114074 - Physics 1P

Noam Soker

January 27, 2017

Abstract

1 Intro

What is mass? Mass is kind of energy.
Course goals:

1. Pleasure of lecturer

2. Recognize definitions in mechanics

3. Recognize physical ideas

Energy is a conserved value as a result of time symmetry.

2 Mechanics

Movement of bodies
Galileo Galilei (1564-1642)
Nicolaus Copernicus (1473-1543)
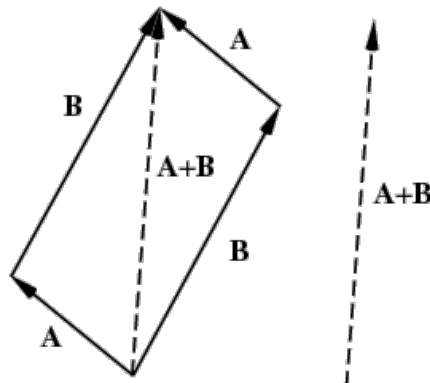Isaac Newton (1642-1727)
  Johannes Kepler
  Tycho Brahe
Albert Einstein

2.1 vectors

Vector is variable with value and direction.

- $\vec{A}$ - vector A

- $|\vec{A}|$ or A - value of vector

- $\hat{A}$ - unit vector in direction of A

Vector sum Both of vectors must be same unit. $\vec{A} + \vec{B}$

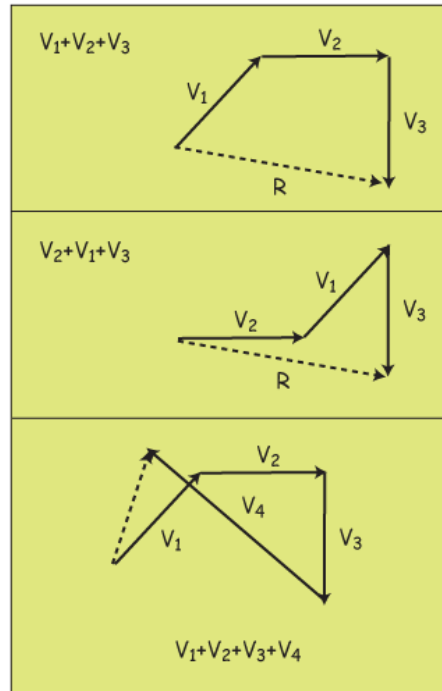$\vec{A} + \vec{B} = \vec{A} + \vec{B}$ - Commutativity

Vector subtraction

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

$-\vec{B}$ - opposite direction

$$\vec{A} + \vec{B} = -(\vec{A} + \vec{B})$$

Vector sum of multiple vectors   Order isn't important! $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ - Associativity

Multiplication by scalar $k \cdot (\vec{A} + \vec{B}) = k \cdot \vec{A} + k \cdot \vec{B}$ - Distributive
A physical quantity must fulfill following conditions to be a vector:

1. To have value and direction independent on coordination system

2. Satisfy parallelogram law of vector sum (Commutativity)

Counter example to 2:   Define turn of 90° around an axis according to right hand. Turn around x and then y is not equal to turn y and then x. $\vec{A} + \vec{B} \neq \vec{A} + \vec{B}$

Vector multiplication   Don't have to be same unit.

1. Scalar multiplication (dot product, inner product) - result is scalar. For $\vec{A}, \vec{B}$ with $\angle \vec{A}\vec{B} = \theta$ then $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = AB \cos \theta$. Product of vector A on projection of vector B on it.
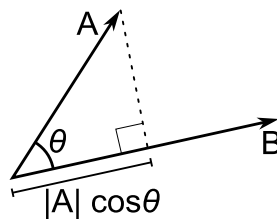
For perpendicular vectors scalar product is 0.

Using components (Cartesian coordinate system): Define unit vector and axis(x, y, z). Take vector $\vec{A}$ with angle θ to x. Take projection of $\vec{A}$ on axis: A_x, A_y, A_z. Then $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$.

$$\vec{A} \cdot \hat{x} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot \hat{x} = A_x\hat{x} \cdot \hat{x} + A_y\hat{y} \cdot \hat{x} + A_z\hat{z} \cdot \hat{x} = A_x + 0 + 0 = A_x$$

2

$$A = \sqrt{\vec{A} \cdot \vec{A}} = ((A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}))^{\frac{1}{2}} = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{A_x}{|\vec{A}|} \cdot \hat{x}$$

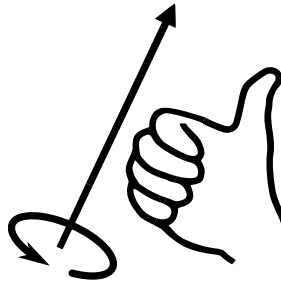$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

$$\hat{A} \cdot \hat{B} = \frac{\vec{A}}{A} \cdot \frac{\vec{B}}{B} = \frac{\vec{A} \cdot \vec{B}}{A \cdot B} = \frac{AB\cos\alpha}{AB} = \cos\alpha$$

2. Vector product (cross product) - $\vec{A} \times \vec{B}$

$$\vec{C} = \vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$
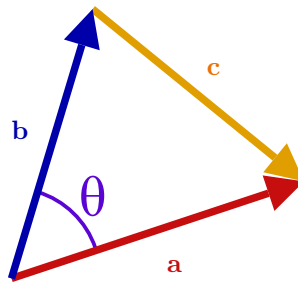
$$\vec{C} = \hat{c}AB\sin\alpha$$

Right hand rule: Four fingers in direction of shortest path between $\vec{A}$ and $\vec{B}$, then thumb shows direction of $\vec{C}$.

$$\vec{A} \times \vec{A} = 0$$

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Law of cosine

$$\mathbf{c} \quad = \quad \mathbf{a} \quad \text{âĹŠ} \, \mathbf{b}$$

$$\vec{C} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

$$(\vec{A} - \vec{B}) \cdot (\vec{A} - \vec{B}) = \vec{C} \cdot \vec{C}$$

$$A^2 + B^2 - 2AB\cos\alpha = C^2$$

Power Rate of doing work

$$W = LF\cos\alpha = \vec{L} \cdot \vec{F}$$

In a short time dt a body passed a short distance dL. So $dW = d\vec{L} \cdot F$

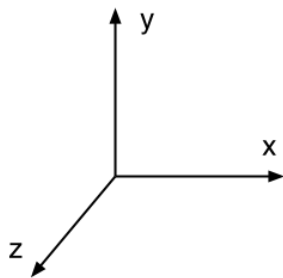Then $power = \frac{dW}{dt} = \frac{d\vec{L}}{dt} \cdot \vec{F} = \vec{v} \cdot \vec{F}$

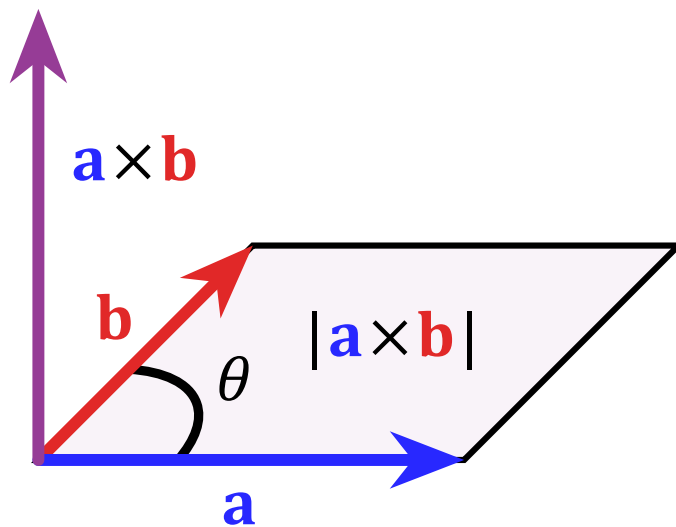**Units** Units are:

Force - N

Energy - J

Power - W

2.2   Cartesian coordinate system

$$\vec{A} \times \vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) =$$
$$= A_xB_x(\hat{x} \times \hat{x}) \qquad +A_xB_y(\hat{x} \times \hat{y}) + A_xB_z(\hat{x} \times \hat{z})+$$
$$+A_yB_x(\hat{y} \times \hat{x}) + A_yB_y(\hat{y} \times \hat{y}) + A_yB_z(\hat{y} \times \hat{z})+$$
$$+ A_zB_x(\hat{z} \times \hat{x}) + A_zB_y(\hat{z} \times \hat{y}) + A_zB_z(\hat{z} \times \hat{z}) =$$
$$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x + A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$
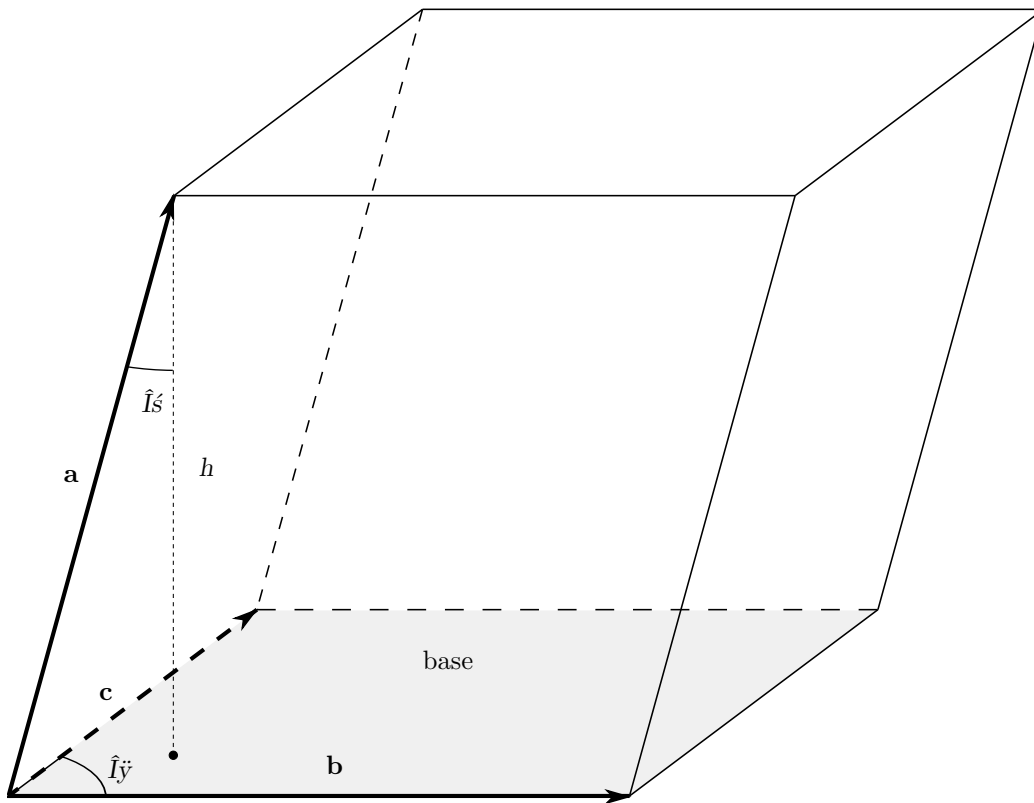
2.3   Usages

Area of parallelogram

h is an altitude.
$S = ah = ab\sin\theta = \mid \vec{a} \times \vec{b} \mid$
$\vec{S} = \vec{a} \times \vec{b}$
If there is a curved surface area vector is perpendicular to surface in every point.

Parallelepiped volume

Two equal parallel parallelogram defined with $\vec{b}, \vec{c}$ and $\vec{a}$ is a vector between them.

$\vec{S} = \vec{b} \times \vec{c}$

$h = \mid \vec{a} \mid \cos\alpha$

$V = S \cdot h = \mid \vec{S} \cdot \vec{a} \mid = \mid (\vec{b} \times \vec{c}) \cdot \vec{a} \mid$

Same order (a,b,c), no matter which are used for cross product gives same result: $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{b} \cdot (\vec{c} \times \vec{a})$. However different order (b,a,c), e.g. $(\vec{a} \times \vec{c}) \cdot \vec{b}$ will give opposite sign.

Law of sines $\vec{C} = \vec{A} + \vec{B}$

$\vec{A} \times \vec{C} = \vec{A} \times \vec{A} + \vec{A} \times \vec{B} = \vec{A} \times \vec{B}$

$\mid \vec{A} \times \vec{C} \mid = \mid \vec{A} \times \vec{B} \mid$

$AC \sin\beta = AB \sin\gamma$

$\frac{C}{\sin\gamma} = \frac{B}{\sin\beta}$

Rate of covering of area   A stick of length L moves with velocity $\vec{v}$. $\vec{l}$ is perpendicular to stick and $\mid \vec{l} \mid = l$ and $\angle \vec{l}\vec{v} = \theta$
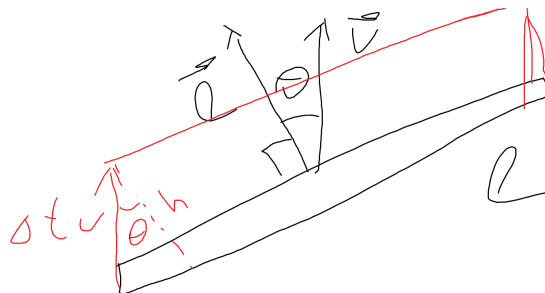
What is area covered by a stick in time Δt?

$$dA = hl = \Delta t v \cos\theta l$$

$$\Delta t \to dt$$

$$\frac{dA}{dt} = vl \cos\theta = \vec{v} \cdot \vec{l}$$

5

Rate of covering of volume   Surface of area $\vec{S} = \vec{a} \times \vec{b}$.

Volume is

$$dV = (vdt)S\cos\theta$$

$$\frac{dV}{dt} = vS\cos\theta = \vec{v} \cdot \vec{S}$$
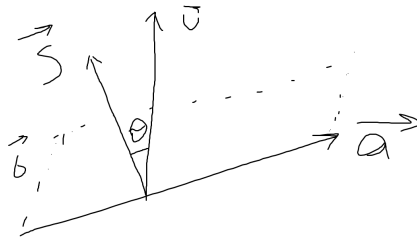
What is mass of air covered by surface per time?

$$\frac{dm_{air}}{dt} = V \cdot \rho_{air} = \vec{S} \cdot \vec{v}\rho_{air}$$

$$\frac{dm_{air}}{dt} = \dot{m}_{air} = \vec{S} \cdot \vec{v}\rho_{air} = \vec{S} \cdot \vec{\Phi}_{air}$$

$\vec{\Phi}_{air} = \left[\frac{kg}{m^2 sec}\right] = $ flux of mass of air.

Torque(moment of force)
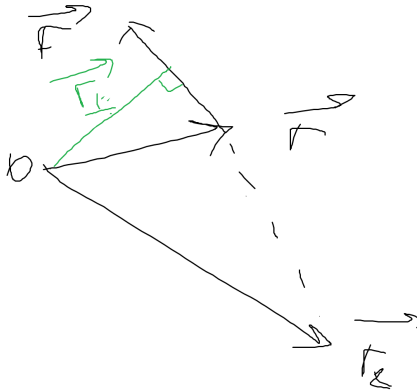
$\vec{N} = \vec{r} \times \vec{F}$
$\vec{F} = \left[\frac{kgm}{s^2}\right]$
$\vec{r} = [m]$
$\vec{N}\left[\frac{kg \cdot m^2}{s^2}\right]$

$$\vec{N} = \vec{r} \times \vec{F} = \vec{r}_1 \times \vec{F} = \vec{r}_\perp \times \vec{F}$$

3   Derivatives of vector

Derivative of position of body by time is velocity

$$\vec{v} = \frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t}$$
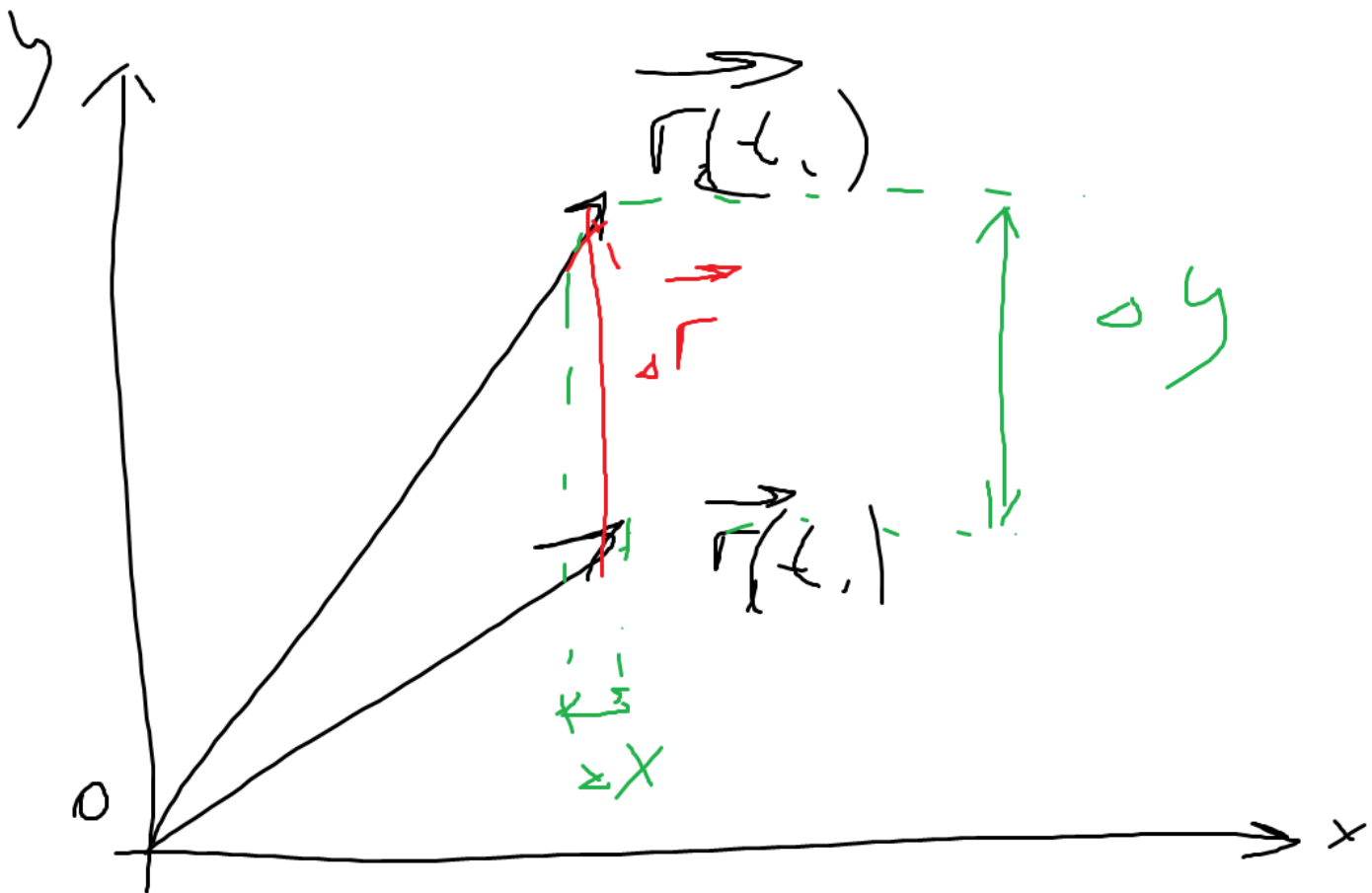
$$\Delta \vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$ where $v_x = \frac{dx}{dt}$

When coordinate system doesn't change with time

3.1   Coordinate system changing in time
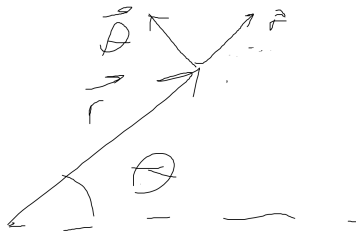
$$\vec{r} = r \cdot \hat{r}$$

$$\left(\text{☆}\right) \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

Circle movement
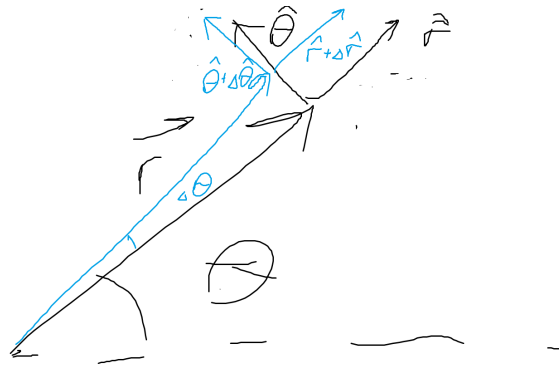
Lets find $\Delta\hat{r}$ and $\Delta\hat{\theta}$. Move parallel beginnings of $\Delta\hat{r}$ to beginning of $\hat{r}$:

$$\sin\alpha \simeq \alpha$$
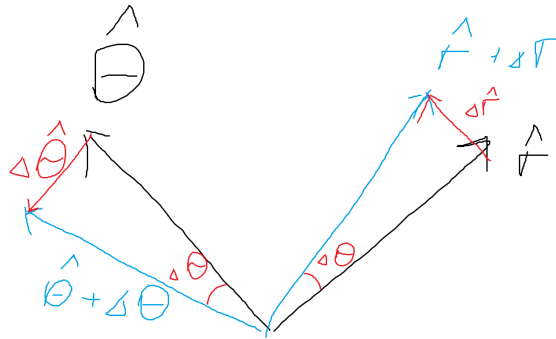
$$\Delta\hat{r} \parallel \hat{\theta} \perp \hat{r}$$

$$|\Delta\hat{r}| = |\hat{r}|\sin\Delta\theta = 1 \cdot \Delta\theta$$

Let's show that change in $\hat{r}$ is perpendicular to $\hat{r}$:

$$\frac{d}{dt}(1) = 0 = \frac{d}{dt}(\hat{r}\cdot\hat{r}) = \frac{d\hat{r}}{dt}\cdot\hat{r} + \hat{r}\cdot\frac{d\hat{r}}{dt}$$

$$\hat{r}\cdot\frac{d\hat{r}}{dt} = 0$$

$$\hat{r} \perp \frac{d\hat{r}}{dt}$$

So $\Delta\hat{r} = (\Delta\theta)\cdot\hat{\theta}$

$$\frac{\Delta\hat{r}}{\Delta t} = \frac{(\Delta\theta)\cdot\hat{\theta}}{\Delta t}$$

When $\Delta t \to 0$

$$\frac{d\hat{r}}{dt} = \frac{(d\theta)\cdot\hat{\theta}}{dt}$$

Denote:

$$\omega = \frac{d\theta}{dt}$$

$$(\star)\dot{\hat{r}} = \frac{d\hat{r}}{dt} = \omega\hat{\theta}$$

$\Delta\hat{\theta}$ Similarly, direction of $\Delta\hat{\theta}$ is $-\hat{r}$ and $|\Delta\hat{\theta}| = \Delta\theta$

$$(\star\star)\dot{\hat{\theta}} = \frac{d\hat{\theta}}{dt} = -\frac{d\theta}{dt}\hat{r} = -\omega\hat{r}$$

$$\left(\text{☆☆}\right)\dot{\vec{r}} = \frac{dr}{dt}\cdot\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}\cdot\hat{r} + r\frac{(d\theta)}{dt}\cdot\hat{\theta}$$

Acceleration

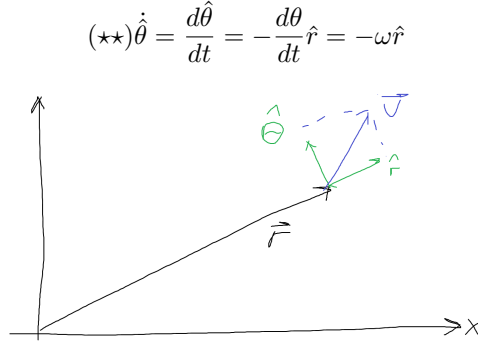$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

Cartesian:

$$\vec{a} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z}$$

Polar:

$$\vec{a} = \frac{d\vec{v}}{dt} \overset{\text{☆☆}}{=} \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}\right) = \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\cdot\frac{d\hat{r}}{dt} + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} + r\frac{d\theta}{dt}\frac{d\hat{\theta}}{dt} =$$

$$= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\cdot\hat{\theta} + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\hat{\theta} + r\frac{d^2\theta}{dt^2}\hat{\theta} - r\frac{d\theta}{dt}^2\hat{r} = \left[\frac{d^2r}{dt^2} - r\frac{d\theta}{dt}^2\right]\hat{r} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\theta}$$

$$\vec{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \frac{1}{r}\left[\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right]\hat{\theta}$$

3.2   Check common cases

1. $\frac{d\theta}{dt} = 0$

$$\vec{a} = \left[\frac{d^2r}{dt^2} - 0\right]\hat{r} + \frac{1}{r}\left[\frac{d}{dt}(0)\right]\hat{\theta} = \frac{d^2\vec{r}}{dt^2}$$

2. $r = const$ and $\frac{d\theta}{dt} = \omega$

$$\vec{a} = \left[0 - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{r} + \frac{1}{r}[0]\hat{\theta} = -r\omega^2\hat{r}$$

Circle motion $r = const$ and $\frac{d\theta}{dt} = \omega = const$
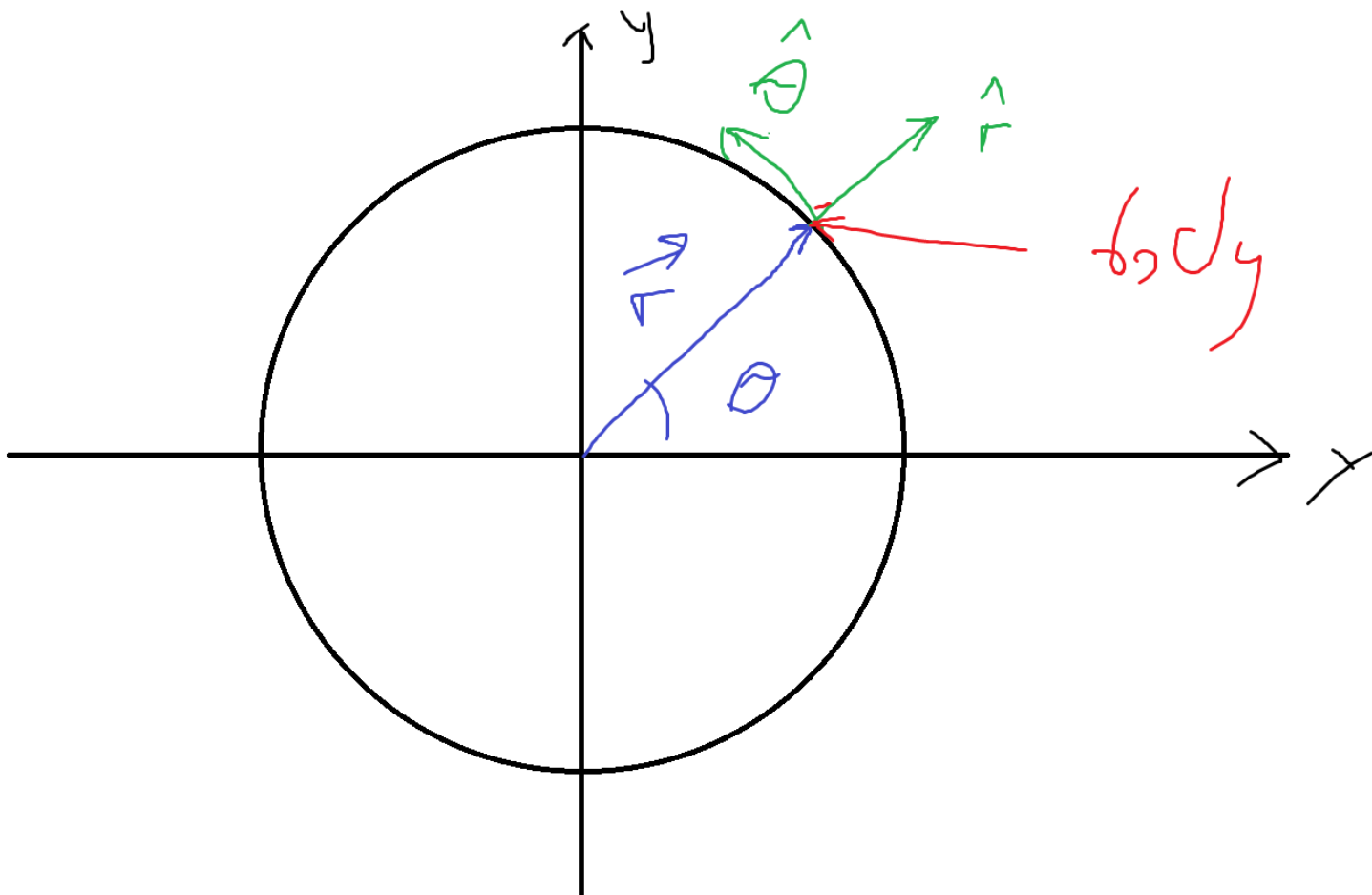
$$\vec{v} = r\frac{d\theta}{dt}\hat{\theta} = r\omega\hat{\theta}$$

Let at $t = 0$ $\theta = 0$. At different time t exist following connections:

$$\theta = \omega t$$

In Cartesian:

$$\vec{r} = x(t)\hat{x} + y(t)\hat{t} = r\cos\theta\hat{x} + r\sin\theta\hat{y}$$

Then:

$$\vec{r} = r\cos(\omega t)\,\hat{x} + r\sin(\omega t)\,\hat{y}$$

$$\vec{v} = \underbrace{-\omega r\sin(\omega t)}_{v_x}\,\hat{x} + \underbrace{\omega r\cos(\omega t)}_{v_y}\,\hat{y}$$

$$v = \sqrt{v_x^2 + v_y^2} = \omega r$$

$$\vec{a} = -\omega^2 r\cos(\omega t)\,\hat{x} - \omega^2 r\sin(\omega t)\,\hat{y} = \omega^2 r\left[\cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y}\right] = \omega^2 r\hat{r}$$

4 Newton laws

Isaak Newton(1643-1727)

4.1 First law of Newton

A body at rest or moving with constant velocity will keep its velocity unless there is force acting on body

4.2   Second law of Newton

The rate of change of momentum of a body, is directly proportional to the force applied.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

If mass is constant:

$$\vec{F} = m\vec{a}$$

m is inertial mass or mass.

4.3   Third law of Newton

When 2 bodies interact, the force first body exerts on the second is equal in magnitude and opposite in direction to force second body exerts on the first.

