114246 - Electromagnetism and Electrodynamics

Adi Nusser

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Abstract

1 Introduction

In this course we use CGS system. Force between two charges is

$$\vec{\mathbf{F}} = \frac{q_1 q_2}{\mathbf{r}^2}\hat{\mathbf{r}}$$

The unit of charge is statcoulomb, or esu.
Field around charge q is

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{F}}}{q'} = \frac{q}{\mathbf{r}^2}\hat{\mathbf{r}}$$

Then force can be written as

$$\vec{\mathbf{F}} = q'\vec{\mathbf{E}}$$

Principle of linearity (superposition) If we have some frame of reference we can rewrite force as

$$\vec{\mathbf{F}}_1 = \frac{q' q_1}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_1\right|^3}\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

And fields can be summed as following:

$$\vec{\mathbf{E}} = \sum_{i=1}^{N} E_i = \sum_{i=1}^{N} \frac{q_i(\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i)}{\left|\vec{\mathbf{r}'} - \vec{\mathbf{r}}_i\right|^3}$$

If charge is continuous define

$$\rho(\vec{\mathbf{r}}) = \frac{\Delta q}{\Delta V}$$

field turns into integral:

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3\mathbf{r}'\, \rho(\vec{\mathbf{r}'}) \frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3}$$

Potential

$$\vec{E} = \frac{q\vec{\mathbf{r}}}{\mathbf{r}^2} = -\vec{\boldsymbol{\nabla}}\frac{q}{r}$$

For some frame of reference

$$\vec{E} = \frac{q(\vec{\mathbf{r}} - \vec{\mathbf{r}'})}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|^3} = -\vec{\boldsymbol{\nabla}}\frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|} = -\vec{\boldsymbol{\nabla}}\Phi$$

Gradient, divergence and Laplacian in spherical coordinates

$$\vec{\boldsymbol{\nabla}} f(r,\theta,\phi) = \hat{\mathbf{r}}\frac{\partial f}{\partial x} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{\partial f}{\partial \phi}\frac{1}{r\sin\theta}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}(\mathbf{r}^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Continuous case

$$\vec{E}(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r' \, \rho\left(\vec{\mathbf{r}'}\right) \frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = -\vec{\boldsymbol{\nabla}} \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

And

$$\Phi = \int \frac{\mathrm{d}^3\mathbf{r}' \, \rho(\mathbf{r}')}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

Gauss theorem

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}(\mathbf{r}) = \oint \vec{\mathbf{A}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \mathbf{A}_n \, \mathrm{d}s$$

Lets apply Gauss theorem on electric field of point charge in origin:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \vec{\boldsymbol{\nabla}} \cdot \frac{q\vec{\mathbf{r}}}{\mathbf{r}^3}$$

Then

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = -\nabla^2 \frac{q}{r}$$

If $r \neq 0$,

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = q\vec{\boldsymbol{\nabla}} \frac{\vec{\mathbf{r}}}{\mathbf{r}^3} = q\vec{\boldsymbol{\nabla}} \frac{\hat{\mathbf{r}}}{\mathbf{r}^2} = \frac{q}{\mathbf{r}^2} \frac{\partial}{\partial r} \left(\mathbf{r}^2 \frac{1}{\mathbf{r}^2} \right) = 0$$

By applying Gauss law:

$$\int\limits_V \mathrm{d}^3\mathbf{r} \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = \oint \vec{\mathbf{E}} \, \mathrm{d}\vec{\mathbf{s}} = \int \mathrm{d}\Omega \, \frac{q\hat{\mathbf{R}}}{\mathbf{R}^2} \cdot \vec{\mathbf{R}} \cdot \mathbf{R}^2 = q \oint \mathrm{d}\Omega = 4\pi q$$

Which is right for a ball of any radius, in particular, $R \to 0$. Thus

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 0$$

while

$$\int\limits_{V_\epsilon} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3\mathbf{r} = 4\pi q$$

Delta function Lets define

$$\begin{cases} \delta^D = 0 & x \neq 0 \\ \int\limits_{x \in V} \delta^D(x) \, \mathrm{d}x = 1 & \end{cases}$$

We can define it as limit of

$$F_\Delta = \begin{cases} \frac{1}{\Delta} & -\frac{\Delta}{2} < x < \frac{\Delta}{2} \\ 0 & otherwise \end{cases}$$

Then $\lim_{\Delta \to 0} F_\Delta = \delta^D$.

Potential of point charge For a charge in point $\mathbf{r}_a$:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi q \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And for a couple of charges

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

Then we can define dencity of charge for a point charge as

$$\rho(\vec{\mathbf{r}}) = \sum_a q_a \delta^D(\vec{\mathbf{r}} - \vec{\mathbf{r}}_a)$$

And then in both cases

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi \rho(\vec{\mathbf{r}})$$

Since

$$\vec{E} = -\vec{\nabla}\Phi$$

$$\vec{\nabla}\left(-\vec{\nabla}\Phi\right) = 4\pi\rho$$

Which means

$$\nabla^2\Phi = -4\pi\rho$$

which is Poisson equation. For bound conditions of $\phi, E \overset{r\to\infty}{\to} 0$ solution is $E = q\frac{\hat{\mathbf{r}}}{\mathbf{r}^2}$.

1.1 Bound conditions

Direchlet bound conditions $\Phi = \Phi_S(\vec{\mathbf{r}})$ for $r \in S$.

Neumann bound condtions We have E_n on S.

1.2 Example of Solutions

Suppose we have Dirichlet bound conditions: $\Phi_S = 0$. Obvious solution is $\Phi = \rho = 0$
Lets show it's unique solution:

$$\int\limits_V E^2 \,\mathrm{d}^3\mathbf{r} = \int |\vec{\nabla}\Phi|^2 \,\mathrm{d}^3\mathbf{r}$$

$$|\nabla\Phi|^2 = \vec{\nabla}\Phi\cdot\vec{\nabla}\Phi = \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right) - \Phi\nabla^2\Phi = \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)$$

$$\int\limits_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)\mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi\cdot\vec{\nabla}\Phi\cdot\mathrm{d}\vec{\mathbf{s}} = 0$$

Thus $E = 0$ and $\Phi = 0$.

General case In general case $\nabla^2\Phi = -4\pi q$. Suppose we have bound conditions $\Phi = \Phi_S \neq 0$. Suppose we have two solutions Φ_1, Φ_2.
Take a look at $\Phi = \Phi_1 - \Phi_2$, which has boundary conditions $\Phi_1 - \Phi_2 = \Phi_S - \Phi_S = 0$, thus solutions are equal, from previous paragraph.

Neumann bound conditions Instead of Φ we now have $E_n = -\vec{\nabla}\Phi\cdot\hat{\mathbf{n}} = 0$.

$$\int\limits_V E^2 \,\mathrm{d}^3\mathbf{r} = \int \vec{\nabla}\left(\Phi\cdot\vec{\nabla}\Phi\right)\mathrm{d}^3\mathbf{r} \overset{\text{Gauss}}{=} \oint \Phi\cdot\vec{\nabla}\Phi\cdot\mathrm{d}\vec{\mathbf{s}} = \oint \Phi\cdot\underbrace{\vec{\nabla}\Phi\cdot\hat{\mathbf{n}}}_{-E_n}\mathrm{d}s = 0$$

Thus $E = 0$. Note that E_n determines E_t.

Earnshaw theorem If in some volume $\rho = 0$, then there is no local maximum or minimum of potential in this volume, since then $\vec{\nabla}\Phi = 0$ and either $\nabla^2\Phi > 0$ or $\nabla^2\Phi < 0$, but $\nabla^2\Phi = \rho = 0$

1.3 Methods of solutions of Poisson equation

Method of image charges Suppose we have infinite grounded plane and a point charge in distance a from it. The bound condition is $\phi(x = 0, y, z) = 0$ And the potential of point charge is $\nabla^2\Phi = -4\pi q\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}_0)$
The way to solve this kind of problems is to add imaginary charge such that we get zero potential on bound (e.g., symmetrically with opposite charge).
If we have two such charges:

$$\Phi = \frac{q}{(x-a)^2 + y^2 + z^2} - \frac{q}{(x+a)^2 + y^2 + z^2}$$

This fulfills bound conditions:

$$\phi(x = 0, y, z) = \frac{q}{a^2 + y^2 + z^2} - \frac{q}{a^2 + y^2 + z^2} = 0$$

Denote density of charge on plane as σ, then $\mathrm{d}q = \sigma\,\mathrm{d}s$. If this case force between plane and charge is $\left|\frac{\sigma q \mathrm{d}s}{R^2}\right|$

How do we find σ? From Gauss law

$$\oint \vec{E}\cdot \mathrm{d}s = 4\pi \int\limits_V \rho \,\mathrm{d}^3\mathbf{r}$$

$$-|E_n|\,\mathrm{d}s = 4\pi\sigma\,\mathrm{d}s$$

$$4\pi\sigma = E_n$$

1.4 Green function

Suppose we have a unit charge in point $\vec{\mathbf{r}'}$. Then potential is $\Phi_{\vec{\mathbf{r}'}}$ such that $\Delta\Phi = -4\pi\delta(\vec{\mathbf{r}} - \vec{\mathbf{r}'})$.

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \Phi_{\vec{\mathbf{r}'}}$$

Thus

$$\nabla^2 G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = -4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)$$

Then we can write G as

$$G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = \frac{1}{|r - r'|} + F\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$$

And F guaranties bound conditions, while $\nabla^2 F = 0$.

