114246 - Electromagnetism and Electrodynamics

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Abstract

1 Introduction

In this course we use CGS system. Force between two charges is

$$\vec{\mathbf{F}} = \frac{q_1 q_2}{\mathbf{r}^2}\hat{\mathbf{r}}$$

The unit of charge is statcoulomb, or esu.
Field around charge q is

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q'} = \frac{q}{\mathbf{r}^2}\hat{\mathbf{r}}$$

Then force can be written as

$$\vec{\mathbf{F}} = q'\vec{\mathbf{E}}$$

Principle of linearity (superposition) If we have some frame of reference we can rewrite force as

$$\vec{\mathbf{F}}_1 = \frac{q' q_1}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_1\right|^3}\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

And fields can be summed as following:

$$\vec{\mathbf{E}} = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} \frac{q_i(\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i)}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i\right|^3}$$

If charge is continuous define

$$\rho(\vec{\mathbf{r}}) = \frac{\Delta q}{\Delta V}$$

field turns into integral:

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3\mathbf{r}'\,\rho(\vec{\mathbf{r}'})\frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3}$$

Potential

$$\vec{E} = \frac{q\vec{\mathbf{r}}}{\mathbf{r}^2} = -\vec{\boldsymbol{\nabla}}\frac{q}{r}$$

For some frame of reference

$$\vec{E} = \frac{q(\vec{\mathbf{r}} - \vec{\mathbf{r}'})}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3} = -\vec{\boldsymbol{\nabla}}\frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|} = -\vec{\boldsymbol{\nabla}}\Phi$$

Gradient, divergence and Laplacian in spherical coordinates

$$\vec{\boldsymbol{\nabla}} f(r,\theta,\phi) = \hat{\mathbf{r}}\frac{\partial f}{\partial x} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{\partial f}{\partial \phi}\frac{1}{r\sin\theta}$$

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{A}}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(\mathbf{r}^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Continuous case

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r' \, \rho\left(\vec{\mathbf{r}'}\right) \frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = -\vec{\boldsymbol{\nabla}} \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

And

$$\Phi = \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

Gauss theorem

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}(\mathbf{r}) = \oint \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \mathbf{A}_n \, \mathrm{d}s$$

Lets apply Gauss theorem on electric field of point charge in origin:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \vec{\boldsymbol{\nabla}} \cdot \frac{q\vec{\mathbf{r}}}{\mathbf{r}^3}$$

Then

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = -\nabla^2 \frac{q}{r}$$

If $r \neq 0$,

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = q\vec{\boldsymbol{\nabla}} \frac{\vec{\mathbf{r}}}{\mathbf{r}^3} = q\vec{\boldsymbol{\nabla}} \frac{\hat{\mathbf{r}}}{\mathbf{r}^2} = \frac{q}{\mathbf{r}^2} \frac{\partial}{\partial r} \left(\mathbf{r}^2 \frac{1}{\mathbf{r}^2}\right) = 0$$

By applying Gauss law:

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \oint \vec{\mathbf{E}} \, \mathrm{d}\vec{\mathbf{s}} = \int \mathrm{d}\Omega \, \frac{q\hat{\mathbf{R}}}{\mathbf{R}^2} \cdot \vec{\mathbf{R}} \cdot \mathbf{R}^2 = q \oint \mathrm{d}\Omega = 4\pi q$$

Which is right for a ball of any radius, in particular, $R \to 0$. Thus

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0$$

while

$$\int\limits_{V_\epsilon} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3\mathbf{r} = 4\pi q$$

Delta function Lets define

$$\begin{cases} \delta^D = 0 & x \neq 0 \\ \int\limits_{x \in V} \delta^D(x) \, \mathrm{d}x = 1 & \end{cases}$$

We can define it as limit of

$$F_\Delta = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} < x < \frac{\Delta}{2} \\ 0 & otherwise \end{cases}$$

Then $\lim_{\Delta \to 0} F_\Delta = \delta^D$.

