114246 - Electromagnetism and Electrodynamics

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Abstract

1 Introduction

In this course we use CGS system. Force between two charges is

$$\vec{\mathbf{F}} = \frac{q_1 q_2}{\mathbf{r}^2}\hat{\mathbf{r}}$$

The unit of charge is statcoulomb, or esu.
Field around charge q is

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q'} = \frac{q}{\mathbf{r}^2}\hat{\mathbf{r}}$$

Then force can be written as

$$\vec{\mathbf{F}} = q'\vec{\mathbf{E}}$$

Principle of linearity (superposition) If we have some frame of reference we can rewrite force as

$$\vec{\mathbf{F}}_1 = \frac{q' q_1}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_1\right|^3}\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

And fields can be summed as following:

$$\vec{\mathbf{E}} = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} \frac{q_i(\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i)}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i\right|^3}$$

If charge is continuous define

$$\rho(\vec{\mathbf{r}}) = \frac{\Delta q}{\Delta V}$$

field turns into integral:

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3\mathbf{r}'\,\rho(\vec{\mathbf{r}'})\frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3}$$

Potential

$$\vec{E} = \frac{q\vec{\mathbf{r}}}{\mathbf{r}^2} = -\vec{\boldsymbol{\nabla}}\frac{q}{r}$$

For some frame of reference

$$\vec{E} = \frac{q(\vec{\mathbf{r}} - \vec{\mathbf{r}'})}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3} = -\vec{\boldsymbol{\nabla}}\frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|} = -\vec{\boldsymbol{\nabla}}\Phi$$

Gradient, divergence and Laplacian in spherical coordinates

$$\vec{\boldsymbol{\nabla}} f(r,\theta,\phi) = \hat{\mathbf{r}}\frac{\partial f}{\partial x} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{\partial f}{\partial \phi}\frac{1}{r\sin\theta}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(\mathbf{r}^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Continuous case

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r' \, \rho\left(\vec{\mathbf{r}'}\right) \frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = -\vec{\boldsymbol{\nabla}} \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

And

$$\Phi = \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

Gauss theorem

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}(\mathbf{r}) = \oint \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \mathbf{A}_n \, \mathrm{d}s$$

Lets apply Gauss theorem on electric field of point charge in origin:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \vec{\boldsymbol{\nabla}} \cdot \frac{q\vec{\mathbf{r}}}{\mathbf{r}^3}$$

Then

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = -\nabla^2 \frac{q}{r}$$

If $r \neq 0$,

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = q\vec{\boldsymbol{\nabla}} \frac{\vec{\mathbf{r}}}{\mathbf{r}^3} = q\vec{\boldsymbol{\nabla}} \frac{\hat{\mathbf{r}}}{\mathbf{r}^2} = \frac{q}{\mathbf{r}^2} \frac{\partial}{\partial r} \left(\mathbf{r}^2 \frac{1}{\mathbf{r}^2}\right) = 0$$

By applying Gauss law:

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \oint \vec{\mathbf{E}} \, \mathrm{d}\vec{\mathbf{s}} = \int \mathrm{d}\Omega \, \frac{q\hat{\mathbf{R}}}{\mathbf{R}^2} \cdot \vec{\mathbf{R}} \cdot \mathbf{R}^2 = q \oint \mathrm{d}\Omega = 4\pi q$$

Which is right for a ball of any radius, in particular, $R \to 0$. Thus

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0$$

while

$$\int\limits_{V_\epsilon} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3\mathbf{r} = 4\pi q$$

Delta function Lets define

$$\begin{cases} \delta^D = 0 & x \neq 0 \\ \int\limits_{x \in V} \delta^D(x) \, \mathrm{d}x = 1 \end{cases}$$

We can define it as limit of

$$F_\Delta = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} < x < \frac{\Delta}{2} \\ 0 & otherwise \end{cases}$$

Then $\lim_{\Delta \to 0} F_\Delta = \delta^D$.

Potential of point charge For a charge in point $\mathbf{r}_a$:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi q \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And for a couple of charges

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

Then we can define dencity of charge for a point charge as

$$\rho(\vec{\mathbf{r}}) = \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And then in both cases

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi\rho(\vec{\mathbf{r}})$$

Since

$$\vec{E} = -\vec{\nabla}\Phi$$
$$\vec{\nabla}\left(-\vec{\nabla}\Phi\right) = 4\pi\rho$$

Which means

$$\nabla^2\Phi = -4\pi\rho$$

which is Poisson equation. For bound conditions of $\phi, E \overset{r\to\infty}{\to} 0$ solution is $E = q\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}$.

1.1 Bound conditions

Direchlet bound conditions $\Phi = \Phi_S(\vec{\mathbf{r}})$ for $r \in S$.

Neumann bound condtions We have E_n on S.

1.2 Example of Solutions

Suppose we have Dirichlet bound conditions: $\Phi_S = 0$. Obvious solution is $\Phi = \rho = 0$
Lets show it's unique solution:

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int |\vec{\nabla}\Phi|^2 \,\mathrm{d}^3\mathbf{r}$$

$$|\nabla\Phi|^2 = \vec{\nabla}\Phi \cdot \vec{\nabla}\Phi = \vec{\nabla}\left(\Phi \cdot \vec{\nabla}\Phi\right) - \Phi\nabla^2\Phi = \vec{\nabla}\left(\Phi \cdot \vec{\nabla}\Phi\right)$$

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi \cdot \vec{\nabla}\Phi\right) \mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi \cdot \vec{\nabla}\Phi \cdot \mathrm{d}\vec{\mathbf{s}} = 0$$

Thus $E = 0$ and $\Phi = 0$.