Bound conditions

1. $G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right) = 0$.for bound S.
2. $\frac{\partial}{\partial\hat{\mathbf{n}}}G(\vec{\mathbf{r}}, \vec{\mathbf{r}'}) = -\frac{4\pi}{S}$. Then number comes from Gauss' law

Green theorem

$$\int\limits_V \left(\phi\vec{\boldsymbol{\nabla}}\psi - \psi\vec{\boldsymbol{\nabla}}\phi\right)\mathrm{d}^3\mathbf{r} = \oint\limits_S \left[\phi\frac{\partial\psi}{\partial\hat{\mathbf{n}}} - \psi\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]\mathrm{d}s$$

Since

$$\int\limits_V \mathrm{d}^3\mathbf{r}\,\vec{\boldsymbol{\nabla}}\cdot\left(\phi\vec{\boldsymbol{\nabla}}\psi\right) = \oint \phi\vec{\boldsymbol{\nabla}}\psi\cdot\mathrm{d}\vec{\mathbf{s}} = \oint \phi\frac{\partial\psi}{\partial\hat{\mathbf{n}}}\,\mathrm{d}s$$

we get

$$\int\limits_V \mathrm{d}^3\mathbf{r}\left(\phi\nabla^2\psi - \psi\nabla^2\phi\right) = \int\limits_V \mathrm{d}^3\mathbf{r}\,\vec{\boldsymbol{\nabla}}\cdot\left(\phi\vec{\boldsymbol{\nabla}}\psi\right) - \vec{\boldsymbol{\nabla}}\phi\cdot\vec{\boldsymbol{\nabla}}\psi - \vec{\boldsymbol{\nabla}}\cdot\left(\phi\vec{\boldsymbol{\nabla}}\psi\right) + \vec{\boldsymbol{\nabla}}\psi\cdot\vec{\boldsymbol{\nabla}}\phi = \oint\limits_S \left[\phi\frac{\partial\psi}{\partial\hat{\mathbf{n}}} - \psi\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]\mathrm{d}s$$

Now for $\psi(\vec{\mathbf{r}}) = G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)$

$$\int\limits_V \mathrm{d}^3\mathbf{r}\left(-\phi\cdot 4\pi\delta\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint\limits_S \mathrm{d}s\left[\phi\frac{\partial G}{\partial\hat{\mathbf{n}}} - G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

Suppose we want $\nabla^2\phi = 4\pi\rho(\vec{\mathbf{r}})$ for some ρ and Dirichlet bound conditions on same surface S:

$$\int\limits_V \mathrm{d}^3\mathbf{r}\left(-4\pi\phi\left(\vec{\mathbf{r}'}\right) - G\nabla^2\phi\right) = \oint\limits_S \mathrm{d}s\left[\phi\frac{\partial G}{\partial\hat{\mathbf{n}}} - G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

Substituting $G = 0$ on the bound surface

$$-4\pi\phi\left(\vec{\mathbf{r}'}\right) + \int\limits_V \mathrm{d}^3\mathbf{r}\,G\cdot 4\pi\rho(\vec{\mathbf{r}}) = \oint\limits_S \mathrm{d}s\,\phi\frac{\partial G}{\partial\hat{\mathbf{n}}}$$

i.e.

$$\phi\left(\vec{\mathbf{r}'}\right) = \int\limits_V G\left(\vec{\mathbf{r}}, \vec{\mathbf{r}'}\right)\rho(\vec{\mathbf{r}})\,\mathrm{d}^3\mathbf{r} - \oint\limits_S \frac{\phi}{4\pi}\cdot\frac{\partial G}{\partial\hat{\mathbf{n}}}\,\mathrm{d}s$$

Neumann bound conditions

$$-4\pi\phi\left(\vec{\mathbf{r}'}\right)+4\pi\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\mathrm{d}^3\mathbf{r}=\oint\mathrm{d}s\left[\phi\left(-\frac{4\pi}{S}\right)-G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\right]$$

However

$$\oint\phi\left(-\frac{4\pi}{S}\right)\mathrm{d}s=-\frac{4\pi}{S}\oint\phi\,\mathrm{d}s=-4\pi\langle\phi\rangle$$

$$\phi\left(\vec{\mathbf{r}'}\right)=\int\rho(\vec{\mathbf{r}})G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\mathrm{d}^3\mathbf{r}+\frac{1}{4\pi}\oint G\frac{\partial\phi}{\partial\hat{\mathbf{n}}}\,\mathrm{d}s+\langle\phi\rangle$$

Example

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}$$

With bound conditions

$$0=G\left(\vec{\mathbf{r}}\to\infty,\vec{\mathbf{r}'}\right)$$

Thus

$$\phi(\vec{\mathbf{r}})=\int\rho(\vec{\mathbf{r}})\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}'}\right|}\,\mathrm{d}^3\mathbf{r}$$

Example Suppose we have an infinite plane with potential $\phi(x=0,y,z)=\phi_S(y,z)$ and $\rho=0$ in one side of space ($x>0$). We are searching for $\phi(x>0,y,z)$.
Define Green function as a solution of previous problem, with point charge and grounded plane:

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=\frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}-\frac{1}{\left[(x+x')^2+(y-y')^2+(z-z')^2\right]^{\frac{1}{2}}}$$

$$\phi\left(\vec{\mathbf{r}'}\right)=\underbrace{\int_V G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)\rho(\vec{\mathbf{r}})\,\mathrm{d}^3\mathbf{r}}_{0}-\oint_S\frac{\phi}{4\pi}\cdot\frac{\partial G}{\partial\hat{\mathbf{n}}}\,\mathrm{d}s$$

$$\frac{\partial G}{\partial\hat{\mathbf{n}}}=-\left.\frac{\partial G}{\partial x}\right|_{x=0}$$

$$\phi=\frac{x'}{2\pi}\int\mathrm{d}z\,\mathrm{d}y\,\frac{\phi_S(y,z)}{\left[x'^2+(y-y')^2+(z-z')^2\right]^{\frac{3}{2}}}$$

Symmetry of Green function

$$G\left(\vec{\mathbf{r}},\vec{\mathbf{r}'}\right)=G\left(\vec{\mathbf{r}'},\vec{\mathbf{r}}\right)$$

1.5 Separation of variables

Suppose we have two planes parallel to y axis with zero potential with distance L between them, and a plane parallel to x axis with potential $V(x)$. We want to solve

$$\nabla^2\phi=0$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Since there is no change on z direction we get

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

Let's use anzatz for the solution $\phi=X(x)Y(y)$:

$$Y\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+X\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

$$\frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}x^2}+\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=0$$

Since we have operands depending on different variables, both are constant.

$$\begin{cases}\frac{1}{Y}\frac{\mathrm{d}^2Y}{\mathrm{d}y^2}=+k^2\\ \frac{1}{X}\frac{\mathrm{d}^2X}{\mathrm{d}X^2}=-k^2\end{cases}$$

(since we want Y to decrease to 0 in infinity, we choose positive value for Y and negative for X).
The solution is

$$\begin{cases}X=a\cdot\sin(kx)+b\cdot\cos(kx)\\ Y=c\cdot e^{ky}+d\cdot e^{-ky}\end{cases}$$

Since $\phi(0,y,z)=\phi(L,y,z)=0$, $b=0$ and k acquires discrete values $k_n=\frac{n\pi}{L}$. Since $\phi(x,y\to\infty,z)=0$, $c=0$.
Now we need to force $\phi(x,y=0,z)=V(x)$:

$$\Phi=X(x)Y(y=0)=V(x)$$

We know that the solution is of form

$$d_na_n\sin(k_nx)e^{-k_ny}$$

Thus we can get a general solution in a form

$$\phi=\sum d_na_n\sin(k_nx)e^{-k_ny}$$

So we want

$$V(x)=\sum\underbrace{d_na_n}_{q_n}\sin(k_nx)$$

which is Fourier series:

$$\int\mathrm{d}x\,V(x)\sin(k_{n'}x)=\sum_n\int\mathrm{d}x\,q_n\sin(k_nx)\sin(k_{n'}x)=\sum_n q_n\frac{L}{2}\delta_{nn'}=\frac{Lq_{n'}}{2}$$

Example Suppose we have to planes with angle θ_0 between them and potential ϕ_0 on both. In cylindrical coordinates

$$\nabla^2\phi=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

By separation of variables

$$\phi=F(R)G(\theta)$$

$$\frac{G(\theta)}{R}\frac{\mathrm{d}F}{\mathrm{d}R}+G\frac{\mathrm{d}^2F}{\mathrm{d}R^2}+\frac{F(R)}{R^2}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=0$$

$$\begin{cases}\frac{1}{G}\frac{\mathrm{d}^2G}{\mathrm{d}\theta^2}=-\nu^2\\ \frac{1}{F}\frac{\mathrm{d}F}{\mathrm{d}R}+\frac{R^2}{F}\frac{\mathrm{d}^2F}{\mathrm{d}R^2}=\nu^2\end{cases}$$

$\nu>0$

$$\begin{cases}G=A\cos(\nu\theta)+B\sin(\nu\theta)\\ F=aR^{-\nu}+bR^{\nu}\end{cases}$$

$\nu=0$

$$\begin{cases}G=\tilde{A}+\tilde{B}\theta\\ F=\tilde{a}+\tilde{b}\ln R\end{cases}$$

We get $\tilde{b}=0$ such that F doesn't diverge in 0. Also there shouldn't be dependence on angle, so $\tilde{B}=0$ and $\tilde{A}\neq0$, thus for $\nu=0$ potential is constant.
For positive ν, $A=0$ and $a=0$ and also want $\sin(\nu\theta_0)=0$ thus

$$\phi(\theta)=\phi_0+\sum_{n=1}a_nR^{\frac{n\pi}{\theta_0}}\sin\left(n\pi\frac{\theta}{\theta_0}\right)$$

If $R\to0$, the most dominant element of sum is $n=1$. Thus

$$\phi\propto R^{\frac{\pi}{\theta_0}}$$

i.e., due to $E \sim -\frac{\partial \phi}{\partial R} \sim R^{\frac{\pi}{\theta_0}-1}$:

$$\begin{cases} E \to 0 & \theta_0 < \pi \\ E \to \infty & \theta_0 > \pi \end{cases}$$