Notes

1. Two forces are exerted on different bodies

2. This law is connected to conservation of momentum

3. Wrong in special relativity

Example

$$m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2r}{dt^2} = m\ddot{\vec{r}}$$

If force is zero:

$$m\frac{d\vec{v}}{dt} = 0$$

$$\vec{v} = \vec{v_0} = const$$

$$\frac{d\vec{r}}{dt} = \vec{v_0}$$

$$d\vec{r} = \vec{v_0}dt$$

Find integral:

$$\int_{\vec{r}_{beg}}^{\vec{r}_{end}} d\vec{r} = \int_{t_{end}}^{t_{beg}} \vec{v}dt$$

$$\left[\vec{r}\right]_{\vec{r}_0}^{\vec{r}} = \left[\vec{v_0}t\right]_{\vec{t}_0}^{\vec{t}}$$

$$\vec{r} - \vec{r}_0 = \vec{v}_t - \vec{v}_0 t$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 \left(t - t_0\right)$$

4.4   Movement

Movement is an equation describing position of body changing with a time

Second law

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}} = m\frac{d^2\vec{r}}{dt^2}$$

It's differential equation.

Example   Constant acceleration on surface Earth.

$$\vec{F} = -mg\hat{y}$$

Using second law for a constant mass:

$$\underbrace{m}_{\substack{\text{gravitational} \\ \text{mass}}} \frac{d^2\vec{r}}{dt^2} = - \underbrace{m}_{\substack{\text{inertial} \\ \text{mass}}} g\hat{y}$$

$$-g\hat{y} = \hat{x}\frac{d^2x}{dt^2} + \hat{y}\frac{d^2y}{dt^2}$$

$$\frac{dx^2}{dt^2} = 0$$

$$\frac{d^2y}{dt^2} = -g$$

Solution is:

$$x_0 = x_0 + v_{x0}t$$

$$y = y_0 + v_{y0}t + -\frac{1}{2}gt^2$$

$$v_y = v_{y0} - gt$$

Gravitational field $-g\hat{y}$
Examples:

$$Gravitational\ force \quad \vec{F} = \underbrace{m}_{\substack{\text{gravitational} \\ \text{mass}}} \cdot \underbrace{(-g\hat{y})}_{\substack{\text{at Earth's} \\ \text{surface}}}$$

$$Electircal\ force \quad \vec{F} = q\vec{E}$$

Solution of movement equations

$$x = x_0 + (v_0 \cos\theta)t$$
$$y = y_0 + (v_0 \sin\theta)t - \frac{1}{2}gt^2$$

Equation of the form of trajectory:

$$y - \left(y_0 + \frac{v_0^2 \sin^2\theta}{2g}\right) = -\frac{g}{2v_0^2 \cos^2\theta} \cdot \left[x - \left(x_0 + \frac{v_0^2 \sin\theta\cos\theta}{g}\right)\right]$$

Maximal height:

$$h_{max} = y_m - y_0 = \frac{v_0^2 \sin^2\theta}{2g}$$

It's possible to get it from energy too:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + mgh$$

$$\frac{1}{2}m(v_{0x}^2 + v_{0y}^2) = \frac{1}{2}m(v_x^2 + v_y^2) + mgh$$

$v_x = v_{0x} = const$ and $v_y = 0$

12

$$\frac{1}{2}mv_{0y}^2 = mgh$$

$$\frac{v_{0y}^2}{2g} = h$$

Range (horizontal distance when returning to same height):

$$R = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}$$

Max range:

$$R \stackrel{\theta=\frac{\pi}{4}}{=} \frac{v_0^2}{g}$$

4.5 Gravitational law of Newton

Force between 2 masses:

$$\vec{F} = -\hat{r}\frac{Gm_1 m_2}{r^2}$$

$G = 6.67 \cdot 10^{-11} \frac{Nm^2}{kg^2} = 6.67 \cdot 10^{-8} \frac{dyn \cdot cm^2}{g^2}$

Electromagnetic force

$$Electrical\ force \quad \vec{F} = q\vec{E} = \hat{r}q\frac{kQ}{r^2}$$
$$Magnetic\ force \quad \vec{F} = q\vec{V} \times \underbrace{\vec{B}}_{\substack{\text{magnetic} \\ \text{field}}}$$

$Field = \frac{force}{charge}$

Interaction	Particle	Mass	Influences on
Gravitation	graviton(?)	0	Planets and stars
Weak	z^0	$99m_p$	decay of neuron
	w^+	$88m_p$	
	w^-	$88m_p$	
Electromagnetism	photon	0	Most of interactions
Strong	gluon	0	Protons in core

Work   Work is change of energy. Denote energy as H, work in one dimension x. F is force, then

$$dH = dxF_x$$

Multiply and divide by small time:

$$dH = \frac{dx}{dt}F_x dt = \frac{dx}{dt}dp_x$$

$F_x dt = dp_x$ - is impulse
Divide by dp_x

$$\frac{dH}{dp_x} = \frac{dx}{dt}$$

5 Frames of reference

Two first laws of Newton are valid only in non-accelerating (inertial) frame of reference. Examples of accelerating frames of references:

- carousel

- accelerating car

Denote inertial system as (I)

Results:

- In accelerating system body is in rest if force is on it giving him acceleration opposite to acceleration of system

- Inertial system is defined as system in which a body for which $\vec{F} = 0$ has $\vec{a} = 0$, where $\vec{a}$ is relative to the system.

Imaginary forces Second law of Newton: $\vec{F} = m\vec{a}_I$
a - acceleration of non-inertial system
a_0 - acceleration of a particle of mass m in accelerating system.

$$a_I = \vec{a} + \vec{a}_0$$

$$\vec{F}_I = m\vec{a}_I = m(\vec{a} + \vec{a}_0)$$

$$\vec{F}_I - m\vec{a}_0 = m\vec{a}$$

Imaginary force: $\vec{F}_0 = -m\vec{a}_0$
Acceleration of a particle relative to accelerating system:

$$m \cdot \underbrace{\vec{a}}_{\substack{\text{acceleration} \\ \text{in non-inertial} \\ \text{system}}} = \underbrace{\vec{F}_I}_{\substack{\text{real force} \\ \text{exerted by car}}} + \underbrace{\vec{F}_0}_{\substack{\text{imaginary force} \\ \text{not exerted by} \\ \text{car}}}$$

Example An elevator accelerating with acceleration a_1. $a_0 = a_1$ is acceleration of the system
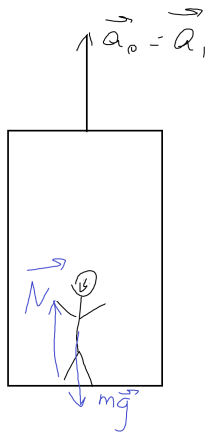
Relative to elevator a person has no acceleration: $\vec{a} = 0$.

$$m\vec{a} = \vec{F}_I + \vec{F}_0$$

$$\vec{a} = 0$$

$$F_I = N - \underbrace{m}_{\substack{\text{gravitational} \\ \text{mass}}} \cdot g$$

14

$$F_0 = - \underbrace{m}_{\substack{\text{inertial} \\ \text{mass}}} \cdot a_1$$

$$0 = N - mg - ma_1$$

$$N = m(g + a_1)$$

We can't distinguish between acceleration of elevator and increase of free-fall acceleration, i.e. gravitational field and acceleration. Same for free-fall and lack of gravitational field.

Numerical example for elevator Elevator accelerates with $\vec{a}_1 = -9.8ms^{-2}$. Then $N = m(g + a_1) = m(9.8 - 9.8) = 0$, i.e. you fill weightlessness.

Example Rotating system. $\vec{\omega}$ is angular velocity perpendicular to the surface of rotation (right hand), i.e. $\odot$

Mass m is in rest relative to rotating system. $\vec{a}_R = 0$ and $\vec{v}_R = 0$
In inertial system it has tangent velocity: $\vec{v}_I = v_I \hat{\theta}$. $\vec{v}_I = \omega r$. r is distance from center of the system. Since $\vec{v}_I \perp \vec{\omega} \perp \vec{r}$:

$$\vec{v}_I = \vec{\omega} \times \vec{r}$$

From an expression for acceleration in polar coordinates:

$$\vec{a}_I = -\hat{r}\omega^2 r = -\hat{r}\frac{v_I^2}{r}$$

15

$$F_I = m\vec{a}_I$$

In rotating system:

$$m\vec{a} = \vec{F}_I + \vec{F}_0$$

Since $\vec{a} = 0$

$$0m = (-\hat{r}\omega^2 r m) + \vec{F}_0$$

$$\vec{F}_0 = +\hat{r} m \omega^2 r$$

F_I is centripetal force - real force inside.
F_0 is centrifugal force - imaginary force out.

5.1

Take a look on expression for acceleration, and try to understand, what each term means:

$$\vec{a} = \left[\underbrace{\frac{d^2 r}{dt^2}}_{\substack{\text{v changes} \\ \text{due to change} \\ \text{of r}}} - \underbrace{r\left(\frac{d\theta}{dt}\right)^2}_{\substack{\text{centripetal} \\ \text{acceleration} \\ \text{even if} \\ \omega = const}} \right]\hat{r} + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + \underbrace{r\frac{d^2\theta}{dt^2}}_{\substack{\text{exists even if} \\ \text{r = const} \\ \text{due to changes} \\ \text{in } \omega}} \right]\hat{\theta}$$

Last term:

$$2\frac{dr}{dt}\frac{d\theta}{dt} = 2 \cdot \left(\substack{\text{change of} \\ \text{radius}} \right) \cdot \omega$$

If there is acceleration in direction θ then there is force in direction $\hat{\theta}$.
If radius increases and there is no forces, then ω decreases to keep $\frac{dr}{dt}\frac{d\theta}{dt}$ constant:

$$a_\theta = 2\frac{dr}{dt} + \frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0 \iff F_\theta = 0$$

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = -\frac{2}{r}\frac{dr}{dt}\frac{d\theta}{dt} = -\frac{2}{r}\frac{dr}{dt}\omega$$

i.e. change in ω is of opposite sign to change in r.
This is connected to conservation of angular momentum.

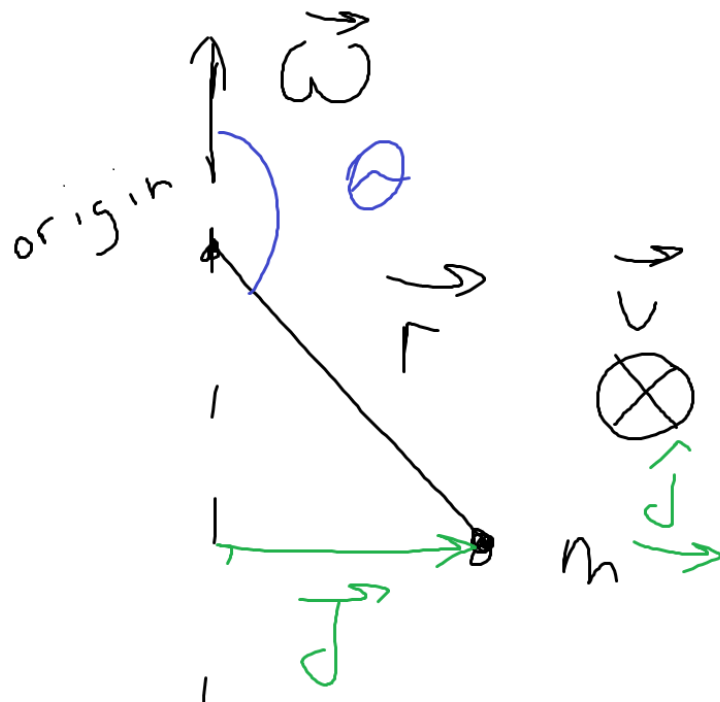**Person in falling elevator** $a_0 = -g$ is system's acceleration
Persons acceleration:

$$\vec{a} = \frac{F_I + F_0}{m} = \frac{F_I}{m} + \frac{F_0}{m}$$

$$\vec{a} = -g + [-(a_0)] = -g + [-(-g)] = 0$$

5.2   Rotating systems

Connections

$$\vec{v} \perp \vec{r} \quad \vec{v} \perp \vec{\omega}$$

$$|\vec{v}| = d\omega = \omega |\vec{r}| \sin\left(\frac{\pi}{2} - \theta\right) = \omega |\vec{r}| \sin\theta$$

$$\vec{v} = \vec{\omega} \times \vec{r}$$

In rotating system there is imaginary force (centrifugal) out of center:

$$\vec{a} = \omega^2 d\hat{d}$$

In inertial system there is real force inside:

$$\vec{a} = -\omega^2 d\hat{d}$$

General expression
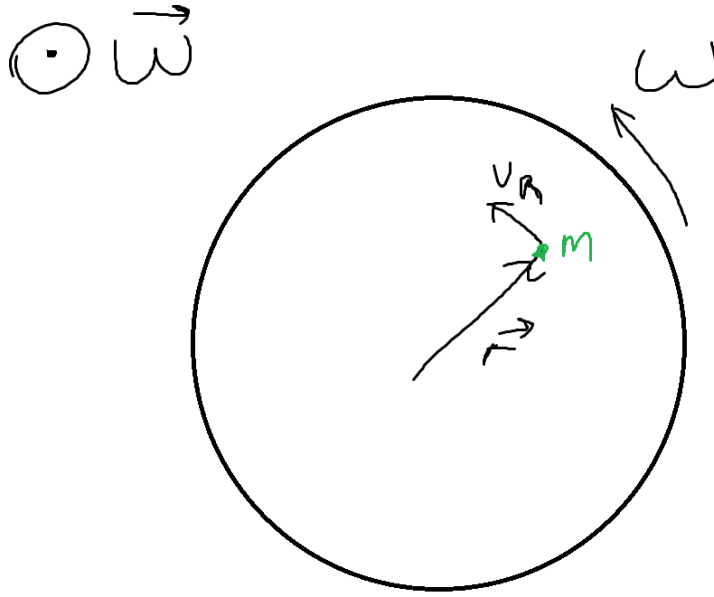
$$|\vec{a}| = \omega^2 d = \omega^2 r \sin\theta = \omega \left(|\vec{\omega} \times \vec{r}|\right) \overset{\substack{\text{since vectors are} \\ \text{perpendicular}}}{=} |\omega \times (\vec{\omega} \times \vec{r})|$$

Direction of centrifugal acceleration   It's imaginary acceleration. Centrifugal acceleration is out of rotation axis and so is perpendicular to $\vec{\omega}$.
In this example it's also perpendicular to $\vec{v}$. So

$$\underbrace{\vec{a}_{cen}}_{imaginary} = -\vec{\omega} \times \vec{v} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$\vec{a}$ should be in direction of $\hat{r}$, i.e. $r = const$ and $\omega = const$

17

1. Rotating system with angular velocity ω

2. Mass m with perpendicular velocity (in the system) v_R.

$$\vec{v}_R = v_R \hat{\theta}$$

Acceleration in inertial system

$$v_I = \underbrace{\omega r}_{\substack{\text{velocity of} \\ \text{a point in} \\ \text{rotating system}}} + \underbrace{v_R}_{\substack{\text{velocity of} \\ \text{mass relative} \\ \text{to a point}}}$$

$$\vec{a}_I = -\omega_I^2 r \hat{r} \overset{\omega_I = \frac{v_I}{r}}{=} -\left(\frac{v_I}{r}\right)^2 r \hat{r} = -\hat{r}\frac{(\omega r + v_r)^2}{r} = \hat{r}\left[\omega^2 r + \frac{v_R^2}{r} + 2\omega v_R\right]$$

A Person in a rotating system sees a mass moving with a velocity v_R. Since relative to him a mass has a tangent speed, he "sees" centripetal acceleration in his frame of reference:

$$\vec{a}_R = -\hat{r}\frac{v_R^2}{r}$$

$$\vec{a}_I = -\hat{r}\frac{v_R^2}{r} - \hat{r}\omega^2 r - \hat{r}2\omega v_r$$

$$\vec{a}_R = -\hat{r}\frac{v_R^2}{r} = \vec{a}_I + \underbrace{\hat{r}\omega^2 r}_{\substack{\text{centrifugal} \\ \text{acceleration}}} + \underbrace{\hat{r}2\omega v_R}_{\substack{\text{Coriolis} \\ \text{acceleration}}}$$
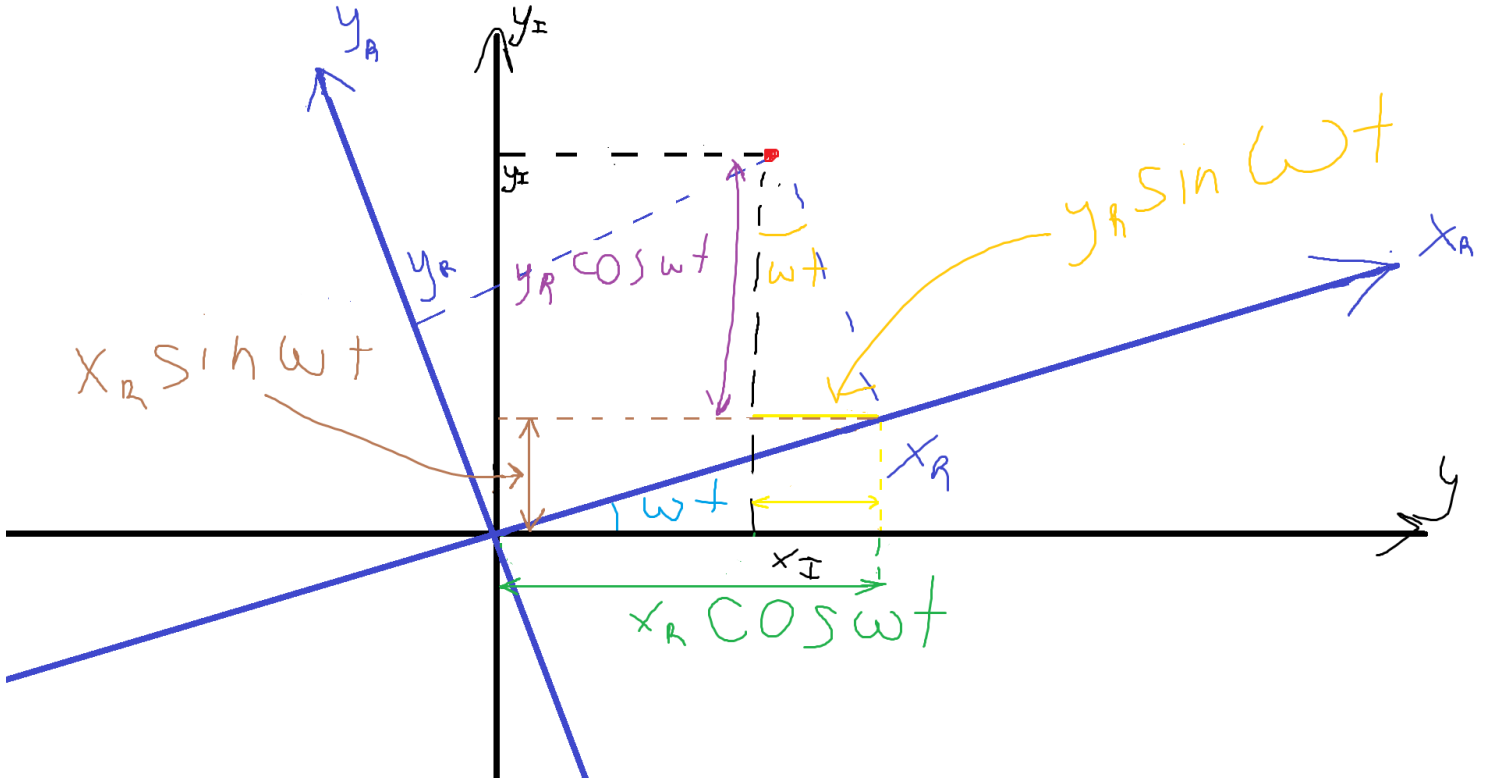
Write according to vectors:

$$\vec{a}_R = \vec{a}_I \underbrace{-\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\substack{\text{centrifugal} \\ \text{acceleration}}} \underbrace{-2\vec{\omega}\times\vec{v}_R}_{\substack{\text{Coriolis} \\ \text{acceleration}}}$$

This result is general for a movement even outside of plane.

5.3 Exact treatment

Given an rotation system R with a constant angular velocity $\vec{\omega} = \omega\hat{z}_I$.
Axis z_R and z_I are same. An angle between x_I and x_R is $\theta = \omega t$.

We can see

$$\begin{cases} x_I = x_R \cos \omega t - y_R \sin \omega t \\ y_I = x_R \sin \omega t + y_R \cos \omega t \end{cases}$$

This is right for any vector and can be used to switch between systems R and I.

$$\begin{cases} (v_R)_{x_I} = \dot{x}_R \cos \omega t - \dot{y}_R \sin \omega t \\ (v_R)_{y_I} = \dot{x}_R \sin \omega t + \dot{y}_R \cos \omega t \end{cases}$$

Note: This isn't velocity in system I, those are components of velocity in system R for axis of system I.

$$\begin{cases} (a_R)_{x_I} = \dot{x}_R \cos \omega t - \dot{y}_R \sin \omega t \\ (a_R)_{y_I} = \ddot{x}_R \sin \omega t + \ddot{y}_R \cos \omega t \end{cases}$$

To find v_I we derive position vector:

$$\frac{d\vec{r}}{dt} = \hat{x}_I \dot{x}_I + \hat{y}_R \dot{y}_R$$

$$\frac{dx_I}{dt} = \dot{x}_I = \underbrace{\dot{x}_R \cos \omega t - \dot{y}_R \sin \omega t}_{(v_R)_{x_I}} \underbrace{-\omega x_R \sin \omega t - \omega y_R \cos \omega t}_{(-\omega y_I)_{x_I}}$$
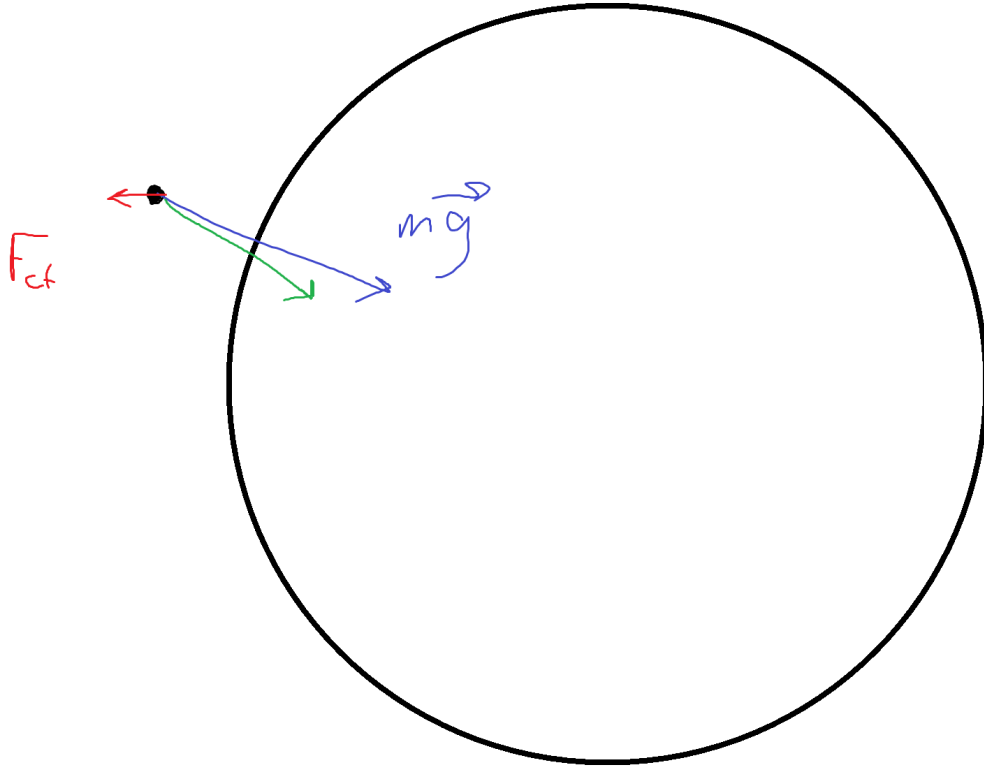
$$\frac{dy_I}{dt} = \dot{y}_I = \underbrace{\dot{x}_R \sin \omega t + \dot{y}_R \cos \omega t}_{(v_R)_{y_I}} \underbrace{+\omega x_R \cos \omega t - \omega y_R \sin \omega t}_{(\omega x_I)_{y_I}}$$

Left term is velocity, right is cross product of ω and $\vec{r}_I$ ($-\omega y_I \hat{x}_I + \omega x_I \hat{y}_I$, ω has only z component, and $\vec{r}$ has only x and y).

$$\vec{v}_I = \left(\frac{d\vec{r}}{dt} \right)_I = \left(\frac{d\vec{r}}{dt} \right)_R + (\vec{\omega} \times \vec{r}_I)_R$$

It is right for any vector. If we replace position with velocity, we acquire an equation for acceleration:

$$\vec{a}_I = \left(\frac{d\vec{v}_I}{dt}\right)_I = \left(\frac{d\vec{v}_I}{dt}\right)_R + (\vec{\omega} \times \vec{v}_I)_R$$

$$= \left\{ \frac{d}{dt}\left[\left(\frac{d\vec{r}}{dt}\right)_R + (\vec{\omega} \times \vec{r})_R \right]\right\}_R + \left\{ \vec{\omega} \times \left[\left(\frac{d\vec{r}}{dt}\right)_R + (\vec{\omega} \times \vec{r})_R \right]\right\}_R \overset{\omega=const}{=}$$

$$= \qquad\qquad \vec{a}_R + \vec{\omega} \times \vec{v}_R + \qquad\qquad \vec{\omega} \times \vec{v}_R + \omega \times (\vec{\omega} \times \vec{v}_R) \qquad\qquad =$$

$$= \vec{a}_R - \vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2\vec{\omega} \times \vec{v}_R$$

Free fall on Earth includes a centrifugal acceleration, so a fall is not exactly in direction to center of Earth, but it is negligible.

Examples

Slow (negligible Coriolis) fall on the surface of the Earth
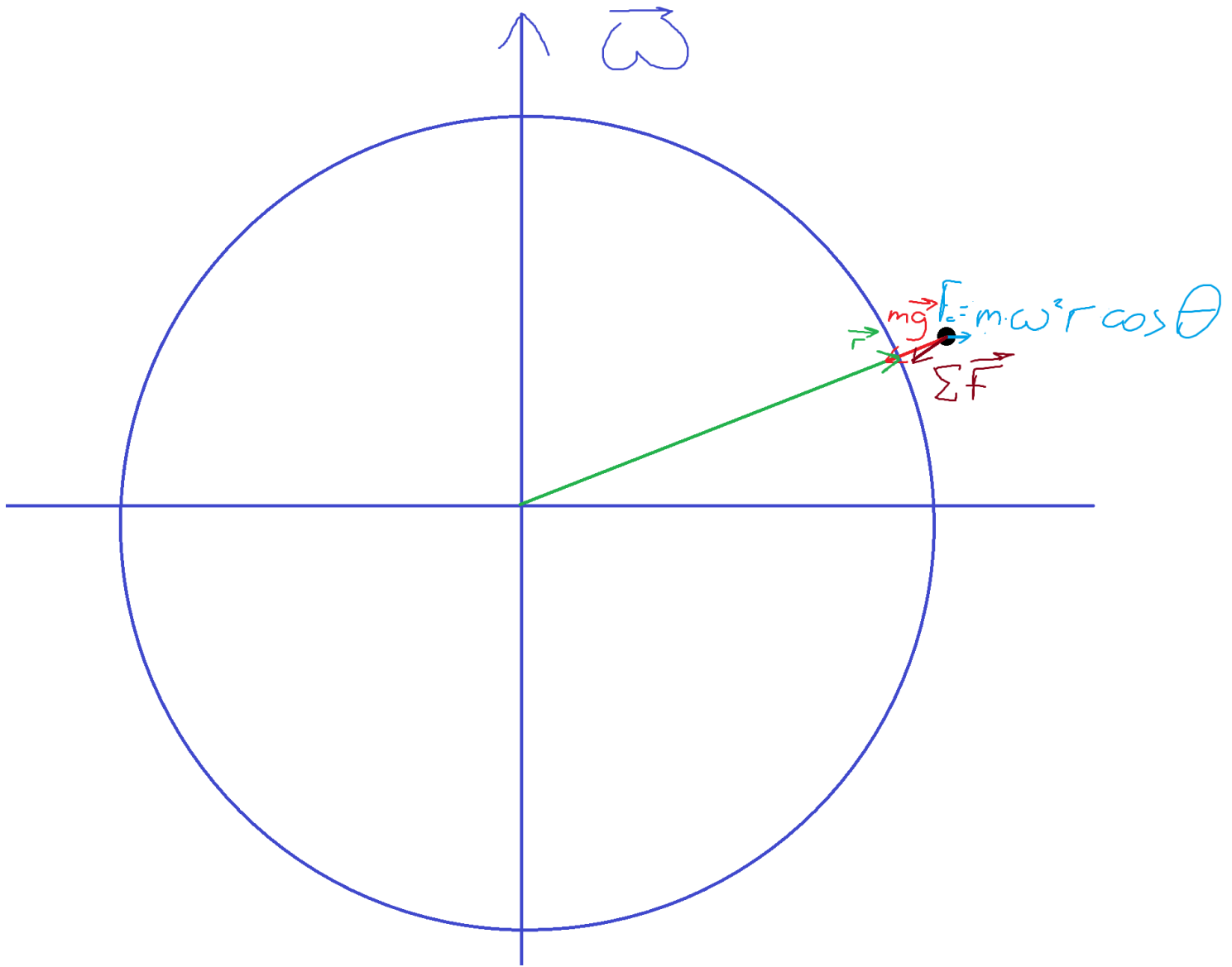
$$mg = 9.8m$$

Then centrifugal acceleration on equator equals to

$$a_c = \omega^2 r = \frac{2\pi}{\underbrace{\left[(23 \cdot 60 + 56) \cdot 60\right]}_{T_{turn}=23h56m}} \cdot \underbrace{6.378 \cdot 10^6}_{Earth's\ radius} = 0.0339ms^{-2}$$

Motion on Earth's surface (Coriolis).

1.
   - At the body moving to the East acts the force in direction $\hat{d}$ - out of rotational axis.
   - At the body moving to the West acts the force in direction $\hat{d}$ - towards rotational axis.

2.
   - For horizontal motion in North hemisphere - Coriolis force's projection on the surface is to the right.
   - For horizontal motion in South hemisphere - Coriolis force's projection on the surface is to the left.

3.
   - At falling body acts Coriolis force to the East (at direction of rotation).
   - At rising body acts Coriolis force to the West.