Potential of point charge For a charge in point $\mathbf{r}_a$:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi q \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And for a couple of charges

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

Then we can define dencity of charge for a point charge as

$$\rho(\vec{\mathbf{r}}) = \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And then in both cases

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi \rho(\vec{\mathbf{r}})$$

Since

$$\vec{E} = -\vec{\nabla}\Phi$$
$$\vec{\nabla}\left(-\vec{\nabla}\Phi\right) = 4\pi\rho$$

Which means

$$\nabla^2\Phi = -4\pi\rho$$

which is Poisson equation. For bound conditions of $\phi, E \overset{r\to\infty}{\to} 0$ solution is $E = q\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}$.

1.1 Bound conditions

Direchlet bound conditions $\Phi = \Phi_S(\vec{\mathbf{r}})$ for $r \in S$.

Neumann bound condtions We have E_n on S.

1.2 Example of Solutions

Suppose we have Dirichlet bound conditions: $\Phi_S = 0$. Obvious solution is $\Phi = \rho = 0$
Lets show it's unique solution:

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int |\vec{\nabla}\Phi|^2 \,\mathrm{d}^3\mathbf{r}$$

$$|\nabla\Phi|^2 = \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right) - \Phi\nabla^2\Phi = \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)$$

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)\mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi\cdot\vec{\nabla}\Phi\cdot\mathrm{d}\vec{\mathbf{s}} = 0$$

Thus $E = 0$ and $\Phi = 0$.

General case In general case $\nabla^2\Phi = -4\pi q$. Suppose we have bound conditions $\Phi = \Phi_S \neq 0$. Suppose we have two solutions Φ_1, Φ_2.
Take a look at $\Phi = \Phi_1 - \Phi_2$, which has boundary conditions $\Phi_1 - \Phi_2 = \Phi_S - \Phi_S = 0$, thus solutions are equal, from previous paragraph.

Neumann bound conditions Instead of Φ we now have $E_n = -\vec{\nabla}\Phi\cdot\hat{\mathbf{n}} = 0$.

$$\int_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)\mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi\cdot\vec{\nabla}\Phi\cdot\mathrm{d}\vec{\mathbf{s}} = \oint \Phi\cdot\underbrace{\vec{\nabla}\Phi\cdot\hat{\mathbf{n}}}_{-E_n}\mathrm{d}s = 0$$

Thus $E = 0$. Note that E_n determines E_t.

Earnshaw theorem If in some volume $\rho = 0$, then there is no local maximum or minimum of potential in this volume, since then $\vec{\nabla}\Phi = 0$ and either $\nabla^2\Phi > 0$ or $\nabla^2\Phi < 0$, but $\nabla^2\Phi = \rho = 0$

1.3 Methods of solutions of Poisson equation

Method of image charges Suppose we have infinite grounded plane and a point charge in distance a from it. The bound condition is $\phi(x = 0, y, z) = 0$ And the potential of point charge is $\nabla^2\Phi = -4\pi q\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$
The way to solve this kind of problems is to add imaginary charge such that we get zero potential on bound (e.g., symmetrically with opposite charge).
If we have two such charges:

$$\Phi = \frac{q}{(x-a)^2 + y^2 + z^2} - \frac{q}{(x+a)^2 + y^2 + z^2}$$

This fulfills bound conditions:

$$\phi(x = 0, y, z) = \frac{q}{a^2 + y^2 + z^2} - \frac{q}{a^2 + y^2 + z^2} = 0$$

Denote density of charge on plane as σ, then $\mathrm{d}q = \sigma\,\mathrm{d}s$. If this case force between plane and charge is $\left|\frac{\sigma q \mathrm{d}s}{R^2}\right|$

How do we find σ? From Gauss law

$$\oint \vec{E} \cdot \mathrm{d}s = 4\pi \int_V \rho \, \mathrm{d}^3\mathbf{r}$$

$$-|E_n| \, \mathrm{d}s = 4\pi\sigma \, \mathrm{d}s$$

$$4\pi\sigma = E_n$$

1.4 Green function

Suppose we have a unit charge in point $\vec{\mathbf{r}'}$. Then potential is $\Phi_{\vec{\mathbf{r}'}}$ such that $\Delta\Phi = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}'})$.