1.6 Solution with Fourier transform

$$f_k = \int \mathrm{d}x\, e^{ixk} F(x) \iff \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e^{-ixk} f_k$$

$$\mathrm{d}^2 F\, x = \int \frac{\mathrm{d}k}{2\pi} (-k^2) e^{-ikx} f_k$$

Since $\nabla^2 \phi = -4\pi\rho$

$$\int (-k^2) \phi_k e^{-ikx} \,\mathrm{d}^3 k = -4\pi \int \rho_k e^{-ikx} \,\mathrm{d}^3 k$$

Now

$$\delta\left(\vec{\mathbf{k}} - \vec{\mathbf{k}'}\right) = C \int e^{-i\vec{\mathbf{k}}x} e^{i\vec{\mathbf{k}'}x} \,\mathrm{d}x$$

Thus, this is orthogonal basis and coefficients have to be equal

$$-k^2 \phi_{\vec{\mathbf{k}}} = -4\pi \rho_{\vec{\mathbf{k}}}$$

$$k^2 \phi_{\vec{\mathbf{k}}} = 4\pi \rho_{\vec{\mathbf{k}}}$$

Example $\rho_k = 0$, then

$$k^2 \phi_{\vec{\mathbf{k}}} = 0$$

i.e., either

$$k^2 = 0 \text{ or } \phi_{\vec{\mathbf{k}}}$$

However, $k_x, k_y, k_z > 0$, else, $e^{-ikx} = e^{|k|x} \to \infty$, which doesn't fulfills boundary conditions, thus $\phi = 0$.

Finite boundary conditions In this case, we have a Fourier series instead of transform.

Example If $\rho \neq 0$;

$$\phi(\vec{x}) = \int \frac{\rho\left(\vec{\mathbf{x}'}\right)}{|x - x'|} \,\mathrm{d}^3 x$$

i.e.

$$\phi_k = \frac{4\pi \rho_k}{k^2}$$

1.7 Multipole expansion

If we are far from a set of charges we can approximate them as a single charge Q:

$$\phi = \frac{Q}{r} = \frac{\sum_i q_i}{r} = \frac{\int \rho(r') \,\mathrm{d}^3 r'}{r}$$

This is monopole approximation.
Now if we denote $f\left(\vec{\mathbf{r}'}\right) = \frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|}$ We can take Taylor expansion of it:

$$f\left(\vec{\mathbf{r}'}\right) = f\left(\vec{\mathbf{r}'} = 0\right) + \sum_{\alpha=1}^{3} \left.\frac{\partial f}{\partial x'_\alpha}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha + \sum_{\alpha,\beta=1}^{3} \left.\frac{\partial^2 f}{\partial x'_\alpha \partial x'_\beta}\right|_{\vec{\mathbf{r}'}=0} x'_\alpha x'_\beta + \dots$$

Then the monopole approximation is first element in the series:

$$f\left(\vec{\mathbf{r}'}\right) \approx f\left(\vec{\mathbf{r}'} = 0\right)$$

$$\phi_0(\vec{\mathbf{r}}) = \frac{Q}{r}$$

Dipole expansion is

$$\phi_1(\vec{\mathbf{r}}) = \int \mathrm{d}^3 r' \, \rho(r) \frac{\partial}{\partial r'} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \cdot \vec{\mathbf{r}'}$$

$$\frac{\partial}{\partial r'} f = \vec{\boldsymbol{\nabla}} f = \sum_\alpha \frac{\partial f}{\partial x_\alpha} \hat{\mathbf{x}}_\alpha$$

i.e.,

$$\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} = \frac{1}{r} + \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} + \mathcal{O}\left(r^2\right)$$

meaning

$$\phi_1 = \int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3} = \frac{\vec{\mathbf{r}}}{r^3} \underbrace{\int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \vec{\mathbf{r}'}}_{\vec{\mathbf{P}}}$$

Example A single point charge in point $\vec{\mathbf{r}'} = \vec{\mathbf{r}}_q$.

$$\phi = \frac{q}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_q|} \approx \phi_1(\vec{\mathbf{r}}) + \phi_2(\vec{\mathbf{r}}) = \frac{q}{r} + \frac{\vec{\mathbf{r}} \cdot P}{r^3} = \frac{q}{r} + \frac{q \vec{\mathbf{r}} \cdot \vec{\mathbf{r}}_q}{r^3}$$

Example Two point charge. Then

$$\vec{\mathbf{P}} = q_1 \vec{\mathbf{r}}_1 + q_2 \vec{\mathbf{r}}_2$$

Quadruple expansion Note that

$$\left.\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right|_{\vec{\mathbf{r}'}=0} = \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r}$$

(since

$$\left.\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right| = -\left.\frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}\right|$$

). Also note that

$$\sum_{\alpha,\beta} \delta_{\alpha,\beta} \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{1}{r} = 0$$

Now

$$\phi_2 = \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \frac{1}{2} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} x'_\alpha x'_\beta \rho(\vec{\mathbf{r}'}) = \frac{1}{2} \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} \left(x'_\alpha x'_\beta - \frac{1}{3} r'^2 \delta_{\alpha,\beta} \right) \rho(\vec{\mathbf{r}'})$$

We can rewrite as

$$\phi_2 = \frac{1}{6} \frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} Q_{\alpha\beta}$$

where

$$Q_{\alpha\beta} = \sum_{\alpha,\beta} \int \mathrm{d}^3 r' \, \rho(\vec{\mathbf{r}'}) \left(r'_\alpha r'_\beta - \frac{1}{3} r'^2 \delta_{\alpha,\beta} \right)$$

and

$$\frac{\partial^2 \frac{1}{r}}{\partial x_\alpha \partial x_\beta} = 3 \frac{x_\alpha x_\beta}{r^5} - \frac{\delta_{\alpha,\beta}}{r^3}$$

Also

$$\operatorname{tr} Q = \sum_{\alpha,\beta} \delta_{\alpha,\beta} Q_{\alpha,\beta} = 0$$

Thus

$$\phi_2 = \frac{1}{2} \frac{\vec{\mathbf{r}}^T Q \vec{\mathbf{r}}}{r^5}$$

Legendre polynomial Define functions $P_l(x)$:

$$G(t,x) = \frac{1}{\sqrt{1-2tx+t^2}} = \sum_{l=0}^{\infty} P_l(x) t^l$$

It can be shown that those are orthogonal functions:

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\,\mathrm{d}x = \frac{2}{2l+1}\delta_{ll'}$$

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\mu}} = \frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}}$$

where $r_>$ and $r_<$ are the bigger and smaller out of r and r' correspondingly, and $\mu = \cos\left(\vec{\mathbf{r}}\cdot\vec{\mathbf{r}}'\right)$. Taking $t = \frac{r_<}{r_>}$ and $x = 2\mu$ we get

$$\frac{1}{\left|\vec{\mathbf{r}}-\vec{\mathbf{r}}'\right|} = \frac{1}{r_>}\frac{1}{\sqrt{1-2\frac{r_<}{r_>}\mu+\frac{r_<^2}{r_>^2}}} = \sum_{l=0}^{\infty}\frac{1}{r_>}\left(\frac{r_<}{r_>}\right)^l P_l(\mu)$$

We get

$$\phi = \int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}}'\right)\sum_{l=0}^{\infty}\frac{1}{r}\left(\frac{r'}{r}\right)^l P_l(\mu)$$

we can rewrite, using spherical harmonics

$$P_l(\mu') = \sum_{n=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}\left(\hat{\mathbf{r}}'\right)$$

Spherical harmonics Spherical harmonics are an analogue of Fourier series in spherical coordinates.

$$G(\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{l,m}Y_{lm}(\theta,\varphi)$$

Y are orthogonal functional basis

$$\int \mathrm{d}\Omega\,Y_{lm}(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

A coefficient is

$$g_{l'm'} = \int \mathrm{d}\Omega\,G(\theta,\varphi)Y^*_{l'm'}(\theta,\varphi)$$

The spherical harmonics are proportional to Legendre polynomials

$$Y_{lm}(\theta,\varphi) \propto P_l^m(\cos\theta)e^{\pm i\varphi}$$

$$\phi(\vec{\mathbf{r}}) = \sum_l \int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}}'\right)\frac{r'^l}{r^{l+1}}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}Y_{lm}(\hat{\mathbf{r}})Y^*_{lm}\left(\hat{\mathbf{r}}'\right) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{Y_{lm}(\hat{\mathbf{r}})}{r^{l+1}}\underbrace{\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}}'\right)r'^l Y^*_{lm}\left(\hat{\mathbf{r}}'\right)}_{\text{multipole moment}}$$

$$\int \mathrm{d}^3r'\,\rho(r',\theta',\varphi')r'^l Y^*_{lm}(\theta',\varphi') = \int \mathrm{d}r'\,r'^{2+l}\,\mathrm{d}\theta'\sin(\theta')\,\mathrm{d}\phi'\,\rho(r',\theta',\varphi')Y^*_{lm}(\theta',\varphi') = \int \mathrm{d}r'\,r'^{2+l}\,\mathrm{d}\Omega'\,\rho(r',\theta',\varphi')Y^*_{lm}(\theta',\varphi')$$

Now, for $l=0$, for example,

$$\phi^{l=0}(r,\theta,\varphi) = 4\pi\frac{Y_{lm}(\hat{\mathbf{r}})}{r}\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}}'\right)r'^l Y^*_{00}\left(\hat{\mathbf{r}}'\right) = 4\pi Y^2_{00}\frac{1}{r}\int \mathrm{d}^3r'\,\rho\left(\vec{\mathbf{r}}'\right) = \frac{Q}{r}$$

for $l=1$:

$$\phi^{l=1}(r,\theta,\varphi) = \sum_{m=-1}^{1}\frac{4\pi}{3}\frac{Y_{1m}(\hat{\mathbf{r}})}{r^2}\int \mathrm{d}r'\,r'^3\,\mathrm{d}\Omega'\,\rho(r',\theta',\varphi')Y^*_{1m}(\theta',\varphi')$$