Vertical throw.
Coriolis also changes the speed of the body, but while the velocity is low it's possible to neglect Coriolis velocity, i.e. $v_R = (v_0 - gt)\hat{r}$ and the height will be $h = v_0 t - \frac{1}{2}gt^2$

$$a_{cor} = -2\vec{\omega} \times \vec{v}_R = 2|\vec{\omega} \times \vec{v}_R| = 2\omega v_R \cos \lambda$$

21

Direction is to the West if $v_R > 0$ else to the East. If we denote by $+x$ direction to the West then:

$$\vec{a}_{cor} = 2\omega v_R \cos \lambda \hat{x}$$

And

$$\vec{v}_{cor} = \int \vec{a}_{cor} dt$$

$$\vec{x} = \int \vec{v}_{cor} dt$$

We can reduce the problem to 1 dimension and substitute v_R:

$$a_{cor} = 2\omega(v_0 - gt) \cos \lambda$$

$$v_{cor} = \int_0^t 2\omega(v_0 - gt')dt' \cos \lambda = 2\omega \cos \lambda \int_0^t (v_0 - gt') \, dt' = 2\omega \cos \lambda \left[v_0 t' - \frac{1}{2} gt'^2 \right]_0^t$$

$$v_{cor} = 2\omega \cos \lambda \left(v_0 t - \frac{1}{2} gt^2 \right)$$

Now find deviation x:

$$\vec{x} = \int_0^{t_f} \vec{v}_{cor} dt$$

$$\vec{x} = 2\omega \cos \lambda \int_0^{t_f} \left(v_0 t - \frac{1}{2} gt^2 \right) dt = 2\omega \cos \left[\frac{1}{2} v_0 t^2 - \frac{1}{6} gt^3 \right]_0^{t_f}$$

Since $t_F = \frac{2v_0}{g}$

$$\vec{x} = 2\omega \cos \lambda \left[\frac{1}{2} v_0 \cdot \left(\frac{2v_0}{g} \right)^2 - \frac{1}{6} g \left(\frac{2v_0}{g} \right)^3 \right]$$

$$x = \frac{4}{3} \omega \cos \lambda \frac{v_0^3}{g^2}$$

Example $v_0 = 200 ms^{-1}$, $\lambda = 32°$
We get $h = 2040.8m \ll R_{Earth}$, then maximal $v_{cor} = 0.252 ms^{-1} \ll v_0$. We obtain deviation at return to the Earth's surface $x = 6.85m$

6   Conservation of energy

$$E_k = \frac{1}{2} mv^2$$

$$\vec{p} = m\vec{v}$$

We start with one dimension y:

$$F = m\ddot{y}$$

Change in velocity is:

$$dv = \ddot{y} dt$$

If v_0 is initial velocity in $t = 0$ then velocity at time t is

$$\int_{v_0}^{v(t)} dv = \int_0^t \ddot{y} dt'$$

$$v = \int_0^t \frac{F}{m} dt' \stackrel{F=const}{=} \frac{F}{m} t$$

We do one more integral to get displacement:

$$y(t) - y_0 = \int_0^t v(t)dt' = \int_0^t v_0 + \frac{F}{m}t' dt' = v_0 t + \frac{1}{2}\frac{F}{m}t^2$$

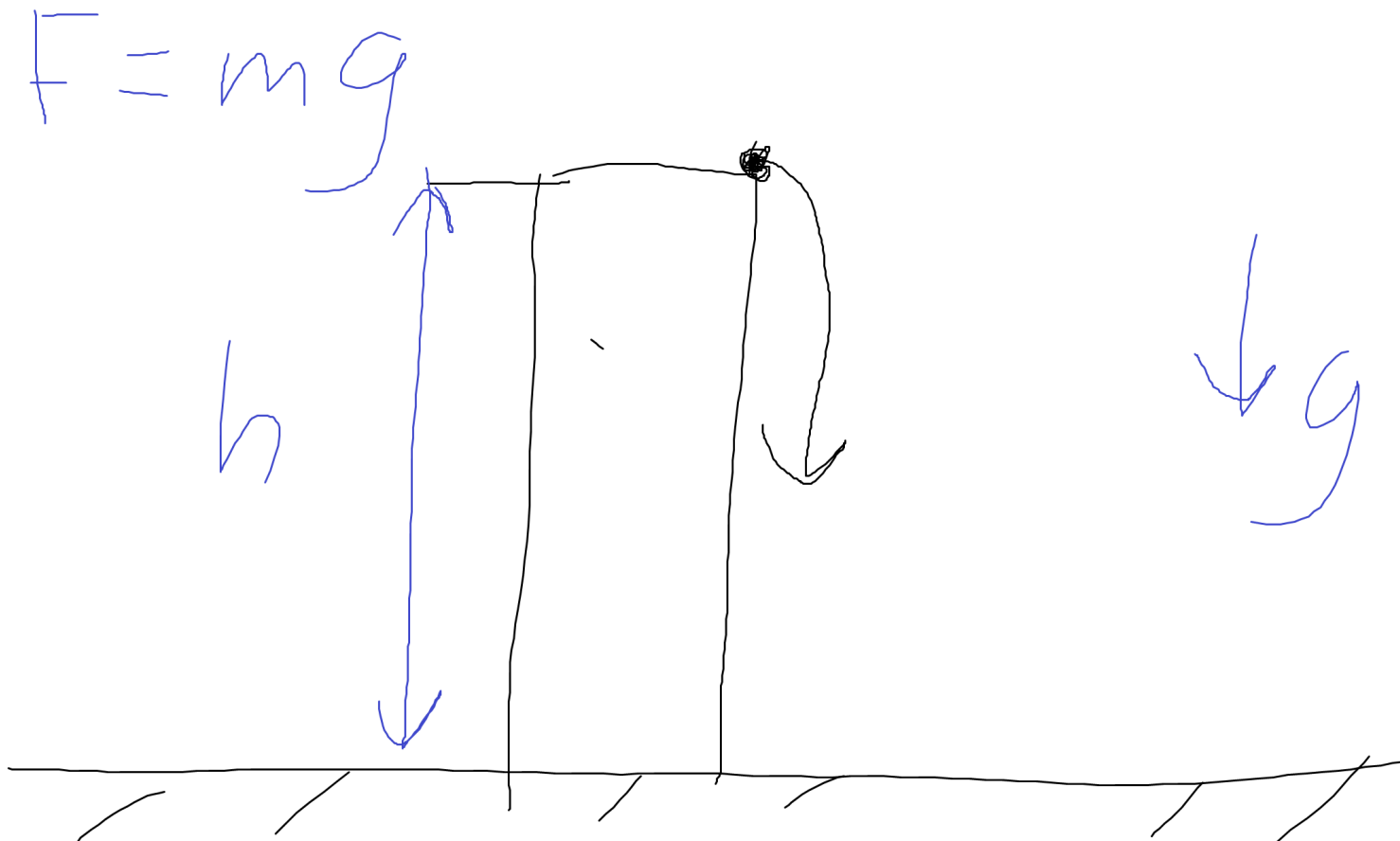From velocity equation we can get $t = \frac{m}{F}(v_t - v_0)$ and then:

$$y - y_0 = \frac{1}{2}\frac{m}{F}\left(v^2 - v_0^2\right)$$

By multiplying we acquire

$$\underbrace{F(y - y_0)}_{work} = \underbrace{\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2}_{change\ in\ energy}$$

6.1 Potential energy

$$W = \underbrace{Fh}_{\substack{work\ of \\ gravitational \\ force}} = \underbrace{mgh}_{\substack{potential \\ energy}}$$

Definition Work done by field's forces.

$$E_{height} = \vec{F}_{grav} \cdot \underbrace{\vec{x}}_{\substack{displacement \\ from\ b\ to\ a}}$$

Definition Potential energy in point b relative to point a is a work needed to move a body from point a to b without acceleration done by force opposite to field's force.

For free-falling body $E = const = mgh = \frac{mv^2}{2} + mgy$ (from conservation of the energy).

Conservation of energy   In a system of particles in which act mutual forces which aren't explicitly dependent on time(explicitly dependent means that constants or law itself change with time), total energy of system is conserved.

Energy conservation $\rightleftharpoons$ symmetry in time

More general results   It is convenient to talk about function of energy:

$$E = \underbrace{K}_{\substack{\text{kinetic} \\ \text{energy}}} + \underbrace{U}_{\substack{\text{potential} \\ \text{energy}}}$$

Example of hight energy as an example:

$$U = mgh \overset{\substack{\text{height energy} \\ \text{equals to work of} \\ \text{force opposite to} \\ \text{field's forces}}}{=} -F_g y = mgy$$

$$U = \left| -\vec{F}_y \right| y = (-\hat{y} F_y) y \hat{y} = F_y y = mgy$$

Derive the equation of energy:

$$0 \overset{\substack{\text{conservation} \\ \text{of energy}}}{=} \frac{dE}{dt} = \frac{d}{dt}(K+U) = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{dU}{dt} = mv\frac{dv}{dt} + \frac{dU}{dy}\frac{dy}{dt} = mva + \frac{dU}{dy} \cdot v$$

We got:

$$0 = v\left(ma + \frac{dU}{dy}\right)$$

If $m \neq 0$

$$ma = -\frac{dU}{dy}$$

This is second law of Newton we derived without usage of forces. The force is:

$$F_g = -\frac{dU}{dy}$$

In 3D we get:

$$\vec{F}_{field} = -\frac{\delta v}{\delta x}\hat{x} - \frac{\delta v}{\delta y}\hat{y} - \frac{\delta v}{\delta z}\hat{z} = -\vec{\nabla}U$$

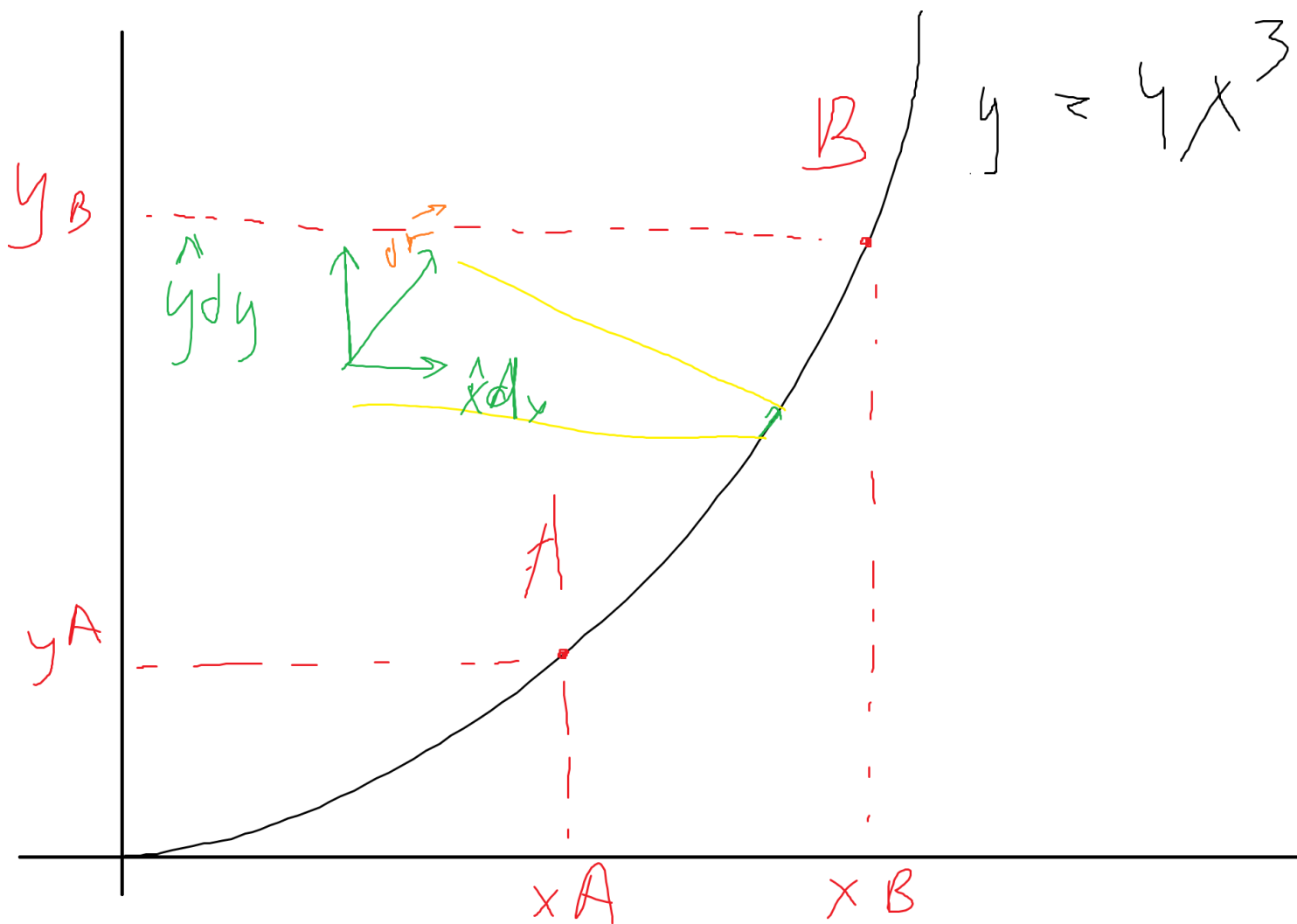**Expanding treatment of force**   Non-constant force in 3D:

$$\underbrace{\Delta W}_{\substack{\text{change} \\ \text{in} \\ \text{work}}} = \vec{F} \cdot \Delta \vec{r} = F\Delta r \cos\alpha$$

Total work is sum of many small intervals:

$$W_{A \to B} = \int_A^B \vec{F} d\vec{r}$$

Example   Position of body is determined by equation $y = 4x^3$. What is work of gravitation in displacement of body from A to B.

$y = 4x^3$

$$\vec{F}_G = mg(-\hat{y})$$

$$W = \int_A^B [mg\,(-\hat{y})] \cdot (\hat{x}dx + \hat{y}dy) = \int_A^B -mgdy \stackrel{\text{limits for x}}{=} \int_{x_A=2}^{x_B=3} (-mg)\,12x^2dx = \left[-mg4x^3\right]_2^3$$

$$\frac{dy}{dx} = 12x^2 \Rightarrow dy = 12x^2dx$$

Kinetic energy   From second law $\vec{F} = m\frac{d\vec{v}}{dt}$.

Mathematical connection $d\vec{r} = \frac{d\vec{r}}{dt}dt = \vec{v}dt$

$$W_{A\to B} = \int_A^B \vec{F} \cdot d\vec{r} = m\underbrace{\int_A^B \frac{d\vec{v}}{dt}}_{\vec{F}} \cdot \underbrace{\vec{v}dt}_{d\vec{r}} = m\int_A^B \frac{1}{2}\left(\frac{dv^2}{dt}\right)dt = m\int_A^B \frac{1}{2}dv^2 = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

Work equals to change in kinetic energy.

$$E_k = \frac{1}{2}mv^2$$

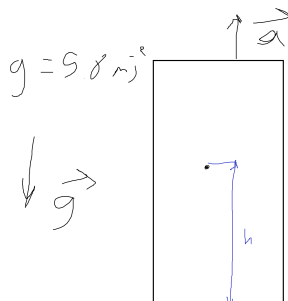$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F}\vec{v}$$

Since

$$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F}\frac{d\vec{r}}{dt}dt = \int \vec{F}\vec{v}dt$$

Example

$$\vec{F}_{cor} = -2\vec{\omega} \times \vec{v}_R m$$

$$P_{cor} = \vec{F}_{cor} \cdot \vec{v}_R \overset{\vec{v}_R \perp \vec{\omega} \times \vec{v}_R}{=} 0$$

Falling mass in elevator

What is final speed of falling mass?

$$g_{eff} = -(a+g)\hat{y}$$
$$v_f = \sqrt{2(g+a)h}$$

What says someone in frame of elevator?

$$U = m(g+a)h = mg_{eff}h = \underbrace{mgh}_{\substack{\text{potential energy} \\ \text{of gravity}}} + \underbrace{mga}_{\substack{\text{potential energy} \\ \text{of acceleration} \\ \text{due to} \\ \text{imaginary force}}}$$

Potential energy   Work between two points equals to potential energy:

$$U_{\vec{r}_B} - U_{\vec{r}_A} = W_{A \to B} = \int_A^B \vec{F}_{ag} d\vec{r}$$

$$\underbrace{\vec{F}_{ag}}_{\substack{\text{Force opposite} \\ \text{to field}}} = -\vec{F}$$

$$U_{\vec{r}_B} - U_{\vec{r}_A} = \int_A^B \left(-\vec{F}\right) d\vec{r} \overset{cortesian}{=} -\int_A^B (F_x dx + F_y dy + F_z dz)$$

$$\vec{F} = -\left(\frac{\delta U}{\delta x}\hat{x} + \frac{\delta U}{\delta y}\hat{y} + \frac{\delta U}{\delta z}\hat{z}\right) = -\vec{\nabla}U$$

Conservative force   A conservative force is a force with the property that the work done in moving a particle between two points is independent of the taken path. Potential energy can be defined only for conservative force.
Also closed path results in zero work for conservative force:

$$\oint \vec{F} d\vec{r} = 0$$

Examples   gravity, electricity, centrifugal in rotating system.

Example   Spring
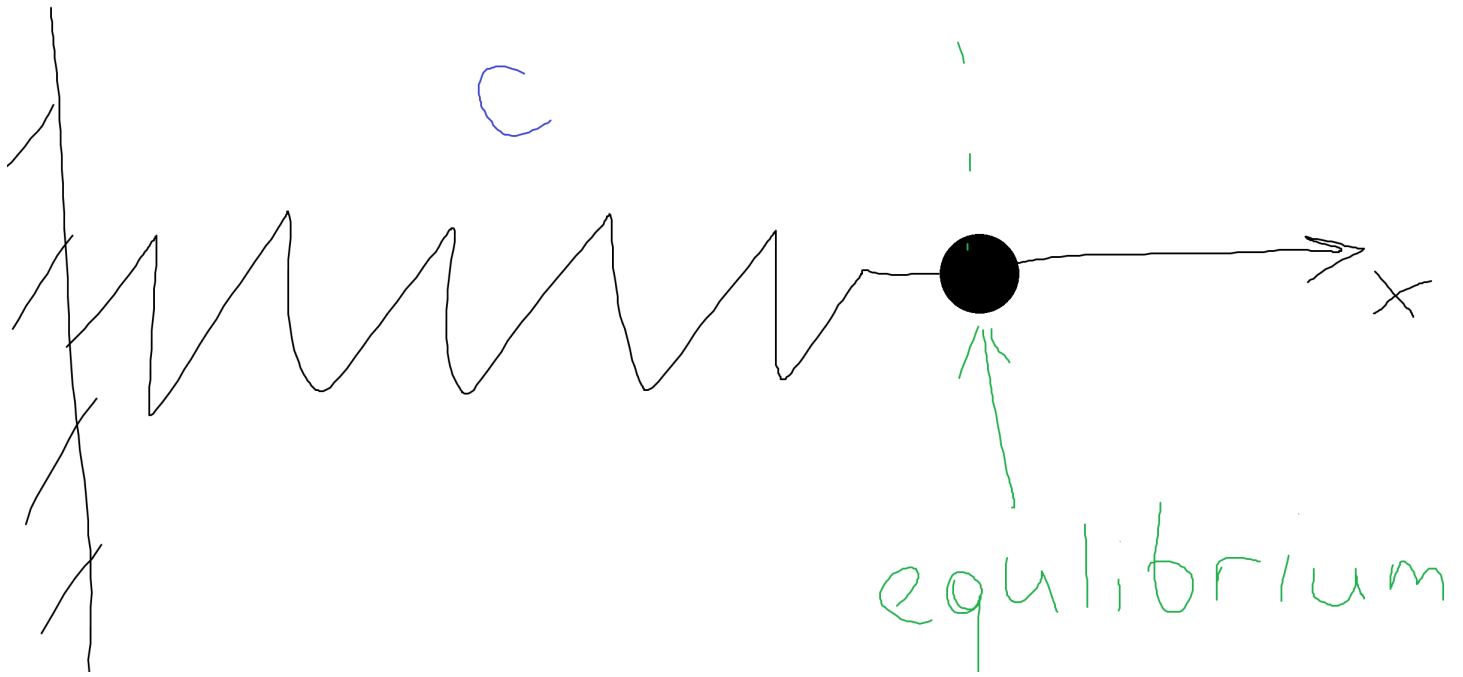
$$\vec{F} = -cx\hat{x}$$

where c is constant of spring.

$$W_{x_1 \to x_2} = \int_{x_1}^{x_2} \underbrace{\vec{F}_{ag}}_{\substack{\text{Force opposite} \\ \text{to spring force}}} \cdot d\vec{r} = \int_{x_1}^{x_2} \left(-\vec{F}_{spring}\right) d\vec{r}$$

Since $\vec{F}_{spring} = -cx\hat{x}$ and $d\vec{r} = dx\hat{x}$

$$W_{x_1 \to x_2} = \int_{x_1}^{x_2} cx dx = \left[cx^2\right]_{x_1}^{x_2} = \frac{1}{2}cx_2^2 - \frac{1}{2}cx_1^2$$
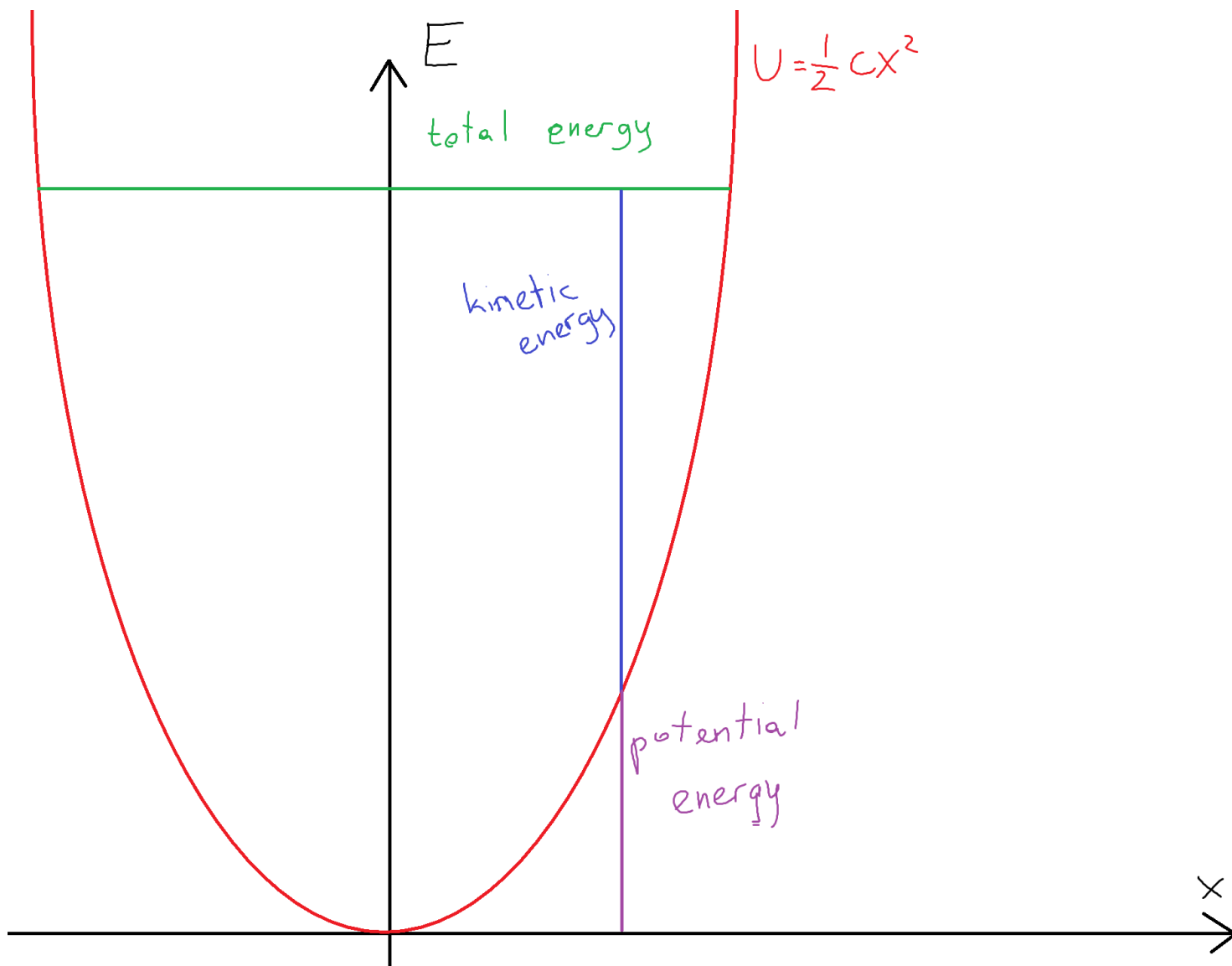
Then

$$U = \frac{1}{2}cx^2$$

and

$$\vec{F} = -\hat{x}\frac{\delta U}{\delta x} = -cx\hat{x}$$

as expected.

Field is a force per unit of mass (or, in case of electricity, charge).

Potential is potential energy per unit of mass (for gravity) or unit of charge (for electricity).
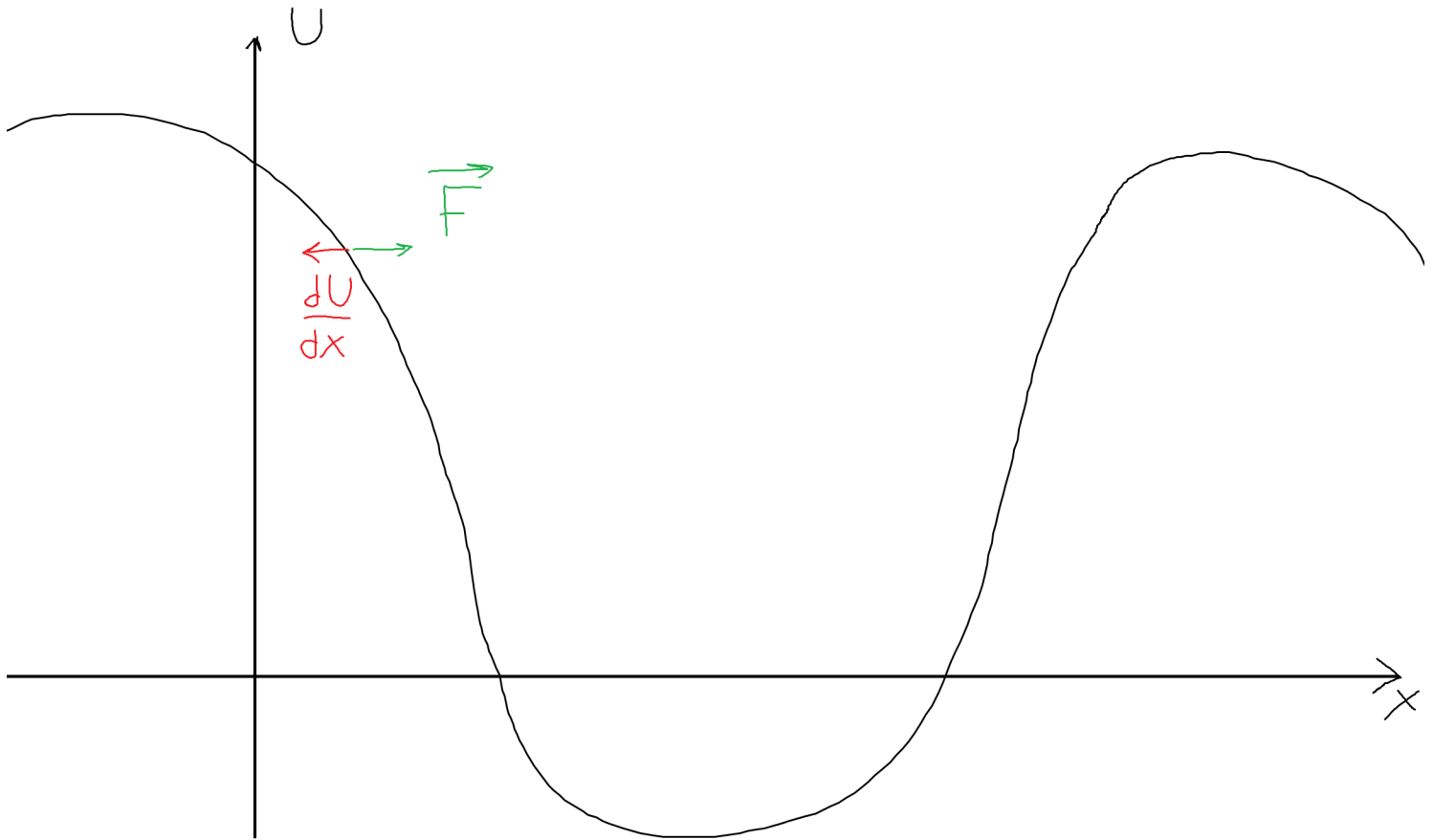
Example Gravitation on Earth's surface is $\vec{F} = -\hat{y}mg$. Then gravitational field is $\vec{g} = -\hat{y}g$ and gravitaional potential is:

$$\phi(\vec{r}) - \phi(\vec{A}) = \int_{\vec{r}}^{\vec{A}} \vec{g} \cdot d\vec{r}$$
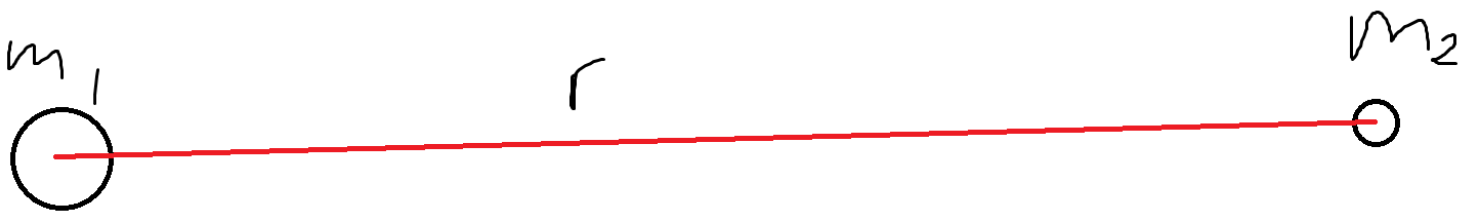
Gradient in 2D in polar coordinates

$$\vec{\nabla}U = \hat{r}\frac{\delta U}{\delta r} + \hat{\theta}\frac{1}{r}\frac{\delta U}{\delta \theta}$$

Potential energy in 1D

- Force acts in direction of low potential

- In points of maximum and minimum of potential force equals to 0.

