\mathcal{B}\quad\phi^{-1}(B)\in\mathcal{F}$.

Remark $\mathcal{C}$ is collection of sets in Ω. $\phi^{-1}(\mathcal{C})=\left\{\phi^{-1}(C):C\in\mathcal{C}\right\}$.

- $$\phi^{-1}\left(\bigcap_{i\in I}B_i\right)=\bigcap\phi^{-1}(B_i)$$
- $$\phi^{-1}\left(\bigcup_{i\in I}B_i\right)=\bigcup\phi^{-1}(B_i)$$
- $$\phi^{-1}\left(B^C\right)=\left[\phi^{-1}(B)\right]^c$$

Lemma 3.4. Let $\sigma(\mathcal{C})=\mathcal{B}$. ϕ is measurable iff $\phi^{-1}(\mathcal{C})\subseteq\mathcal{F}$.

Collary 3.1. $\Omega=\mathbb{R}$, $\mathcal{B}(\mathbb{R})$ then ϕ is measurable iff

$$\forall x\ \phi^{-1}((-\infty,x])\subseteq\mathcal{F}$$

Lemma 3.5. Let $(S,\mathcal{F})$, $(T,\mathcal{T})$, $(\Omega,\mathcal{B})$ be measurable spaces. Let $\phi_1:S\to T$ and $\phi_2 T\to\Omega$ measurable. Then $\phi_2\circ\phi_1$ is measurable.

Proof. Let $B\in\mathcal{B}$. Then $\phi_2^{-1}(B)\in\mathcal{T}$, and thus $\phi_1^{-1}\big(\phi_2^{-1}(B)\big)\in\mathcal{F}$, meaning $(\phi_2\circ\phi_1)^{-1}(B)\in\mathcal{F}$. □

Lemma 3.6. $\Omega=\mathbb{R}$. Then $\{\phi|\phi$ is $\mathcal{F},\mathcal{B}$-measurable$\}$ is an algebra over $\mathbb{R}$.

Proof. Using previous lemma and the fact $+$ is continuous, and thus measurable, we define $\Psi(s)=(\phi_1(s),\phi_2(s))$. Ψ is measurable. Take a look at

$$\Psi^{-1}\big((-\infty,x_1]\times(-\infty,x_2]\big)=\{s:\phi_1(s)\in(-\infty,x_1],\ \phi_2(s)\in(-\infty,x_2]\}$$

□

Notation

$$\phi:(S,\mathcal{F})\to(\Omega,\mathcal{B})$$

We write $\phi\in\mathcal{F}$ for ϕ is $\mathcal{F},\mathcal{B}$ measurable.

Constructions preserved by measurability

Proposition 3.2. If $\{\phi_n\}_{n=1}^{\infty}$ measurable maps $(S, \mathcal{F}) \to (\Omega, \mathcal{B})$, then $\liminf \phi_n$, $\limsup \phi_n$, $\inf \phi_n$, $\sup_n$ are also measurable.

Proof. For example, fpr infimum, we need to show that

$$\left\{ s | \inf_n \phi_n(s) \leq c \right\} \in \mathcal{F}$$

or alternatively,

$$\left\{ s | \inf_n \phi_n(s) > c \right\} \in \mathcal{F}$$

which is just countable intersection:

$$\bigcap_n \{s : \phi_n(s) > c\}$$

Same for $\limsup$, which is just infimum of supremum:

$$\limsup \phi_n = \inf_m \left(\sup_{n \geq m} \phi_n \right)$$

□

Recall

$$S_n = \text{number of 1's until n}$$

We can view s_n as a composition of projection and sum:

$$\omega \mapsto \big(\pi_1(\omega), \ldots \pi_n(\omega)\big) \mapsto \sum_{i=1}^{n} \pi_i(\omega)$$

Both are continuous (projection from the definition of product topology) and thus measurable, and so is $\frac{S_n}{n}$.

4 Random variables

Definition 4.1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. $X : \Omega \to (S, \mathcal{S})$ measurable is called a random variable.

Notation

$$\{\omega : X(\omega) \in A\} = X^{-1}(A)$$

We use notation like $X \in A$.

Basic constructions with random variables

Definition 4.2. Given a probability space $(\Omega, \mathcal{F}, P)$ and measurable $(S, \mathcal{S})$, X induces measure $\mathcal{L}_X$ on $(S, \mathcal{S})$ via

$$\mathcal{L}_X(E) = P(X \in E)$$

$\mathcal{L}_X$ is called marginal distribution of X or law of X.

Proposition 4.1. $\mathcal{L}_X$ is countably additive set function.