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \Phi_{\vec{\mathbf{r}'}}$$

Thus

$$\nabla^2 G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = -4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

Then we can write G as

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \frac{1}{|r - r'|} + F\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$$

And F guaranties bound conditions, while $\nabla^2 F = 0$.

Bound conditions

1. $G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = 0$.for bound S.
2. $\frac{\partial}{\partial \hat{\mathbf{n}}} G(\vec{\mathbf{r}}, \vec{\mathbf{r}'}) = -\frac{4\pi}{S}$. Then number comes from Gauss' law

Green theorem

$$\int_V \left(\phi \vec{\boldsymbol{\nabla}} \psi - \psi \vec{\boldsymbol{\nabla}} \phi\right) \mathrm{d}^3\mathbf{r} = \oint_S \left[\phi \frac{\partial \psi}{\partial \hat{\mathbf{n}}} - \psi \frac{\partial \phi}{\partial \hat{\mathbf{n}}}\right] \mathrm{d}s$$

Since

$$\int_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \left(\phi \vec{\boldsymbol{\nabla}} \psi\right) = \oint \phi \vec{\boldsymbol{\nabla}} \psi \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \phi \frac{\partial \psi}{\partial \hat{\mathbf{n}}} \, \mathrm{d}s$$

we get

$$\int_V \mathrm{d}^3\mathbf{r} \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) = \int_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \left(\phi \vec{\boldsymbol{\nabla}} \psi\right) - \vec{\boldsymbol{\nabla}} \phi \cdot \vec{\boldsymbol{\nabla}} \psi - \vec{\boldsymbol{\nabla}} \cdot \left(\phi \vec{\boldsymbol{\nabla}} \psi\right) + \vec{\boldsymbol{\nabla}} \psi \cdot \vec{\boldsymbol{\nabla}} \phi = \oint_S \left[\phi \frac{\partial \psi}{\partial \hat{\mathbf{n}}} - \psi \frac{\partial \phi}{\partial \hat{\mathbf{n}}}\right] \mathrm{d}s$$

Now for $\psi(\vec{\mathbf{r}}) = G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$

$$\int_V \mathrm{d}^3\mathbf{r} \left(-\phi \cdot 4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint_S \mathrm{d}s \left[\phi \frac{\partial G}{\partial \hat{\mathbf{n}}} - G \frac{\partial \phi}{\partial \hat{\mathbf{n}}}\right]$$

Suppose we want $\nabla^2\phi = 4\pi\rho(\vec{\mathbf{r}})$ for some ρ and Dirichlet bound conditions on same surface S:

$$\int_V \mathrm{d}^3\mathbf{r} \left(-4\pi\phi\left(\vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint_S \mathrm{d}s \left[\phi \frac{\partial G}{\partial \hat{\mathbf{n}}} - G \frac{\partial \phi}{\partial \hat{\mathbf{n}}}\right]$$

Substituting $G = 0$ on the bound surface

$$-4\pi\phi\left(\vec{\mathbf{r}'}\right) + \int_V \mathrm{d}^3\mathbf{r} \, G \cdot 4\pi\rho(\vec{\mathbf{r}}) = \oint_S \mathrm{d}s \, \phi \frac{\partial G}{\partial \hat{\mathbf{n}}}$$

i.e.

$$\phi\left(\vec{\mathbf{r}'}\right) = \int_V G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) \rho(\vec{\mathbf{r}}) \, \mathrm{d}^3\mathbf{r} - \oint_S \frac{\phi}{4\pi} \cdot \frac{\partial G}{\partial \hat{\mathbf{n}}} \, \mathrm{d}s$$

Neumann bound conditions

$$-4\pi\phi\left(\vec{\mathbf{r}'}\right)+4\pi\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\mathrm{d}^3\mathbf{r}=\oint\mathrm{d}s\left[\phi\left(-\frac{4\pi}{S}\right)-G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