General case In general case $\nabla^2\Phi = -4\pi q$. Suppose we have bound conditions $\Phi = \Phi_S \neq 0$. Suppose we have two solutions Φ_1, Φ_2.
Take a look at $\Phi = \Phi_1 - \Phi_2$, which has boundary conditions $\Phi_1 - \Phi_2 = \Phi_S - \Phi_S = 0$, thus solutions are equal, from previous paragraph.

Neumann bound conditions Instead of Φ we now have $E_n = -\vec{\nabla}\Phi \cdot \hat{\mathbf{n}} = 0$.

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi \cdot \vec{\nabla}\Phi\right) \mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi \cdot \vec{\nabla}\Phi \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \Phi \cdot \underbrace{\vec{\nabla}\Phi \cdot \hat{\mathbf{n}}}_{-E_n} \mathrm{d}s = 0$$

Thus $E = 0$. Note that E_n determines E_t.

Earnshaw theorem If in some volume $\rho = 0$, then there is no local maximum or minimum of potential in this volume, since then $\vec{\nabla}\Phi = 0$ and either $\nabla^2\Phi > 0$ or $\nabla^2\Phi < 0$, but $\nabla^2\Phi = \rho = 0$

1.3 Methods of solutions of Poisson equation

Method of image charges Suppose we have infinite grounded plane and a point charge in distance a from it. The bound condition is $\phi(x = 0, y, z) = 0$ And the potential of point charge is $\nabla^2\Phi = -4\pi q\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$
The way to solve this kind of problems is to add imaginary charge such that we get zero potential on bound (e.g., symmetrically with opposite charge).
If we have two such charges:

$$\Phi = \frac{q}{(x-a)^2 + y^2 + z^2} - \frac{q}{(x+a)^2 + y^2 + z^2}$$

This fulfills bound conditions:

$$\phi(x = 0, y, z) = \frac{q}{a^2 + y^2 + z^2} - \frac{q}{a^2 + y^2 + z^2} = 0$$

Denote density of charge on plane as σ, then $\mathrm{d}q = \sigma \,\mathrm{d}s$. If this case force between plane and charge is $\left|\frac{\sigma q \mathrm{d}s}{R^2}\right|$

How do we find σ? From Gauss law

$$\oint \vec{E} \cdot \mathrm{d}s = 4\pi \int\limits_V \rho \, \mathrm{d}^3\mathbf{r}$$

$$-|E_n| \, \mathrm{d}s = 4\pi\sigma \, \mathrm{d}s$$

$$4\pi\sigma = E_n$$

1.4 Green function

Suppose we have a unit charge in point $\vec{\mathbf{r}'}$. Then potential is $\Phi_{\vec{\mathbf{r}'}}$ such that $\Delta\Phi = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}'})$.

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \Phi_{\vec{\mathbf{r}'}}$$

Thus

$$\nabla^2 G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = -4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

Then we can write G as

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \frac{1}{|r - r'|} + F\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$$

And F guaranties bound conditions, while $\nabla^2 F = 0$.

Bound conditions

1. $G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = 0$.for bound S.
2. $\frac{\partial}{\partial \hat{\mathbf{n}}} G(\vec{\mathbf{r}}, \vec{\mathbf{r}'}) = -\frac{4\pi}{S}$. Then number comes from Gauss' law

Green theorem

$$\int\limits_V \left(\phi \vec{\boldsymbol{\nabla}}\psi - \psi \vec{\boldsymbol{\nabla}}\phi\right) \mathrm{d}^3\mathbf{r} = \oint\limits_S \left[\phi \frac{\partial\psi}{\partial\hat{\mathbf{n}}} - \psi \frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right] \mathrm{d}s$$

Since

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \left(\phi \vec{\boldsymbol{\nabla}}\psi\right) = \oint \phi \vec{\boldsymbol{\nabla}}\psi \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \phi \frac{\partial\psi}{\partial\hat{\mathbf{n}}} \, \mathrm{d}s$$

we get

$$\int\limits_V \mathrm{d}^3\mathbf{r} \left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \left(\phi\vec{\boldsymbol{\nabla}}\psi\right) - \vec{\boldsymbol{\nabla}}\phi \cdot \vec{\boldsymbol{\nabla}}\psi - \vec{\boldsymbol{\nabla}} \cdot \left(\phi\vec{\boldsymbol{\nabla}}\psi\right) + \vec{\boldsymbol{\nabla}}\psi \cdot \vec{\boldsymbol{\nabla}}\phi = \oint\limits_S \left[\phi\frac{\partial\psi}{\partial\hat{\mathbf{n}}} - \psi\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right] \mathrm{d}s$$

Now for $\psi(\vec{\mathbf{r}}) = G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$

$$\int\limits_V \mathrm{d}^3\mathbf{r} \left(-\phi \cdot 4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint\limits_S \mathrm{d}s \left[\phi\frac{\partial G}{\partial\hat{\mathbf{n}}} - G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

Suppose we want $\nabla^2\phi = 4\pi\rho(\vec{\mathbf{r}})$ for some ρ and Dirichlet bound conditions on same surface S:

$$\int\limits_V \mathrm{d}^3\mathbf{r} \left(-4\pi\phi\left(\vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint\limits_S \mathrm{d}s \left[\phi\frac{\partial G}{\partial\hat{\mathbf{n}}} - G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

