106349 - Advanced probability

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Abstract

1 Introduction. Summary of course through an example. Branching process

We have an individual that gives a birth to a random number of offsprings – random variable X. X define a distribution, i.e., $P : \mathbb{Z}^+ \to [0,1]$, i.e., $P(X = k) \in [0,1]$, and $\sum_{k=0}^{\infty} P(X = k) = 1$.

Definition 1.1. $f_X(\theta) = \sum_{k=0}^{\infty} \theta^k P(X = k)$ – moment-generating function.

The series is absolutely convergent for $\theta \in [-1,1]$ since k sums to 1. For $\theta \in (-1,1)$, f_x is analytic, thus we can differentiate it term-by-term:

$$f'_X(\theta) = \sum_{k \geq 1} \theta^{k-1} P(X = k)$$

Since, f_X is analytic, knowing it means knowing $P(X = k)$ and vice versa.
Note that $f_X(0) = P(X = 0)$ and $f_X(1) = 1$. Also

$$f'_X(1) = \sum_{k \geq 0}^{\infty} kP(X = k) = \mathbb{E}X = \mu$$

$$\lim_{\theta \to 1} \frac{f_X(1) - f_X(\theta)}{1 - \theta} = \lim_{\theta \to 1} \frac{1 - f_X(\theta)}{1 - \theta}$$

Note also that f_X is convex, since second derivative is positive.

Size of n^{th} generation Let $\left(X_r^{(n)}\right)_{n,r=1\infty}$, where n is generation and r is offspring number (index) in n^{th} generation.
Assume $X_r^{(n)}$ are i.i.d. (independent, identically distributed) random variables.
Identically distributed means

$$P(X_n^r = k) = P(X = k)$$

.
Independence means

$$P\left(\forall i < J \; X_{r_i}^{n_i} = k\right) = \prod_{i=1}^{J} P\left(X_{r_i}^{n_i} = k\right)$$

.
Define $z_1 = X_1^1$. $z_2 = \sum_{r=1}^{z_1} X_r^2$ an so on:

$$z_{n+1} = \sum_{r=1}^{z_n} X_r^n$$

We want to study asymptotics of z_n.
Given U and V taking values in $\mathbb{Z}^+$,

$$\mathbb{E}][U|V = k] = \sum_{j=0}^{\infty} jP(U = j|V = k)$$

, where

$$P(U = j|V = k) = \frac{P(U = j, V = k)}{P(V = k)}$$

If U, V are independent, $P(U=j|V=k)=P(U=j)$ and thus $\mathbb{E}[U|V=k]=\mathbb{E}U$.

Definition 1.2. Define random variable $\mathbb{E}[U|V]$ such that

$$\mathbb{E}[U|V]=\mathbb{E}[U|V=k]$$

if $V=k$.

Definition 1.3 (Tower property)**.**

$$\mathbb{E}\big[\mathbb{E}[U|V]\big]=\mathbb{E}U$$

Define

$$f_n=\sum_{k=0}^{\infty}\sum_{k=0}^{\infty}\theta^k P(z_n=k)=\mathbb{E}\theta^{z_n}$$

.

Theorem 1.1.

$$f_{n+1}(\theta)=f_n(f_X(\theta))$$

or

$$f_n(\theta)=\underbrace{f\circ f\circ\cdots\circ f}_{n\text{ times}}(\theta))$$

Proof. Use tower property with $U^{z_{n+1}}$ and $V=\theta^{z_n}$. By tower property

$$\mathbb{E}[\theta^{z_{n+1}}]=\mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big]$$

$$\mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big]=\sum_{k=0}^{\infty}P(z_n=k)\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}=k]$$

What is $\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}=k]$?

$$\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}=k]=\mathbb{E}\Big[\theta^{\sum_{j=1}^{k}X_j^{n+1}}|\theta^{z_n}=k\Big]\overset{\text{independence}}{=}\mathbb{E}\Big[\theta^{\sum_{j=1}^{z_n}X_j^{n+1}}\Big]\overset{\text{independence}}{=}\prod_{j=1}^{k}\mathbb{E}\Big[\theta^{X_j^{n+1}}\Big]\overset{\text{i.d.}}{=}(f_X(\theta))^k$$

Thus

$$\mathbb{E}\big[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]\big]=\sum_{k=0}^{\infty}P(z_n=k)(f_X(\theta))^k=f_n(f(\theta))$$

Also we can say

$$\mathbb{E}[\theta^{z_{n+1}}|z_n]=(f_X(\theta))^{z_n}$$

$\square$

Study of z_n What is $\pi_n=P(z_n=0)=f_n(0)=f(\pi_{n-1})$, probability that population is extinguished. Since $z_{n-1}=0\Rightarrow z_n=0$, i.e. π_n is non-decreasing.
Let $P(z_n=0\text{ for some n})=\pi$.
We hope that $\{z_n=0\}$ such that

$$\bigcup_n\{z_n=0\}=\{z_n=0\text{ for some n}\}$$

i.e., $\pi=\lim_{n\to\infty}\pi_n$. We call π the extinction probability.