Denote

$$\vec{\mathbf{P}} = \int \mathrm{d}^3 r'\, \vec{\mathbf{r}'} \rho\Big(\vec{\mathbf{r}'}\Big)$$

Previously, we got

$$\phi_1 = \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{P}}}{r^3}$$

Now, for $m = 0$:

$$\frac{4\pi}{3} \frac{\overbrace{\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta}^{Y_{10}}}{r^2} \int \underbrace{\mathrm{d}r'\, {r'}^3\, \mathrm{d}\Omega'}_{\mathrm{d}^3 r' r'} \rho(r', \theta', \varphi') \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta' = \frac{4\pi}{3} \frac{\frac{3}{4\pi}\cos\theta}{r^2} \int \mathrm{d}^3 r'\, \rho(r', \theta', \varphi') z' = \frac{\cos\theta}{r^2} \int \mathrm{d}^3 r'\, \rho(r', \theta', \varphi') z' = \frac{\cos\theta}{r^2} P_z$$

Dipole inside constant electrical field Given field $\vec{\mathbf{E}}$ and dipole moment $\vec{\mathbf{P}}$:

$$U = -\vec{\mathbf{E}} \cdot \vec{\mathbf{P}} = -\vec{\mathbf{E}} \cdot (q\vec{\mathbf{r}} + (-q)(-\vec{\mathbf{r}})) = -\vec{\mathbf{E}}(2\vec{\mathbf{r}})q$$

The torque is

$$\vec{\mathbf{N}} = \vec{\mathbf{r}}_1 \times q_1 \vec{\mathbf{E}} + \vec{\mathbf{r}}_2 \times q_2 \vec{\mathbf{E}} = (q_1 \vec{\mathbf{r}}_1 + q_2 \vec{\mathbf{r}}_2) \times \vec{\mathbf{E}} = \vec{\mathbf{P}} \times \vec{\mathbf{E}}$$

We could acquire it as

$$\left| -\frac{\partial U}{\partial \theta} \right| = \left| PE \frac{\partial \cos\theta}{\partial \theta} \right| = |PE \sin\theta|$$

Dipole inside electrical field Now

$$U = q_1 \phi^{ext}(\vec{\mathbf{r}}_1) + q_2 \phi^{ext}(\vec{\mathbf{r}}_2)$$

We can approximate potential in a second point:

$$\phi^{ext}(\vec{\mathbf{r}}_2) = \phi^{ext}(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) + \frac{1}{2}\frac{\partial \phi^{ext}}{\partial x_\alpha x_\beta}(x_{2,\alpha} - x_{1,\alpha})(x_{2,\beta} - x_{1,\beta}) + \mathcal{O}\big((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^3\big)$$

For constant field we get the same results. Else we get

$$U = q_1 \phi^{ext}(\vec{\mathbf{r}}_1) + q_2 \left[\phi^{ext}(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) + \underbrace{\frac{1}{2}\frac{\partial \phi^{ext}}{\partial x_\alpha x_\beta}(x_{2,\alpha} - x_{1,\alpha})(x_{2,\beta} - x_{1,\beta}) + \mathcal{O}\big((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^3\big)}_{0} \right] =$$

$$= \mathrm{const} - \vec{\mathbf{E}} \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) q = \mathrm{const} - \vec{\mathbf{E}} \cdot \vec{\mathbf{P}}$$

Lets calculate force on dipole:

$$\vec{\mathbf{F}} = q_1 \vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + q_2 \vec{\mathbf{E}}(\vec{\mathbf{r}}_2)$$

Approximating E with Taylor series

$$E(\vec{\mathbf{r}}_2) = \vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + \sum_{\alpha,\beta} \hat{\mathbf{x}}_\alpha \frac{\partial E_\alpha}{\partial x_\beta}(x_{2\beta} - x_{1\beta}) + \mathcal{O}\Big((\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)^2\Big) \approx E(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1)$$

$$\vec{\mathbf{F}} = q_1 \vec{\mathbf{E}}(\vec{\mathbf{r}}_1) + q_2 \Big[E(\vec{\mathbf{r}}_1) + \vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) \Big] = -q\vec{\boldsymbol{\nabla}}E \cdot (\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1) = \Big(\vec{\mathbf{P}} \cdot \vec{\boldsymbol{\nabla}}\Big) E$$

Thus

$$U = -\vec{\mathbf{P}} \cdot \vec{\mathbf{E}}$$

In general case

$$\vec{\mathbf{P}} = \int \mathrm{d}^3 r'\, \rho(\vec{\mathbf{r}}')\vec{\mathbf{r}}' = \sum q_i \vec{\mathbf{r}}'_i$$

Now

$$U = \sum_i q_i \phi^{ext}(\vec{\mathbf{r}}_i) \approxeq \sum_i q_i \left[\phi^{ext}(\vec{\mathbf{r}}_0) \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot (\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_0) + \frac{1}{2}\frac{\partial^2 \phi^{ext}}{\partial x_\alpha \partial x_\beta}(x_{i\alpha} - x_{0\alpha})(x_{i\beta} - x_{0\beta}) \right] =$$

$$= \sum_i q_i \phi^{ext}(0) + \vec{\boldsymbol{\nabla}}\phi^{ext} \cdot \sum_i q_i \vec{\mathbf{r}}_i + \frac{1}{2}\frac{\partial^2 \phi^{ext}}{\partial x_\alpha \partial x_\beta} \sum_i q_i x_{i\alpha} x_{i\beta}$$

$$\frac{1}{2}\frac{\partial^2\phi^{ext}}{\partial x_\alpha \partial x_\beta}\sum_i q_i x_{i\alpha}x_{i\beta} = \sum_{\alpha,\beta}\frac{1}{6}\frac{\partial^2\phi^{ext}}{\partial x_\alpha \partial x_\beta}D_{\alpha\beta}$$

where

$$D_{\alpha\beta} = \sum q_i\left(3x_{i\alpha}x_{i\beta} - \mathbf{r}_i^2\delta_{\alpha,\beta}\right)$$

Self energy of system of particles For two particles the energy is

$$U = \frac{q_1 q_2}{|\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1|} = q_1\phi_2(\vec{\mathbf{r}}_1) = \frac{1}{2}\left(q_1\phi(\vec{\mathbf{r}}_1) + q_2\phi(\vec{\mathbf{r}}_2)\right)$$

i.e.

$$U = \frac{1}{2}\sum q_i\phi(r_i) = \frac{1}{2}\int \mathrm{d}^3r\,\rho(\vec{\mathbf{r}})\phi(\vec{\mathbf{r}})$$

Substituting $\nabla^2\phi = -4\rho(\vec{\mathbf{r}})$:

$$U = -\frac{1}{8\pi}\int \mathrm{d}^3r\,\nabla^2\phi\phi$$

Since

$$\vec{\boldsymbol{\nabla}}\cdot\phi\vec{\boldsymbol{\nabla}}\phi = |\vec{\boldsymbol{\nabla}}\phi|^2 + \phi\nabla^2\phi$$

we get

$$U = -\frac{1}{8\pi}\int \mathrm{d}^3r\,\underbrace{\vec{\boldsymbol{\nabla}}\cdot\phi\vec{\boldsymbol{\nabla}}\phi}_{0} - \underbrace{|\vec{\boldsymbol{\nabla}}\phi|^2}_{E^2} = \frac{1}{8\pi}\int \mathrm{d}^3r\,E^2$$

Infinite energy Suppose all internal energy comes from own electric field:

$$U \sim \frac{e^2}{R_0} \sim m_e c^2$$

i.e.

$$R_0 \sim \frac{e^2}{mc^2}$$

However this is not proper quantum limit, since it lacks Plank constant. The proper one is Compton wavelength:

$$\lambda \sim \frac{\hbar}{m_e c}$$

Thus

$$\frac{\lambda}{R_0} \sim \frac{\bar{h}c}{e^2} \sim 137$$

Pure dipole If we take an charge system such that it's potential is dipole's one in every place in space, we get field equal to

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{3(\vec{\mathbf{P}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{\mathbf{P}}}{r^3} - 4\pi\hat{r}(\vec{\mathbf{P}}\cdot\hat{\mathbf{r}})\delta(\vec{\mathbf{r}})$$

2 Magnetostatics

The force between two currents is

$$\mathrm{d}F = \kappa I I'\frac{\mathrm{d}\vec{\mathbf{l}}' \times (\mathrm{d}\vec{\mathbf{l}}\,\vec{\mathbf{r}})}{r^3}$$

Difference between magnetic and electric field

$$\begin{cases}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{B}} = 0\\ \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{A}} \Rightarrow \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}} = \frac{4\pi}{c}\vec{\mathbf{j}}\end{cases} \iff \begin{cases}\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{E}} = 4\pi\rho\\ \vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}}\phi \Rightarrow \vec{\boldsymbol{\nabla}}\times\vec{\mathbf{E}} = 0\end{cases}$$

where j is current density, such that $\int\vec{\mathbf{j}}\cdot\mathrm{d}\vec{\mathbf{s}}$ is equal to charge passing through surface in unit time.

$$\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{j}} = \frac{c}{4\pi}\vec{\boldsymbol{\nabla}}\cdot\left(\vec{\boldsymbol{\nabla}}\times\vec{\mathbf{B}}\right) = 0$$

Suppose we have constant charge, then $\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = 0$ and thus

$$0 = \oint \vec{\mathbf{j}} = \int\limits_V \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} \, \mathrm{d}^3 r \xrightarrow{V \to 0} \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} \cdot \Delta V$$

In general case

$$\frac{\partial Q}{\partial t} = -\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$$