Substituting $G = 0$ on the bound surface

$$-4\pi\phi\left(\vec{\mathbf{r}'}\right) + \int\limits_V \mathrm{d}^3\mathbf{r} \, G \cdot 4\pi\rho(\vec{\mathbf{r}}) = \oint\limits_S \mathrm{d}s \, \phi\frac{\partial G}{\partial\hat{\mathbf{n}}}$$

i.e.

$$\phi\left(\vec{\mathbf{r}'}\right) = \int\limits_V G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)\rho(\vec{\mathbf{r}}) \, \mathrm{d}^3\mathbf{r} - \oint\limits_S \frac{\phi}{4\pi} \cdot \frac{\partial G}{\partial\hat{\mathbf{n}}} \, \mathrm{d}s$$

Neumann bound conditions

$$-4\pi\phi\left(\vec{\mathbf{r'}}\right)+4\pi\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)\mathrm{d}^3\mathbf{r}=\oint\mathrm{d}s\left[\phi\left(-\frac{4\pi}{S}\right)-G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

However

$$\oint\phi\left(-\frac{4\pi}{S}\right)\mathrm{d}s=-\frac{4\pi}{S}\oint\phi\,\mathrm{d}s=-4\pi\langle\phi\rangle$$

$$\phi\left(\vec{\mathbf{r'}}\right)=\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)\mathrm{d}^3\mathbf{r}+\frac{1}{4\pi}\oint G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\mathrm{d}s+\langle\phi\rangle$$

Example

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)=\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r'}}\right|}$$

With bound conditions

$$0=G\left(\vec{\mathbf{r}}\to\infty,\vec{\mathbf{r'}}\right)$$

Thus

$$\phi(\vec{\mathbf{r}})=\int\rho(\vec{\mathbf{r}})\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r'}}\right|}\mathrm{d}^3\mathbf{r}$$

Example Suppose we have an infinite plane with potential $\phi(x=0,y,z)=\phi_S(y,z)$ and $\rho=0$ in one side of space ($x>0$). We are searching for $\phi(x>0,y,z)$.
Define Green function as a solution of previous problem, with point charge and grounded plane:

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)=\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}-\frac{1}{\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}$$

$$\phi\left(\vec{\mathbf{r'}}\right)=\underbrace{\int_V G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)\rho(\vec{\mathbf{r}})\,\mathrm{d}^3\mathbf{r}}_{0}-\oint_S\frac{\phi}{4\pi}\cdot\frac{\partial G}{\partial\hat{\mathbf{n}}}\mathrm{d}s$$

$$\frac{\partial G}{\partial\hat{\mathbf{n}}}=-\left.\frac{\partial G}{\partial x}\right|_{x=0}$$

$$\phi=\frac{x'}{2\pi}\int\mathrm{d}z\,\mathrm{d}y\,\frac{\phi_S(y,z)}{\left[x'^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}$$

Symmetry of Green function

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r'}}\right)=G\left(\vec{\mathbf{r'}},\vec{\mathbf{r}}\right)$$

1.5 Separation of variables

Suppose we have two planes parallel to y axis with zero potential with distance L between them, and a plane parallel to x axis with potential $V(x)$. We want to solve

$$\nabla^2\phi=0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Since there is no change on z direction we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Let's use anzatz for the solution $\phi=X(x)Y(y)$:

$$Y\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

Since we have operands depending on different variables, both are constant.

$$\begin{cases}\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=+k^2\\ \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}X^2}=-k^2\end{cases}$$

(since we want Y to decrease to 0 in infinity, we choose positive value for Y and negative for X).
The solution is

$$\begin{cases}X=a\cdot\sin(kx)+b\cdot\cos(kx)\\ Y=c\cdot e^{ky}+d\cdot e^{-ky}\end{cases}$$

Since $\phi(0,y,z)=\phi(L,y,z)=0$, $b=0$ and k acquires discrete values $k_n=\frac{n\pi}{L}$. Since $\phi(x,y\to\infty,z)=0$, $c=0$.
Now we need to force $\phi(x,y=0,z)=V(x)$:

$$\Phi=X(x)Y(y=0)=V(x)$$

We know that the solution is of form

$$d_na_n\sin(k_nx)e^{-k_ny}$$

Thus we can get a general solution in a form

$$\phi=\sum d_na_n\sin(k_nx)e^{-k_ny}$$

So we want

$$V(x)=\sum\underbrace{d_na_n}_{q_n}\sin(k_nx)$$

which is Fourier series:

$$\int\mathrm{d}x\,V(x)\sin(k_{n'}x)=\sum_n\int\mathrm{d}x\,q_n\sin(k_nx)\sin(k_{n'}x)=\sum_n q_n\frac{L}{2}\delta_{nn'}=\frac{Lq_{n'}}{2}$$

Example Suppose we have to planes with angle θ_0 between them and potential ϕ_0 on both. In cylindrical coordinates

$$\nabla^2\phi=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

By separation of variables

$$\phi=F(R)G(\theta)$$

$$\frac{G(\theta)}{R}\frac{\mathrm{d}F}{\mathrm{d}R}+G\frac{\mathrm{d}^2F}{\mathrm{d}R^2}+\frac{F(R)}{R^2}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=0$$

$$\begin{cases}\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=-\nu^2\\ \frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}R}+\frac{R^2}{F}\frac{\mathrm{d}^2F}{\mathrm{d}R^2}=\nu^2\end{cases}$$

$\nu>0$

$$\begin{cases}G=A\cos(\nu\theta)+B\sin(\nu\theta)\\ F=aR^{-\nu}+bR^{\nu}\end{cases}$$