- – Maximum point: non-stable equilibrium. $\frac{d^2U}{dx^2} < 0$
  – Minimum point: stable equilibrium. $\frac{d^2U}{dx^2} > 0$

Potential energy between two bodies

$$F = -\hat{r}\frac{Gm_1m_2}{r^2}$$

where m_1, m_2 - (gravitational) masses and G - gravitational constant.

$$U(\vec{r}) - U(\vec{A}) = \int_{\vec{A}}^{\vec{r}} \left(-\vec{F}\right) \cdot d\vec{r} \overset{d\vec{r} = \hat{r}dr}{=} \int_{r_A}^{r} \frac{Gm_1m_2}{r^2} dr = \left[-\frac{Gm_1m_2}{r}\right]_{r_A}^{r} = -\frac{Gm_1m_2}{r} - \left(-\frac{Gm_1m_2}{r_A}\right)$$

We choose reference point $r_A \to \infty$ since $U(\infty) = 0$ and get:
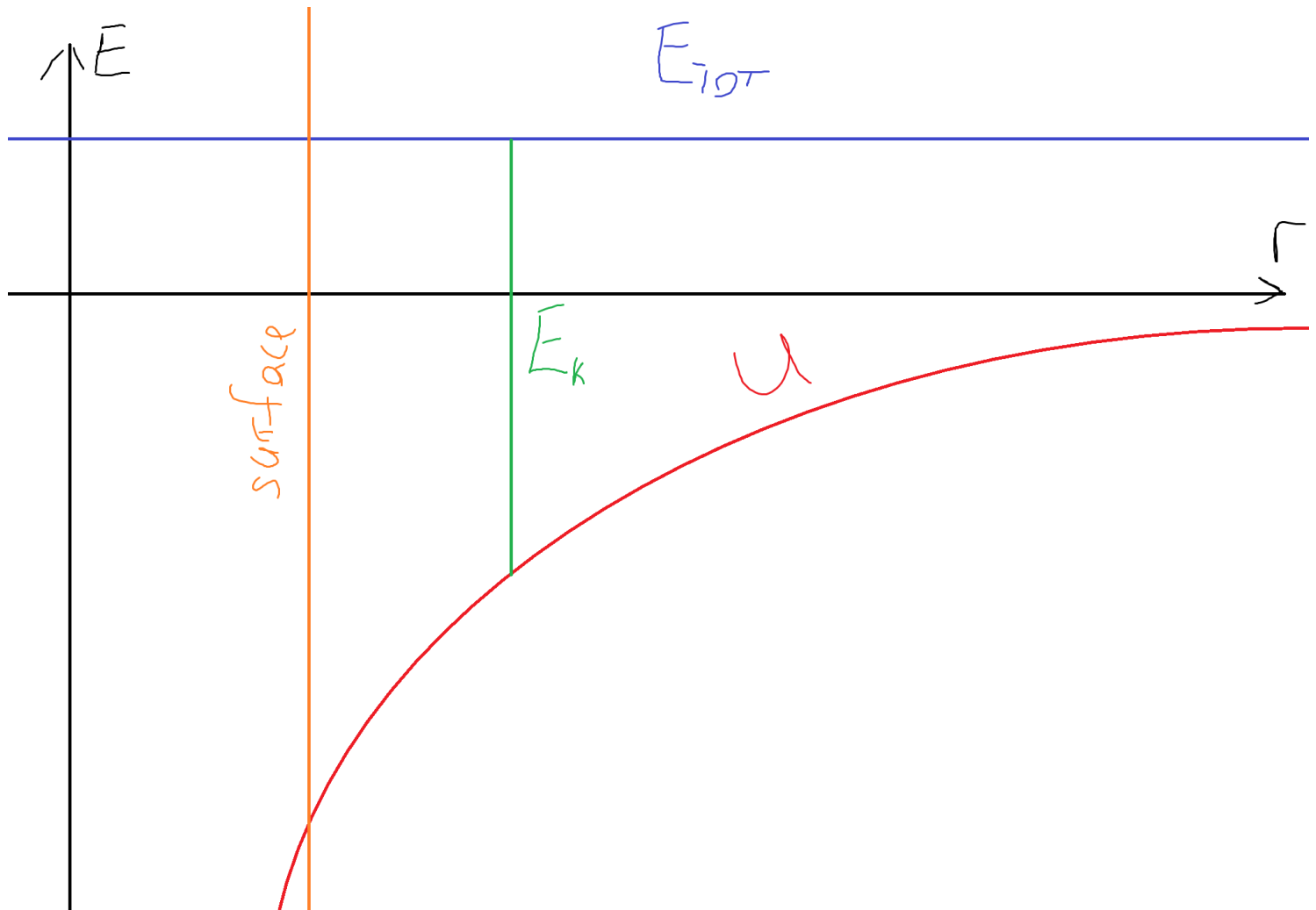
$$U(r) = -\frac{Gm_1m_2}{r}$$

At Earth's surface:

$$U(r) = -\frac{GM_\oplus}{R_\oplus}m$$

At height h over Earth's surface:

$$U(r_h) = -\frac{GM_\oplus}{R_\oplus + h}m = -\frac{GM_\oplus}{R_\oplus\left(1 + \frac{h}{R_\oplus}\right)}m = -\frac{GM_\oplus}{R_\oplus}m + \underbrace{\frac{GM_\oplus}{R_\oplus^2}}_{g} mh$$

i.e.

$$U(r_h) = U_{floor} + gmh$$

6.2 Conservation of impulse

Let's show that internal forces don't change total impulse. For N bodies:

$$\underbrace{\vec{p}}_{\text{total impulse}} = \sum_{i=1}^{n} \underbrace{m_i\vec{v}_i}_{\substack{\text{impulse of each} \\ \text{mass}}}$$

According to the third law:

$$F_{ij} = F_{ji}$$

Derive total impulse:

$$\frac{d\vec{p}}{dt} = \sum_{i=1}^{n} \frac{d}{dt}(m_i\vec{v}_i)$$

For two particles:

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(m_1\vec{v}_1) + \frac{d}{dt}(m_1\vec{v}_1)$$

30

Now substitute $\frac{d}{dt}(m_1\vec{v}_1) = \vec{F}_{21}$ and $\frac{d}{dt}(m_2\vec{v}_2) = \vec{F}_{12}$:

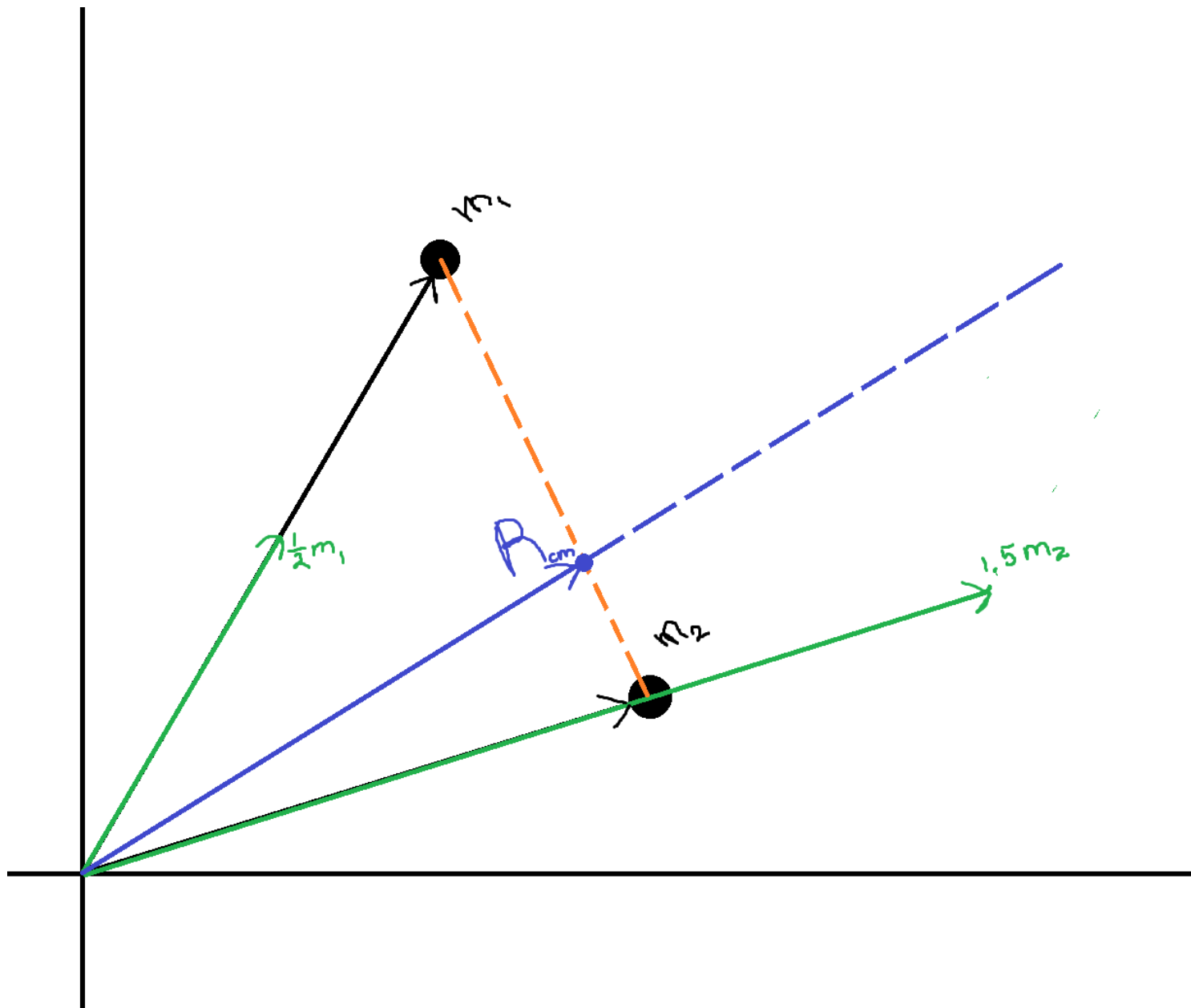$$\frac{d\vec{p}}{dt} = \vec{F}_{12} + \vec{F}_{21} = 0$$

6.3   Mass center

Definition

$$\vec{R}_{cm} = \frac{\sum_{i=1}^{N} \vec{r}_i m_i}{\sum_{i=1}^{N} m_i}$$

Example   A second law for a system of masses:

$$\vec{F} = \left(\sum_{i=1}^{N} m_i\right) \ddot{\vec{R}}_{cm}$$

Example

Velocity of mass center

$$\dot{\vec{R}}_{cm} = \frac{\sum \dot{\vec{r}}_i m_i}{\sum m_i} = \frac{\sum \vec{v}_i m_i}{\sum m_i}$$
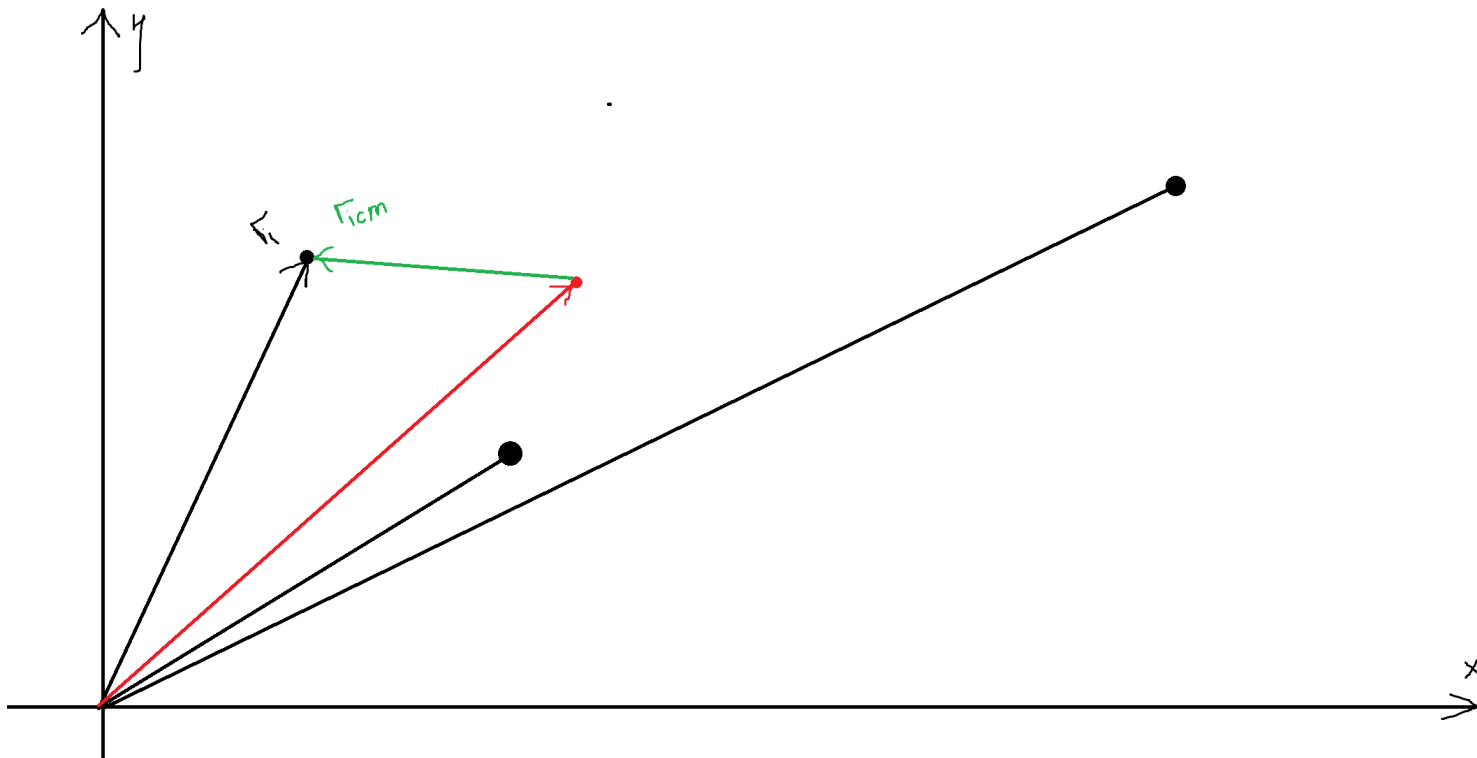
Acceleration of mass center

$$\sum m_i \ddot{\vec{R}}_{cm} = \sum \ddot{\vec{r}}_i m_i = \sum \vec{F}_i$$

Or:

$$\underbrace{M_{total}}_{\substack{\text{total mass} \\ (\sum m_i)}} \cdot \ddot{\vec{R}}_{ct} = \underbrace{\sum \vec{F}_i}_{\substack{\text{sum of forces on} \\ \text{particles}}} \overset{3rd\,law}{=} \vec{F}_{external}$$

System of center of mass

$$\ddot{\vec{r}}_{icm} = \vec{r}_i - \vec{R}_{cm}$$

Then impulse of center of mass relative to its own system is

$$\vec{p}_{cm\ to\ cm} = \sum p_{icm} = \sum \dot{\vec{r}}_{icm} m_i$$

$$\vec{p}_{cm\ to\ cm} = \sum \left(\dot{\vec{r}}_i - \dot{\vec{R}}_{cm} \right) m_i = \sum \vec{v}_i m_i - vec\dot{R}_{cm} \sum m_i = 0$$

Example Mass of $m_1 = 0.4kg$ moves with $v_1 = 6\frac{m}{s}$ and collides with mass $m_2 = 0.1kg$ in rest. From conservations of impulse and energy:

$$m_1 v_{10} + m_2 v_{20} = m_1 v_1 + m_2 v_2$$

$$\frac{1}{2} m_1 v_{10}^2 + \frac{1}{2} m_2 v_{20}^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

1. First solution is trivial: $v_1 = 6ms^{-1}$ and $v_2 = 0ms^{-1}$, but isn't interesting.

2. Second solution is $v_1 = 3.6ms^{-1}$ and $v_2 = 9.6ms^{-1}$.

Velocity of mass center before collision:

$$\vec{v}_{cm} = \frac{\vec{v}_{10} m_1 + \vec{v}_{20} m_2}{m_1 + m_2} = 4.8ms^{-1}$$

And after:

$$\vec{v}_{cm} = \frac{\vec{v}_1 m_1 + \vec{v}_2 m_2}{m_1 + m_2} = 4.8ms^{-1}$$

Impulse of system before and after is:
$$\vec{p} = m_1(\vec{v}_1 - \vec{cm}) + m_2(\vec{v}_2 - \vec{v}_{cm}) = 0$$
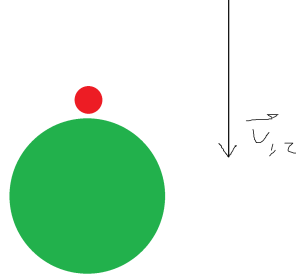Relative to center of mass:

$$\vec{v}_{10cm} = \vec{v}_{10} - \vec{cm} = 1.2ms^{-1} \quad \vec{v}_{20cm} = \vec{v}_{20} - \vec{cm} = -4.8ms^{-1}$$

$$\vec{v}_{1cm} = \vec{v}_1 - \vec{cm} = -1.2ms^{-1} \quad \vec{v}_{2cm} = \vec{v}_2 - \vec{cm} = 4.8ms^{-1}$$

**Example**

System of center of mass

Two balls, heavy and light are falling with same velocity. What happens?

1. Heavy ball collides with floor. Effectively center of mass of ball and floor is the floor, so it returns with same speed and opposite direction.

2. Now light ball collides with heavy ball. Similarly, center of mass of two balls is the heavy one, so the light ball returns with the same speed relative to heavy ball and opposite direction. However, relative to the floor, the speed is triple, since the big ball has a velocity relative to the floor.

If we have more than one ball, the process returns on itself couple of times.
This effect is used in gravitational slingshot:
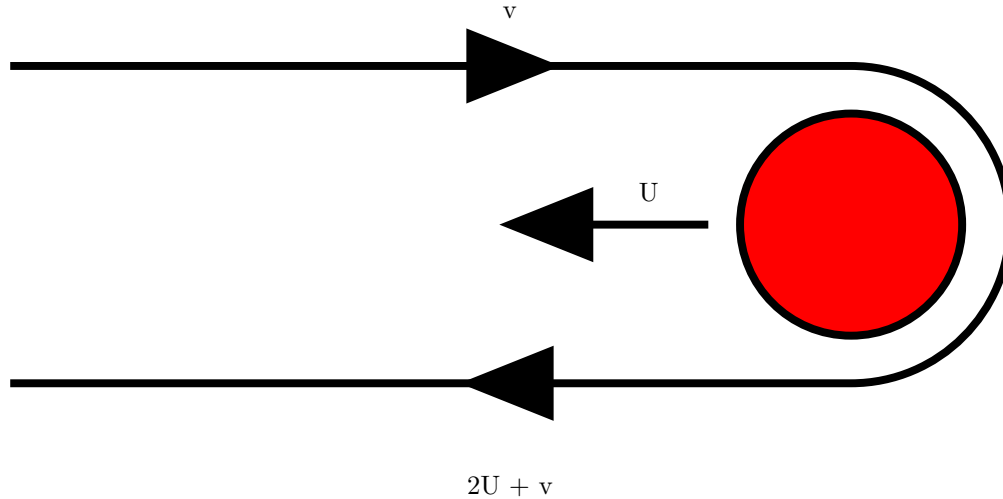
Here instead of collision we use gravitational force.

Collision of two bodies in 3D For two bodies collision happens in plane formed by two vectors of speed. We have total of 4 variables: $v_1, \theta_1, v_2, \theta_2$ or $v_{1x}, v_{1y}, v_{2x}, v_{2y}$. This is elastic collision i.e. kinetic energy is conserved. There are only three equations: conservation of impulse for each of axis and conservation of energy. So to acquire full solution we need some new factor.
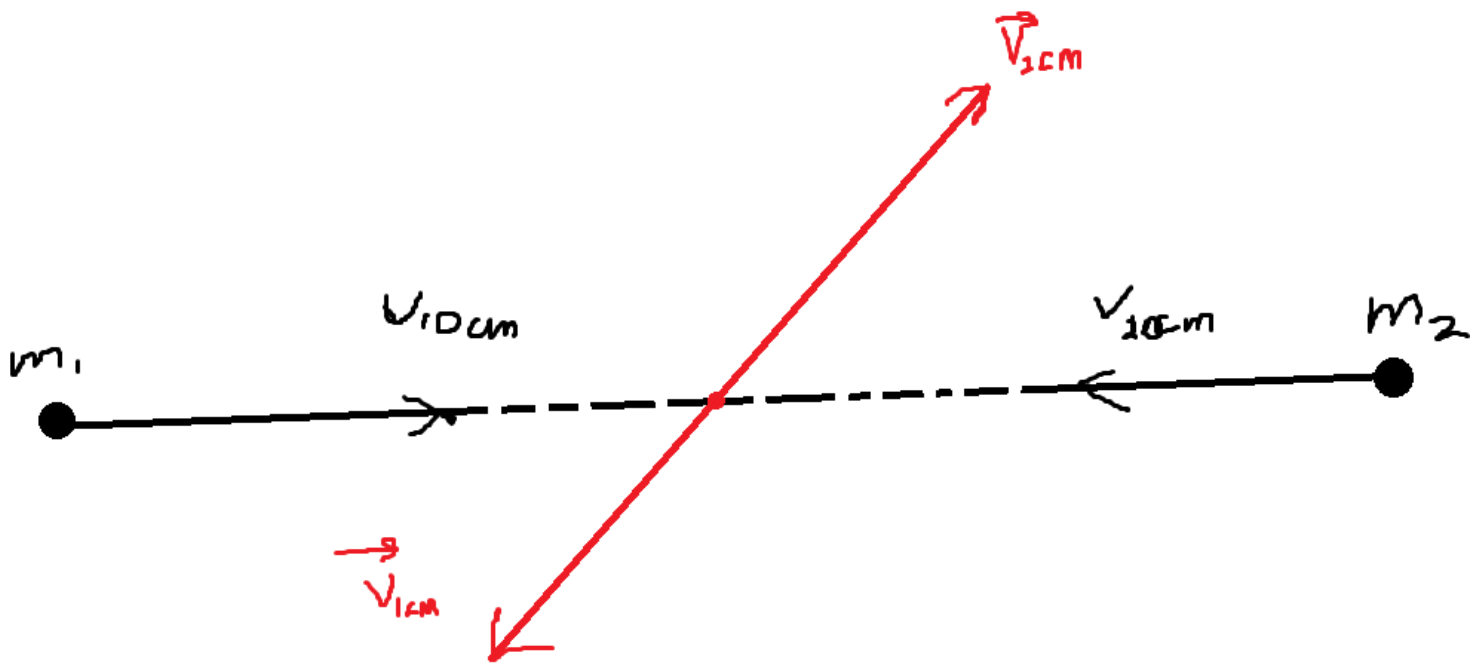
7 Systems with changing mass

Second Law of Newton

$$\vec{F} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt} = \dot{m}\vec{v} + m\vec{a}$$

$$\vec{F} = \frac{d\vec{p}}{dt}$$

- $\vec{a}$ doesn't have to be in same direction as $\vec{v}$.

- You should be accurate in calculating $\dot{m}\vec{v}$ since $\vec{v}$ depends on frame of reference

Example Plank of area $\vec{S}$ moves with velocity $\vec{v}$ and gathers dust which density is ρ. At $t = 0$ mass of plank is M_0 and velocity is $\vec{v}_0$. What is acceleration of plank and dependency of velocity of time.
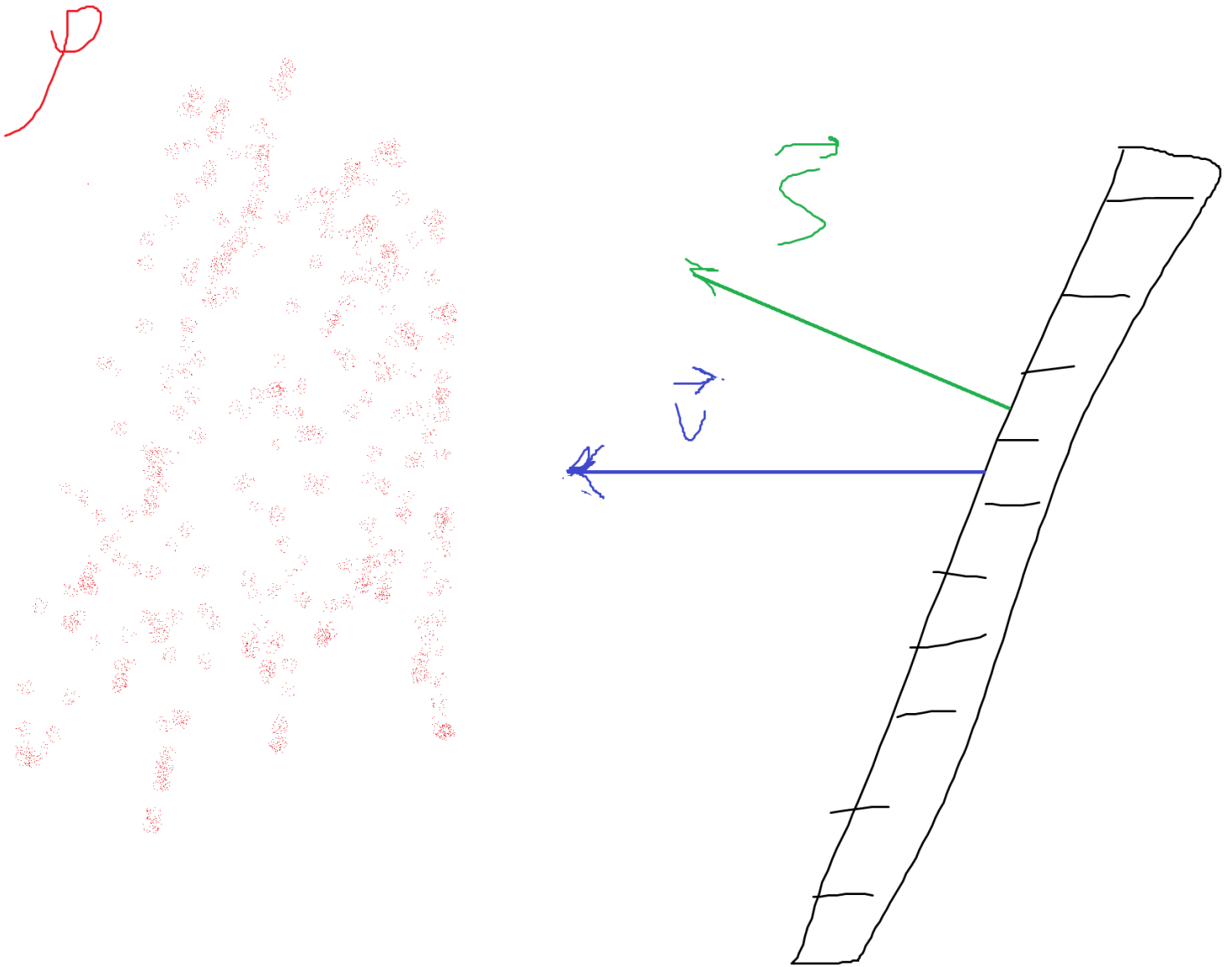
Volume of parallelepiped is $\Delta V = (\Delta t v) S \cos \theta = \vec{v} \cdot \vec{S} \Delta t$

Mass gathered in time Δt is $\Delta m = \Delta V \rho$. And speed of mass change is

$$\dot{m} = \frac{dm}{dt} = \frac{\Delta m}{\Delta t} = \frac{\vec{v} \cdot \vec{S} \Delta t}{\Delta t} \rho = \vec{v} \vec{S} \rho = v S \rho \cos \theta$$

Since $\rho, S, \cos \theta$ are constant, denote $k = S \rho \cos \theta$ meaning $\dot{m} = kv$.

There are three ways to solve the problem:

1. System of plank and cloud (reference point of cloud). There is no external forces, the momentum is conserved.

$$0 = F_{ext} = \frac{d}{dt} \left(m_{total} \vec{v}_{total} \right)$$

$$0 = \dot{m}v + m\dot{v} + \dot{m}_{cloud} \underbrace{v_{cloud}}_{0} + m_{cloud} \underbrace{\dot{v}_{cloud}}_{0}$$

$$\frac{\dot{v}}{v} = -\frac{\dot{m}}{m}$$

$$\frac{1}{v} \frac{dv}{dt} = -\frac{1}{m} \frac{dm}{dt}$$

$$\frac{dv}{v} = \frac{dm}{m}$$

Now solve integral with initial conditions

$$\int_{v_0}^{v} \frac{dv'}{v'} = -\int_{m_0}^{m} \frac{dm'}{m'}$$

$$[\ln v']_{v_0}^{v} = -[\ln m']_{m_0}^{m}$$

$$\ln \frac{v}{v_0} = \ln \frac{m_0}{m}$$

$$\frac{v}{v_0} = \frac{m_0}{m}$$

$$m = \frac{v_0 m_0}{v}$$

From $\frac{\dot{v}}{v} = -\frac{\dot{m}}{m}$ we acquire:

$$\dot{v} = \frac{dv}{dt} = -\frac{\dot{m}}{m} v \overset{\dot{m}=kv}{=} -\frac{kv}{\frac{m_0 v_0}{v}} v = -\frac{k}{m_0 v_0} v^3$$

We need to solve differential equation $\frac{dv}{dt} = -\frac{k}{m_0 v_0} v^3$:

$$\frac{dv}{v^3} = -\frac{k}{m_0 v_0} dt$$

. Integral is:

$$\int_{v_0}^{v(t)} \frac{dv}{v^3} = -\int_{0}^{t} \frac{k}{m_0 v_0} dt'$$

$$\left[-\frac{1}{2} v^{-2} \right]_{v_0}^{v(t)} = -\frac{k}{m_0 v_0} t$$

$$-\frac{1}{2} \frac{1}{v^2} + \frac{1}{2} \frac{1}{v_0^2} = -\frac{k}{m_0 v_0} t$$

With solution:

$$v = \left(\frac{2k}{m_0 v_0} t + \frac{1}{v_0^2} \right)^{-\frac{1}{2}}$$

2. Force on plank changes velocity of cloud in frame of reference of cloud in $t = 0$. There is a change in impulse of cloud.

$$\underbrace{f_c}_{\substack{\text{force of plank} \\ \text{on cloud}}} = \frac{dp_{cloud}}{dt} = \frac{p_{end} - p_{begin}}{\Delta t}$$

Initial velocity of cloud is 0 and terminal velocity is v (frame of reference of cloud in $t = 0$): $p_{begin} = \Delta m \cdot 0$ and $p_{begin} = \Delta m \cdot v$.

So $f_c = \frac{\Delta m v}{\Delta t} = \dot{m} v$ and $F_p = -f_c = -\dot{m} v$.