However

$$\oint\phi\left(-\frac{4\pi}{S}\right)\mathrm{d}s=-\frac{4\pi}{S}\oint\phi\,\mathrm{d}s=-4\pi\langle\phi\rangle$$

$$\phi\left(\vec{\mathbf{r}'}\right)=\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\mathrm{d}^3\mathbf{r}+\frac{1}{4\pi}\oint G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\mathrm{d}s+\langle\phi\rangle$$

Example

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}$$

With bound conditions

$$0=G\left(\vec{\mathbf{r}}\to\infty,\vec{\mathbf{r}'}\right)$$

Thus

$$\phi(\vec{\mathbf{r}})=\int\rho(\vec{\mathbf{r}})\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}\mathrm{d}^3\mathbf{r}$$

Example Suppose we have an infinite plane with potential $\phi(x=0,y,z)=\phi_S(y,z)$ and $\rho=0$ in one side of space $(x>0)$. We are searching for $\phi(x>0,y,z)$.
Define Green function as a solution of previous problem, with point charge and grounded plane:

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}-\frac{1}{\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}$$

$$\phi\left(\vec{\mathbf{r}'}\right)=\underbrace{\int_V G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\rho(\vec{\mathbf{r}})\,\mathrm{d}^3\mathbf{r}}_{0}-\oint_S\frac{\phi}{4\pi}\cdot\frac{\partial G}{\partial\hat{\mathbf{n}}}\mathrm{d}s$$

$$\frac{\partial G}{\partial\hat{\mathbf{n}}}=-\left.\frac{\partial G}{\partial x}\right|_{x=0}$$

$$\phi=\frac{x'}{2\pi}\int\mathrm{d}z\,\mathrm{d}y\,\frac{\phi_S(y,z)}{\left[x'^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}$$

Symmetry of Green function

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=G\left(\vec{\mathbf{r}'},\vec{\mathbf{r}}\right)$$

1.5 Separation of variables

Suppose we have two planes parallel to y axis with zero potential with distance L between them, and a plane parallel to x axis with potential $V(x)$. We want to solve

$$\nabla^2\phi=0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Since there is no change on z direction we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Let's use anzatz for the solution $\phi=X(x)Y(y)$:

$$Y\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

Since we have operands depending on different variables, both are constant.

$$\begin{cases}\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=+k^2\\ \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}X^2}=-k^2\end{cases}$$

(since we want Y to decrease to 0 in infinity, we choose positive value for Y and negative for X).
The solution is

$$\begin{cases}X=a\cdot\sin(kx)+b\cdot\cos(kx)\\ X=c\cdot e^{ky}+d\cdot e^{-ky}\end{cases}$$

Since $\phi(0,y,z)=\phi(L,y,z)=0$, $b=0$ and k acquires discrete values $k_n=\frac{n\pi}{L}$. Since $\phi(x,y\to\infty,z)=0$, $c=0$.
Now we need to force $\phi(x,y=0,z)=V(x)$:

$$\Phi=X(x)Y(y=0)=V(x)$$

We know that the solution is of form

$$da\sin(k_nx)e^{-k_ny}$$

Thus we can get a general solution in a form

$$\phi=\sum d_na_n\sin(k_nx)e^{-k_ny}$$

So we want

$$V(x)=\sum\underbrace{d_na_n}_{q_n}\sin(k_nx)$$

which is Fourier series:

$$\int\mathrm{d}x\,V(x)\sin(k_{n'}x)=\sum_n\int\mathrm{d}x\,q_n\sin(k_nx)\sin(k_{n'}x)=\sum_n q_n\frac{L}{2}\delta_{nn'}=\frac{Lq_{n'}}{2}$$

Example Suppose we have to planes with angle θ_0 between them and potential ϕ_0 on both. In cylindrical coordinates

$$\nabla^2\phi=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

By separation of variables

$$\phi=F(R)G(\theta)$$

$$\frac{G(\theta)}{R}\frac{\mathrm{d}F}{\mathrm{d}R}+G\frac{\mathrm{d}^2F}{\mathrm{d}R^2}+\frac{F(R)}{R^2}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=0$$

$$\begin{cases}\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=-\nu^2\\ \frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}R}+\frac{R^2}{F}\frac{\mathrm{d}^2F}{\mathrm{d}R^2}=\nu^2\end{cases}$$