$\nu=0$

$$\begin{cases}G=\tilde{A}+\tilde{B}\theta\\ F=\tilde{a}+\tilde{b}\ln R\end{cases}$$

We get $\tilde{b}=0$ such that F doesn't diverge in 0. Also there shouldn't be dependence on angle, so $\tilde{B}=0$ and $\tilde{A}\neq0$, thus for $\nu=0$ potential is constant.
For positive ν, $A=0$ and $a=0$ and also want $\sin(\nu\theta_0)=0$ thus

$$\phi(\theta)=\phi_0+\sum_{n=1}a_nR^{\frac{n\pi}{\theta_0}}\sin\left(n\pi\frac{\theta}{\theta_0}\right)$$

If $R\to0$, the most dominant element of sum is $n=1$. Thus

$$\phi\propto R^{\frac{\pi}{\theta_0}}$$

i.e., due to $E \sim -\frac{\partial \phi}{\partial R} \sim R^{\frac{\pi}{\theta_0}-1}$:

$$\begin{cases} E \to 0 & \theta_0 < \pi \\ E \to \infty & \theta_0 > \pi \end{cases}$$

1.6 Solution with Fourier transform

$$f_k = \int \mathrm{d}x\, e^{ixk} F(x) \iff \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{-ixk} f_k$$

$$\mathrm{d}^2 F\, x = \int \frac{\mathrm{d}k}{2\pi} (-k^2) e^{-ikx} f_k$$

Since $\nabla^2 \phi = -4\pi\rho$

$$\int (-k^2)\phi_k e^{-ikx}\, \mathrm{d}^3 k = -4\pi \int \rho_k e^{-ikx}\, \mathrm{d}^3 k$$

Now

$$\delta\left(\vec{\mathbf{k}} - \vec{\mathbf{k}'}\right) = C \int e^{-i\vec{\mathbf{k}}x} e^{i\vec{\mathbf{k}'}x}\, \mathrm{d}x$$

Thus, this is orthogonal basis and coefficients have to be equal

$$-k^2 \phi_{\vec{\mathbf{k}}} = -4\pi\rho_{\vec{\mathbf{k}}}$$

$$k^2 \phi_{\vec{\mathbf{k}}} = 4\pi\rho_{\vec{\mathbf{k}}}$$

Example $\rho_k = 0$, then

$$k^2 \phi_{\vec{\mathbf{k}}} = 0$$

i.e., either

$$k^2 = 0 \text{ or } \phi_{\vec{\mathbf{k}}}$$

However, $k_x, k_y, k_z > 0$, else, $e^{-ikx} = e^{|k|x} \to \infty$, which doesn't fulfills boundary conditions, thus $\phi = 0$.

Finite boundary conditions In this case, we have a Fourier series instead of transform.

Example If $\rho \neq 0$;

$$\phi(\vec{x}) = \int \frac{\rho\left(\vec{\mathbf{x}'}\right)}{|x - x'|} \mathrm{d}^3 x$$

i.e.

$$\phi_k = \frac{4\pi\rho_k}{k^2}$$

1.7 Multipole expansion

If we are far from a set of charges we can approximate them as a single charge Q:

$$\phi = \frac{Q}{r} = \frac{\sum_i q_i}{r} = \frac{\int \rho(r')\, \mathrm{d}^3 r'}{r}$$

This is monopole approximation.
Now if we denote $f\left(\vec{\mathbf{r}'}\right) = \frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|}$ We can take Taylor expansion of it:

$$f\left(\vec{\mathbf{r}'}\right) = f\left(\vec{\mathbf{r}'} = 0\right) + \sum_{\alpha=1}^{3} \left.\frac{\partial f}{\partial x'_\alpha}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha + \sum_{\alpha,\beta=1}^{3} \left.\frac{\partial^2 f}{\partial x'_\alpha \partial x'_\beta}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha x'_\beta + \dots$$

Then the monopole approximation is first element in the series:

$$f\left(\vec{\mathbf{r}'}\right) \approx f\left(\vec{\mathbf{r}'} = 0\right)$$

$$\phi_0(\vec{\mathbf{r}}) = \frac{Q}{r}$$

Dipole expansion is

$$\phi_1(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r'\, \rho(r) \frac{\partial}{\partial r'} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \cdot \vec{\mathbf{r}'}$$

$$\frac{\partial}{\partial r'} f = \vec{\boldsymbol{\nabla}} f = \sum_\alpha \frac{\partial f}{\partial x_\alpha} \hat{\mathbf{x}}_\alpha$$

i.e.,

$$\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} = \frac{1}{r} + \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} + \mathcal{O}\left(r^2\right)$$

meaning

$$\phi_1 = \int \mathrm{d}^3 r'\, \rho(\vec{\mathbf{r}'}) \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} = \frac{\vec{\mathbf{r}}}{r^3} \underbrace{\int \mathrm{d}^3 r'\, \rho(\vec{\mathbf{r}'}) \vec{\mathbf{r}'}}_{vaP}$$

Example A single point charge in point $\vec{\mathbf{r}'} = \vec{\mathbf{r}}_q$.

$$\phi = \frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_q|} \approx \phi_1(\vec{\mathbf{r}}) + \phi_2(\vec{\mathbf{r}}) = \frac{q}{r} + \frac{\vec{\mathbf{r}} \cdot P}{r^3} = \frac{q}{r} + \frac{q\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}_q}{r^3}$$