In a very short time change in mass is negligible (the force which cloud exerts on plank is exerted only on particles which were already on plank after previous moment). So we can use second law of Newton:

$$m\dot{v} = F_p = -\dot{m}v \Rightarrow \dot{v} = -\frac{\dot{m}}{m} v = -\frac{k}{m} v^2$$

3. We choose frame of reference in which at time t velocity of plank $v(t) = 0$.

Just like in 1, total impulse is conserved.

$$0 = \frac{d}{dt} p_{total} = \frac{d}{dt} (mv' + m_{cloud} v'_{cloud})$$

where v' is a velocity in our frame of reference.

$$0 = \dot{m}v' + m\underbrace{\dot{v}'}_{0} + \dot{m}_{cloud}v'_{cloud} + m_{cloud}\underbrace{\dot{v}'_{cloud}}_{0}$$

Velocity of cloud doesn't changes (the velocity of frame of reference is constant) and also velocity of plank doesn't changes according to definition. But $v'_{cloud} \neq 0$, and $v'_{cloud} = -v$. Since the reference point isn't accelerating, $\dot{v}' = \dot{v}$. By substituting $\dot{m}_{cloud} = -\dot{m}$ acquire:
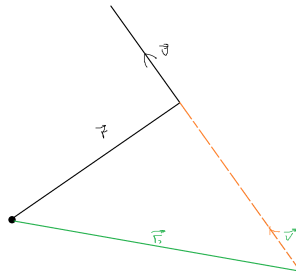
$$m\dot{v} + \dot{m}v = 0$$

8 Angular momentum

Angular momentum is a conserved value due to rotation symmetry.
Angular momentum of single particle relative to origin (in inertial frame of reference)

$$\vec{J} = \vec{r} \times \vec{p} = \vec{r} \times (m\vec{v})$$

$$\vec{J} = \vec{J}_2$$

.

- Both value and direction are conserved.

- Units of angular momentum are $kg \cdot \frac{m^2}{s} = J \cdot s$.

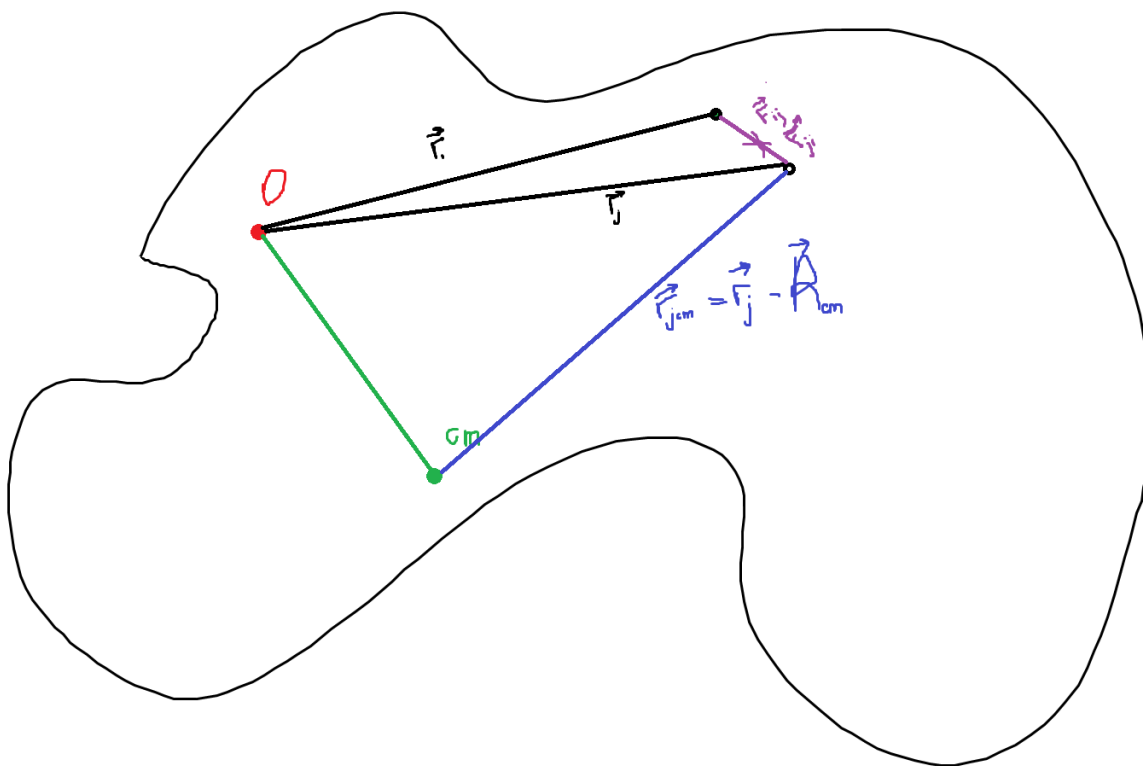**Torque** $\vec{N} = \vec{r} \times \vec{F}$.
Units of torque are $N \cdot m$.
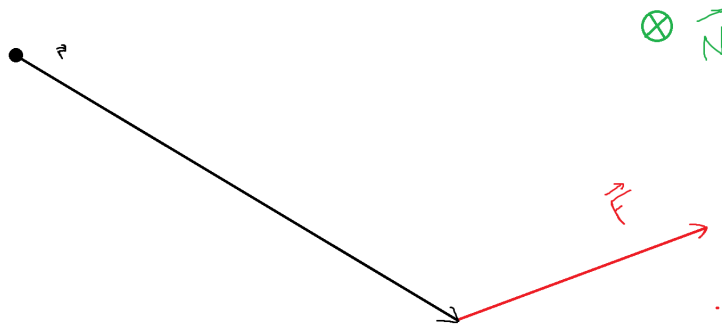
$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \underbrace{\frac{d\vec{r}}{dt} \times \vec{p}}_{0} + \underbrace{\vec{r} \times \frac{d\vec{p}}{dt}}_{\vec{N}} = \vec{N}$$

Central force $\vec{F} = F\hat{r}$. No torque.

8.1 Multi-particle systems

$$\vec{N}_{i\ on\ j} = \vec{r}_j \times \vec{F}_{i\ on\ j} = \vec{r}_\perp \times \vec{F}_{i\ on\ j} = -\vec{r}_\perp \times \vec{F}_{j\ on\ i} = -\vec{r}_i \times \vec{F}_{j\ on\ i} = -\vec{N}_{j\ on\ i}$$

Internal torques cancel each other. To calculate change in angular momentum we need to consider only external forces.

$$\vec{N} = \sum_{\substack{\text{all} \\ \text{masses} \\ \text{in} \\ \text{system}}} \vec{r}_i \times \vec{F}_{i\ ext}$$

A torque is around of some point so it depends on choice of origin.

Example Constant gravitational force: $\vec{g} = const$. Total torque is

$$\vec{N}_0 = \sum \vec{r}_i \times (m_i \vec{g}) \overset{\vec{g}=const}{=} \sum (\vec{r}_i m_i) \times \vec{g}$$

Substitute $\vec{r}_i = \vec{R}_{cm} + \vec{r}_{icm}$

$$\vec{N}_0 = \left[\sum \left(\vec{R}_{cm} + \vec{r}_{icm} \right) m_i \right] \times \vec{g} = \left(\vec{R}_{cm} \sum m_i \right) \times \vec{g} + \underbrace{\left(\sum \vec{r}_{icm} m_i \right)}_{0} \times \vec{g} = \left(\vec{R}_{cm} M \right) \times \vec{g}$$

In constant gravity field torque is equals to torque on point mass in center of mass: $\vec{N}_0 = \left(\vec{R}_{cm} M_{total} \right) \times \vec{g}$. Torque around center of mass is 0.

Non-costant gravity field Torque around center of mass isn't 0. Moon-Earth system:

The torque against Earth rotation is more than torque in direction of rotation, meaning angular momentum decreases and the Earth rotation slows. Since angular momentum is conserved, Moon is drifting away from Earth.
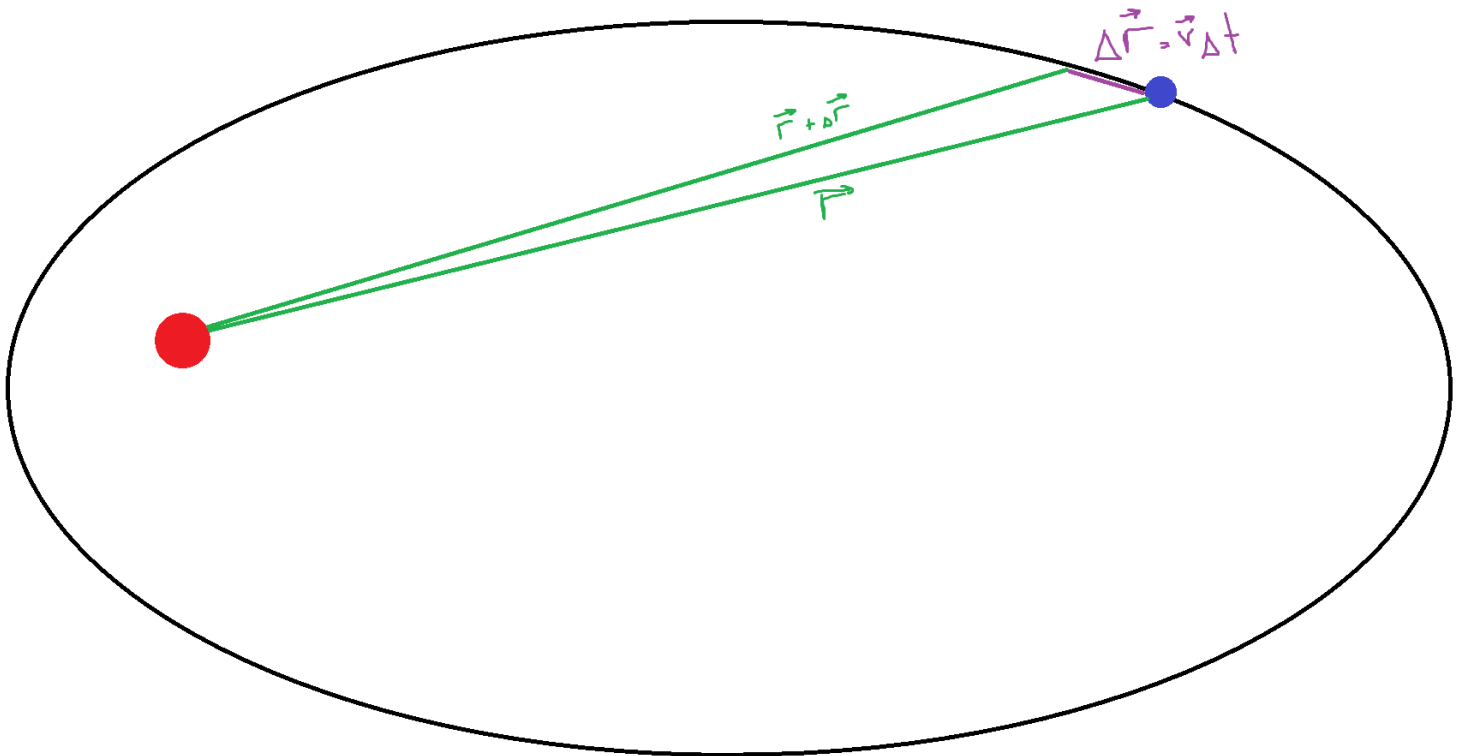
Angular momentum of the body

$$\vec{J}\sum_{i=1}^{N} m_i \vec{r}_i \times \vec{v}_i = \sum m_i \left(\vec{r}_i - \vec{R}_{cm} \right) \times \vec{v}_i + \sum \vec{R}_{cm} \times (m_i \times \vec{v}_i) =$$

$$= \sum \vec{r}_{icm} \times (m_i \vec{v}_i) + \vec{R}_{cm} \times \sum m_i \vec{v}_i = \vec{J}_{cm} + \vec{R}_{cm} \times \vec{p}$$

$$\vec{J}_{cm} = \vec{I}_{cm}\vec{\omega}$$

where I is moment of inertia and ω is angular velocity.

Example   Movement of planets around Sun.

Area of triangle is $\vec{S} = \frac{1}{2}\vec{r} \times \Delta\vec{r}$. Derive the area:

$$\frac{d\vec{S}}{dt} = \frac{1}{2}\frac{\vec{r} \times \Delta\vec{r}}{\Delta t} = \frac{1}{2}\vec{r} \times \frac{\Delta\vec{r}}{\Delta t} = \frac{1}{2}\vec{r}\vec{v} = \frac{1}{2}\frac{\vec{r} \times (m\vec{v})}{m} = \frac{\vec{r} \times \vec{p}}{2m} = \frac{\vec{J}}{2m}$$

We acquired that $\frac{d\vec{S}}{dt} = const$, meaning in same time dt a planet covers same area, which is second law of Kepler.

Example Angular acceleration at the process of crash.

$J = mvr = mv_0 r_0 = const$ (there is only central force). Then $v = v_0 \frac{r_0}{r} = \frac{J}{mr}$. If r decreases, then v increases, since $\vec{J}$ is constant and kinetic energy increases too.

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{J}{mr}\right)^2$$

$$E_k = \frac{J^2}{2mr^2}$$

Change in kinetic energy is caused by force directed inside and its work is:

$$U_c = -\int_\infty^r F_{centrifugal}dr = -\int_\infty^r \frac{mv^2}{r}dr$$

We substitute $v = \frac{J}{mr}$, i.e. $F_{centrifugal} = \frac{J^2}{mr^3}$. Then the work is:

$$U_c = -\int_\infty^r \frac{J^2}{mr^3}dr = -\left[\frac{J^2}{2mr^2}\right]_\infty^r = -\frac{J^2}{2mr^2}$$

which is equal to change in kinetic energy. **Note:** this works only when angular momentum is constant.

8.2 Gravity

M_* - central mass. $m \ll M_*$. Force of gravity is $\vec{F} = -\hat{r}\frac{GM_*}{r^2}m$ and gravitational energy $U_g = -\frac{GM_*m}{r}$.
Total energy is

$$E = \frac{1}{2}mv^2 + \left(-\frac{GM_*m}{r}\right)$$

where $v = v_r\hat{r} + v_\theta\hat{\theta}$, when $v_\theta = \frac{J}{mr}$ since radial velocity doesn't affects angular momentum. Substituting:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(\frac{J}{mr}\right)^2 - \frac{GM_*m}{r} = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{J^2}{2mr^2}}_{\substack{\text{centrifugal} \\ \text{potential}}} - \frac{GM_*m}{r}$$

Define effective potential which depends only on r and not on θ: $U_{eff} = \frac{J^2}{2mr^2} - \frac{GM_*m}{r}$. So:

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}(r)$$

This equation is only in one dimension r and not describes all the trajectory.

Different trajectories of bodies Kinetic energy is positive to the right of the blue line, which is total potential, and equals to 0 (i.e. $\dot{r} = 0$) at the line:
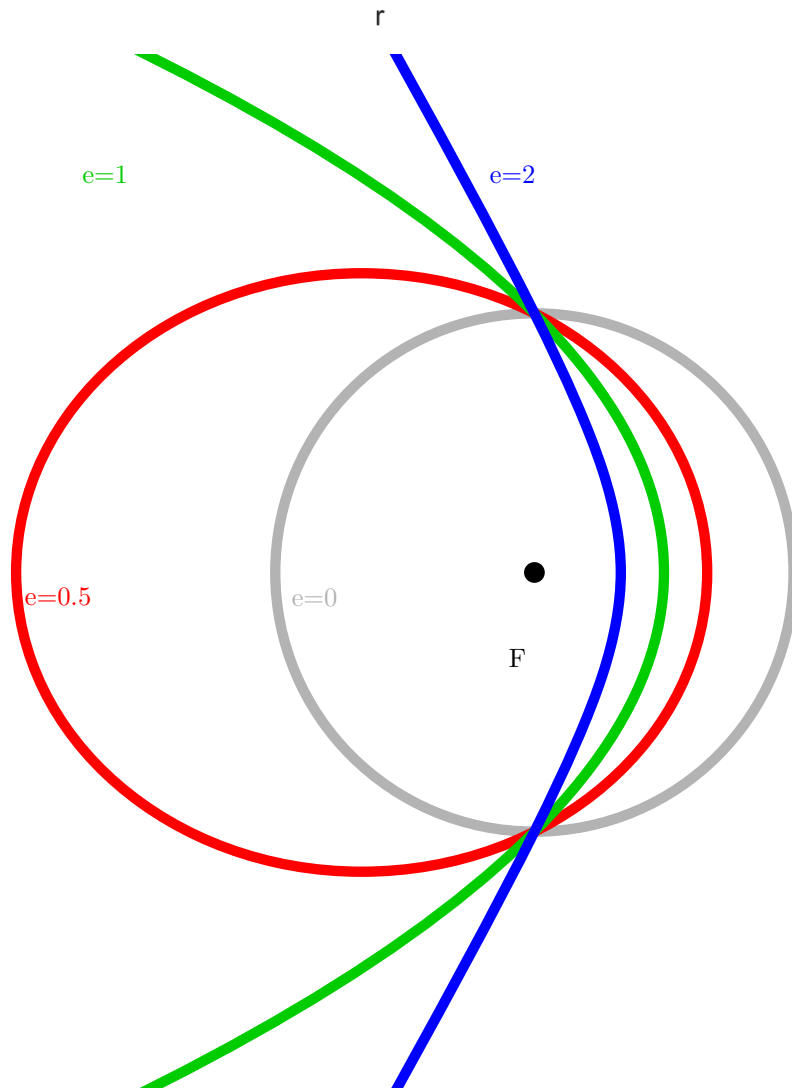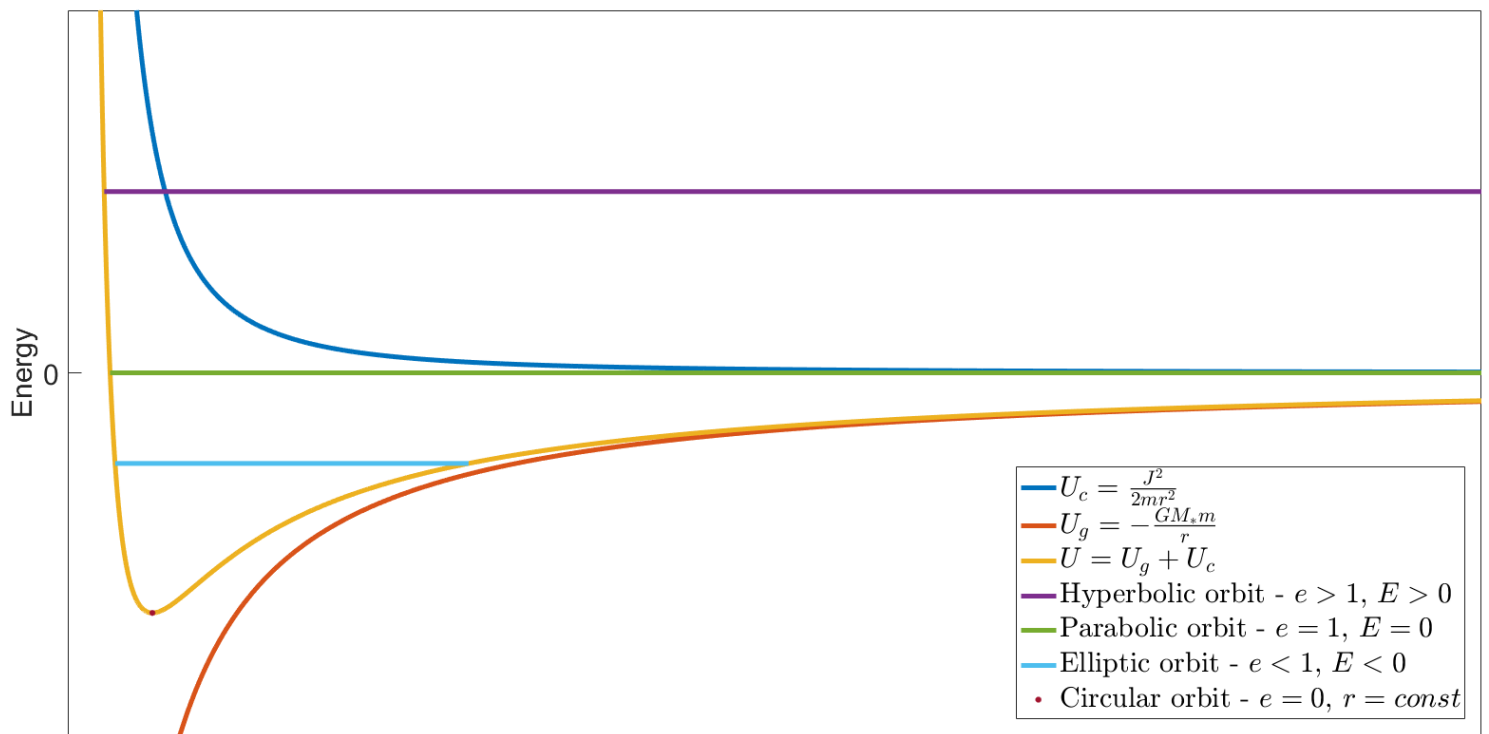
Note: Hyperbolic orbit works also with repelling, for example electrical force.

Circular orbit $\dot{r} = 0$ and $\vec{v} = v_\theta \hat{\theta}$.

Centrifugal force equals to gravitational force:

$$m\frac{v^2}{r} = \frac{GM_* m}{r^2}$$

Velocity is $v = \sqrt{\frac{GM_*}{r}}$, angular velocity is $\omega = \frac{v}{r} = \sqrt{\frac{GM_*}{r^3}}$ and cycle time is $T = \frac{2\pi}{\omega} = \frac{2\pi r}{v} = 2\pi \cdot \frac{r^{\frac{3}{2}}}{\sqrt{GM_*}}$

Kepler's laws

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.

2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
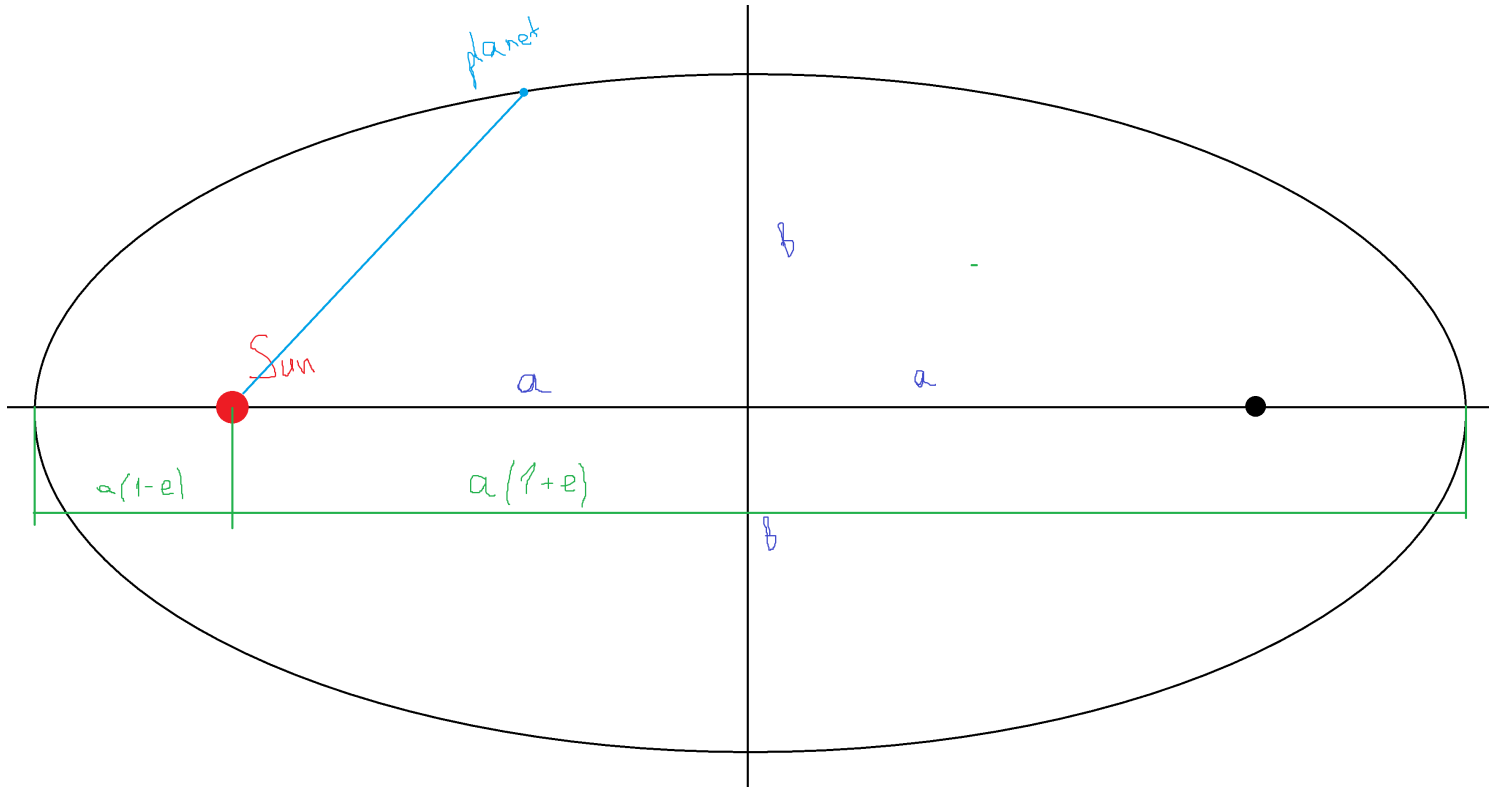
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

$b = a\sqrt{1 - e^2}$.
Ellipse equation is

$$r = \frac{a\left(1 - e^2\right)}{1 - e\cos\theta}$$

Physical solution of movement of m around M_*. From second law:

$$m\vec{a} = \vec{F}$$

Force is $\vec{F} = -\hat{r}\frac{GM_* m}{r^2}$ and acceleration $\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\hat{\theta}$
Movement equation in direction $\hat{\theta}$:

$$m\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = F_\theta = 0$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

$$r^2\dot{\theta} = const$$

Multiplying by m:

$$mr^2\dot{\theta} = const = J$$

Since $r\dot{\theta} = r\frac{d\theta}{dt} = v_\theta$:

42

$J = mrv_\theta = const$

Movement equation in direction $\hat{r}$:

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = -\frac{GM_*m}{r^2}$$

Substituting $\dot{\theta} = \frac{J}{mr^2}$:

$$\ddot{r} - r\frac{J^2}{m^2r^4} = -\frac{GM_*}{r^2}$$

We get the equation:

$$\frac{J^2}{m^2r^4}\left[\frac{d^2r}{d\theta^2} - \frac{2}{r}\left(\frac{dr}{d\theta}\right)^2\right] - r\frac{J^2}{m^2r^4} = -\frac{GM_*}{r^2}$$

The solution of this differential equation is:

$$\frac{1}{r} = \frac{GM_*}{\left(\frac{J}{m}\right)^2}\left[1 - \left(1 + \frac{2\left(\frac{E}{m}\right)\left(\frac{J}{m}\right)^2}{G^2M_*^2}\right)^{\frac{1}{2}}\cos\theta\right]$$

When eccentricity:

$$e = \left[1 + \frac{2\left(\frac{E}{m}\right)\left(\frac{J}{m}\right)^2}{G^2M_*^2}\right]^{\frac{1}{2}}$$

Additional relations

1. Circular orbit has $v = \sqrt{\frac{GM_*}{r}}$. So from $(e = 0)$:

$$\frac{E}{m} = \frac{1}{m}\left[\frac{1}{2}mv^2 + \left(-G\frac{M_*m}{r}\right)\right] = \frac{1}{2}v^2 - \frac{GM_*}{r}$$

   by substituting we get

$$E = E_k + U_G = \frac{1}{2}U_G = -E_k$$

2. In elliptical orbit E_k and U_G are changing in time. It is possible to define average values over orbit:

$$E = \frac{1}{2}\bar{U}_G = -\bar{E}_k$$

   Calculation shows:

$$\bar{U}_G = -\frac{GM_*m}{a}$$

$$E = -\frac{1}{2}\frac{GM_*m}{a}$$

3. $e = \frac{r_{max}-r_{min}}{r_{max}+r_{min}}$, where $r_{max} = a(1+e)$ and $r_{min} = a(1-e)$.