$\nu>0$

$$\begin{cases}G=A\cos(\nu\theta)+B\sin(\nu\theta)\\ F=aR^{-\nu}+bR^{\nu}\end{cases}$$

$\nu=0$

$$\begin{cases}G=\tilde{A}+\tilde{B}\theta\\ F=\tilde{a}+\tilde{b}\ln R\end{cases}$$

We get $\tilde{b}=0$ such that F doesn't diverge in 0. Also there shouldn't be dependence on angle, so $\tilde{B}=0$ and $\tilde{A}\neq 0$, thus for $\nu=0$ potential is constant.
For positive ν, $A=0$ and $a=0$ and also want $\sin(\nu\theta_0)=0$ thus

$$\phi(\theta)=\phi_0+\sum_{n=1}a_nR^{\frac{n\pi}{\theta_0}}\sin\left(n\pi\frac{\theta}{\theta_0}\right)$$

If $R\to 0$, the most dominant element of sum is $n=1$. Thus

$$\phi\propto R^{\frac{\pi}{\theta_0}}$$

i.e., due to $E \sim -\frac{\partial \phi}{\partial R} \sim R^{\frac{\pi}{\theta_0}-1}$:

$$\begin{cases} E \to 0 & \theta_0 < \pi \\ E \to \infty & \theta_0 > \pi \end{cases}$$

1.6 Solution with Fourier transform

$$f_k = \int \mathrm{d}x\, e^{ixk} F(x) \iff \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{-ixk} f_k$$

$$\mathrm{d}^2 F\, x = \int \frac{\mathrm{d}k}{2\pi} (-k^2) e^{-ikx} f_k$$

Since $\nabla^2 \phi = -4\pi\rho$

$$\int (-k^2)\phi_k e^{-ikx}\, \mathrm{d}^3 k = -4\pi \int \rho_k e^{-ikx}\, \mathrm{d}^3 k$$

Now

$$\delta\left(\vec{\mathbf{k}} - \vec{\mathbf{k}'}\right) = C \int e^{-i\vec{\mathbf{k}}x} e^{i\vec{\mathbf{k}'}x}\, \mathrm{d}x$$

Thus, this is orthogonal basis and coefficients have to be equal

$$-k^2 \phi_{\vec{\mathbf{k}}} = -4\pi\rho_{\vec{\mathbf{k}}}$$

$$k^2 \phi_{\vec{\mathbf{k}}} = 4\pi\rho_{\vec{\mathbf{k}}}$$

Example $\rho_k = 0$, then

$$k^2 \phi_{\vec{\mathbf{k}}} = 0$$

i.e., either

$$k^2 = 0 \text{ or } \phi_{\vec{\mathbf{k}}}$$

However, $k_x, k_y, k_z > 0$, else, $e^{-ikx} = e^{|k|x} \to \infty$, which doesn't fulfills boundary conditions, thus $\phi = 0$.

Finite boundary conditions In this case, we have a Fourier series instead of transform.

Example If $\rho \neq 0$;

$$\phi(\vec{x}) = \int \frac{\rho\left(\vec{\mathbf{x}'}\right)}{|x - x'|}\, \mathrm{d}^3 x$$

i.e.

$$\phi_k = \frac{4\pi\rho_k}{k^2}$$

1.7 Multipole expansion

If we are far from a set of charges we can approximate them as a single charge Q:

$$\phi = \frac{Q}{r} = \frac{\sum_i q_i}{r} = \frac{\int \rho(r')\, \mathrm{d}^3 r'}{r}$$