Example Two point charge. Then

$$\vec{\mathbf{P}} = q_1 \vec{\mathbf{r}}_1 + q_2 \vec{\mathbf{r}}_2$$

Quadruple expansion Note that

$$\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\Bigg|_{\vec{\mathbf{r}'}=0} = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r}$$

(since

$$\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\Bigg| = -\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\Bigg|$$

). Also note that

$$\sum_{\alpha,\beta} \delta_{\alpha,\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r} = 0$$

Now

$$\phi_2 = \sum_{\alpha,\beta} \int \mathrm{d}^3 r'\, \frac{1}{2} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} x'_\alpha x'_\beta \rho(\vec{\mathbf{r}'}) = \frac{1}{2} \sum_{\alpha,\beta} \int \mathrm{d}^3 r'\, \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} \left(x'_\alpha x'_\beta - \frac{1}{3} {r'}^2 \delta_{\alpha,\beta} \right) \rho(\vec{\mathbf{r}'})$$

We can rewrite as

$$\phi_2 = \frac{1}{6} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} Q_{\alpha\beta}$$

where

$$Q_{\alpha\beta} = \sum_{\alpha,\beta} \int \mathrm{d}^3 r'\, \rho(\vec{\mathbf{r}'}) \left(x'_\alpha x'_\beta - \frac{1}{3} {r'}^2 \delta_{\alpha,\beta} \right)$$

and

$$\frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} = 3 \frac{x_\alpha x_\beta}{r^5} - \frac{\delta_{\alpha,\beta}}{r^3}$$

Also

$$\operatorname{tr} Q = \sum_{\alpha,\beta} \delta_{\alpha,\beta} Q_{\alpha,\beta} = 0$$

Legendre polynomial Define functions $P_l(x)$:

$$G(t,x)=\frac{1}{\sqrt{1-2tx+t^2}}=\sum_{l=0}^{\infty}P_l(x)t^l$$

It can be shown that those are orthogonal functions:

$$\int_{-1}^{1}P_l(x)P_{l'}(x)\,\mathrm{d}x=\frac{2}{2l+1}\delta_{ll'}$$

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}=\frac{1}{\sqrt{r^2+r'^2-2rr'\mu}}=\frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}}$$

where $r_>$ and $r_<$ are the bigger and smaller out of r and r' correspondingly, and $\mu=\cos\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{r}'}\right)$.
Taking $t=\frac{r_<}{r_>}$ and $x=2\mu$ we get

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}=\frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}}=\sum_{l=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l P_l(\mu)$$

We get

$$\phi=\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}'}\right)\sum_{l=0}^{\infty}\frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\mu)$$

we can rewrite, using spherical harmonics

$$P_l(\mu')=\sum_{n=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}\left(\hat{\mathbf{r}'}\right)$$

Spherical harmonics Spherical harmonics are an analogue of Fourier series in spherical coordinates.

$$G(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}g_{l,m}Y_{lm}(\theta,\varphi)$$

Y are orthogonal functional basis

$$\int \mathrm{d}\Omega\,Y_{lm}(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi)=\delta_{ll'}\delta_{mm'}$$

A coefficient is

$$g_{l'm'}=\int \mathrm{d}\Omega\,G(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi)$$

The spherical harmonics are proportional to Legendre polynomials

$$Y_{lm}(\theta,\varphi)\propto P_l^m(\cos\theta)e^{\pm i\varphi}$$

$$\phi(\vec{\mathbf{r}})=\sum_l\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}'}\right)\frac{r'^l}{r^{l+1}}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}\left(\hat{\mathbf{r}'}\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\hat{\mathbf{r}})}{r^{l+1}}\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}'}\right)r'^lY^*_{lm}\left(\hat{\mathbf{r}'}\right)$$

$$\int \mathrm{d}^3r'\,\rho(r',\theta',\varphi')r'^lY^*_{lm}(\theta',\varphi')=\int \mathrm{d}r'\,r'^{2+l}\,\mathrm{d}\theta'\sin(\theta')\,\mathrm{d}\phi'\,\rho(r',\theta',\varphi')Y^*_{lm}(\theta',\varphi')=\int \mathrm{d}r'\,r'^{2+l}\,\mathrm{d}\Omega'\,\rho(r',\theta',\varphi')Y^*_{lm}(\theta',\varphi')$$

Now, for $l=0$, for example,

$$\phi^{l=0}(r,\theta,\varphi)=4\pi\frac{Y_{lm}(\hat{\mathbf{r}})}{r}\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}'}\right)r'^lY^*_{00}\left(\hat{\mathbf{r}'}\right)=4\pi Y_{00}^2\frac{1}{r}\int \mathrm{d}^3r'\,\rho(\vec{\mathbf{r}'})=\frac{Q}{r}$$

for $l=1$:

$$\phi^{l=1}(r,\theta,\varphi)=\sum_{m=-1}^{1}\frac{4\pi}{3}\frac{Y_{1m}(\hat{\mathbf{r}})}{r^2}\int \mathrm{d}r'\,r'^3\,\mathrm{d}\Omega'\,\rho(r',\theta',\varphi')Y^*_{1m}(\theta',\varphi')$$

Denote

$$\vec{\mathbf{P}} = \int \mathrm{d}^3 r'\, \vec{\mathbf{r}'}\rho\left(\vec{\mathbf{r}'}\right)$$