4. For ellipse, substituting $\frac{E}{m} = -\frac{1}{2}\frac{GM_*}{a}$ in expression for e we get:

$$\frac{J}{m} = \sqrt{GM_*a(1-e^2)}$$

   and

$$\left(\frac{J}{m}\right)^2\frac{1}{GM_*} = a(1-e^2)$$

5. For ellipse

$$\frac{1}{2}\frac{GM_*}{a} = \frac{v^2}{2} - \frac{GM_*}{r}$$

   or

$$\frac{1}{a} = \frac{2}{43r} - \frac{v^2}{GM_*}$$

Third law of Kepler   We found that $\frac{d\vec{S}}{dt} = \frac{\vec{J}}{2m}$ which is implied from conservation of angular momentum.
Integral on time of cycle:

$$\oint dS = \int_0^T \underbrace{\frac{J}{2m}}_{\text{const}} dt = \frac{J}{2m}T$$

By substituting $b = a\sqrt{1-e^2}$ and $\frac{J}{m}$ we get:

$$T^2 = \left(\frac{2m}{J}\right)^2 S^2 = \frac{4}{GM_*a(1-e^2)}\pi^2 a^2 \underbrace{\left[a^2\left(1-e^2\right)\right]}_{\text{ellipse area}}$$

And we get third law:

$$T^2 = \frac{4\pi^2}{GM_*}a^3$$

Reduced mass   When $m_2 \lll m_1$:
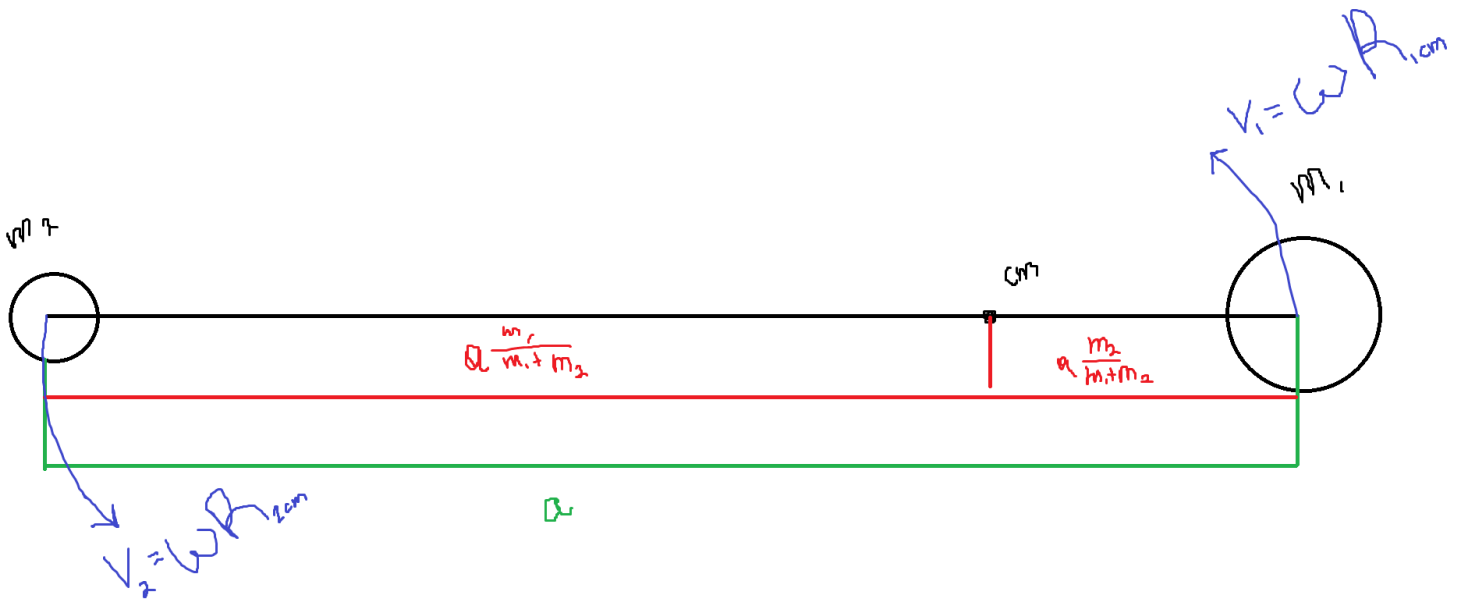
$$\vec{R}_{1cm} = \frac{m_2}{m_1+m_2}a\hat{r}$$

$$\vec{R}_{2cm} = \frac{m_1}{m_1+m_2}a\left(-\hat{r}\right)$$

We are at circular orbit. Lets compare centrifugal force with gravitational force:

$$|F_{cen}| = |F_g|$$

At second mass:

$$m_2\omega^2 R_{2cm} = \frac{Gm_1m_2}{a^2}$$

By substituting R_{2cm}:

$$\omega^2 \frac{m_1}{m_1+m_2}a = \frac{Gm_1}{a^2}$$

$$\omega = \sqrt{\frac{G\left(m_1+m_2\right)}{a^3}}$$

Now substitute $T = \frac{2\pi}{\omega}$:

$$T^2 = \frac{2\pi a^3}{G\left(m_1+m_2\right)}$$

Which is third law of Kepler.
Angular momentum:

44

$$\left|\vec{J}\right| = m_1 R_{1cm}^2 \omega + m_2 R_{2cm}^2 \omega$$

Direction of $\vec{J}$ is perpendicular to plane of movement.

$$J = \frac{m_1 m_2^2}{(m_1 + m_2)^2} a^2 \omega + \frac{m_2 m_1^2}{(m_1 + m_2)^2} a^2 \omega$$

Acquiring:

$$J = \frac{m_1 m_2}{(m_1 + m_2)} a^2 \omega$$

Define reduced mass - $\mu = \frac{m_1 m_2}{(m_1 + m_2)}$ resulting in:

$$J = \mu a^2 \omega$$

Kinetic energy is:

$$E_k = \frac{1}{2} m_1 \left(R_{1cm}\omega\right)^2 + \frac{1}{2} m_2 \left(R_{2cm}\omega\right)^2$$

Acquiring:

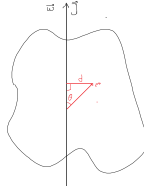$$E_k = \frac{1}{2} \mu a^2 \omega^2$$

And total energy:

$$E = \frac{1}{2} \mu a^2 \omega^2 - \frac{G m_1 m_2}{a}$$

$$E = \frac{1}{2} \mu a^2 \omega^2 - \frac{G (m_1 + m_2) \mu}{a}$$

When $m_1 \ll m_2$ we replace m with μ and M_* with $m_1 + m_2$.

Rigid body We talk about rigid body rotating around constant axis. ω is same for each part of body. We consider only case when $\vec{\omega}$ and $\vec{J}$ are in same direction.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

In point p there is mass m and its contribution to total angular momentum is:

$$\vec{j}_p = m_p \vec{r}_p \times \vec{v}_p$$

We work with $\hat{\omega} = \hat{J}$ which is not always true. Then

$$[\vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i)]_{\hat{j}} = |\vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i)| \sin\theta \hat{J} = (r_i \sin\theta)^2 \vec{\omega} = d^2 \vec{\omega}$$

Total angular momentum is:

$$\vec{J} = \sum_i \vec{j}_i = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \vec{r}_i \times (\vec{\omega}_i \times \vec{r}_i) = \sum_i m_i d_i^2 \vec{\omega}$$

where $d_i = r_i \sin\theta_i$ is distance from rotation axis. We get

$$\vec{J} = \vec{\omega} \sum_i m_i d_i^2$$

And kinetic energy of the body:

$$E_k = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\vec{\omega} \times \vec{r}_i)(\vec{\omega} \times \vec{r}_i) = \sum_i \frac{1}{2} m_i d_i^2 \omega^2$$

45

We got:

$$E_k = \frac{1}{2}\left(\sum_i m_i d_i^2\right)\omega^2$$

Conclusion For our case we got:

$$\vec{J} = I\vec{\omega}$$

$$E_k = \frac{1}{2}I\omega^2$$

Where I is moment of inertia which we defined as:

$$I = \sum m_i d_i^2$$

Moment of inertia depends on choice of axis of rotation (both place and direction).
For contiguous body we replace mass with small mass $m_i \rightarrow dm = \rho dV$.

$$I = \int \underbrace{R^2}_{\substack{\text{distance of} \\ \text{dV from} \\ \text{axis}}} \underbrace{\rho dV}_{\text{dm}}$$

Parallel axis For constant ω, for two parallel axis I and I_c, which goes throw center of the body, with distance l between them for body of total mass M moment of inertia is

$$I_c = \sum_i m_i r_{icm}^2$$

$$I = I_c + Mr^2$$

And kinetic energy is:

$$K = \underbrace{K_c}_{\substack{\text{kinetic energy} \\ \text{as a result of} \\ \text{movement} \\ \text{relative to} \\ \text{center of mass}}} + \underbrace{\frac{1}{2}mv_{cm}^2}_{\substack{\text{kinetic energy} \\ \text{as a result of} \\ \text{movement of} \\ \text{body}}}$$

So angular momentum is:

$$\vec{J} = I\vec{\omega} = I_c\vec{\omega} + Ml^2\vec{\omega}$$

And more general expression for kinetic energy:

$$K = \frac{1}{2}I\vec{\omega} = \underbrace{\frac{1}{2}I_c\omega^2}_{\substack{\text{kinetic energy} \\ \text{as a result of} \\ \text{movement} \\ \text{relative to} \\ \text{center of mass}}} + \underbrace{\frac{1}{2}Ml^2\omega^2}_{\frac{1}{2}mv_{cm}^2}$$

Kinetic energy

$$K = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum_i m_i \left(\underbrace{\vec{v}_{icm}}_{\substack{\text{velocity of point} \\ \text{i relative to cm}}} + \underbrace{\vec{v}_{cm}}_{\text{velocity of cm}} \right)^2$$

$$K = \frac{1}{2}\sum_i m_i \left(\vec{v}_{icm} + \vec{v}_{cm}\right)\cdot\left(\vec{v}_{icm} + \vec{v}_{cm}\right) = \underbrace{\frac{1}{2}\sum_i m_i v_{icm}^2}_{\substack{\text{kinetic energy} \\ \text{as a result of} \\ \text{movement} \\ \text{relative to} \\ \text{center of mass}}} + \frac{1}{2}\sum_i m_i v_{cm}^2 + \sum_i \underbrace{m_i \vec{v}_{icm}}_{\substack{\text{0 since } \vec{p} \\ \text{relative to cm is} \\ 0}} \vec{v}_{cm}$$

46

Perpendicular axis   Two axis in plane of body perpendicular to each other. Moment of inertia relative to them is I_1 and I_2, while I_z is moment of inertia relative to third axis perpendicular to the plane of the body.
Since $x^2 + y^2 = r^2$: $I_z = I_1 + I_2$.

Examples of values of I

Examples for the ring

1. Axis perpendicular to plane of ring passing through center of the ring:

$$I_c = MR^2$$

2. Axis in plane of ring passing through center of the ring:

$$I_1 = \frac{1}{2}MR^2$$

3. Axis perpendicular to plane of ring passing through surface of the ring:

$$I = I_c + Ml^2 = 2MR^2$$

Examples for the hard rod
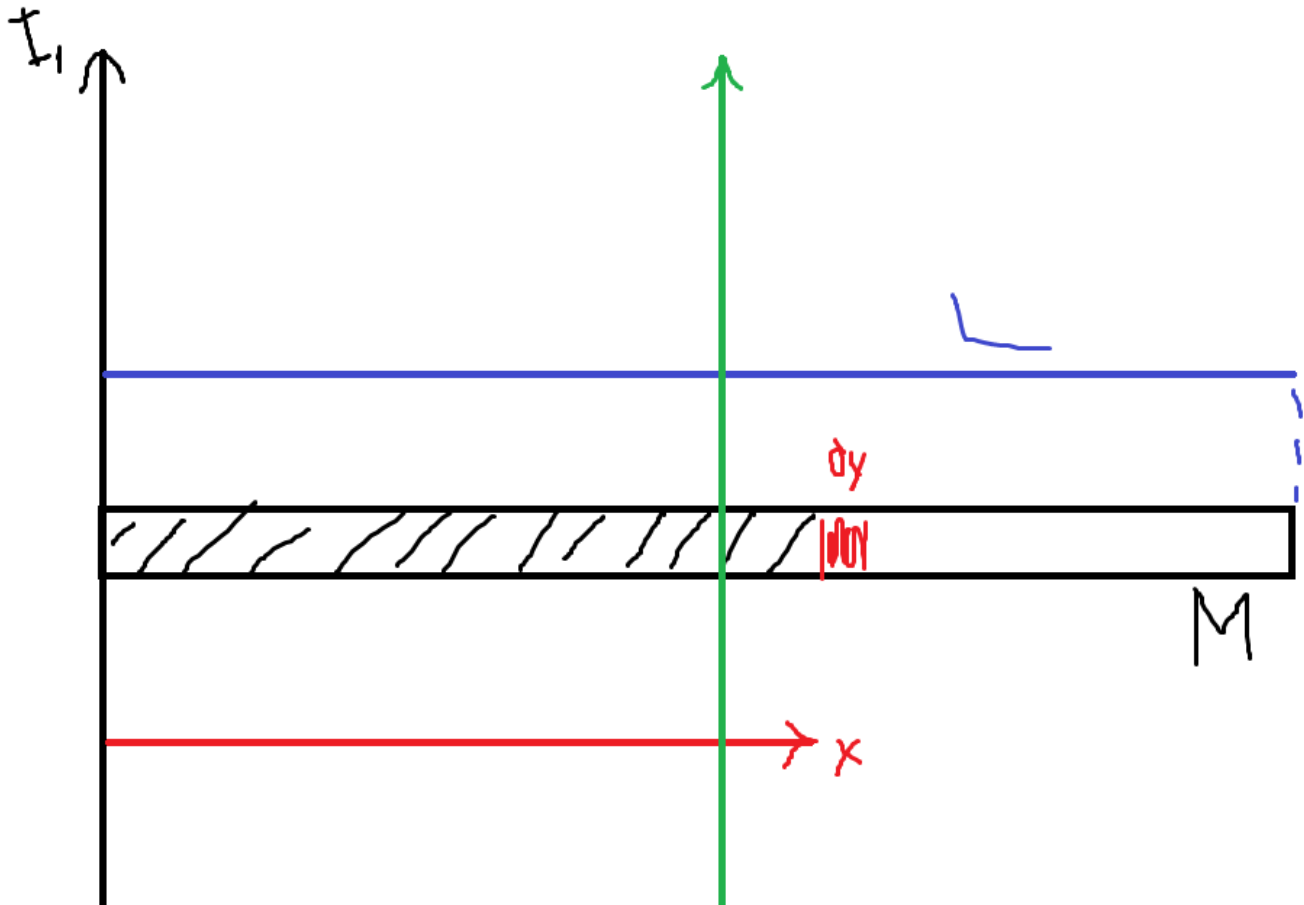
1. Axis in the end of the rod (I_1):

$$I = \sum_i m_i d_i^2$$

For continuous body $\sum \to \int$, $d_i \to x$, $m_i \to \frac{M}{L}dx$:

$$I = \int_{x=0}^{x=L} x^2 dm = \int_{x=0}^{x=L} x^2 \frac{M}{L}dx = \frac{1}{3}ML^2$$

47

2. Axis in the middle of the rod (I_2):

$$\frac{1}{12}ML^2$$

Examples for the disc

1. Axis perpendicular to plane of disc passing through center of the ring:

$$I_c = \frac{1}{2}MR^2$$

2. Axis in plane of disc passing through center of the ring:

$$I_1 = \frac{1}{4}MR^2$$

Examples for the rectangular plate   Of size $a \times b$.

1. Axis in plane of plate passing through center of the ring:

$$I_x = \frac{1}{12}Ma^2$$

$$I_y = \frac{1}{12}Mb^2$$

2. Axis perpendicular to plane of ring passing through center of the ring:

$$I_z = I_x + I_y$$

Examples for the sphere   With constant density $\rho = \frac{M}{\frac{4\pi}{3}R^3}$.
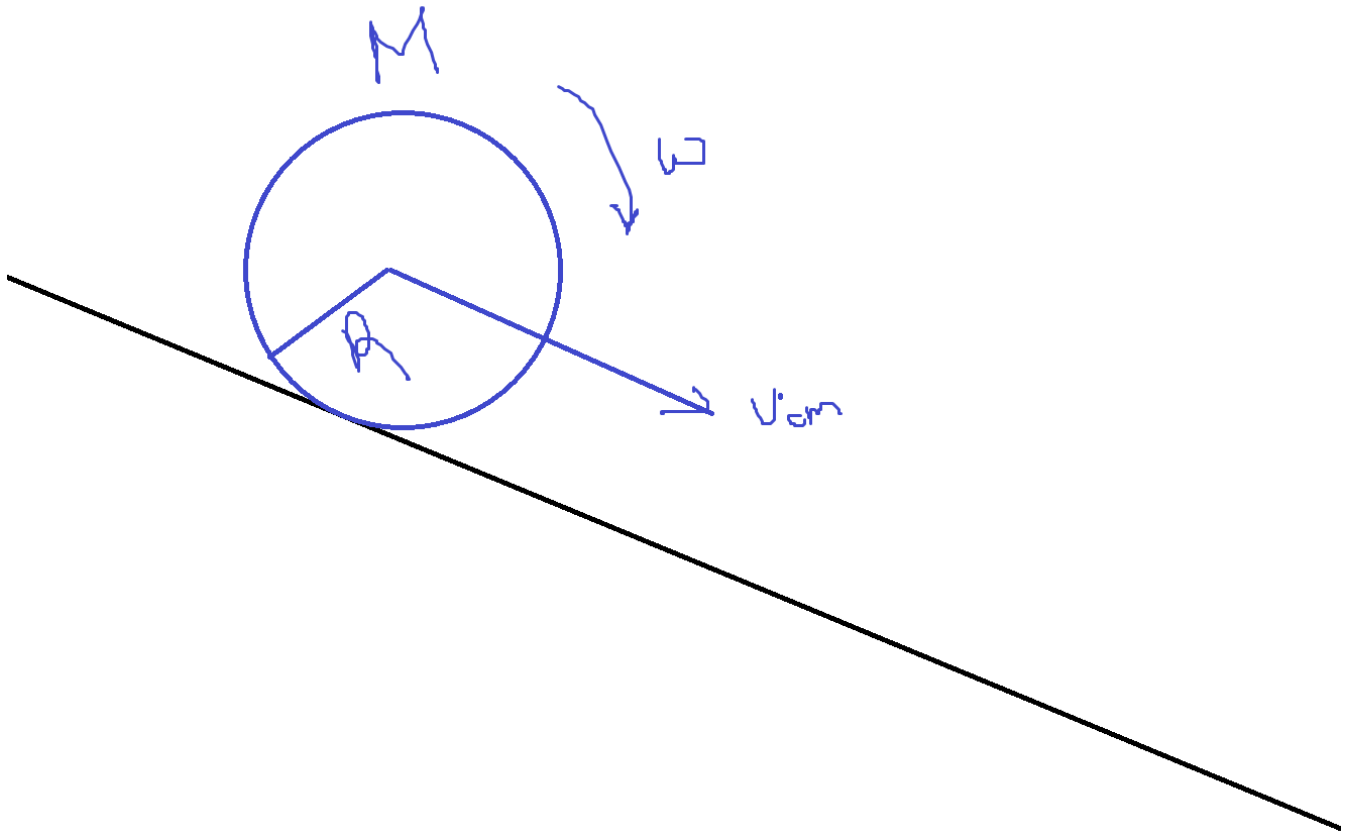
1. Axis passing through center of the sphere:

$$I_z = \frac{2}{5}MR^2$$

Problems with constant axis of rotation   Equation of the movement is:

$$\frac{d\vec{J}}{dt} = \vec{N} = \vec{r} \times \vec{F}$$

which is equivalent to second law.

Kinetic energy of rolling body is:

$$K = K_c + \frac{1}{2}mv_{cm}^2 = \frac{1}{2}I_c\omega^2 + \frac{1}{2}mR^2\omega^2 =$$
$$= \frac{1}{2}I_c\left(\frac{v_c m}{R}\right)\frac{1}{2}mv_{cm}^2 = \frac{1}{2}\left(\frac{I_c}{R^2} + M\right)v_{cm}^2$$

8.3   Torque and change in angular momentum

$$\frac{dK}{dt} = \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) \overset{\text{I} = \atop \text{const}}{=\!=\!=} \frac{1}{2}I\frac{d}{dt}\omega^2 = \frac{1}{2}\cdot I\left(\dot{\vec{\omega}}\vec{\omega} + \vec{\omega}\dot{\vec{\omega}}\right) = I\dot{\vec{\omega}}\vec{\omega} = \vec{N}\cdot\vec{\omega}$$
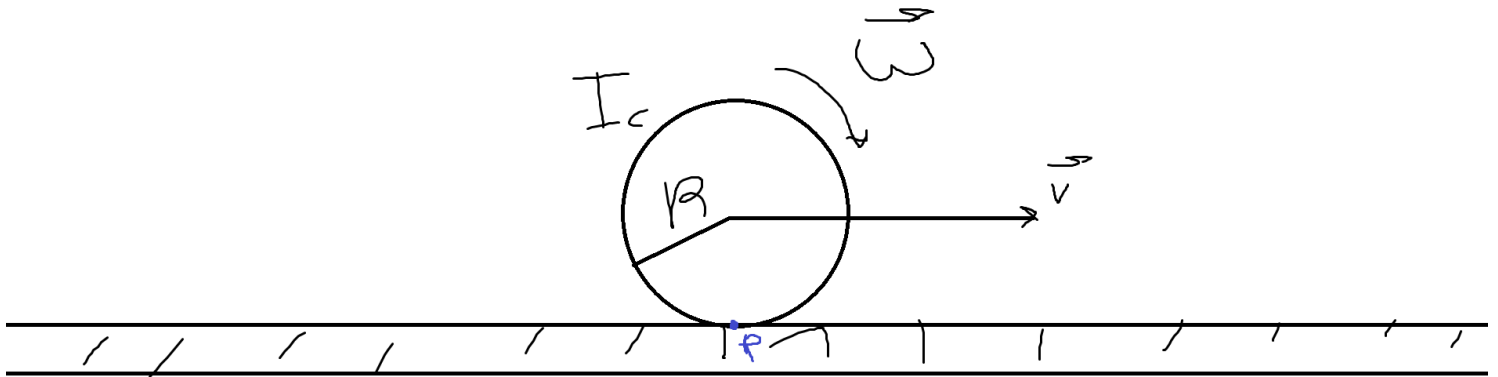
In point of contact p there is no relative motion of body and a surface so there is only **static friction**. The condition for full rolling:

$$\omega R = v_c$$

In a given moment the body rotates around point p with angular velocity ω.
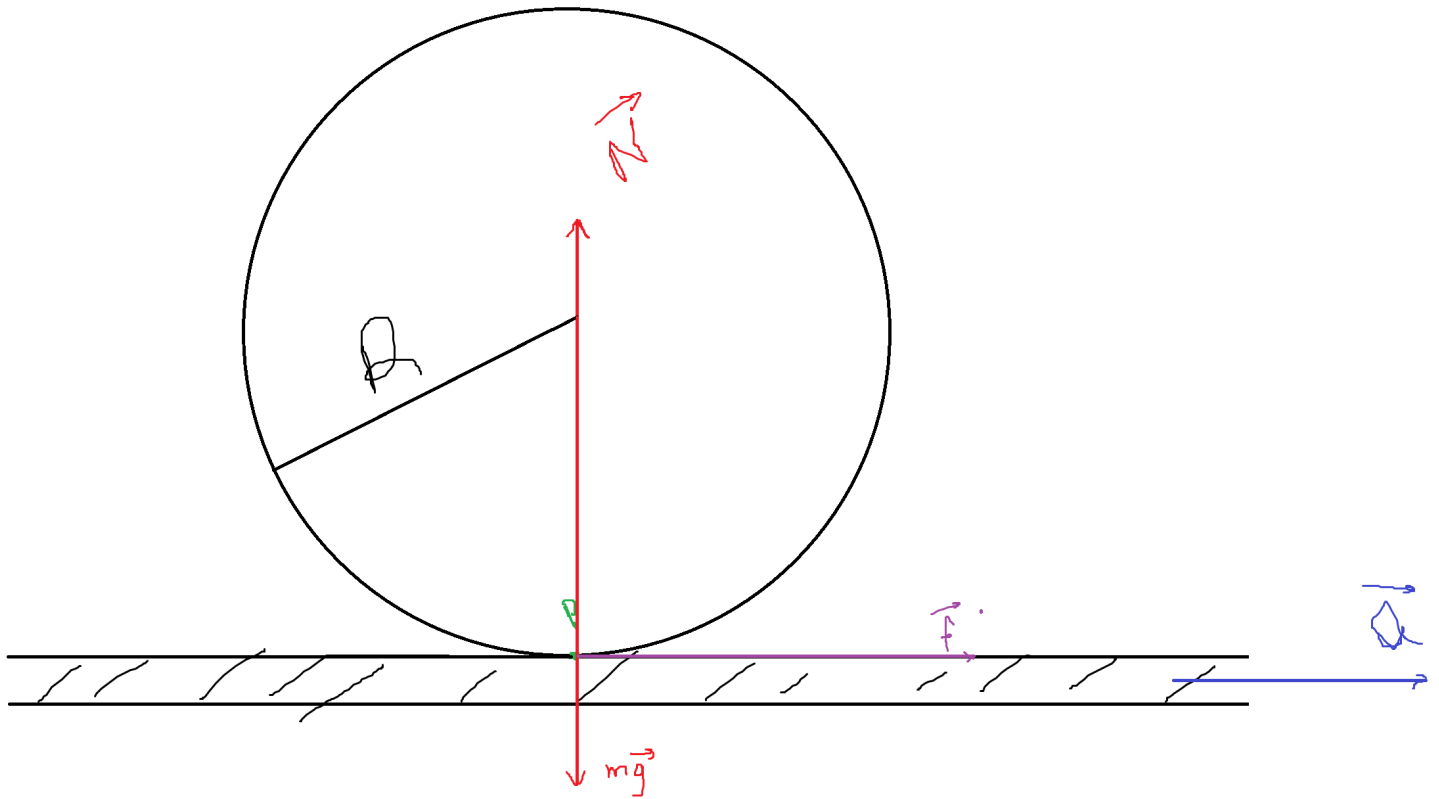
There is no moments for these forces:

$$\vec{N}_N = 0$$
$$\vec{N}_{mg} = 0$$
$$\vec{N}_f = 0$$

But there is still rotation since point p accelerates.
Lets work around center of mass:

$$M\frac{dv_c}{dt} = f$$

Since f is the only force in x direction. It's torque is

$$\vec{N}_{cf} = Rf$$

.
Motion equation for rolling (i.e. $\frac{d\omega}{dt}$):

$$I_c\frac{d\omega}{dt} = RF$$

By substituting $I_c = \frac{1}{2}MR^2$ for full cylinder:

$$\frac{1}{2}MR^2\frac{d\omega}{dt} = Rf$$
$$\dot{v}_c = \frac{1}{2}R\dot{\omega}$$

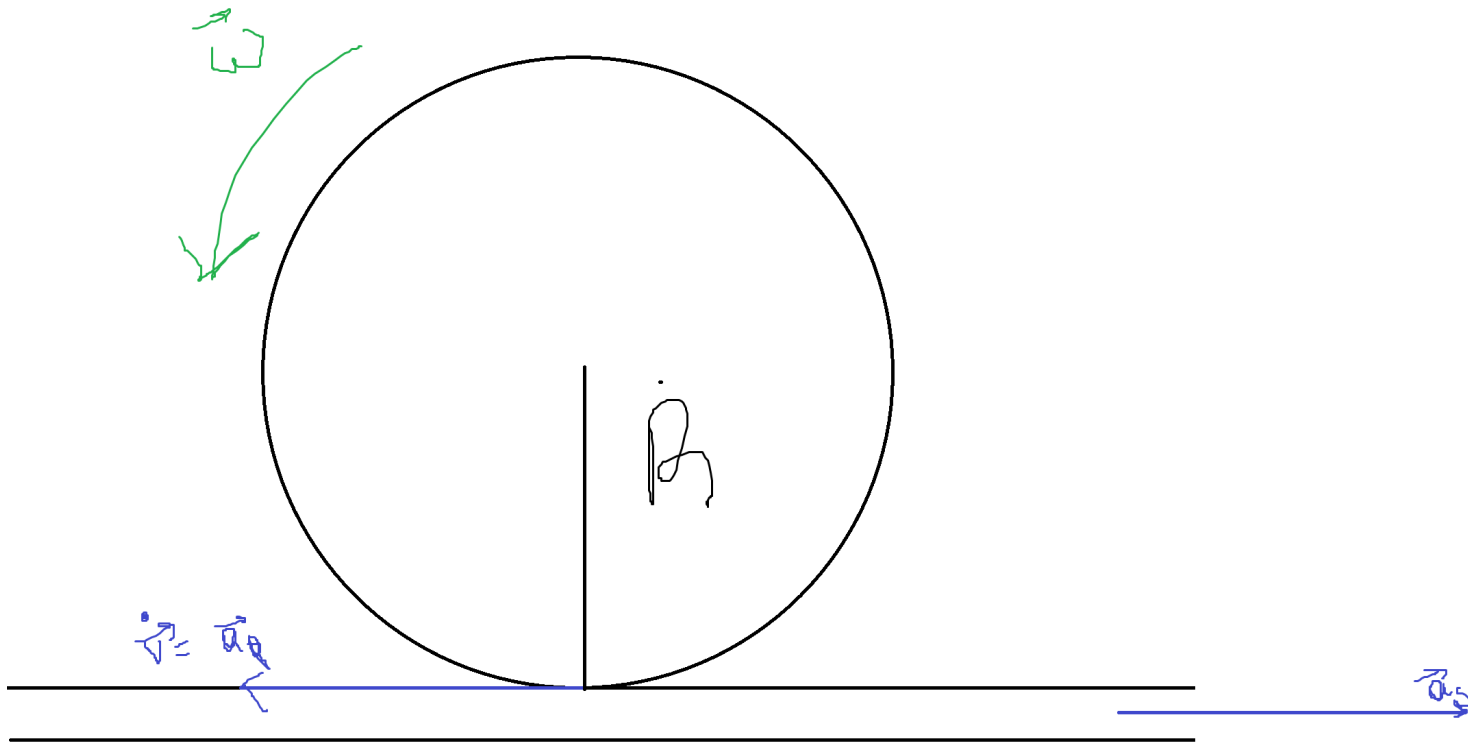Surface acceleration is

$$a_s = a_p = \underbrace{\frac{dv_c}{dt}}_{\substack{\text{acceleration due} \\ \text{to motion}}} + \underbrace{R\frac{d\omega}{dt}}_{\substack{\text{acceleration due} \\ \text{to rotation}}}$$

From

$$a = \frac{dv_c}{dt} + R\frac{d\omega}{dt}$$

we get $a_c = \dot{v}_c = \frac{1}{3}a$ and $f = \frac{1}{3}Ma$

Value	Linear	Angular
Position	$\vec{r}$	$\vec{\theta}$
Velocity	$\vec{v} = \dot{\vec{r}}$	$\vec{\omega} = \dot{\vec{\theta}}$
Acceleration	$\vec{a}\ddot{\vec{r}}$	$\vec{\alpha} = \ddot{\vec{\theta}}$
Force	$\vec{F}$	$\vec{N}$
Mass	M	I
Momentum	$\vec{p}$	$\vec{J}$
Second law	$\vec{F} = m\vec{a}$	$\vec{N} = I\vec{\alpha}$
Kinetic energy	$K = \frac{mv^2}{2}$	$K = \frac{I\omega^2}{2}$
Work	$W = \int \vec{F}d\vec{x}$	$W = \int \vec{N}d\vec{\theta}$
Impulse	$\int \vec{F}dt$	$\int \vec{N}dt$
Power	$\vec{F} \cdot \vec{v}$	$\vec{N} \cdot \vec{\omega}$

9 Harmonic oscillator

Cyclic motion in non-closed trajectory.