This is monopole approximation.
Now if we denote $f\left(\vec{\mathbf{r}'}\right) = \frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|}$ We can take Taylor expansion of it:

$$f\left(\vec{\mathbf{r}'}\right) = f\left(\vec{\mathbf{r}'} = 0\right) + \sum_{\alpha=1}^{3} \left.\frac{\partial f}{\partial x'_\alpha}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha + \sum_{\alpha,\beta=1}^{3} \left.\frac{\partial^2 f}{\partial x'_\alpha \partial x'_\beta}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha x'_\beta + \dots$$

Then the monopole approximation is first element in the series:

$$f\left(\vec{\mathbf{r}'}\right) \approx f\left(\vec{\mathbf{r}'} = 0\right)$$

$$\phi_1(\vec{\mathbf{r}}) = \frac{Q}{r}$$

Dipole expansion is

$$\phi_2(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r' \, \rho(r) \frac{\partial}{\partial r'} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \cdot \vec{\mathbf{r}'}$$

$$\frac{\partial}{\partial r'} f = \vec{\boldsymbol{\nabla}} f = \sum_\alpha \frac{\partial f}{\partial x_\alpha} \hat{\mathbf{x}}_\alpha$$

i.e.,

$$\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} = \frac{1}{r} + \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} + \mathcal{O}\left(r^2\right)$$

meaning

$$\phi_1 = \int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} = \frac{\vec{\mathbf{r}}}{r^3} \underbrace{\int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \vec{\mathbf{r}'}}_{vaP}$$

Example A single point charge in point $\vec{\mathbf{r}'} = \vec{\mathbf{r}}_q$.

$$\phi = \frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_q|} \approx \phi_1(\vec{\mathbf{r}}) + \phi_2(\vec{\mathbf{r}}) = \frac{q}{r} + \frac{\vec{\mathbf{r}} \cdot P}{r^3} = \frac{q}{r} + \frac{q\vec{\mathbf{r}} \cdot \vec{\mathbf{r}}_q}{r^3}$$

Example Two point charge. Then

$$\vec{\mathbf{P}} = q_1 \vec{\mathbf{r}}_1 + q_2 \vec{\mathbf{r}}_2$$

Quadruple expansion Note that

$$\left.\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right|_{\vec{\mathbf{r}'}=0} = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r}$$

(since

$$\left|\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right| = -\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \left|\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right|$$

). Also note that

$$\sum_{\alpha,\beta} \delta_{\alpha,\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r} = 0$$

Now

$$\phi_3 = \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \frac{1}{2} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} x'_\alpha x'_\beta \rho(\vec{\mathbf{r}'}) = \frac{1}{2} \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} \left(x'_\alpha x'_\beta - \frac{1}{3} {r'}^2 \delta_{\alpha,\beta} \right) \rho(\vec{\mathbf{r}'})$$

We can rewrite as

$$\phi_3 = \frac{1}{6} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} Q_{\alpha\beta}$$

where

$$Q_{\alpha\beta} = \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \left(x'_\alpha x'_\beta - \frac{1}{3} {r'}^2 \delta_{\alpha,\beta} \right)$$

and

$$\frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} = 3 \frac{x_\alpha x_\beta}{r^5} - \frac{\delta_{\alpha,\beta}}{r^3}$$

Also

$$\operatorname{tr} Q = \sum_{\alpha,\beta} \delta_{\alpha,\beta} Q_{\alpha,\beta} = 0$$

Legendre polynomial Define functions $P_l(x)$:

$$G(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l$$

It can be shown that those are orthogonal functions:

$$\int_{-1}^{1} P_l(x)P_{l'}(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{ll'}$$

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\mu}} = \frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}}$$

where $r_>$ and $r_<$ are the bigger and smaller out of r and r' correspondingly, and $\mu = \cos\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)$.
Taking $t = \frac{r_<}{r_>}$ and $x = 2\mu$ we get

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|} = \frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}} = \sum_{l=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l P_l(\mu)$$

We get

$$\phi = \int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}'}\right)\sum_{l=0}^{\infty}\frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\mu)$$

we can, using spherical harmonics

$$P_l(\mu') = \sum_{n=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}(\hat{\mathbf{r}}')$$