Previously, we got

$$\phi_1 = \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{P}}}{r^3}$$

Now, for $m = 0$:

$$\frac{4\pi}{3}\frac{\overbrace{\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta}^{Y_{10}}}{r^2}\int \underbrace{\mathrm{d}r'\, {r'}^3\, \mathrm{d}\Omega'}_{\mathrm{d}^3 r' r'}\rho(r',\theta',\varphi')\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta' = \frac{4\pi}{3}\frac{\frac{3}{4\pi}\cos\theta}{r^2}\int \mathrm{d}^3 r'\, \rho(r',\theta',\varphi')z' = \frac{\cos\theta}{r^2}\int \mathrm{d}^3 r'\, \rho(r',\theta',\varphi')z' = \frac{\cos\theta}{r^2}P_z$$

Dipole inside constant electrical field Given field $\vec{\mathbf{E}}$ and dipole moment $\vec{\mathbf{P}}$:

$$U = -\vec{\mathbf{E}} \cdot \vec{\mathbf{P}} = -\vec{\mathbf{E}} \cdot (q\vec{\mathbf{r}} + (-q)(-\vec{\mathbf{r}})) = -\vec{\mathbf{E}}(2\vec{\mathbf{r}})q$$

The torque is

$$\vec{\mathbf{N}} = \vec{\mathbf{r}}_1 \times q_1\vec{\mathbf{E}} + \vec{\mathbf{r}}_2 \times q_2\vec{\mathbf{E}} = (q_1\vec{\mathbf{r}}_1 + q_2\vec{\mathbf{r}}_2) \times \vec{\mathbf{E}} = \vec{\mathbf{P}} \times \vec{\mathbf{E}}$$

We could acquire it as

$$\left|-\frac{\partial U}{\partial \theta}\right| = \left|PE\frac{\partial \cos\theta}{\partial\theta}\right| = |PE\sin\theta|$$

Dipole inside electrical field Now

$$U = q_1\phi^{ext}(\vec{\mathbf{r}}_1) + q_2\phi^{ext}(\vec{\mathbf{r}}_2)$$

We can approximate potential in a second point:

$$\phi^{ext}(\vec{\mathbf{r}}_2) = \phi^{ext}(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) + \frac{1}{2}\frac{\partial\phi^{ext}}{\partial x_\alpha x_\beta}(x_{2,\alpha} - x_{1,\alpha})(x_{2,\beta} - x_{1,\beta}) + \mathcal{O}\left((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^3\right)$$

For constant field we get the same results. Else we get

$$U = q_1\phi^{ext}(\vec{\mathbf{r}}_1) + q_2\left[\phi^{ext}(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) + \underbrace{\frac{1}{2}\frac{\partial\phi^{ext}}{\partial x_\alpha x_\beta}(x_{2,\alpha} - x_{1,\alpha})(x_{2,\beta} - x_{1,\beta}) + \mathcal{O}\left((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^3\right)}_{0}\right] =$$
$$= \text{const} - \vec{\mathbf{E}} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)q = \text{const} - \vec{\mathbf{E}} \cdot \vec{\mathbf{P}}$$

Lets calculate force on dipole:

$$\vec{\mathbf{F}} = q_1\vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + q_2\vec{\mathbf{E}}(\vec{\mathbf{r}}_2)$$

Approximating E with Taylor series

$$E(\vec{\mathbf{r}}_2) = \vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + \sum_{\alpha,\beta}\hat{\mathbf{x}}_\alpha\frac{\partial E_\alpha}{\partial x_\beta}(x_{2\beta} - x_{1\beta}) + \mathcal{O}\left((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^2\right) \approx E(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)$$

$$\vec{\mathbf{F}} = q_1\vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + q_2\left[E(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)\right] = -q\vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) = \left(\vec{\mathbf{P}} \cdot \vec{\boldsymbol{\nabla}}\right)E$$

Thus

$$U = -\vec{\mathbf{P}} \cdot \vec{\mathbf{E}}$$

In general case

$$\vec{\mathbf{P}} = \int \mathrm{d}^3 r'\, \rho(\vec{\mathbf{r}}')\vec{\mathbf{r}}' = \sum q_i\vec{\mathbf{r}}_i'$$

Now

$$U = \sum_i q_i\phi^{ext}(\vec{\mathbf{r}}_i) \approxeq \sum_i q_i\left[\phi^{ext}(\vec{\mathbf{r}}_0)\vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_0) + \frac{1}{2}\frac{\partial^2\phi^{ext}}{\partial x_\alpha\partial x_\beta}(x_{i\alpha} - x_{0\alpha})(x_{i\beta} - x_{0\beta})\right] =$$
$$= \sum_i q_i\phi^{ext}(0) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot \sum_i q_i\vec{\mathbf{r}}_i + \frac{1}{2}\frac{\partial^2\phi^{ext}}{\partial x_\alpha\partial x_\beta}\sum_i q_i x_{i\alpha}x_{i\beta}$$

$$\frac{1}{2}\frac{\partial^2\phi^{ext}}{\partial x_\alpha \partial x_\beta}\sum_i q_i x_{i\alpha}x_{i\beta} = \sum_{\alpha,\beta}\frac{1}{6}\frac{\partial^2\phi^{ext}}{\partial x_\alpha \partial x_\beta}D_{\alpha\beta}$$

where

$$D_{\alpha\beta} = \sum q_i\left(3x_{i\alpha}x_{i\beta} - \mathbf{r}_i^2\delta_{\alpha,\beta}\right)$$