Substituting $Q = \int\limits_V \mathrm{d}^3 r \, \rho$ and $\oint \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = \int\limits_V \mathrm{d}^3 r \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}$, we get

$$\frac{\partial}{\partial t} \int \mathrm{d}^3 r \, \rho = -\int\limits_V \mathrm{d}^3 r \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}$$

In limit $V \to 0$ we get continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} = 0$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \right) = \frac{4\pi}{c} \vec{\mathbf{j}}$$

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} \right) = \vec{\boldsymbol{\nabla}} \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} \right) - \nabla^2 \vec{\mathbf{A}}$$

From gauge invariance we are free to choose $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$. Thus we acquired

$$\nabla^2 \vec{\mathbf{A}} = -\frac{4\pi}{c} \vec{\mathbf{j}}$$

The solution is

$$\vec{\mathbf{A}} = \frac{1}{c} \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'}) \, \mathrm{d}^3 r'}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|}$$

with border conditions $\vec{\mathbf{B}} \to 0$ in infinity. Lets check whether $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}$:

$$\vec{\boldsymbol{\nabla}}_r \cdot \int \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'}) \, \mathrm{d}^3 r'}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} = \int \vec{\boldsymbol{\nabla}}_r \cdot \left(\frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'}) \, \mathrm{d}^3 r'}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} \right) = \int \mathrm{d}^3 r' \, \vec{\mathbf{j}}(\vec{\mathbf{r}'}) \vec{\boldsymbol{\nabla}}_r \cdot \frac{1}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} = -\int \mathrm{d}^3 r' \, \vec{\mathbf{j}}(\vec{\mathbf{r}'}) \vec{\boldsymbol{\nabla}}_{r'} \cdot \frac{1}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} =$$

$$= -\int \mathrm{d}^3 r' \, \vec{\boldsymbol{\nabla}}_{r'} \cdot \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} - \frac{1}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} \underbrace{\vec{\boldsymbol{\nabla}}_{r'} \cdot \vec{\mathbf{j}}(\vec{\mathbf{r}'})}_{0} = -\oint \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}'})}{\left| \vec{\mathbf{r}} - \vec{\mathbf{r}'} \right|} \cdot \mathrm{d}\vec{\mathbf{s}} = 0$$

Stokes' theorem For some vector field $\vec{\mathbf{V}}$ and closed curve Γ:

$$\int \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{V}} \right) \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{V}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

In our case we get

$$\int\limits_S \left(\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} \right) \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

$$\frac{4\pi}{c} \int\limits_S \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

We denote $I = \int\limits_S \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$, which is current.

$$\frac{4\pi}{c} I = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}}$$

2.1 Gauge transformation

Suppose we have $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}} \psi$. We get $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}'$. Suppose $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = \chi$. Then

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} + \nabla^2 \psi$$

If we want $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = 0$, we get $\chi + \nabla^2 \psi = 0$. Thus we can find ψ such that $\nabla^2 \psi = -\chi$ and we get $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}}' = 0$.

If

$$\vec{\mathbf{A}} = \frac{1}{c}\int \mathrm{d}^3 r' \, \frac{\vec{\mathbf{j}}(\vec{\mathbf{r}})}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

we can get

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}} = \frac{1}{c}\int \mathrm{d}^3 r' \, \vec{\mathbf{j}}(\vec{\mathbf{r}}) \times \frac{\left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$$

2.2 Biot–Savart law

Suppose we have current I on some curve l'. Then magnetic field due to this current in point $\vec{\mathbf{r}}$ is

$$\mathrm{d}\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{I}{c}\,\mathrm{d}\vec{\mathbf{l}'} \times \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$$

Example Suppose we have infinite wire with current $I\hat{\mathbf{z}}$. Then the magnetic fiend due to the wire is

$$\mathrm{d}\vec{\mathbf{B}} = \frac{I}{c}\frac{\mathrm{d}z\,\hat{\mathbf{z}} \times \left(\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = \frac{I}{c}\frac{\mathrm{d}z\,R\hat{\mathbf{y}}}{\left[R^2 + z^2\right]^{\frac{3}{2}}}$$

Thus

$$B = \frac{I}{c}\int_{-\infty}^{\infty} \frac{\mathrm{d}z\,R}{\left[R^2 + z^2\right]^{\frac{3}{2}}} = \frac{2I}{cR}$$

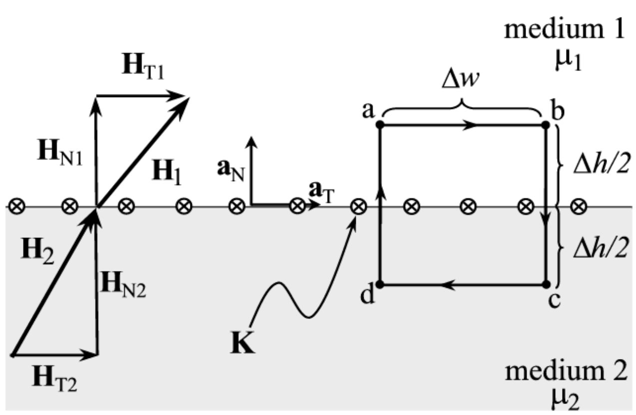
Alternatively

$$\frac{4\pi}{c}I = \oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}} = B \cdot 2\pi R$$

$$B = \frac{2I}{cR}$$

Meissner effect Superconducting material actively excludes magnetic fields from its interior. Thus $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0$ and there are currents only on the edge.

Example Suppose we have a boundary between two materials, with fields $\vec{\mathbf{B}}_1$, $\vec{\mathbf{B}}_2$.



Taking integral on small box on boudary we get

$$0 = \oint \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}}\,\mathrm{d}^3 r = \oint \vec{\mathbf{B}} \cdot \mathrm{d}s = \left(\vec{\mathbf{B}}_{1,\perp} - \vec{\mathbf{B}}_{2,\perp}\right)\Delta s$$

i.e.

$$\vec{\mathbf{B}}_{1,\perp} = \vec{\mathbf{B}}_{2,\perp}$$

What happens with $B_{\parallel}$?

$$\oint \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{l}} = \frac{4\pi}{c} \int \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}} = I = \frac{4\pi}{c} K \,\mathrm{d}l$$

$$\left(B_{2,\parallel} - B_{1,\parallel}\right) = \frac{4\pi}{c} \vec{\mathbf{K}} \,\mathrm{d}l$$

$$\hat{\mathbf{n}} \times (\vec{\mathbf{B}}_2 - \vec{\mathbf{B}}_1) = \frac{4\pi}{c} \vec{\mathbf{K}}$$

where $\vec{\mathbf{K}}$ is surface current density.

Magnetic scalar potential It's impossible to write $\vec{\mathbf{B}} = -\vec{\boldsymbol{\nabla}}\phi_m$. However, if there is some area in which there is no current, we can try to do so, since $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = 0$. Using Biot-Savart law

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{I}{c} \oint \mathrm{d}\vec{\mathbf{r}'} \times \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} = \frac{I}{c} \underbrace{\oint \mathrm{d}r' \times \underbrace{\vec{\mathbf{U}}(\vec{\mathbf{r}})}_{\frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}}}_{\vec{\mathbf{H}}}$$

For some constant vector $\vec{\mathbf{k}}$:

$$\vec{\mathbf{k}} \cdot \vec{\mathbf{H}} = \vec{\mathbf{k}} \cdot \oint \mathrm{d}\vec{\mathbf{r}'} \times \vec{\mathbf{U}}(\vec{\mathbf{r}}) = \oint \mathrm{d}\vec{\mathbf{r}'} \cdot \left(\vec{\mathbf{U}} \times \vec{\mathbf{k}}\right) = \int \mathrm{d}\vec{\mathbf{s}} \cdot \vec{\boldsymbol{\nabla}} \times \left(\vec{\mathbf{U}} \times \vec{\mathbf{k}}\right) = \int \mathrm{d}\vec{\mathbf{s}} \cdot \left[(\vec{\mathbf{k}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{U}} - \vec{\mathbf{k}}(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{U}})\right]$$

where

$$(\vec{\mathbf{k}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{U}} = \sum_{\alpha,\beta} k_\alpha \frac{\partial U_\beta}{\partial x_\alpha} \hat{\mathbf{x}}_\beta$$

We can rewrite as

$$(\vec{\mathbf{k}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{U}} = \sum_{\alpha} (k_\alpha \hat{\mathbf{x}}_\alpha) \cdot \sum_{\alpha,\beta} \left(\frac{\partial U_\beta}{\partial x_\alpha} \hat{\mathbf{x}}_\beta \hat{\mathbf{x}}_\alpha\right)$$

Denote matrix of derivatives $\frac{\partial U_\beta}{\partial x_\alpha} \hat{\mathbf{x}}_\beta \hat{\mathbf{x}}_\alpha$ as $\vec{\boldsymbol{\nabla}}\vec{\mathbf{U}}$, i.e., dyadic or outer product of $\vec{\boldsymbol{\nabla}}$ and $\vec{\mathbf{U}}$. Then

$$\oint \mathrm{d}\vec{\mathbf{r}'} \times \vec{\mathbf{U}}(\vec{\mathbf{r}}) = \int \vec{\boldsymbol{\nabla}}\vec{\mathbf{U}} \cdot \mathrm{d}\vec{\mathbf{s}} - \int \mathrm{d}s \, \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{U}}$$

In our case $\vec{\mathbf{U}} = \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$ and since $\vec{\mathbf{r}} \neq \vec{\mathbf{r}'}$, $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{U}} = 0$. Finally, we got

$$\vec{\mathbf{B}}(\vec{\mathbf{r}}) = \frac{I}{c} \int \vec{\boldsymbol{\nabla}}_{r'} \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} \cdot \mathrm{d}\vec{\mathbf{s}'} = -\frac{I}{c} \int \vec{\boldsymbol{\nabla}}_{r} \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} \cdot \mathrm{d}\vec{\mathbf{s}'} = -\frac{I}{c} \vec{\boldsymbol{\nabla}}_{r} \int \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} \cdot \mathrm{d}\vec{\mathbf{s}'}$$

That means

$$\phi_m(\vec{\mathbf{r}}) = \frac{I}{c} \vec{\boldsymbol{\nabla}}_{r} \int \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3} \cdot \mathrm{d}\vec{\mathbf{s}'}$$

such that

$$\vec{\mathbf{B}} = -\vec{\boldsymbol{\nabla}}\phi_m(\vec{\mathbf{r}})$$

Geometrical meaning of ϕ_m Suppose $\vec{\mathbf{r}} = 0$. Then

$$\phi_m(\vec{\mathbf{r}}) = \frac{I}{c} \vec{\boldsymbol{\nabla}}_{r} \int -\frac{\vec{\mathbf{r}'}}{r'^3} \cdot \mathrm{d}\vec{\mathbf{s}'}$$

Note that $\frac{\vec{\mathbf{r}'} \cdot \mathrm{d}\vec{\mathbf{s}'}}{r'^3}$ is solid angle Ω of current loop, i.e.

$$\phi_m = \frac{I}{c} \Omega$$

2.3 Multipole expansion of magnets

First of all for $\vec{\mathbf{E}}$ monopole is sum of charges, however there is no such thing as magnetic monopole.