9.1 Spring
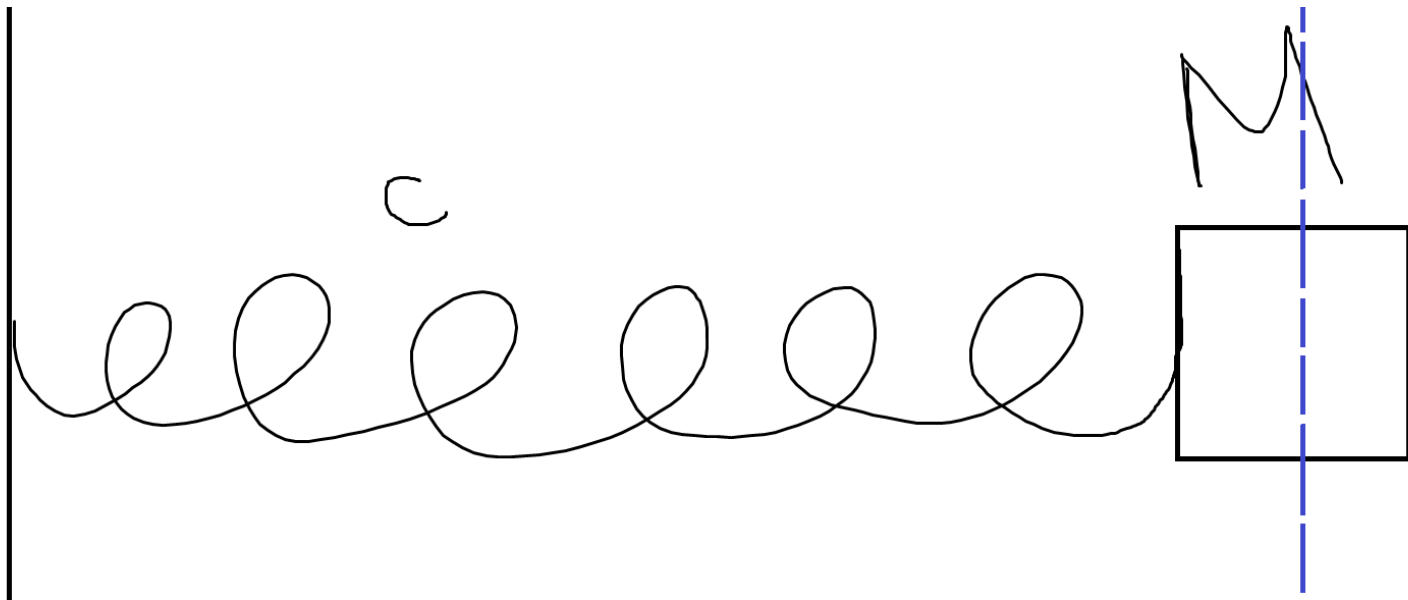
$x\hat{x}$ distance from equilibrium and C is spring constant. Force acting on body is:

$$F = -Cx\hat{x}$$

From second law motion equation is:

$$M\ddot{x} = -Cx$$

General solution is

$$x = A\sin(\omega_0 t + \phi) = A\cos\left(\omega_0 t + \phi - \frac{\pi}{2}\right)$$

or

$$x = B\sin\omega_0 t + D\sin\omega_0 t$$

$A = \sqrt{B^2 + D^2}$ is amplitude. The solution is

$$\dot{x} = \frac{d}{dt}[A\sin(\omega_0 t + \phi)] = A\omega_0\cos(\omega_0 t + \phi)$$

$$\ddot{x} = \frac{d}{dt}[A\omega_0\cos(\omega_0 t + \phi)] = -\omega_0^2 x$$

By substituting:

$$M\left(\omega_0^2 x\right) - Cx$$

Acquiring ω_0;

$$\omega_0 = \sqrt{\frac{C}{M}}$$

Note $\omega_0 = \left[\frac{rad}{s}\right]$, while number of cycles per second is $f_0 = \frac{\omega_0}{2\pi}$ and cycle time is $T = \frac{1}{f_0}$.

Energy conservation   Potential energy is $U = \frac{1}{2}Cx^2$. In regular problems this is a potential energy around equilibrium. Total energy is:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}cx^2$$

By deriving:

$$0 = mv\dot{v} + cx\dot{x}$$

which is the same equation by reduction of $\dot{x}$.
Another way to get it is to take energy when $x = 0$:

$$\frac{1}{2}mv^2 = \frac{1}{2}CA^2 - \frac{1}{2}cx^2$$

$$v = \frac{dx}{dt} = \sqrt{\frac{C}{m}}\sqrt{A^2 - x^2}$$

$$\frac{dx}{\sqrt{A^2 - x^2}} = \omega_0 dt$$

It can solved by integration and the solution of this equation is just same.

9.2 Damped Harmonic Oscillator

in addition to returning force there is damping force $f = -bv$, where $v = -\dot{x}$ and b is constant. Damping force coverts kinetic energy to heat (or something else) while returning force converts kinetic energy to potential and back. From second law:

$$m\ddot{x} = -Cx - d\dot{x}$$

Define $\tau = \frac{m}{b}[s]$ or $\frac{1}{\tau} = \frac{b}{m}$. Also $\omega_0^2 = \frac{C}{m}$. By dividing by m:

$$\ddot{x} + \frac{1}{\tau}\dot{x} + \omega_0^2 x = 0$$

We acquired homogeneous linear differential equation with solution:

$$x = x_0 e^{-\frac{t}{2\tau}}\sin\left(\omega_d t + \phi\right)$$

where ϕ and x_0 are initial conditions and

$$\omega_d = \left[w_0^2 - \frac{1}{(2\tau)^2}\right]^{\frac{1}{2}}$$

If $\omega_0^2 < \frac{1}{(2\tau)^2}$, then there is no oscillation.
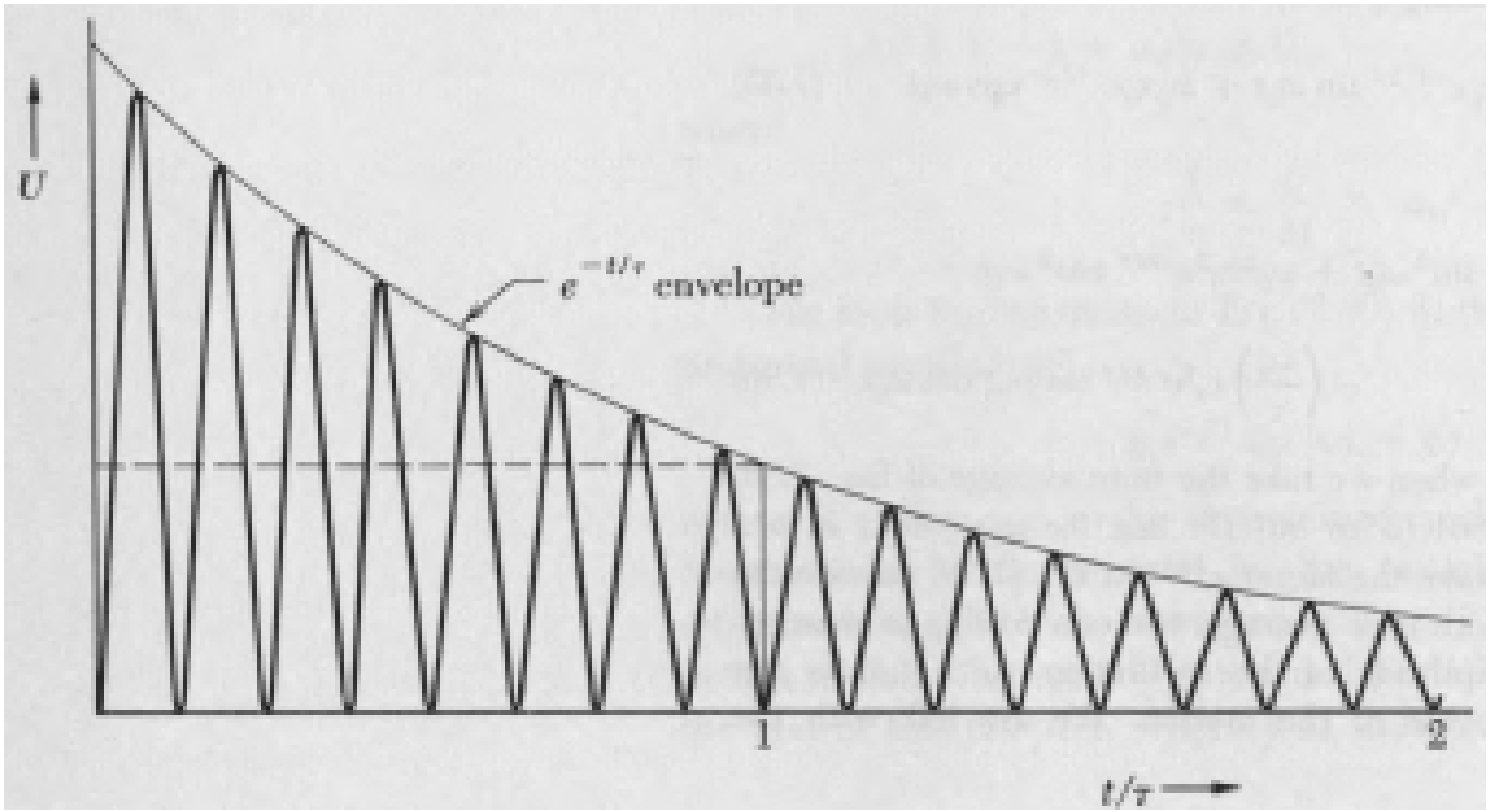
Potential energy   Around stable equilibrium points potential energy is approximately parabola $U = const \cdot (x - x_0)^2$

Average energy   Average energy for non-damped is

$$< E >= \frac{1}{t} \int_t E dt$$

Lets calculate an average energy for full cycle T:

$$x = A \sin (\omega_0 t + \phi)$$
$$\dot{x} = A \omega_0 \cos (\omega_0 t + \phi)$$

Kinetic energy is

$$< K >= \frac{1}{T} \int_0^T \left(\frac{1}{2} m \dot{x}^2 \right) dt = \frac{1}{2T} m \omega^2 A^2 \int_0^T \cos^2 (\omega_0 t + \phi) dt$$

Cycle time is $T = \frac{2\pi}{\omega_0}$. A an ϕ are constant and depend on initial state. Also, an average of sin is equal to average of cos *for full cycle*. Thus

$$\frac{1}{T} \int_0^T \cos^2 (\omega_0 t + \phi) dt = \frac{1}{T} \int_0^T \sin^2 (\omega_0 t + \phi) dt =$$
$$= \frac{1}{2T} \int_0^T \left[\cos^2 (\omega_0 t + \phi) + \sin^2 (\omega_0 t + \phi) \right] dt = \frac{1}{2}$$

By substituting in $< K >$:

$$< K >= \frac{1}{4} m \omega_0^2 A^2$$

Now, for potential energy:

$$< U >= \frac{1}{T} \int_0^T c x^2 dt = \frac{1}{T} \int_0^T \frac{1}{2} c A^2 \sin^2 (\omega_0 t + \phi) = \frac{1}{4} c A^2 = \frac{1}{4} m \omega_0^2 A^2$$

.
We acquired

$$< U >=< K >$$

, which is called virial theorem.

Damped oscillator   For damped oscillator average energy isn't constant.

$$\left(\substack{\text{power} \\ \text{of waste}} \right) = -\frac{d}{dt} < E >$$

If there are many (more than 3) oscillation in τ, then

$$-\frac{d}{dt} < E >= \frac{< E(t) >}{\tau}$$

Quality factor Q

$$Q = \frac{\text{energy stored}}{\text{energy lost in one oscilation}} = \frac{2\pi E(t)}{\left(\frac{-d<E>}{dt} \right) T}$$

or

$$Q = \omega_0 \tau$$

High Q means high loss of energy. In time τ energy is decreases by factor $\frac{1}{e}$.

9.3 Driven oscillator

Driving force which depends on time:
$$F(t) = F_0 \sin \omega_0 t$$

There is also system (spring) with undamped frequency

$$\omega_0 = \frac{c}{m}$$

Also there is damping

$$\tau = \frac{m}{b}$$

Also define

$$\alpha = \frac{F_0}{m}$$

The equation of motion is:

$$m\ddot{x} = F(t) - b\dot{x} - Cx$$

After F starts to action system it goes through complex changes and stands on steady-state solution

$$x = x_0 \sin(\omega t + \phi')$$

Note: x_0 and ϕ depends on parameters of system and not on initial conditions which decay:

$$\tan \phi' = -\frac{\frac{\omega}{\tau}}{\omega_0^2 - \omega^2}$$

$$x_0 = \frac{\frac{F_0}{m}}{\sqrt{(w_0^2 - w^2) + \left(\frac{\omega}{\tau}\right)^2}}$$

Phase ϕ' - what is phase difference between system's oscillation and driving force. Driving force is maximal in time $\omega t = \frac{\pi}{2}$. x is maximal in $\omega t + \phi' = \frac{\pi}{2}$ or $\omega t = \frac{\pi}{2} - \phi'$

Amplitude x_0 is maximal for $\omega = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}$. If there is no damping, i.e. $Q \to \infty$, then for $\omega = \omega_0$ $x_0 \to \infty$.

Examples

- $\omega \ll \omega_0$ then $\phi \to 0$:

$$x_0 \to \frac{\alpha_0}{\omega_0^2} = \frac{m\alpha_0}{c} = \frac{F_0}{c}$$

  Spring is much more important for x_0 than mass

- Resonance. $\omega = \omega_0$, i.e $\phi \to -\frac{\pi}{2}$

$$x_0 = \frac{\alpha_0 \tau}{\omega_0} = \frac{F_0}{b} \frac{1}{\omega_0}$$
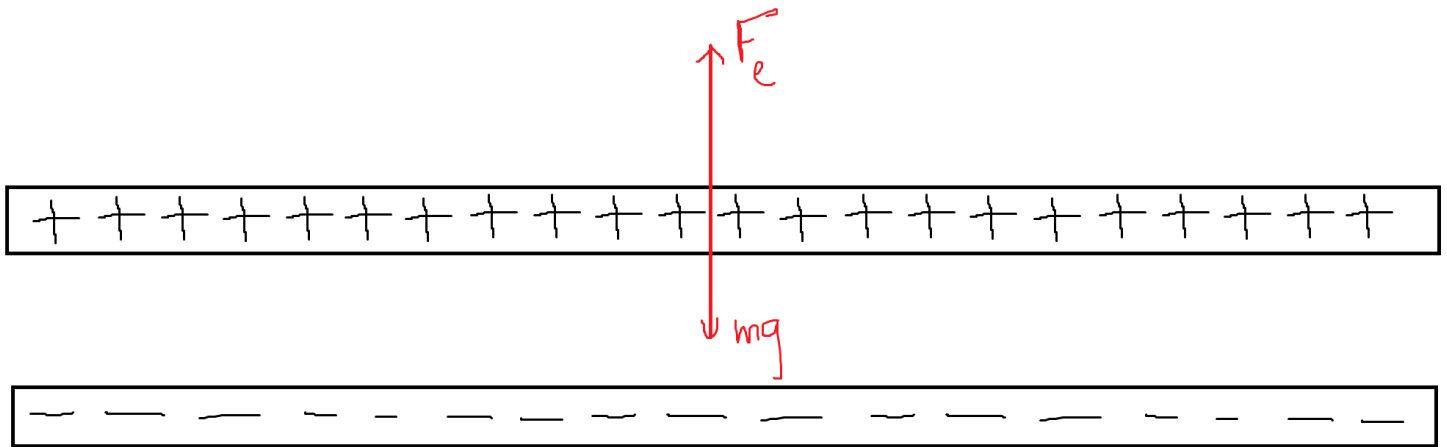
  If $b \to 0$ then $x_0 \to \infty$

- $\omega \gg \omega_0$ then $\phi \to -\pi$

$$x_0 \to \frac{\alpha_0}{\omega^2} = \frac{F_0}{m\omega^2}$$

10 Special relativity

Motivation Lack of agreement between Newton's laws and electromagnetic theory (Maxwell equations)

Example of disagreement   Given this system:

This is what sees observer in resting frame of reference. The observer in fast moving system sees something else. Since system is moving it can be described as current and there is new force (magnetic) downwards:
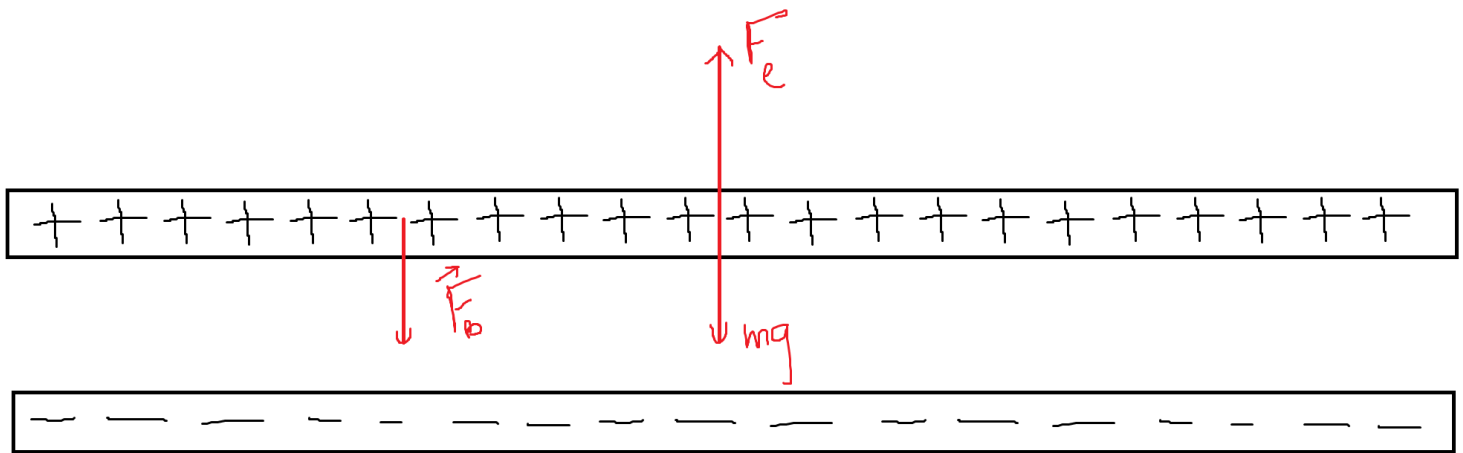
Solution of special relativity   The system is still in equilibrium, since F_E grows since density of electrical charge in a rod grows because of Lorentz contraction.

Michelson-Morley experiment   is another example of problem. Special relativity explains there is no ether.

Notes on known in 1904

- Electromagnetic radiation is moving at speed of light

- Speed of light is finite. It was first measured in 1676.

10.1   Assumption of special relativity

- Speed of light in vacuum is constant and denoted c.

- Symmetry. Our space is isotropic and homogeneous. The laws of physics are invariant (i.e. identical) in all inertial systems (non-accelerating frames of reference).

Experiment 1
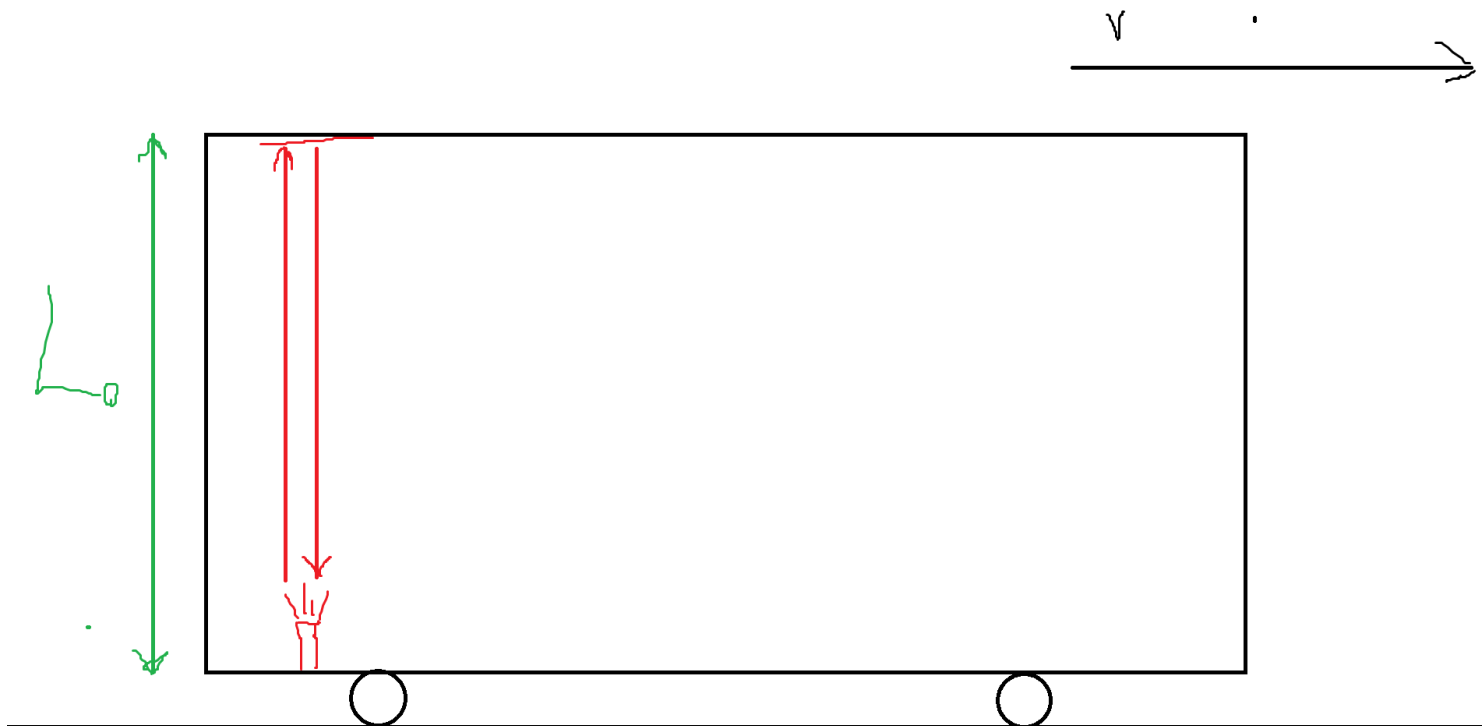
Time that takes light to get to the mirror and back is $t' = \frac{L_0}{c}$. Then train passes vt' until it light gets to the mirror. And since speed of light is constant, light passes ct'.
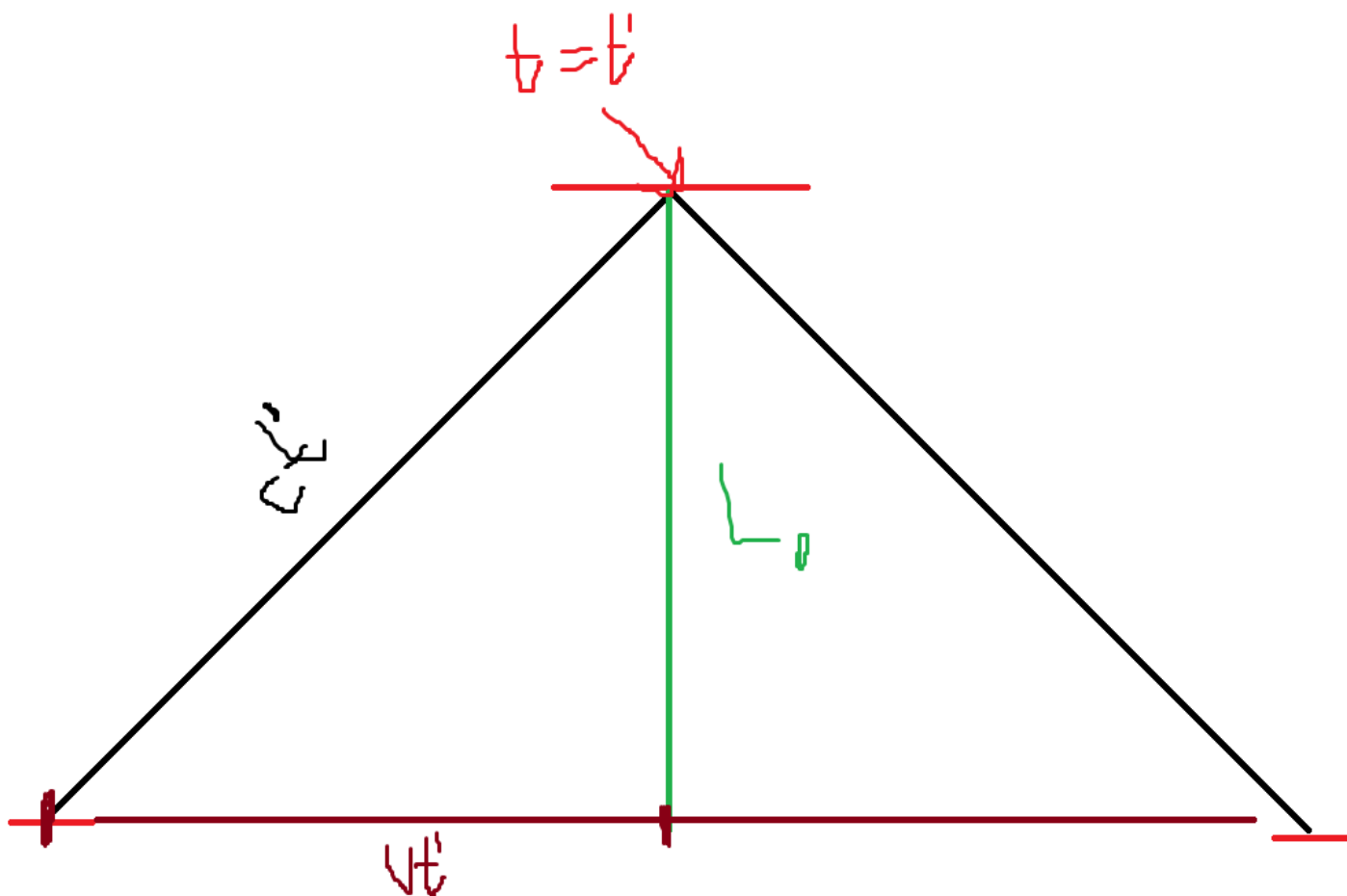
We get the following:

$$(vt')^2 + L_0^2 = (ct')^2$$

57

$$t'^2 \left(c^2 - v^2\right) = L_0^2$$

$$t' = \frac{L_0}{\sqrt{c^2 - v^2}} = \frac{\frac{L_0}{c}}{sqrt1 - \frac{v^2}{c^2}}$$

Since $\frac{L_0}{c} = t_0$:

$$t' = \frac{L_0}{\sqrt{c^2 - v^2}} = \frac{t_0}{sqrt1 - \frac{v^2}{c^2}}$$

Define $\gamma = \frac{1}{sqrt1 - \frac{v^2}{c^2}} \geq 1$.

Observer in train used single clock. Single clock to measure difference between two events is proper time. Event is a point in four-dimensional space: (x, y, z, w).

Proper time is the shortest time between two events.

10.2 Lorentz transformation

For two frame of reference that synchronized their clock at $t = t' = 0$ when they are at same place:
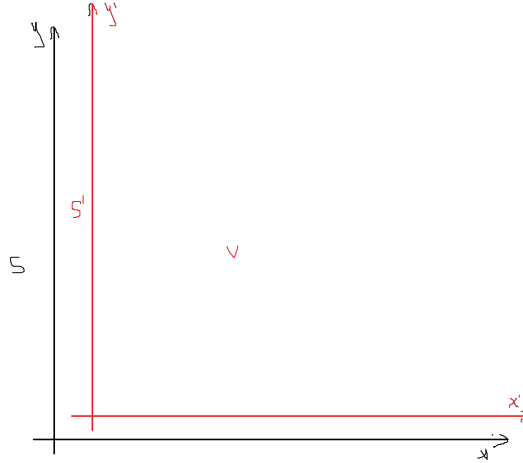
In frame of reference S'. At $t' = 0$ a frame S' turns on light which is spread as $x'^2 + y'^2 + z'^2 = (ct')^2$.

For stationary frame of reference, $x^2 + y^2 + z^2 = (ct)^2$, since speed of light is constant. By substituting Galileo transformation $x' = x - vt$ we get wrong equation: $x^2 - 2xvt + y^2 + z^2 = (ct)^2$ (inconsistent with what we got from constant speed of light).

Define new transformation It should be:

- linear (to have inverse transformation). General form is

$$\begin{cases} x' = \alpha x + \epsilon t \\ y' = y \\ z' = z \\ t' = \lambda x - \eta t \end{cases}$$

- $y' = y$ and $z' = z$ due to symmetry.

- At low speed $\frac{v}{c} \ll 1$ we get back to Newton-Galileo

We can find $\alpha, \epsilon, \delta, \eta$ by substituting equations for S and S':

$$\begin{cases} x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \\ y' = y \\ z' = z \\ t' = \frac{t - \frac{v}{c^2} x}{\sqrt{1 - \frac{v^2}{c^2}}} \end{cases}$$

Define

$$\beta = \frac{v}{c}$$
58

and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Then

$$
\begin{cases}
x' = \gamma\,(x - \beta ct) \\
y' = y \\
z' = z \\
t' = \gamma\left(t - \frac{\beta x}{c}\right)
\end{cases}
$$

Inverse transformation is acquired by replacing v with $-v$ (symmetry of observers)

$$
\begin{cases}
x = \gamma\,(x' + \beta ct) \\
y = y' \\
z = z' \\
t = \gamma\left(t' + \frac{\beta x'}{c}\right)
\end{cases}
$$

Experiment
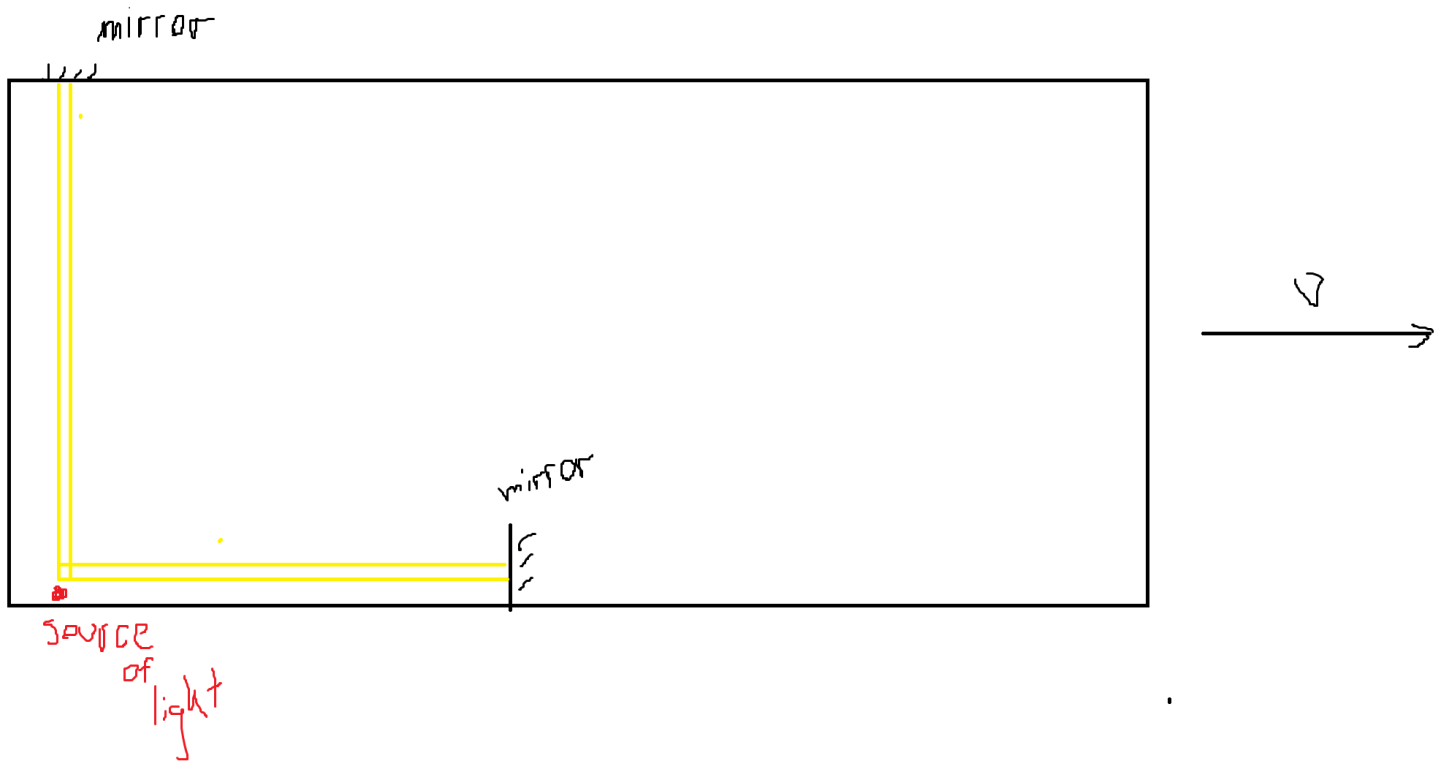
In Newtonian mechanics velocity of light would be $v + c$, but in relativity it's constant: c.
In frame of reference of lab, light passed $L + vt'$. Then

$$
ct' = L + vt'
$$

$$
t' = \frac{L}{c - v}
$$

When light returns from mirror,

$$
t'_{back} = \frac{L}{c + v}
$$

And total time according to lab:

$$
t'_{total} = \frac{L}{c - v} + \frac{L}{c + v} = \frac{2Lc}{c^2 - v^2} = \frac{2\frac{L}{c}}{1 - \frac{v^2}{c^2}}
$$

From previous experiment:

59

$$t'_{total} = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting all together:

$$\frac{2\frac{L}{c}}{1 - \frac{v^2}{c^2}} = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting $t_0 = \frac{2L_0}{c}$ we get:

$$L = \sqrt{1 - \frac{v^2}{c^2}} L_0$$

Lorentz contraction   Rod of length L_0 is resting in frame S. L_0 is a proper length (length in rest frame). Clock are synchronized at origin.