Self energy of system of particles For two particles the energy is

$$U = \frac{q_1 q_2}{|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|} = q_1\phi_2(\vec{\mathbf{r}}_1) = \frac{1}{2}\big(q_1\phi(\vec{\mathbf{r}}_1) + q_2\phi(\vec{\mathbf{r}}_2)\big)$$

i.e.

$$U = \frac{1}{2}\sum q_i\phi(r_i) = \frac{1}{2}\int \mathrm{d}^3r\,\rho(\vec{\mathbf{r}})\phi(\vec{\mathbf{r}})$$

Substituting $\nabla^2\phi = -4\rho(\vec{\mathbf{r}})$:

$$U = -\frac{1}{8\pi}\int \mathrm{d}^3r\,\nabla^2\phi\phi$$

Since

$$\vec{\boldsymbol{\nabla}}\cdot\phi\vec{\boldsymbol{\nabla}}\phi = |\vec{\boldsymbol{\nabla}}\phi|^2 + \phi\nabla^2\phi$$

we get

$$U = -\frac{1}{8\pi}\int \mathrm{d}^3r\,\underbrace{\vec{\boldsymbol{\nabla}}\cdot\phi\vec{\boldsymbol{\nabla}}\phi}_{0} - \underbrace{|\vec{\boldsymbol{\nabla}}\phi|^2}_{E^2} = \frac{1}{8\pi}\int \mathrm{d}^3r\,E^2$$

Infinite energy Suppose all internal energy comes from own electric field:

$$U \sim \frac{e^2}{R_0} \sim m_e c^2$$

i.e.

$$R_0 \sim \frac{e^2}{mc^2}$$

However this is not proper quantum limit, since it lacks Plank constant. The proper one is Compton wavelength:

$$\lambda \sim \frac{\hbar}{m_e c}$$

Thus

$$\frac{\lambda}{R_0} \sim \frac{\bar{h}c}{e^2} \sim 137$$

Pure dipole If we take an charge system such that it's potential is dipole's one in every place in space, we get field equal to

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{3(\vec{\mathbf{P}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{\mathbf{P}}}{r^3} - 4\pi\hat{r}(\vec{\mathbf{P}}\cdot\hat{\mathbf{r}})\delta(\vec{\mathbf{r}})$$

2 Magnetostatics

The force between two currents is

$$\mathrm{d}F = \kappa II'\frac{\mathrm{d}\vec{\mathbf{l}}' \times (\mathrm{d}\vec{\mathbf{l}}\,\vec{\mathbf{r}})}{r^3}$$

Difference between magnetic and electric field

$$\begin{cases}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{B}} = 0\\ \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} \Rightarrow \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}} = \frac{4\pi}{c}\vec{\mathbf{j}}\end{cases} \iff \begin{cases}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}} = 4\pi\rho\\ \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\phi \Rightarrow \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} = 0\end{cases}$$

where j is current density, such that $\int\vec{\mathbf{j}}\cdot\mathrm{d}\vec{\mathbf{s}}$ is equal to charge passing through surface in unit time.

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{j}} = \frac{c}{4\pi}\vec{\boldsymbol{\nabla}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}}\right) = 0$$

Suppose we have constant charge, then $\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = 0$ and thus

$$0 = \oint \vec{\mathbf{j}} = \int\limits_V \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\, \mathrm{d}^3 r \xrightarrow{V\to 0} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} \cdot \Delta V$$

In general case

$$\frac{\partial Q}{\partial t} = -\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$$

Substituting $Q = \int\limits_V \mathrm{d}^3 r\, \rho$ and $\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = \int\limits_V \mathrm{d}^3 r\, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}$, we get

$$\frac{\partial}{\partial t} \int \mathrm{d}^3 r\, \rho = -\int\limits_V \mathrm{d}^3 r\, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}$$

In limit $V \to 0$ we get continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} = 0$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}\right) = \frac{4\pi}{c} \vec{\mathbf{j}}$$

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}\right) = \vec{\boldsymbol{\nabla}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}\right) - \nabla^2 \vec{\mathbf{A}}$$

From gauge invariance we are free to choose $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$. Thus we acquired

$$\nabla^2 \vec{\mathbf{A}} = -\frac{4\pi}{c} \vec{\mathbf{j}}$$

The solution is

$$\vec{\mathbf{A}} = \frac{1}{c} \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})\, \mathrm{d}^3 r'}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

with border conditions $\vec{\mathbf{B}} \to 0$ in infinity. Lets check whether $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}$:

$$\vec{\boldsymbol{\nabla}}_r \cdot \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})\, \mathrm{d}^3 r'}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} = \int \vec{\boldsymbol{\nabla}}_r \cdot \left(\frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})\, \mathrm{d}^3 r'}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \right) = \int \mathrm{d}^3 r'\, \vec{\mathbf{j}}(\vec{\mathbf{r}'}) \vec{\boldsymbol{\nabla}}_r \cdot \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} = -\int \mathrm{d}^3 r'\, \vec{\mathbf{j}}(\vec{\mathbf{r}'}) \vec{\boldsymbol{\nabla}}_{r'} \cdot \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} =$$

$$= -\int \mathrm{d}^3 r'\, \vec{\boldsymbol{\nabla}}_{r'} \cdot \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} - \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \underbrace{\vec{\boldsymbol{\nabla}}_{r'} \cdot \vec{\mathbf{j}}(\vec{\mathbf{r}'})}_{0} = -\oint \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \cdot \mathrm{d}\vec{\mathbf{s}} = 0$$

Stokes' theorem For some vector field $\vec{\mathbf{V}}$ and closed curve Γ:

$$\int \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{V}}\right) \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{V}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

In our case we get

$$\int\limits_S \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}}\right) \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

$$\frac{4\pi}{c} \int\limits_S \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

We denote $I = \int\limits_S \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$, which is current.

$$\frac{4\pi}{c} I = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

2.1 Gauge transformation

Suppose we have $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}}\psi$. We get $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}'$. Suppose $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = \chi$. Then

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} + \nabla^2 \psi$$

If we want $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = 0$, we get $\chi + \nabla^2 \psi = 0$. Thus we can find ψ such that $\nabla^2 \psi = -\chi$ and we get $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = 0$.