Exercise Suppose there exists magnetic monopole $\vec{\mathbf{B}} = \frac{Q_B \hat{\mathbf{r}}}{r^2}$. Solve the problem of movement

$$m\ddot{r} = \vec{\mathbf{F}}_B = \frac{q}{c}\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

Write down the scalar potential

$$\phi_m = \frac{I}{c}\int \mathrm{d}s' \cdot \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}'}}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|^3}$$

First order approximation $r' \approx 0$:

$$\phi_m \approx \frac{I}{c}\int \mathrm{d}s' \cdot \frac{\vec{\mathbf{r}}}{r^3} = \frac{I}{c}\frac{\vec{\mathbf{r}}}{r^3} \cdot \int \mathrm{d}s' = \vec{\mathbf{M}} \cdot \frac{\vec{\mathbf{r}}}{r^3}$$

where

$$\vec{\mathbf{M}} = \frac{I}{c}\vec{\mathbf{S}}$$

is magnetic moment.
Alternatively we could use vector potential:

$$\vec{\mathbf{A}} = \frac{1}{c}\int \mathrm{d}^3r' \, \frac{\vec{\mathbf{j}}\left(\vec{\mathbf{r}'}\right)}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$$

Taking Taylor series of $\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|}$ around 0, we get

$$\vec{\mathbf{A}} \approx \frac{1}{cr}\int \vec{\mathbf{j}}\left(\vec{\mathbf{r}'}\right) \mathrm{d}^3r' + \frac{\vec{\mathbf{r}}}{cr^3} \cdot \int \vec{\mathbf{r}'}\vec{\mathbf{j}}\left(\vec{\mathbf{r}'}\right) \mathrm{d}^3r'$$

($\frac{1}{\left|\vec{\mathbf{r}} - \vec{\mathbf{r}'}\right|} \approx \frac{1}{r} + \frac{\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'}}{r^3}$, thus we have a matrix under integral). Note since

$$\frac{\partial x'_\beta}{\partial x'_\alpha} = \delta_{\alpha\beta}$$

is identity matrix, we can rewrite

$$\int \vec{\mathbf{j}} \, \mathrm{d}^3r' = \int j_\alpha \cdot \frac{\partial x'_\beta}{\partial x'_\alpha}\hat{\mathbf{x}}_\beta \, \mathrm{d}^3r' = \int \frac{\partial}{\partial x'_\alpha}\left(j_\alpha x'_\beta \hat{\mathbf{x}}_\beta\right) - x'_\beta \hat{\mathbf{x}}_\beta \underbrace{\frac{\partial j_\alpha}{\partial x'_\alpha}}_{\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}} = 0} \mathrm{d}^3r'$$

In matrix form:

$$\int \vec{\mathbf{j}} \, \mathrm{d}^3r' = \int \vec{\mathbf{j}} \cdot \frac{\partial x'_\beta}{\partial x'_\alpha} \, \mathrm{d}^3r' = \int \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\mathbf{j}}\vec{\mathbf{x}'}\right) - \left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\right)\vec{\mathbf{r}'} \, \mathrm{d}^3r'$$

Now, since there are no currents (magnetostatics)

$$\int \frac{\partial}{\partial x'_\alpha}\left(j_\alpha x'_\beta \hat{\mathbf{x}}_\beta\right) \mathrm{d}^3r' = \int \mathrm{d}s'_\alpha \, j_\alpha x'_\beta \hat{\mathbf{x}}_\beta = 0$$

$$\int \vec{\boldsymbol{\nabla}} \cdot \left(\vec{\mathbf{j}}\vec{\mathbf{x}'}\right) = \int \mathrm{d}s' \, \vec{\mathbf{j}}\vec{\mathbf{x}'} = 0$$

Thus

$$\int \vec{\mathbf{j}} \, \mathrm{d}^3r' = 0$$

For second-order part, take one element of it, and adding and subtracting $\frac{1}{2}\int x'_\beta \hat{\mathbf{x}}'_\beta j_\alpha \, \mathrm{d}^3r'$, thus factoring into symmetric and antisymmetric part, where antisymmetric part is curl:

$$\int \vec{\mathbf{r}'}\vec{\mathbf{j}}\left(\vec{\mathbf{r}'}\right) \mathrm{d}^3r' \bigg|_\alpha = \int j_\beta \hat{\mathbf{x}}'_\beta x'_\alpha \, \mathrm{d}^3r' = \frac{1}{2}\int j_\beta \hat{\mathbf{x}}'_\beta x'_\alpha \, \mathrm{d}^3r' + \frac{1}{2}\int x'_\beta \hat{\mathbf{x}}'_\beta j_\alpha \, \mathrm{d}^3r' + \frac{1}{2}\int j_b \hat{\mathbf{x}}'_\beta x'_\alpha \, \mathrm{d}^3r' - \frac{1}{2}\int x'_\beta \hat{\mathbf{x}}'_\beta j_\alpha \, \mathrm{d}^3r'$$

$$\int \vec{\mathbf{r}}'\vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) \mathrm{d}^3r' = \frac{1}{2}\int \vec{\mathbf{r}}'\vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) - \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right)\vec{\mathbf{r}}' \mathrm{d}^3r' + \frac{1}{2}\int \vec{\mathbf{r}}'\vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) + \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right)\vec{\mathbf{r}}' \mathrm{d}^3r'$$

$$\hat{\mathbf{x}}'_\beta \frac{\partial}{\partial x'_\gamma}\left(x'_\alpha x'_\beta j_\gamma\right) = j_\alpha x'_\beta \hat{\mathbf{x}}'_\beta + j_\beta x'_\alpha \hat{\mathbf{x}}'_\beta + x'_\alpha x'_\beta \hat{\mathbf{x}}'_\beta \underbrace{\frac{\partial j_\gamma}{\partial x'_\gamma}}_{0}$$

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\vec{\mathbf{r}}\vec{\mathbf{r}} = \vec{\mathbf{r}}\vec{\mathbf{j}} \cdot \vec{\boldsymbol{\nabla}}\vec{\mathbf{r}}) + (\vec{\mathbf{j}} \cdot \vec{\boldsymbol{\nabla}}\vec{\mathbf{r}})\vec{\mathbf{r}} + \underbrace{\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\vec{\mathbf{r}}}_{0}$$

Thus

$$\frac{1}{2}\int j_\beta \hat{\mathbf{x}}'_\beta x'_\alpha \mathrm{d}^3r' + \frac{1}{2}\int x'_\beta \hat{\mathbf{x}}'_\beta j_\alpha \mathrm{d}^3r' = \frac{1}{2}\int \hat{\mathbf{x}}'_\beta \frac{\partial}{\partial x'_\gamma}\left(x'_\alpha x'_\beta j_\gamma\right) \mathrm{d}^3r' = \frac{\hat{\mathbf{x}}'_\beta}{2}\oint x'_\alpha x'_\beta \vec{\mathbf{j}} \cdot \mathrm{d}\vec{\mathbf{s}}$$

$$\frac{1}{2}\int \vec{\mathbf{r}}'\vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) - \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right)\vec{\mathbf{r}}' \mathrm{d}^3r' = \frac{1}{2}\int \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{j}}\vec{\mathbf{r}}\vec{\mathbf{r}} \mathrm{d}^3r' = \frac{1}{2}\oint \vec{\mathbf{j}}\vec{\mathbf{r}}\vec{\mathbf{r}} \mathrm{d}^3r'$$

If there are no currents inside, the integral is 0.
From second part we can acquire (from bac-cab identity):

$$A_2 = \frac{1}{2cr^3}\int \vec{\mathbf{r}} \cdot \left[\vec{\mathbf{r}}'\vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) + \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right)\vec{\mathbf{r}}'\right] \mathrm{d}^3r' = \left[\frac{1}{2c}\int \vec{\mathbf{r}}' \times \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) \mathrm{d}^3r'\right] \times \frac{\vec{\mathbf{r}}'}{r^3} = \vec{\mathbf{M}} \times \frac{\vec{\mathbf{r}}}{r^3}$$