Length measurements   We mark position of both ends of rod at some time t' (according to two different clocks in S):

$$\begin{cases} x_1 = \gamma'\left(x'_1 + vt'\right) \\ x_2 = \gamma'\left(x'_2 + vt'\right) \end{cases}$$

Acquiring

$$L_0 = x_2 - x_1 = \gamma'\left(x'_2 - x'_1\right)$$

10.3   Relativity of simultaneity

Let's write Lorentz transformation for both events:

$$t'_1 == \gamma'\left(t_1 - \frac{v}{c^2}x_1\right)$$

$$t'_2 == \gamma'\left(t_2 - \frac{v}{c^2}x_2\right)$$

Suppose $t'_1 = t'_2$:

$$\frac{v}{c^2}\left(x_2 - x_1\right) = t_2 - t_1$$

So if $x_2 > x_1$ then $t_2 > t_1$, i.e. if in one frame of reference two events are simultaneous, in another frame of reference the "backward" event happens first.

Invariant

$$\left(ct'\right)^2 - \left(x'^2 + y'^2 + z'^2\right) = (ct)^2 - \left(x^2 + y^2 + z^2\right)$$

Lets substitute Lorentz transformation to the left side:

$$\left(ct'\right)^2 - x'^2 - y'^2 - z'^2 = \frac{(ct)^2 - 2tvx + \left(\frac{vx}{c}\right)^2}{1 - \frac{v^2}{c^2}} - \frac{x^2 - 2tvx + v^2t^2}{1 - \frac{v^2}{c^2}} - y^2 - z^2 =$$

$$= (ct)^2\left[\frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right] - x^2\left[\frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}}\right] - y^2 - z^2 = (ct)^2 - \left(x^2 + y^2 + z^2\right)$$

If $\left(c\Delta t\right)^2 - \left(\Delta x\right)^2 - \left(\Delta y\right)^2 - \left(\Delta z\right)^2 < 0$ then events switch their order, and no information can pass from one to the other.

Time dilatation Proper time τ - is time measured in same place (with same clock), i.e Δx (and of course Δy, Δz) is 0. By substituting in Lorentz transformation:

$$t_2' = \gamma \left(t_2 - \frac{v}{c^2} x_2 \right)$$

$$t_1' = \gamma \left(t_1 - \frac{v}{c^2} x_1 \right)$$

$$\Delta t' \overset{x_2 \overset{=}{\equiv} x_1}{=} \gamma (t_2 - t_1)$$

Or simply

$$\Delta t' = \frac{\Delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Example Muon decay

$$\mu^- \to e^- + \bar{v}_e + v_\mu$$

$$\mu^+ \to e^+ + v_e + \bar{v}_\mu$$

In cosmic radiation in is created at hight of 6km and lives for around $2 \times 10^{-6} s$

Example Neutron

$$n \to p + e^- + \bar{v}_e$$

Half life of neutron is $t_{\frac{1}{2}} = 10.6s$ Distance from Earth to Pluto $L_0 = 330(c \cdot 60s) = 5.94 \times 10^{12} m$.
Say we accelerate neutron to velocity of $v = \frac{220}{221} c$ on Earth to the Pluto's direction. Which part of neutrons will get to the Pluto?

Answer The distance that neutron passes $L_0 = 330 c \cdot min$. The time required to pass this distance $t_0 = \frac{L_0}{v} = \frac{330 \cdot 221}{220 v} = 331.5 min$. But neutrons decay according to proper time.

$$\tau = t_{Earth} \sqrt{1 - \frac{v^2}{c^2}} = (331.5 min) \sqrt{1 - \left(\frac{220}{221} \right)^2} = 331.5 \frac{21}{221} = 31.5 min$$

$$N = N_0 2^{-\frac{31.5}{10.6}} \approx N_0 2^{-3} = \frac{N_0}{8}$$

Now when neutron measures distance from Earth to Pluto (which are moving relative to him):

$$L = \frac{L_0}{\gamma} = L_0 \sqrt{1 - \frac{v^2}{c^2}} = (330 c \cdot min) \frac{21}{221}$$

Time which will take to pass this distance is

$$t = \frac{L}{v} = \frac{330 \times \frac{21}{221}}{\frac{220}{221} c} = 31.5 min$$

10.4 Lorentz transformation for velocity

Particle moves with velocity u in frame of reference S. Lets derive x' by t:

$$\frac{dx'}{dt} = \frac{d}{dt} [\gamma (x - vt)] = \gamma \frac{dx}{dt} - \gamma v = \gamma u_x - \gamma v$$

If u_x is negative then the number is greater than c, but this is **not** velocity.
And time's derivative is:

$$\frac{dt'}{dt} = \frac{d}{dt} \left[\gamma \left(t - \frac{v}{c^2} x \right) \right] = \gamma - \gamma \frac{v}{c^2} u_x$$

Now we can calculate velocity:

$$u_x' = \frac{dx'}{dt'} = \frac{dx'}{dt'} \frac{dt}{dt} = \frac{\gamma u_x - \gamma v}{\gamma \left(1 - \frac{v}{c^2} u_x \right)} = \frac{u_x - v}{1 - \frac{v}{c^2} u_x}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy'}{dt'}\frac{dt}{dt} = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

$$u'_z = \frac{dz'}{dt'} = \frac{dz'}{dt'}\frac{dt}{dt} = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

Inverse transformation: replace $-$ with $+$.

Example $u_y = c$ Then $u'_x = -v$ and $u_y = \sqrt{1 - \frac{v^2}{c^2}}c$ and $|u'| = \sqrt{(-v)^2 + 1 - v^2} = c$.

Velocity addition Two particle move with two different velocities in opposite directions u_1, u_2 in direction x in frame of reference S. Say S' is frame of reference of first particle. Then $v = u_1$. Substitute it and get:

$$u'_2 = \frac{-u_2 - u_1}{1 + \frac{u_1 u_2}{c^2}} = -\frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}}$$

This is velocity addition rule.

Example $u_1 = 0.8c$ and $u_2 = 0.9c$, then relative velocity will be $u'_1 = \frac{0.8c + 0.9c}{1 + \frac{0.8 \cdot 0.9c^2}{c^2}} = \frac{1.7}{1.72}c < c$

10.5   Doppler effect

Is the change in frequency of a wave for an observer moving relative to its source. If a wave moves in medium it's important who is moving and who isn't. For light in vacuum it's not important.
Frame of reference S sends two pulses to frame S'. At $t = 0$ both are situated in $x = 0$ and $x' = 0$ and a pulse is sent and received immediately.
Now let's calculate when a second pulse, sent at $t = \tau$ is received.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = -\gamma v\tau$$

$$t' = \frac{1 - \frac{v}{c^2}x}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\tau$$

Now the time that takes the signal to get to the receiver is

$$\Delta t' = \frac{x'}{c} = \frac{\gamma v\tau}{c}$$

Then total time is

$$\tau' = t' + \Delta t' = \gamma\tau\left(1 + \frac{v}{c}\right) = \tau\sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

We can see that as sender is moving from receiver, then a frequency of receiving signal is decreased, and if it's moving toward, frequency is increased:

$$f' = \frac{1}{\tau} = f\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

Decrease of frequency is called redshift.

Photons Photon is particle of light. It's energy is

$$E_\gamma = \underbrace{h}_{\substack{\text{Planck's} \\ \text{constant}}} \underbrace{f}_{\text{frequency}}$$

If you're move away from source of light, photon energy is decreased:

$$E'_\gamma = E_\gamma\sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}$$

General case   If there is angle θ between sender and receiver, then

$$\tau' = \tau \frac{1 + \beta \cos \theta}{\sqrt{1 - \beta^2}}$$

And frequency is opposite to time:

$$f' = f \frac{sqrt1 - \beta^2}{1 + \beta \cos \theta}$$

10.6   Quantities

- Conserved quantity - physical quantity that if a closed system is not affected by external forces, cannot change.

- Invariant - quantity which remains unchanged in different frames of reference.

Examples

- Mass is invariant but not conserved.

- Proper time - invariant.

- Energy and momentum are conserved.

- Electrical charge is both invariant and conserved.

10.7   Energy and momentum in relativity
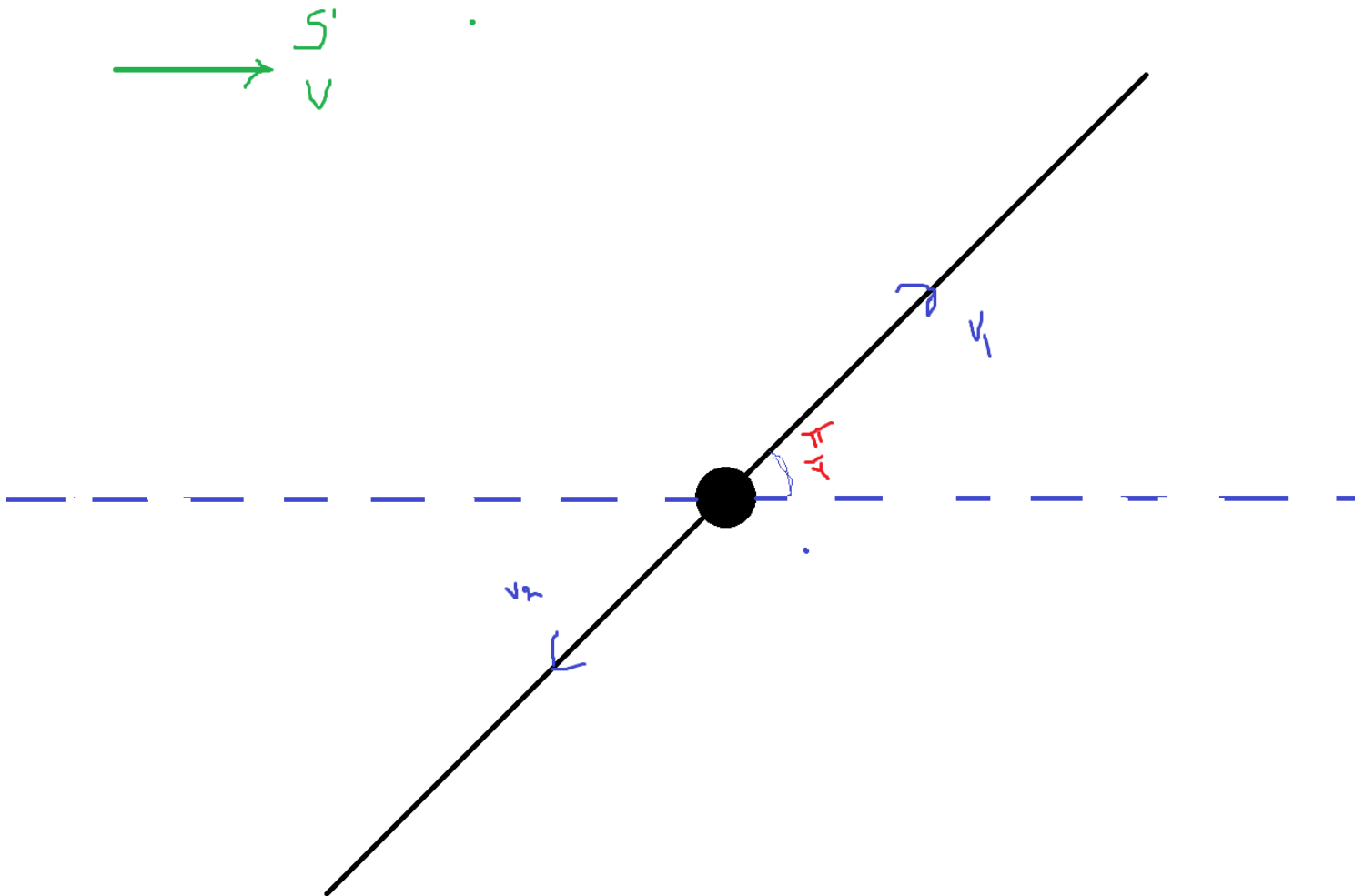
A mass m is decayed into two equal parts moving in direction $\hat{x} + \hat{y}$ with $v_x, v_y = v$. Frame of reference S' moves with same velocity V in direction $\hat{x}$. Before decay, momentum in S' in direction $\hat{y}$ is 0. Velocities after decay are

$$v'_{1y} = \frac{v}{\gamma \left(1 - \frac{v^2}{c^2}\right)} = \frac{v}{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}$$

$$v'_{2y} = -\frac{v_{2y}}{\gamma \left(1 - \frac{vv_{2y}}{c^2}\right)} = -\frac{v}{\gamma \left(1 + \frac{v^2}{c^2}\right)} = \frac{-v}{1 + \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}}$$

Since momentum before decay is 0, then after decay it must be 0 too. That means that classical momentum expression (mv) isn't good anymore.

We need to find new expression for momentum:

1. Same formulation for all inertial frames of references

2. Switch between frame of references using Lorentz transformation

3. If $v \ll c$, we get Newtonian mechanics.

Clues:

1. $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

2. δy is equal for all observers moving in direction $\hat{x}$

3. $\Delta \tau$ - proper time is identical for all observers: $\Delta \tau = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$

Based on 2 and 3 we create new quantity

$$\frac{\Delta y}{\Delta \tau} = \frac{\Delta y}{\Delta t} \frac{\Delta t}{\Delta \tau} = v_y \frac{1}{\frac{d\tau}{dt}} = v_y = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma v_y$$

.

Relative momentum   acquired from clues:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma m \vec{v}$$

10.8   Energy

$$K = E_k = \begin{pmatrix} \text{Work to get} \\ \text{particle} \\ \text{from v=0} \\ \text{to v=v} \end{pmatrix}$$

Substitute force from Newtonian mechanics ($\vec{F} = \frac{\vec{p}}{t}$):

$$K = W = \int_0^{x_f} \vec{F}\,(dx\hat{x}) = \int_0^{x_f} \frac{d}{dt}\left[\frac{mv}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}\right] dx \overset{\substack{\text{switch} \\ \text{of inte-} \\ \text{gration} \\ \text{variable}}}{=} \int_0^{t_f} \left[\frac{m\frac{dv}{dt}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} - \frac{1}{2}\frac{mv\left(\frac{-2v}{c^2}\right)\frac{dv}{dc}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}\right]\frac{dv}{dt}\,dt$$

$$K = \int_0^{t_f} \frac{mv\frac{dv}{dt}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}}\,dt = \int_0^{t_f} \frac{d}{dt}\left[\frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}\right]dt$$

By substituting $v_0 = 0$ and $v_f = v$:

$$K = \frac{mc^2}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} - mc^2$$

$$K = \gamma mc^2 - mc^2$$

When γmc^2 is total energy of the body and mc^2 is rest energy of the body.

$$Erest = mc^2$$

$$E = \gamma mc^2$$

Back to Newton $v \ll c$:

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 1 + \frac{v^2}{2c^2}$$

Then

$$E_k = \left(1 + \frac{v^2}{2c^2}\right)mc^2 - mc^2 = \frac{1}{2}mv^2$$

Exact development of momentum

$$1 = \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} = \frac{1}{1 - \frac{v^2}{c^2}} - \frac{\frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} = \gamma^2 - \frac{v^2}{c^2}\gamma^2$$

Since LHS is invariant, then RHS is also invariant. m^2c^4 is also invariant since both m and c are invariant. By multiplying we get:

$$m^2c^4 = m^2c^4\left(\gamma^2 - \frac{v^2}{c^2}\gamma^2\right)$$

$$\left(mc^2\right)^2 = \left(\gamma mc^2\right)^2 - \left(\gamma mvc\right)^2$$

It looks like another expression:

$$\left(c\Delta\tau\right)^2 = \left(c\Delta t\right)^2 - \left(\Delta\vec{r}\right)^2$$

By using connections between symmetry in time and conservation of energy and also between symmetry in space and conservation of moment we get $\vec{p} = \gamma m\vec{v}$ and $E = \gamma mc^2$ and also following expression:

$$m^2c^4 = E^2 - (pc)^2$$

Also

$$E = \sqrt{p^2c^2 - m^2c^4}$$

If $v \ll c$ then $p \ll mc$ and

$$E \simeq mc^2 + \underbrace{\frac{p^2}{2m}}_{\substack{\text{Kinetic} \\ \text{enegrgy in} \\ \text{Newtonian} \\ \text{mechanics}}}$$

Opposite, if $p \gg mc$ then

$$E \approx pc$$

For photon, $m = 0$,

$$E = pc$$

And

$$p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

In two different frames of references S and S':

$$m^2c^4 = E^2 - p^2c^2 = E'^2 - \left(p'c\right)^2$$

When

$$E = \gamma_p mc^2$$

$E' = \gamma'mc^2$ Denote

$$\gamma'_p = \frac{1}{\sqrt{1 - \frac{u'^2_p}{c^2}}}$$

65

10.9 Transformation of energy and momentum

Proper time is $\Delta\tau = \Delta t \left(1 - \frac{u^2}{c^2}\right)^{\frac{1}{2}}$. Velocity

$$u_x = \frac{dx}{d\tau} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\left(\frac{dt}{d\tau}\right)^{\frac{1}{2}}$$

.

Then $u_x = \frac{dx}{d\tau}\frac{1}{\gamma_p}$ and:

$$\begin{cases} E = \gamma_p mc^2 = mc^2\frac{dt}{d\tau} = & \frac{mc^2}{d\tau}dt \\ p_x = \gamma_p mu_x = m\frac{dx}{d\tau} = & \frac{mc^2}{d\tau}dx \\ p_y = m\frac{dy}{d\tau} = & \frac{mc^2}{d\tau}dy \\ p_z = m\frac{dz}{d\tau} = & \frac{mc^2}{d\tau}dz \end{cases}$$

Since m, τ and c are equal in all frames of reference, $E, \vec{p}$ are proportional to $t, \vec{p}$.
Transformation for energy and momentum is:

$$\begin{cases} p'_x = \gamma\left(p_x - \frac{v}{c^2}E\right) \\ p'_y = p_y \\ p'_z = p_z \\ E' = \gamma\left(tE - vp_x\right) \end{cases}$$

Note:

$$\vec{p} = \vec{u}\frac{E}{c^2}$$

and

$$pc = \sqrt{E^2 - m^2c^4}$$

Example Two photons with energy $E_\gamma = 0.9 MeV$ collide and produce e^+ and e^-. What is momentum of e^-?
First of all $\vec{p}_{e^-} + \vec{p}_{e^+}$.
Since $E_{e^-} + E_{e^+} = 2E_\gamma$ and $E_{e^+} = \sqrt{p_{e^+}^2 c^4 + m_e^2 c^4} = \sqrt{p_{e^-}^2 c^4 + m_e^2 c^4} = E_{e^-}$, then $E_{e^-} = 0.9 MeV$.

$$p_{e^-} = \frac{\sqrt{E_{e^-}^2 - m_e c^4}}{c} = \sqrt{(0.9)^2 - (0.5m)^2}\frac{MeV}{c} = 0.74\frac{MeV}{c}$$

Why can't single photon produce electron and positron? If that happens, than in frame of reference of center of mass of e^+ and e^- total momentum is 0. But photon has nonzero momentum, so momentum isn't conserver.
Note: $\gamma \rightarrow e^+ = e^-$ happens in the vicinity of nucleus of atoms, since nucleus takes "excess" of momentum.

Example Two particles with parameters m_1, E_1, p_1, v_1 and m_2, E_2, p_2, v_2 coliide and produce third particle which is in rest.

$$\vec{p_1} + \vec{p_2} = \vec{p_3}$$

$$\vec{p_1} = \vec{p_2}$$

$$\frac{m_1\vec{v}_1}{\sqrt{1 - \frac{v_1}{c}}} = -\frac{m_2\vec{v}_2}{\sqrt{1 - \frac{v_2}{c}}}$$

By applying Lorentz transformation for frame of reference S'. Before collision:

$$p'_{1x} + p'_{2x} = \gamma\left(p_{1x} + p_{2x}\right) - \frac{\gamma v}{c^2}\left(E_1 + E_2\right)$$

After collision:

$$p'_{1x} + p'_{2x} = p'_{3x} = \gamma p_{3x} - \frac{\gamma v}{c^2}E_3$$

We got $E_1 + E_2 = E_3$.

$$E_3 = m_3 c^2 = E_1 + E_2 = \frac{m_1 c^2}{\sqrt{1 - \frac{v_1^2}{c^2}}} + \frac{m_2 c^2}{\sqrt{1 - \frac{v_2^2}{c^2}}}$$

Obviously, $m_3 > m_1 + m_2$.

$$m_1^2 c^4 = E_1^2 - p_1^2 c^2$$
$$m_2^2 c^4 = E_2^2 - p_2^2 c^2$$
$$m_3^2 c^4 = E_3^2 - p_3^2 c^2$$

But

$$(m_1 + m_2)^2 \, c^4 \neq (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2 c^2$$

Velocity of center of mass   Sum of momentum in frame of reference of center of mass is 0:

$$p_{1cm} = -p_{2cm}$$

We want to find velocity of center of mass, so we apply transformation of Lorentz:

$$p_{1cm} = \gamma_{cm}\left(p_1 - \frac{E_1}{c^2}v_c m\right)$$

$$p_{2cm} = \gamma_{cm}\left(p_2 - \frac{E_2}{c^2}v_c m\right)$$

$$\gamma_{cm}\left(p_1 - \frac{E_1}{c^2}v_c m\right) = -\gamma_{cm}\left(p_2 - \frac{E_2}{c^2}v_c m\right)$$

$$p_1 + p_2 = v_{cm}\left(\frac{E_2}{c^2} + \frac{E_1}{c^2}\right)$$

$$v_{cm} = \frac{p_1 + p_2}{\frac{E_2}{c^2} + \frac{E_1}{c^2}}$$

Or more generally

$$v_{cm} = \frac{\sum_i p_i}{\sum_i \frac{E_i}{c^2}}$$

So $\frac{E_i}{c^2}$ is replacing Newtonian m_i.

DeBroglie Wavelength   Since $p = \frac{h}{\lambda}$ we can define $\lambda = \frac{h}{p}$ for particles, which is very small.

Threshold energy

$$p + p \rightarrow p + p + p + \bar{p}$$

For resulting particles velocity is very small, i.e. momentum is negligible. So

$$E_{0,cm} = 2\frac{m_p c^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$E_c m = 4 m_p c^2$$

Then minimal velocity needed for reaction for each proton is:

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2}$$

I.e $v_p = \frac{\sqrt{3}}{2}c$.
Now lets look at two particles, one is in rest and second is colliding with it. We know that $v_{cm} = \frac{\sqrt{3}}{2}$ and $v_{2cm} = \frac{\sqrt{3}}{2}$. Then v_2 in lab frame of reference:

$$v_2 = \frac{v_c m + v_{2cm}}{1 + \frac{v_c m \cdot v_{2cm}}{c^2}} = \frac{4\sqrt{3}}{7}c$$

$$E_2 = \frac{m_p c^2}{\sqrt{1 - \frac{v^2}{c^2}}} = 7 m_p c^2$$

Threshold kinetic energy when one of the protons is in rest is $6 m_p c^2$.
Center of mass frame of reference is optimal, since after collision all the particles are in rest.

11 General relativity

11.1 Equivalence principle

It isn't possible to know if we're in free fall towards mass or weightless in space.

11.2 Conclusions

Time is slowed down near mass   In special relativity

$$dt^2 + dr^2 - r^2 d\Phi^2 = dt^2 - dx^2 - dy^2$$

Near spherical mass in general relativity proper time of body at coordinate r is:

$$d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) dt^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2 d\phi^2$$

where t,r,ϕ are time, radial distance and angle of far observer.
To calculate r we take circumference of circle divided by 2π.
For body in rest time is progresses slower as body is deeper in potential well

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r}} dt$$

When $r = \frac{2GM}{c^2}$ which is called Schwarzschild radius, denominator becomes 0, and then time stops.

Difference of radius is more the difference of circumferences divided by 2π Radial distance which is measured by body in rest is given by

$$ds^2 = \frac{dr^2}{1 - \frac{2GM}{c^2 r}}$$

Then measured distance between r and $r + dr$ is

$$ds = \frac{dr}{\sqrt{1 - \frac{2GM}{c^2 r}}} > dr$$

It's curved geometry (not Euclidean).

Light is affected by gravity   Light is attracted to mass even though it's massless. Effect of gravitational lensing is based no this.

Light is slowed down in gravitational according to distant observer   Local observer will still measure c.

Gravitational Doppler effect   Light which is sent to farther radius is experiences redshift.

**Schwarzschild radius** $R_s = \frac{2GM}{c^2}$ is event radius. No information can get outside.

$$R_s \cong 3 \left(\frac{M}{M_\odot}\right) km$$

Black hole   Mass which is concentrated inside Schwarzschild radius is a black hole which is characterized by mass, angular momentum and charge.
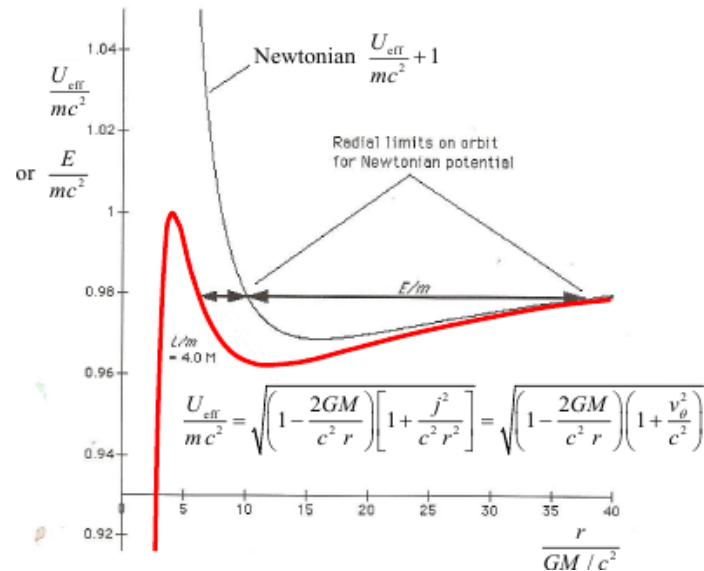
Distant observer sees fall in black hole taking infinite time   Body is stopped on Schwarzschild radius asymptotically.

Circular orbit of light   exist unique non-stable orbit $r = \frac{3}{2} R_s$ in which light is rotating around black hole.

Elliptic orbit precession

MERCURY'S ORBIT
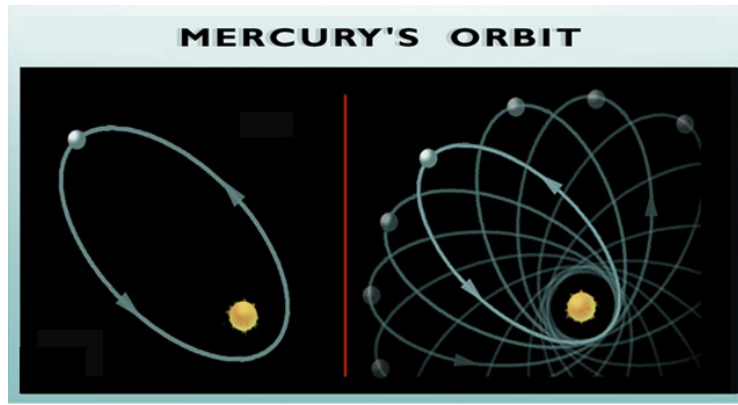
Effective potential in general relativity is different:

$$\frac{U_{eff}}{mc^2} = \sqrt{\left(1 - \frac{2GM}{c^2\,r}\right)\left[1 + \frac{j^2}{c^2\,r^2}\right]} = \sqrt{\left(1 - \frac{2GM}{c^2\,r}\right)\left(1 + \frac{v_\theta^2}{c^2}\right)}$$

Results are:

1. There are to circular orbits: one stable (Newtonian) and one more unstable.

2. There is no stable orbit on distance less then $r = 6\frac{GM}{c^2}$

3. Angular cycle time is less then radial cycle time:

$$T_r = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}\left(1 + \frac{3}{2}\frac{GM}{c^2 a} + \dots\right) = T_{Newton}\left(1 + \frac{3}{2}\frac{v_{Kepler}^2}{c^2} + \dots\right)$$

$$T_\theta = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}\left(1 - \frac{3}{2}\frac{GM}{c^2 a} + \dots\right) = T_{Newton}\left(1 - \frac{3}{2}\frac{v_{Kepler}^2}{c^2} + \dots\right) < T_r$$

Around rotating mass even bodies without angular momentum rotate it is called rotational frame-dragging, since mass is dragging spacetime.

It is possible that body falls into black hole and it will lose mass if the body has significant angular momentum in opposite direction.