

106349 - Advanced probability

Nick Crawford

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Abstract

1 Introduction. Summary of course through an example. Branching process

We have an individual that gives a birth to a random number of offsprings – random variable X . X define a distribution, i.e., $P : \mathbb{Z}^+ \rightarrow [0, 1]$, i.e., $P(X = k) \in [0, 1]$, and $\sum_{k=0}^{\infty} P(X = k) = 1$.

Definition 1.1. $f_X(\theta) = \sum_{k=0}^{\infty} \theta^k P(X = k)$ – moment-generating function.

The series is absolutely convergent for $\theta \in [-1, 1]$ since k sums to 1. For $\theta \in (-1, 1)$, f_X is analytic, thus we can differentiate it term-by-term:

$$f'_X(\theta) = \sum_{k \geq 1} \theta^{k-1} P(X = k)$$

Since, f_X is analytic, knowing it means knowing $P(X = k)$ and vice versa.

Note that $f_X(0) = P(X = 0)$ and $f_X(1) = 1$. Also

$$f'_X(1) = \sum_{k \geq 0} k P(X = k) = \mathbb{E}X = \mu$$

$$\lim_{\theta \rightarrow 1} \frac{f_X(1) - f_X(\theta)}{1 - \theta} = \lim_{\theta \rightarrow 1} \frac{1 - f_X(\theta)}{1 - \theta}$$

Note also that f_X is convex, since second derivative is positive.

Size of n^{th} generation Let $(X_r^{(n)})_{n,r=1}^{\infty}$, where n is generation and r is offspring number (index) in n^{th} generation.

Assume $X_r^{(n)}$ are i.i.d. (independent, identically distributed) random variables.

Identically distributed means

$$P(X_n^r = k) = P(X = k)$$

Independence means

$$P(\forall i < J \ X_{r_i}^{n_i} = k) = \prod_{i=1}^J P(X_{r_i}^{n_i} = k)$$

Define $z_1 = X_1^1$. $z_2 = \sum_{r=1}^{z_1} X_r^2$ and so on:

$$z_{n+1} = \sum_{r=1}^{z_n} X_r^n$$

We want to study asymptotics of z_n .

Given U and V taking values in \mathbb{Z}^+ ,

$$\mathbb{E}[U|V = k] = \sum_{j=0}^{\infty} j P(U = j|V = k)$$

, where

$$P(U = j|V = k) = \frac{P(U = j, V = k)}{P(V = k)}$$

If U, V are independent, $P(U = j|V = k) = P(U = j)$ and thus $\mathbb{E}[U|V = k] = \mathbb{E}U$.

Definition 1.2. Define random variable $\mathbb{E}[U|V]$ such that

$$\mathbb{E}[U|V] = \mathbb{E}[U|V = k]$$

if $V = k$.

Definition 1.3 (Tower property).

$$\mathbb{E}[\mathbb{E}[U|V]] = \mathbb{E}U$$

Define

$$f_n = \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \theta^k P(z_n = k) = \mathbb{E}\theta^{z_n}$$

Theorem 1.1.

$$f_{n+1}(\theta) = f_n(f_X(\theta))$$

or

$$f_n(\theta) = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}(\theta)$$

Proof. Use tower property with $U^{z_{n+1}}$ and $V = \theta^{z_n}$. By tower property

$$\mathbb{E}[\theta^{z_{n+1}}] = \mathbb{E}[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]]$$

$$\mathbb{E}[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]] = \sum_{k=0}^{\infty} P(z_n = k) \mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k]$$

What is $\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k]$?

$$\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n} = k] = \mathbb{E}[\theta^{\sum_{j=1}^k X_j^{n+1}}|\theta^{z_n} = k] \stackrel{\text{independence}}{=} \mathbb{E}[\theta^{\sum_{j=1}^{z_n} X_j^{n+1}}] \stackrel{\text{independence}}{=} \prod_{j=1}^k \mathbb{E}[\theta^{X_j^{n+1}}] \stackrel{\text{i.d.}}{=} (f_X(\theta))^k$$

Thus

$$\mathbb{E}[\mathbb{E}[\theta^{z_{n+1}}|\theta^{z_n}]] = \sum_{k=0}^{\infty} P(z_n = k) (f_X(\theta))^k = f_n(f(\theta))$$

Also we can say

$$\mathbb{E}[\theta^{z_{n+1}}|z_n] = (f_X(\theta))^{z_n}$$

□

Study of z_n What is $\pi_n = P(z_n = 0) = f_n(0) = f(\pi_{n-1})$, probability that population is extinguished. Since $z_{n-1} = 0 \Rightarrow z_n = 0$, i.e. π_n is non-decreasing.