If the loop is in plane, $\int \vec{\mathbf{r}}' \times \mathrm{d}^3r'$ is double area:

$$\oint \vec{\mathbf{r}} \times \mathrm{d}\vec{\mathbf{r}} = 2\int \mathrm{d}\vec{\mathbf{s}}$$

$$\vec{\mathbf{c}} \cdot \oint(\vec{\mathbf{r}} \times \mathrm{d}r) = \oint \vec{\mathbf{c}} \cdot (\vec{\mathbf{r}} \times \mathrm{d}\vec{\mathbf{r}}) = c\oint(\vec{\mathbf{c}} \times \vec{\mathbf{r}}) \cdot \mathrm{d}\vec{\mathbf{r}}\, b = \int \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{c}} \times \vec{\mathbf{r}} \cdot \mathrm{d}\vec{\mathbf{c}} = \int 2 \cdot \mathrm{d}s$$

due to

$$\vec{\boldsymbol{\nabla}} \times \left(\vec{\mathbf{a}} \times \vec{\mathbf{b}}\right) = \vec{\mathbf{a}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{b}}\right) - \vec{\mathbf{b}}\left(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{a}}\right) + \left(\vec{\mathbf{b}} \cdot \vec{\boldsymbol{\nabla}}\right)\vec{\mathbf{a}} - \left(\vec{\mathbf{a}} \cdot \vec{\boldsymbol{\nabla}}\right)\vec{\mathbf{b}}$$

and thus

$$\vec{\mathbf{M}} = \frac{1}{2c}\int \vec{\mathbf{r}}' \times \vec{\mathbf{j}}\left(\vec{\mathbf{r}}'\right) \mathrm{d}^3r' = \frac{I}{2c}\oint \vec{\mathbf{r}}' \times \mathrm{d}\vec{\mathbf{l}}' = \frac{I}{c}\vec{\mathbf{S}}$$

Connection with angular momentum

$$\vec{\mathbf{j}} = \rho_e \vec{\mathbf{v}}_e$$

Substituting

$$\vec{\mathbf{M}} = \frac{1}{2v}\int \vec{\mathbf{r}} \times \rho_e \vec{\mathbf{v}}_e \mathrm{d}^3r$$

This looks pretty much like angular momentum

$$\vec{\mathbf{L}} = \int \vec{\mathbf{r}} \times \rho_m \vec{\mathbf{v}}_m \mathrm{d}^3r$$

If $\rho_m \propto \rho_e$ and $v_m \propto v_e$ (e.g., there is one kind of current carriers), $\vec{\mathbf{L}} \propto \vec{\mathbf{M}}$. For $v_e = v_m$, we get

$$\vec{\mathbf{M}} = \frac{q}{2mc}\vec{\mathbf{L}}$$

We define gyromagnetic relation $\frac{M}{L}$.

Bohr magneton

$$M_B = \frac{e\hbar}{2mc}$$

Electromagnetic force

$$\vec{\mathbf{F}} = q\vec{\mathbf{E}} + q\frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}}$$

If $\vec{\mathbf{E}} = 0$, power of force

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{B}} = \left(q\frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}}\right) \cdot \vec{\mathbf{v}} = 0$$

So for magnetic force:

$$\mathrm{d}\vec{\mathbf{F}} = \frac{1}{c}\sum_{\alpha\in \mathrm{d}l} q_\alpha \vec{\mathbf{v}}_\alpha \times \vec{\mathbf{B}} = \frac{I}{c}\,\mathrm{d}\vec{\mathbf{l}} \times \vec{\mathbf{B}}$$

Suppose we have long wire with $\vec{\mathbf{I}} = I\hat{\mathbf{x}}$ and external field $\vec{\mathbf{B}} = B\vec{\mathbf{y}}$. If ve move wire in direction $-\hat{\mathbf{z}}$, we need to spend energy, however magnetic field doesn't apply work. Where the energy goes? Since now particles have velocity in direction $\hat{\mathbf{z}}$, the field accelerates them.

Energy of $\vec{\mathbf{B}}$ In case of electric field we got energy $\frac{E^2}{8\pi}$. Unsurprisingly for magnetic field we get $\frac{B^2}{8\pi}$.
Suppose we have infinite plane of current $\mathrm{d}I = K\,\mathrm{d}l$ and one parallel plane with current in opposite direction.
What is work per unit area required to move the plane by distance h?
Now since the field B_0 is constant inside and 0 outside, the work is exactly the energy of the magnetic field.
What is the field of the lower plane?

$$\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} = \frac{4\pi}{c}\vec{\mathbf{j}}$$

$$\int \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{B}} \cdot \mathrm{d}\vec{\mathbf{s}} = \oint \vec{\mathbf{B}} \cdot \mathrm{d}l$$

$$\frac{4\pi}{c}KL_0 = 2BL_0$$

Thus

$$B = \frac{2\pi}{c}K$$

Since force is $\mathrm{d}\vec{\mathbf{F}} = \frac{I}{c}\,\mathrm{d}\vec{\mathbf{l}}_I \times \vec{\mathbf{B}}$ where $\vec{\mathbf{l}}_I$ is direction of current. Then force on upper plane is

$$\mathrm{d}F = \frac{I}{c}\,\mathrm{d}l_0 = \frac{K}{c}\,\mathrm{d}l\,\mathrm{d}l_0\,B$$

And thus the work is

$$\mathrm{d}W = \underbrace{h\,\mathrm{d}l\,\mathrm{d}l_0}_{\text{volume element}}\,\frac{K}{c}B = \mathrm{d}V\,\frac{B^2}{2\pi} = \mathrm{d}V\,\frac{B_0^2}{8\pi}$$

i.e., the energy per volume is $\frac{B^2}{8\pi}$.

Force of $\vec{\mathbf{B}}$ on system of currents

$$\vec{\mathbf{F}} = \frac{q}{c}\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

(for comparison, Coriolis force is $\vec{\mathbf{F}} = -2\vec{\boldsymbol{\Omega}} \times \vec{\mathbf{v}}$).
Then

$$\mathrm{d}\vec{\mathbf{F}} = \frac{I}{c}\,\mathrm{d}l \times \vec{\mathbf{B}}$$

i.e.,

$$\vec{\mathbf{F}} = \frac{I}{c}\oint \mathrm{d}l \times \vec{\mathbf{B}}$$

Taylor series of $\vec{\mathbf{B}}$ is

$$\vec{\mathbf{B}} \approx \vec{\mathbf{B}}(0) + (\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\Big|_{\vec{\mathbf{r}}=0}$$

Thus

$$\vec{\mathbf{F}} = \frac{I}{c}\oint \mathrm{d}\vec{\mathbf{r}} \times \left[\vec{\mathbf{B}}(0) + (\vec{\mathbf{r}} \cdot \vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\Big|_{\vec{\mathbf{r}}=0}\right] = \frac{I}{c}\oint \mathrm{d}\vec{\mathbf{r}}$$

Since

$$\oint \mathrm{d}\vec{\mathbf{r}} \times \vec{\mathbf{u}} = \int (\vec{\boldsymbol{\nabla}}\vec{\mathbf{u}}) \cdot \mathrm{d}\vec{\mathbf{s}} - \int \mathrm{d}\vec{\mathbf{s}}\,(\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{u}})$$

we get

$$\vec{\mathbf{F}} = \frac{I}{c}\int \vec{\boldsymbol{\nabla}}\Big((\vec{\mathbf{r}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\Big)\cdot \mathrm{d}s = \frac{I}{c}\int \mathrm{d}s\,\vec{\boldsymbol{\nabla}}\cdot\Big[(\vec{\mathbf{r}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\Big] = \frac{I}{c}\int \mathrm{d}s\,\vec{\boldsymbol{\nabla}}\cdot\vec{\mathbf{B}} = 0$$

In case of magnetic dipole:

$$\vec{\mathbf{F}} = \frac{I}{c}\int \vec{\boldsymbol{\nabla}}\Big((\vec{\mathbf{r}}\cdot\vec{\boldsymbol{\nabla}})\vec{\mathbf{B}}\Big)\cdot \mathrm{d}s = \vec{\mathbf{M}}\cdot\vec{\boldsymbol{\nabla}}\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}}\Big(\vec{\mathbf{M}}\cdot\vec{\mathbf{B}}\Big)$$

Then the energy of magnetic dipole moment in external magnetic field is

$$U = \vec{\mathbf{M}}\cdot\vec{\mathbf{B}}$$

Landau's method Looking at average in time:

$$F = \Big\langle \sum \frac{q}{c}\vec{\mathbf{v}}_\alpha \times \vec{\mathbf{B}}\Big\rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}\sum \frac{q}{c}\vec{\mathbf{r}}_\alpha \times \vec{\mathbf{B}}\Big\rangle \to 0$$

Since if X doesn't diverges

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{X} = \frac{1}{T}\int_0^T \frac{\mathrm{d}X}{\mathrm{d}t}\,\mathrm{d}t = \frac{X(t)-X(0)}{T}$$

Now, the torque is

$$\vec{\mathbf{N}} = \Big\langle \sum \frac{q}{c}\vec{\mathbf{r}}_i \times \Big(\vec{\mathbf{v}}\times\vec{\mathbf{B}}\Big)\Big\rangle = \Big\langle \sum \frac{q}{c}\vec{\mathbf{v}}\Big(\vec{\mathbf{r}}\cdot\vec{\mathbf{B}}\Big) - \vec{\mathbf{B}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{r}})\Big\rangle$$

Similarly,

$$\vec{\mathbf{v}}\cdot\vec{\mathbf{r}} = \frac{\mathrm{d}r^2}{\mathrm{d}t}$$

on average gives 0, i.e.,

$$\vec{\mathbf{N}} = \Big\langle \sum \frac{q}{c}\vec{\mathbf{v}}(\vec{\mathbf{r}}\cdot\vec{\mathbf{B}})\Big\rangle$$

$$\vec{\mathbf{v}}(\vec{\mathbf{r}}\cdot\vec{\mathbf{B}}) = \underbrace{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\vec{\mathbf{r}}(\vec{\mathbf{r}}\cdot\vec{\mathbf{B}})\Big]}_{0} + \underbrace{\frac{1}{2}\vec{\mathbf{v}}(\vec{\mathbf{r}}\cdot\vec{\mathbf{B}}) - \frac{1}{2}\vec{\mathbf{r}}(\vec{\mathbf{v}}\cdot\vec{\mathbf{B}})}_{\frac{1}{2}\vec{\mathbf{v}}\times(\vec{\mathbf{r}}\times\vec{\mathbf{B}})}$$

Define

$$\vec{\mathbf{M}} = \frac{1}{2c}\sum q\vec{\mathbf{r}}\times\vec{\mathbf{B}}$$

Then

$$\vec{\mathbf{N}} = \vec{\mathbf{M}}\times\vec{\mathbf{B}} = \frac{1}{2c}\sum q\vec{\mathbf{r}}\times\vec{\mathbf{v}}$$

Larmor precession The angular momentum vector $\vec{\mathbf{M}}$ precesses about the external field axis with an angular frequency known as the Larmor frequency, $\omega = -\gamma B$
where ω is the angular frequency, $\vec{\mathbf{B}}$ is the magnitude of the applied magnetic field. γ is the gyromagnetic ratio of system.