Theorem 1.2. If $\mu=\mathbb{E}>1$ then π is a unique root of $\pi=f(\pi)$ and $\pi\in[0,1)$. If $\mu\le 1$, $\pi=1$.
If we look at $f(\pi)$ and π, they intersect in 1, and they can intersect in two points since $f(x)$ is convex. There is second intersection iff $f'(1)=\mu>1$.

Construction of X_n^r Construct set Ω, $f_{n,r}:\Omega\to\mathbb{Z}^+$ and $\mathcal{F}$ a collection of subsets of Ω with $P:\mathcal{F}\to[0,1]$.
Let $\Omega=\mathbb{Z}^+\times\mathbb{Z}^+$, $\mathcal{F}=\{0,1\}^{\Omega}$.
The problem is when we have infinitely number of variables.

Example Example of not well-behaved triple $(\Omega, \mathcal{F}, P)$. $\Omega = \mathbb{N}$. Now $\mathcal{F} = \{C \subset \mathbb{N} : C \text{ has density}\}$.
C has density means

$$\frac{|C \cap \mathbb{N}|}{n} \overset{n\to\infty}{\to} \rho(C)$$

However, for $C(m) = \{1, 2, \dots, m\}$, $\forall m \quad \rho(c_m)$, and

$$\rho\Big(\bigcup C_m\Big) = 1$$

Thus $(\mathbb{N}, \mathcal{F}, \rho)$ is not a good probability space, since it doesn't fulfills this $\pi_n \to \pi$ property. Note we can define other probabilities on naturals, for example

$$P(\{i\}) = 2^{-i}$$

Asymptotics of z_i Assuming $\pi \in (0, 1)$, what is behavior of z_n?

Definition 1.4. z_n is a Markov chain if

$$P(z_{n+1} = j | z_i = k_i \quad \forall i \leq n) = P(z_{n+1} = j | z_n = k_n)$$

We can use to compute expectation:

$$\mathbb{E}[z_{n+1} | z_i = k_i \quad \forall i < n] = E[z_{n+1} | z_n = k_n]$$

Then, since $E\Big[\sum_{i=1}^{J} X_i^n\Big] = J\mu$

$$E[z_{n+1} | z_n] = \mu z_n$$

Let $M_n = \frac{z_n}{\mu^n}$ then $\mathbb{E}[M_n] = 1$. Also

$$\mathbb{E}[M_{n+1} | z_0, \dots, z_n] = M_n$$

This is a definition of martingale with respect to $z_0, \dots, z_n$.
Let $(\Omega, \mathcal{F}, P)$ we say S happens almost surely (a.s.) if

$$P(\{w \in \Omega : S \text{ is true for w}\}) = 1$$

Theorem 1.3 (Martingale convergence theorem)**.** If M_n is a positive martingale then $\lim_{n\to\infty} M_N = M_\infty$ exists a.s. and

- $\mu \leq 1$. $M_\infty = 0$ a.s. That means $\mathbb{E}M_\infty = 0$ but $\mathbb{E}M_{=}1$, i.e.,

$$\mathbb{E}\Big[\liminf_{n\to\infty} M_n\Big] < [\liminf_{n\to\infty} \mathbb{E}[M_n]$$

- $\mu > 1$. If $M_\infty > 0$ with positive probability then $z_n \sim \mu^n M_\infty$.

Lemma 1.1 (Fatov's lemma)**.**

$$\mathbb{E}\Big[\liminf_{n\to\infty} M_n\Big] \leq [\liminf_{n\to\infty} \mathbb{E}[M_n]$$

Theorem 1.4.

$$\mathbb{E}[M_\infty] = 1 \iff \mu > 1 \quad \text{and } \mathbb{E}[X \log(X)] < \infty$$

2 Overview of measure theory

Notation

- S is a set.
- $\mathcal{A}$ is algebra of subsets of S
  1. $S \in \mathcal{A}$
  2. $$E \in \mathcal{A} \Rightarrow E^C \in \mathcal{A}$$
     , where $E^C = S \setminus E$
  3. $$E_1, E_2 \in \mathcal{A} \Rightarrow E_1 \cup E_2 \in \mathcal{A}$$
     meaning
     $$E_1, E_2 \in \mathcal{A} \Rightarrow E_1 \cap E_2 \in \mathcal{A}$$

- $\mathcal{F}$ is a σ-algebra if lest item works for countable union.
- $E\Delta F = E \setminus F \cup F \setminus E$

Definition 2.1. A measurable space is a pair $\{S, \mathcal{F}\}$.

Proposition 2.1. If we have $(\mathcal{F}_i)_{i\in I}$, then $\bigcap_{i\in I} \mathcal{F}$ is also a σ-algebra.