Let $P(z_n = 0 \text{ for some } n) = \pi$.

We hope that $\{z_n = 0\}$ such that

$$\bigcup_n \{z_n = 0\} = \{z_n = 0 \text{ for some } n\}$$

i.e., $\pi = \lim_{n \rightarrow \infty} \pi_n$. We call π the extinction probability.

Theorem 1.2. If $\mu = \mathbb{E} > 1$ then π is a unique root of $\pi = f(\pi)$ and $\pi \in [0, 1]$. If $\mu \leq 1$, $\pi = 1$.

If we look at $f(\pi)$ and π , they intersect in 1, and they can intersect in two points since $f(x)$ is convex. There is second intersection iff $f'(1) = \mu > 1$.

Construction of X_n^r Construct set Ω , $f_{n,r} : \Omega \rightarrow \mathbb{Z}^+$ and \mathcal{F} a collection of subsets of Ω with $P : \mathcal{F} \rightarrow [0, 1]$.

Let $\Omega = \mathbb{Z}^+ \times \mathbb{Z}^+$, $\mathcal{F} = \{0, 1\}^\Omega$.

The problem is when we have infinitely number of variables.

Example Example of not well-behaved triple (Ω, \mathcal{F}, P) . $\Omega = \mathbb{N}$. Now $\mathcal{F} = \{C \subset \mathbb{N} : C \text{ has density}\}$. C has density means

$$\frac{|C \cap \mathbb{N}|}{n} \xrightarrow{n \rightarrow \infty} \rho(C)$$

However, for $C(m) = \{1, 2, \dots, m\}$, $\forall m \quad \rho(C_m)$, and

$$\rho\left(\bigcup C_m\right) = 1$$

Thus $(\mathbb{N}, \mathcal{F}, \rho)$ is not a good probability space, since it doesn't fulfill this $\pi_n \rightarrow \pi$ property. Note we can define other probabilities on naturals, for example

$$P(\{i\}) = 2^{-i}$$

Asymptotics of z_i Assuming $\pi \in (0, 1)$, what is behavior of z_n ?

Definition 1.4. z_n is a Markov chain if

$$P(z_{n+1} = j | z_i = k_i \quad \forall i \leq n) = P(z_{n+1} = j | z_n = k_n)$$

We can use to compute expectation:

$$\mathbb{E}[z_{n+1} | z_i = k_i \quad \forall i < n] = E[z_{n+1} | z_n = k_n]$$

Then, since $E\left[\sum_{i=1}^J X_i^n\right] = J\mu$

$$E[z_{n+1} | z_n] = \mu z_n$$

Let $M_n = \frac{z_n}{\mu^n}$ then $\mathbb{E}[M_n] = 1$. Also

$$\mathbb{E}[M_{n+1} | z_0, \dots, z_n] = M_n$$

This is a definition of martingale with respect to z_0, \dots, z_n .

Let (Ω, \mathcal{F}, P) we say S happens almost surely (a.s.) if

$$P(\{w \in \Omega : S \text{ is true for } w\}) = 1$$

Theorem 1.3 (Martingale convergence theorem). If M_n is a positive martingale then $\lim_{n \rightarrow \infty} M_n = M_\infty$ exists a.s. and

- $\mu \leq 1$. $M_\infty = 0$ a.s. That means $\mathbb{E}M_\infty = 0$ but $\mathbb{E}M_n = 1$, i.e.,

$$\mathbb{E}\left[\liminf_{n \rightarrow \infty} M_n\right] < \liminf_{n \rightarrow \infty} \mathbb{E}[M_n]$$

- $\mu > 1$. If $M_\infty > 0$ with positive probability then $z_n \sim \mu^n M_\infty$.

Lemma 1.1 (Fatov's lemma).

$$\mathbb{E}\left[\liminf_{n \rightarrow \infty} M_n\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[M_n]$$

Theorem 1.4.

$$\mathbb{E}[M_\infty] = 1 \iff \mu > 1 \quad \text{and} \quad \mathbb{E}[X \log(X)] < \infty$$