If $(S, \mathcal{S})$ is $\mathbb{R}, \mathcal{B}$. By uniqueness theorem, $\mathcal{L}_X$ if defined by

$$F_X(x) = \mathcal{L}_X\big((-\infty, x]\big) = P\big(X \in (-\infty, x]\big)$$

Proposition 4.2. $\mathcal{L}_X \mapsto F_X$ is 1-1 and onto.

Proof. Uniqueness:
If μ, ν exists such that

$$\mu\big((-\infty, x]\big) = F_X(x) = \nu\big((-\infty, x]\big)$$

then, since they agree on π-system, and thus are equal by uniqueness theorem.
Existence $\mu\big((-\infty, x]\big)$ fulfills Carathéodory's extension theorem requirements, thus there exists unique extension. □

We assume there exists Lebesgue measure on Borel sets $([0, 1], \mathcal{B}, \lambda)$.

Definition 4.3. A coupling of X,Y is $(\Omega, \mathcal{F}, P)$ and $\tilde{X}, \tilde{Y} : \Omega \to \mathbb{R}$ such that $\mu_{\tilde{X}} = \mu_X$ and $\mu_{\tilde{Y}} = \mu_Y$.

Theorem 4.3 (Skorokhod's representation (of a random variable X))**.** Given μ_X, μ_Y can we construct $(\Omega, \mathcal{F}, P)$ and $\tilde{X}, \tilde{Y} : \Omega \to \mathbb{R}$ such that $\mu_{\tilde{X}} = \mu_X$ and $\mu_{\tilde{Y}} = \mu_Y$, i.e., a coupling of X,Y.

Proof. Given increasing, right continuous F such that $F(-\infty) = 0$ and $F(\infty) = 1$. We want to define $X : [0,1] \to \mathbb{R}$ s.t. $F_X = F$.

$$X^-(\omega) = \inf\{x : F(x) \geq \omega\}$$

(We could also choose $X^+(\omega) = \inf\{x : F(x) \geq \omega\}$, which is a bit different)
We want to show that

$$\{\omega : X^-(\omega) \leq x\} = \{\omega \leq F(x)\}$$

$X^-(\omega) \leq x$ means that $F(x) \geq \omega$ (by definition), verifying $\{\omega : X^-(\omega) \leq x\} \subseteq \{\omega \leq F(x)\}$
$F(x) \geq \omega$ means $X^-(\omega) \leq x$ (since F is increasing), finishing the proof.
Note, that X^+ and X^- disagree only on countable number of points.

□

F_X is cumulative distribution function (CDF) or distribution function of X.
We ask the question: what are set properties distinguish F_X?

Proposition 4.4 (Properties of CDF)**.**

1. F_X is non-decreasing
2. F_X is right continuous
3. $F_X(-\infty) = 0$

Proof.

1. If $x < y$, $(-\infty, x] \subseteq (-\infty, y]$, thus, from monotonicity of measure $F_X(x) \leq F_X(y)$
2. we want to show
$$\lim_{x \downarrow x_0} F(x) = F(x_0)$$
Since if $E_n \downarrow E$, then $\mu(E_n) \to \mu(E)$
We can look on sequence $\{x_n\}$:
$$\omega \in \bigcap_n \{X \in (-\infty, x_n)\} \Rightarrow \forall n \quad X(\omega) \leq x_n \Rightarrow X(\omega) \leq x \Rightarrow \bigcap_n \{X \in (-\infty, x_n)\} \subseteq \{X \in (-\infty, x)\}$$
The other direction is obvious.
3. Since
$$\bigcap_x X^{-1}\big((-\infty, x]\big) = \emptyset$$

□

4.1 Independence

$(\Omega, \mathcal{F}, P)$ probability space. Let $\{\mathcal{J}_i\}_{i\in I}$ be a collection of sub-σ-algebras of $\mathcal{F}$.

Definition 4.4 (Independence of σ-algebras)**.** Say $\{\mathcal{J}_i\}_{i\in I}$ are independent if

$$\forall i_1, \dots i_k \, \forall j \quad G_{ij} \in \mathcal{J}_i \quad P\left(\bigcap_{j=1}^{k} G_{ij}\right) = \prod_{j=1}^{k} P(G_{ij})$$

Definition 4.5 (Independence of random variables)**.** Say $\{X_i\}_{i\in I}$ are independent if $\sigma(X_i)$ are independent.

Definition 4.6 (Independence of sets)**.** Say $\{E_i\}_{i\in I}$ are independent if random variables $\mathbb{1}_{E_i}$ are independent.