Definition 2.2. Let C be a collection of subsets of S. $\sigma(C)$ is a smallest σ-algebra containing C (σ-algebra generated by C). It is easy to construct one

$$I = \{\mathcal{F} : \mathcal{F} \supset C\}$$

and then

$$\sigma(C) = bigcap_{\mathcal{F}\in I}\mathcal{F}$$

Definition 2.3. Let $\{S, \mathcal{F}\}$ be a topological space. $\mathcal{B}(X)$ (Borel σ-algebra) is defined as σ-algebra generated by open sets. We denote $\mathcal{B} = \mathcal{B}(\mathbb{R})$.

Exercise

$$\pi(\mathbb{R}) = \{(-\infty, x],\ x \in \mathbb{R}\}$$

Show that $\sigma(\pi(\mathbb{R})) = B$

Definition 2.4. Additive set function on a collection of sets $\mathcal{F}$ is

$$\mu : \mathcal{F} \to [0, \infty)$$

$$\forall E, F \in \mathcal{F}\ E \cap F = \emptyset \quad \mu(E \cup F) = \mu(E) + \mu(F)$$

We say μ is σ-additive if same holds of countable infinite sets

$$\forall \{E_i\}_{i=1}^{\infty}\ E_i \cap E_j = \emptyset \quad \mu(E \cup F) = \sum_{i=1}^{\infty} \mu(E_i)$$

Definition 2.5. A triple $(S, \mathcal{F}, \mu)$ is a measure space if $\mathcal{F}$ is a σ-algebra on S and μ is σ-additive on $\mathcal{F}$.

Definition 2.6. $(S, \mathcal{F}, \mu)$ is finite if $\mu(S) < \infty$
$(S, \mathcal{F}, \mu)$ is σ-finite if

$$\exists \{E_i,\ \mu(E_i) < \infty\}_{i=1}^{\infty} \quad S = \bigcup_{i=1}^{\infty} E_i$$

Definition 2.7. If $\mu(S) = 1$, $(S, \mathcal{F}, \mu)$ is probability space.

Definition 2.8. E is null if $\mu(E) = 0$.

Definition 2.9. ϕ is said to be true almost everywhere with respect of μ if

$$\mu(\{X : \phi(X) = \text{False}\}) = 0$$

2.1 Results from measure theory

Definition 2.10. A collection of sets $\mathcal{D}$ is called a π-system if $E, F \in D \Rightarrow E \cap F \in \mathcal{D}$

Theorem 2.2 (Uniqness)**.** Let $\mathcal{D}$ be a π-system generating a σ-algebra $\mathcal{F}$. Let μ_1 and μ_2 be two finite measures on $\mathcal{F}$ which agree on $\mathcal{D}$. Then $\mu_1 = \mu_2$.

Collary 2.1. $(S, \mathcal{F}, P_1)$, $(S, \mathcal{F}, P_2)$ probability spaces, $P1 = P2$ on π-system $\mathcal{D}$, then $P_1 = P_2$.

Theorem 2.3 (Carathéodory's extension theorem)**.** Let $\mathcal{A}$ be an algebra of sets. $\mu_0 : \mathcal{A} \to \mathbb{R}^+$ σ-additive set function on $\mathcal{A}$. Then exists unique extension $\bar{\mu} : \sigma(\mathcal{A}) \to \mathbb{R}^+$ such that $\bar{m}u = \mu_0$.

Homework Lebesgue on $\mathbb{R}$. $\mathcal{A} = \{\text{open set}\}$. If we have

$$O = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

then

$$\mu_0(O) = \sum_{i=1}^{\infty} b_i - a_i$$

Check that μ_0 is well defined and σ-additive.

Lemma 2.1. $(S, \mathcal{F}, \mu)$ measure space. $A, B \in \mathcal{F}$, then

$$\mu(A \cup B) \leq \mu(A) + \mu(B)$$

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mu(F_i)$$

If $mu(S) < \infty$

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

From that we get inclusion-exclusion:

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A)_i - \sum_{i \neq j} \mu(A_i \cap A_j) + \cdots + (-1)^{n-1} \mu\left(\bigcap_{i=1}^{n} A_i\right)$$

Exercise Proof the lemma

Lemma 2.2. If $F_n \subseteq F_{n+1}$ then

$$\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{n \to \infty} \mu(F_n)$$

If $\mu(S) < \infty$ and $F_n \supseteq F_{n+1}$ then

$$\mu\left(\bigcap_{i=1}^{\infty} F_i\right) = \lim_{n \to \infty} \mu(F_n)$$

Proof. Assume $\mu(S) < \infty$. Define $F_\infty = \bigcup_{i=1}^{\infty} F_i$. Let $G_n = F_n \setminus F_{n+1}$. Then

$$F_\infty = \bigcup_{i=1}^{\infty} G_i$$

Meaning

$$\mu(F_\infty) = \sum_{i=1}^{\infty} G_i$$

$$\mu(F_n) = \sum_{k=1}^{n} G_k$$

Since measure is finite, the tail of series tends to 0, thus

$$\mu(F_\infty) - \mu(F_n) = \sum_{k=n}^{\infty} G_k \to 0$$

□

Exercise Proof unconditionally