If

$$\vec{\mathbf{A}} = \frac{1}{c}\int \mathrm{d}^3 r' \, \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}})}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

we can get

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} = \frac{1}{c}\int \mathrm{d}^3 r' \, \vec{\mathbf{j}}(\vec{\mathbf{r}}) \times \frac{\left(\vec{\mathbf{r}'} - \vec{\mathbf{r}}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$$

2.2 Biot–Savart law

Suppose we have current I on some curve l'. Then magnetic field due to this current in point $\vec{\mathbf{r}}$ is

$$\mathrm{d}\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{I}{c}\,\mathrm{d}\vec{\mathbf{l}'} \times \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$$

Example Suppose we have infinite wire with current $I\hat{\mathbf{z}}$. Then the magnetic fiend due to the wire is

$$\mathrm{d}\vec{\mathbf{B}} = \frac{I}{c}\frac{\mathrm{d}z\,\hat{\mathbf{z}} \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = \frac{I}{c}\frac{\mathrm{d}z\,R\hat{\mathbf{y}}}{\left[R^2 + z^2\right]^{\frac{3}{2}}}$$

Thus

$$B = \frac{I}{c}\int_{-\infty}^{\infty} \frac{\mathrm{d}z\,R}{\left[R^2 + z^2\right]^{\frac{3}{2}}} = \frac{2I}{cR}$$

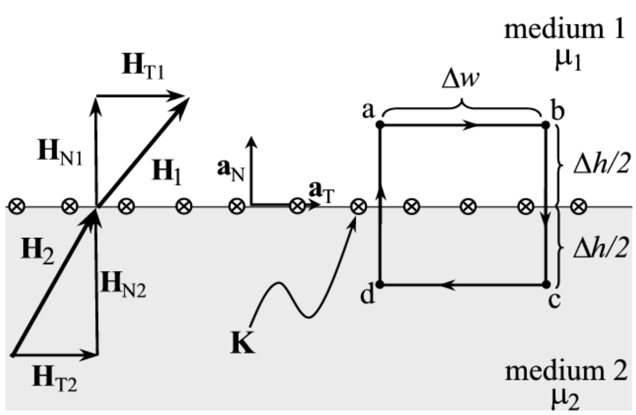
Alternatively

$$\frac{4\pi}{c}I = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}} = B \cdot 2\pi R$$

$$B = \frac{2I}{cR}$$

Meissner effect Superconducting material actively excludes magnetic fields from its interior. Thus $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0$ and there are currents only on the edge.

Example Suppose we have a boundary between two materials, with fields $\vec{\mathbf{B}}_1$, $\vec{\mathbf{B}}_2$.



Taking integral on small box on boudary we get

$$0 = \oint \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}\,\mathrm{d}^3 r = \oint \vec{\mathbf{B}} \cdot \mathrm{d}s = \left(\vec{\mathbf{B}}_{1,\perp} - \vec{\mathbf{B}}_{2,\perp}\right)\Delta s$$

i.e.

$$\vec{\mathbf{B}}_{1,\perp} = \vec{\mathbf{B}}_{2,\perp}$$

What happens with $B_\parallel$?

$$\oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}} = \frac{4\pi}{c} \int \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = I = \frac{4\pi}{c} K \,\mathrm{d}l$$

$$\left(B_{2,\parallel} - B_{1,\parallel}\right) = \frac{4\pi}{c} \vec{\mathbf{K}} \,\mathrm{d}l$$

$$\hat{\mathbf{n}} \times (\vec{\mathbf{B}}_2 - \vec{\mathbf{B}}_1) = \frac{4\pi}{c} \vec{\mathbf{K}}$$

Magnetic scalar potential It's impossible to write $\vec{\mathbf{B}} = -\vec{\boldsymbol{\nabla}}\phi_m$. However, if there is some area in which there is no current, we can try to do so, since $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0$. Using Biot-Savart law

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{I}{c} \oint \mathrm{d}\vec{\mathbf{r}'} \times \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = \frac{I}{c} \underbrace{\oint \mathrm{d}r' \times \underbrace{\vec{\mathbf{U}}(\vec{\mathbf{r}})}_{\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}}}_{\vec{\mathbf{H}}}$$

For some constant vector $\vec{\mathbf{k}}$:

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{H}} = \vec{\mathbf{k}} \cdot \oint \mathrm{d}r' \times \vec{\mathbf{U}}(\vec{\mathbf{r}}) = \oint \mathrm{d}\vec{\mathbf{r}} \times \left(\vec{\mathbf{k}} \cdot \vec{\mathbf{U}}\right) = \int \mathrm{d}\vec{\mathbf{s}} \cdot \vec{\boldsymbol{\nabla}} \times \left(\vec{\mathbf{U}} \times \vec{\mathbf{k}}\right) = \int \mathrm{d}\vec{\mathbf{s}} \cdot \left[(\vec{\mathbf{k}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{U}} - \vec{\mathbf{k}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{U}})\right]$$