3 Electric field in insulating matter

Exercise Suppose molecules are neutral conducting balls. What happens when appears external field?

Field due to polarization Define polarization with vector $\vec{\mathbf{P}}$ and then electric dipole of volume ΔV is

$$\vec{\mathbf{p}} = \vec{\mathbf{P}}\Delta V$$

$$\Phi_{dipole} = \frac{\vec{\mathbf{P}}\cdot\vec{\mathbf{r}}}{r^3}$$

$$\phi_{total} = -\int \mathrm{d}^3r'\,\frac{\vec{\mathbf{r}}-\vec{\mathbf{r}}'}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}\cdot\vec{\mathbf{P}}(r)$$

$$\vec{\mathbf{E}}_p = -\vec{\boldsymbol{\nabla}}_r \int \mathrm{d}^3r' \, \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \cdot \vec{\mathbf{P}}(\vec{\mathbf{r}}')$$

$$\vec{\mathbf{E}}_p = -\int \mathrm{d}^3r' \, \vec{\mathbf{P}}(\vec{\mathbf{r}}') \cdot \vec{\boldsymbol{\nabla}}_r \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} = \int \mathrm{d}^3r' \, \vec{\mathbf{P}}(\vec{\mathbf{r}}') \cdot \vec{\boldsymbol{\nabla}}_{r'} \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} = -\int \mathrm{d}^3r' \, \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \vec{\boldsymbol{\nabla}}_{r'} \cdot \vec{\mathbf{P}}(\vec{\mathbf{r}}') + \oint \mathrm{d}\vec{\mathbf{s}} \cdot \vec{\mathbf{P}}(\vec{\mathbf{r}}') \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}$$

Here $\rho_b(\vec{\mathbf{r}})b = -\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{P}}$ is induced charge density inside the matter.

$$\vec{\mathbf{E}}_p = \int \mathrm{d}^3r' \, \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \rho_b(\vec{\mathbf{r}}) + \oint \mathrm{d}s' \, \sigma_b(\vec{\mathbf{r}}') \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3}$$

where $\sigma_b = \hat{\mathbf{n}} \cdot \vec{\mathbf{P}}$.
The total field thus is

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_{free} + \vec{\mathbf{E}}_p = \int \mathrm{d}^3r' \, \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \left(\rho_{free}(\vec{\mathbf{r}}) - \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{P}} \right) + \oint \mathrm{d}s' \, \frac{\vec{\mathbf{r}} - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}'|^3} \left(\sigma_{free} + \hat{\mathbf{n}} \cdot \vec{\mathbf{P}} \right)$$

3.1 Equations for field in matter

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}} = 4\pi\rho_{free} - 4\pi\vec{\boldsymbol{\nabla}}\vec{\mathbf{P}}$$

$$\hat{\mathbf{n}} \cdot \left(\vec{\mathbf{E}}_2 - \vec{\mathbf{E}}_1 \right) = 4\pi \left[\sigma_{free} + \hat{\mathbf{n}}(\vec{\mathbf{P}}_1 - \vec{\mathbf{P}}_2) \right]$$

We define electic displacement $\vec{\mathbf{D}}$:

$$\vec{\mathbf{D}} = \vec{\mathbf{E}} + 4\pi\vec{\mathbf{P}} \Rightarrow \vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} = 4\pi\rho_{free}$$

$$\hat{\mathbf{n}} \cdot (\vec{\mathbf{D}}_2 - \vec{\mathbf{D}}_1) = 4\pi\sigma_{free}$$

Connection between $\vec{\mathbf{P}}$, $\vec{\mathbf{E}}$, $\vec{\mathbf{D}}$ Define susceptibility χ_E

$$\vec{\mathbf{P}} = \chi_E \vec{\mathbf{E}} \Rightarrow \vec{\mathbf{E}} + 4\pi\vec{\mathbf{P}} = \epsilon\vec{\mathbf{E}}$$

when $\epsilon = 1 + 4\pi\xi_E$.
Note that in general case, $\vec{\mathbf{D}} = \epsilon\vec{\mathbf{E}}$, where ϵ is matrix which also might depend on $\vec{\mathbf{r}}$.
Thus our equations are:

$$\begin{cases} \vec{\boldsymbol{\nabla}}\vec{\mathbf{D}} = 4\pi\rho \\ \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = 0 \\ E_{t,1} = E_{t,2} \end{cases}$$

Uniqueness of solution Suppose there are two solutions, $\vec{\mathbf{D}}_1$ and $\vec{\mathbf{D}}_2$, define $\vec{\mathbf{D}} = \vec{\mathbf{D}}_1 - \vec{\mathbf{D}}_2$.

$$\int \vec{\mathbf{D}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3r = -\int \vec{\mathbf{D}} \cdot \vec{\boldsymbol{\nabla}}\phi \, \mathrm{d}^3r = -\int \left[\vec{\boldsymbol{\nabla}} \cdot \left(\phi\vec{\mathbf{D}} \right) - \phi\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} \right] \mathrm{d}^3r$$

Now

$$\vec{\mathbf{D}} \cdot \vec{\mathbf{E}} = \vec{\mathbf{E}}\epsilon\vec{\mathbf{E}}$$

if ϵ is scalar, $\vec{\mathbf{D}} \cdot \vec{\mathbf{E}} = \epsilon E^2$.
Since $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} = 0$, we get

$$\int \vec{\mathbf{D}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3r = -\int \vec{\boldsymbol{\nabla}} \cdot \left(\phi\vec{\mathbf{D}} \right) \mathrm{d}^3r = -\oint \vec{\phi}\vec{\mathbf{D}} \cdot \mathrm{d}\vec{\mathbf{s}}$$

If $\vec{\mathbf{D}} \cdot \hat{\mathbf{n}} = 0$ or $\phi = 0$ on the boundary, we get $\int \vec{\mathbf{D}} \cdot \vec{\mathbf{E}} \, \mathrm{d}^3r = 0$. Now if ϵ is positive (or positive defined in case of matrix), we conclude that $\vec{\mathbf{E}} = 0$. This provides us with two kinds of boundary conditions - $\vec{\mathbf{D}} \cdot \hat{\mathbf{n}}$ or ϕ should be given on boundary.

Force on test charge Note that force on test charge in dielectric matter is $\vec{\mathbf{E}}q$.

Example Suppose we have matter with $\epsilon > 1$ inside other matter with $\epsilon = 1$. Also there is point charge inside first one (located in origin).Then

$$\vec{\mathbf{D}} = \frac{q}{r^2}\hat{\mathbf{r}}$$

$$\vec{\mathbf{E}} = \frac{\vec{\mathbf{D}}}{r^2}\hat{\mathbf{r}} = \frac{q}{\epsilon r^2}\hat{\mathbf{r}}$$

However

$$\vec{\mathbf{p}} = \frac{\vec{\mathbf{D}} - \vec{\mathbf{E}}}{4\pi}$$

and outside of origin

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{p}} = 0$$

Surface charge There is discontinuity in normal part of $\vec{\mathbf{E}}$ and similarly there is discontinuity in normal part of $\vec{\mathbf{D}}$

ϵ of conductor Since in conductor there is no field, we an choose $\epsilon = \infty$

Example Suppose we have dieletric matter in $x > 0$ and a charge q in $x = -d$ in vacuum.
First of all,$\rho_d = 0$, since there is charges inside of dielectric. Thus the only source of field is σ_b.

$$\sigma_b = \hat{\mathbf{n}} \cdot \vec{\mathbf{P}} = \hat{\mathbf{n}} \cdot \left(\chi_E \vec{\mathbf{E}}\right) = \chi_E \hat{\mathbf{n}} \cdot \vec{\mathbf{E}}$$

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_q + \vec{\mathbf{E}}_{\sigma_b}$$

$$\vec{\mathbf{E}}_q \cdot \hat{\mathbf{n}} = -\frac{qd}{(R^2 + d^2)^{\frac{3}{2}}}$$

$$\hat{\mathbf{n}} \cdot \vec{\mathbf{E}}_{\sigma_b} = -2\pi\sigma_b$$

$$\sigma_b = \chi_E \left[-\frac{qd}{(R^2 + d^2)^{\frac{3}{2}}} - 2\pi\sigma_b\right]$$

Thus

$$\sigma_b = -\frac{\chi_E}{1 + 2\pi\chi_E}\frac{qd}{(R^2 + d^2)^{\frac{3}{2}}}$$

and

$$q_b = \int \sigma_b \, \mathrm{d}s = -\frac{2\pi\chi_e}{1 + 2\pi\chi_e} q$$

And thus in limit $\chi_e \to \infty$ we get $q_b \to -q$.

3.2 Magnetic field in matter

Describe molecules as magnetic dipole. Then their dipole moment $\vec{\mu}$ will try to turn into direction $\vec{\mathbf{B}}$.