Connection with elementary probability theory

Lemma 4.1 (Checking independence)**.** Let $\mathcal{F}_1$, $\mathcal{F}_2$ be σ-algebras, $\mathcal{A}_1$, $\mathcal{A}_2$ π-systems such that $\sigma(\mathcal{A}_\infty) = \mathcal{F}_i$. Then $\mathcal{F}_1$, $\mathcal{F}_2$ are independent iff

$$\forall A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2 \quad P(A_1 \cap A_2)P(A_1)P(A_2)$$

Proof. Given $E \in \mathcal{F}$ let $P_E(A) = P(A \cap E)$, a measure on $(\Omega, \mathcal{F})$.
Given $A_1 \in \mathcal{A}_1$ consider $P_{A_1}\big|_{\mathcal{F}_2}$.

$$\forall\ A_2 \in \mathcal{A}_2\ P_{A_1}(A_2) = P(A_1)P(A_2)$$

Thus P_{A_1} and $P(A_2) \times P$ are measures on $\mathcal{F}_2$ agreeing on $\mathcal{A}_2$.

$$\forall A_1 \in \mathcal{A}_1, E_2 \in \mathcal{F}_2 \quad P(A_1 \cap E_2) = P(A_1)P(E_2)$$

Next iterate argument argument for $\mathcal{F}_1$

$$\forall\ E_2 \in \mathcal{F}_2\ P_{E_2} = P(E_2)P$$

By uniqueness

$$\forall E_i \in \mathcal{F}_i \quad P(E_1 \cap E_2) = P(E_1)P(E_2)$$

$\square$

Collary 4.1. To check $X_1, \ldots, X_k$ are independent random variables it suffices

$$\forall x \in \mathbb{R}^k \quad P(X_i \leq x_i) = \prod_{i=1}^{k} P(X_1 \leq x_i)$$

Lemma 4.2 (Second Borel-Cantelli lemma)**.** If $\sum_i P(E_i) = \infty$ and $\{E_i\}_{i=1}^\infty$ are independent, then

$$P(E_n \text{ i.o.}) = 0$$

Proof.

$$\{E_i \text{ i.o.}\}^C = \left\{E_i^C \text{ eventually}\right\}$$

It's enough to show

$$P\left(\bigcap_{i \geq n} E_i^C\right) = 0$$

or, by truncating

$$P\left(\bigcap_{i \geq n}^{k} E_i^C\right) = 0$$

By independence

$$P\left(\bigcap_{i \geq n} E_i^C\right) \prod_{i=n}^{k} P(E_i^C) = \prod_{i=n}^{k} [1 - P(E_i)] \leq e^{-\sum_{i=n}^{k} P(E_i)}$$

Since the sum tends to infinity, the exponent tends to 0. $\square$

Example Let $\{X_i\}_{i=1}^\infty$ be i.i.d. $Exp(1)$ random variables, i.e.

$$P(X_i > x) = e^{-x}$$

We are interested in growth rate of $X_n \leq f(n)$.
If $f(n) = \alpha \log(n)$

$$P(X_n > f(n)) = e^{-f(n)} = n^{-\alpha}$$

Thus, from Lemmas 3.3, 4.2

$$P(X_n \geq \alpha \log(n) \text{ i.o.}) = \begin{cases} 0 & \alpha > 1 \\ 1 & \alpha \leq 1 \end{cases}$$

Define $L = \limsup_{n \to \infty} \frac{X_n}{\log(n)}$, then

$$P(L \geq 1) = P(X_n \geq \log(n) \text{ i.o.}) = 1$$

Finally, if we look at $E = \bigcup_{k=1}^{\infty} \left\{L \geq 1 + \frac{1}{k}\right\}$,

$$P(E) \leq \sum_{k=1}^{\infty} P\left(L \geq 1 + \frac{1}{k}\right) \leq \sum_{k=1}^{\infty} P\left(X_n \geq \left(1 + \frac{1}{2k}\right)\log(n)\right) = 0$$

thus $P(L \leq 1) = 1$.

Method of generation of i.i.d. uniform $[a, b]$ variables We write $\omega = \sum_{i=1}^{\infty} \frac{\omega_i}{2^i}$

$$\begin{cases} w^{(1)} = \omega_1\omega_3\omega_6\omega_{10}\omega_{15}\dots \\ w^{(2)} = \omega_2\omega_5\omega_9\omega_{14}\dots \\ w^{(3)} = \omega_4\omega_8\omega_{13}\omega_{19}\dots \\ \vdots \end{cases}$$