

# 236621 – Algorithms for Submodular Optimization

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## 1 Introduction

We are looking on  $f : 2^N \rightarrow \mathbb{R}$  for some set  $N = \{1, \dots, n\}$

**Definition 1.1.**  $f$  is submodular if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad (1.1)$$

**Definition 1.2.** Return of  $u$  wrt  $A$  is  $f(A \cup \{u\}) - f(A)$

**Definition 1.3 (Diminishing returns).**  $f$  has diminishing returns if for  $A \subseteq B$

$$f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B) \quad (1.2)$$

**Proposition 1.1.**  $f$  is submodular iff  $f$  has diminishing returns

*Proof.*  $\Rightarrow$ :

Let  $A \subseteq B \subseteq N$  and  $u \notin B$ . Lets use submodularity property on  $A \cup \{u\}$  and  $B$ :

$$f(A \cup \{u\}) + f(B) \geq f(A \cup \{u\} \cup B) + f((A \cup \{u\}) \cap B) = f(B \cup \{u\}) + f(A) \quad (1.3)$$

Thus

$$f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B) \quad (1.4)$$

□

$\Leftarrow$ :

We'll proof by induction over  $|A \cup B| - |A \cap B|$ , i.e., size of symmetric difference.

Basis:  $|A \cup B| - |A \cap B| = 0$ , then  $A = B$ , and then submodular property is fulfilled.

Step: assume  $|A \cup B| - |A \cap B| = k$ . WLOG let  $u \in A$  such that  $u \notin B$ .

$$f(A) + f(B) = f(A) - f(A \setminus \{u\}) + f(A \setminus \{u\}) + f(B) \geq \quad (1.5)$$

$$\geq f(A) - f(A \setminus \{u\}) + f(A \setminus \{u\} \cup B) + f(A \setminus \{u\} \cap B) \geq \quad (1.6)$$

$$\geq f(A \cup B) - f(A \cup B \setminus \{u\}) + f(A \cup B \setminus \{u\}) + f(A \cap B) = f(A \cup B) + f(A \cap B) \quad (1.7)$$

**Definition 1.4 (Monotonous function).**  $f$  is non-decreasing monotonous if  $\forall A \subseteq B \subseteq N$ ,  $f(A) \leq f(B)$ .

**Definition 1.5 (Symmetric function).**  $f$  is symmetric if  $\forall S \subseteq N$ ,  $f(S) \leq f(N \setminus S)$ .

**Definition 1.6 (Normalized function).**  $f$  is normalized if  $f(\emptyset) = 0$ .

## Examples

**Linear function**  $\forall n \in N$  exists weight  $w_n$  and

$$f(S) = \sum_{u \in S} w_u + b \quad (1.8)$$

Such  $f$  is submodular.

**Budget additive function (clipped linear function)**  $\forall n \in N$  exists weight  $w_n$  and

$$f(S) = \min \left\{ \sum_{u \in S} w_u, b \right\} \quad (1.9)$$

Such  $f$  is submodular.

**Coverage function** Given set  $X$  and  $n$  subsets  $S_1, S_2, \dots, S_n \subset X$  define

$$f(S) = \left| \bigcup_{i \in S} S_i \right| \quad (1.10)$$

This  $f$  is obviously submodular.

**Graph cuts** Let  $G = (V, E)$  be a graph and  $w : E \rightarrow \mathbb{R}^+$  weights of edges. Given a cut  $S \subseteq V$  define  $\delta(S)$  to be sum of weights of all edges going through the cut.  $\delta : 2^V \rightarrow \mathbb{R}^+$  is submodular, normalized, and symmetric.

**Rank function** Let  $v_1, \dots, v_n \in \mathbb{R}^d$  vectors, and

$$f(S) = \text{rank}(S) = \dim \text{span}(\{v_i | i \in S\}) \quad (1.11)$$

## 2 Submodular optimization

Given world  $N$ , submodular function  $f : 2^N \rightarrow \mathbb{R}^+$ , and a family of feasible solutions  $\mathcal{I} \subseteq 2^N$

$$\max f(S) \tag{2.1}$$

$$\text{s.t. } S \in \mathcal{I} \tag{2.2}$$

**Note** Most of submodular functions (except for logarithm of determinant of submatrix) are nonnegative. We use the condition to have properly defined multiplicative approximation.

**Note** How  $f$  is given in input? Obviously, not as a list of values, since it's exponential in  $|N|$ . Thus we represent  $f$  with black box, and same applies for constraints. Usually, constraints are simple.

### 2.1 Examples of submodular optimization problems

**Example**  $f$  is submodular and there are no constraints. It generalizes MAX-CUT, MAX-DICUT

**Example**  $f$  is submodular and there is size constraint:

$$\max f(S) \tag{2.3}$$

$$\text{s.t. } |S| \leq k \tag{2.4}$$

. It generalizes MAX-K-COVER.

**Submodular welfare** Given  $k$  players and  $n$  goods, each player has submodular, monotone, non-negative value function  $T_i$  over subsets of  $N$ . The goal is to maximize total value by partitioning the goods.

## 3 Maximization of the submodular function with cardinality constraints

$$\max f(S) \tag{3.1}$$

$$\text{s.t. } |S| \leq k \tag{3.2}$$

**Greedy algorithm** If  $f$  is monotonic, greedy algorithm is an optimal approximating algorithm.

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**Algorithm 3.1** Nemhauser-Wolsey-Fisher

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```

1: procedure GREEDY( $N$ )
2:    $A \leftarrow \emptyset$ 
3:   for  $i = 1$  to  $k$  do
4:     Let  $u_i \in N$  maximize  $f(A_{i-1} \cup \{u_i\}) - f(A_{i-1})$ 
5:      $A_i \leftarrow A_{i-1} \cup \{u_i\}$ 
6:   end for
7:   return  $A_k$ 
8: end procedure

```

---

**Lemma 3.1.** For submodular  $f : 2^N \rightarrow \mathbb{R}_+$ ,

$$f(A \cup B) - f(A) \leq \sum_{b_i \in B} f(A \cup \{b_i\}) - f(A) \tag{3.3}$$

*Proof.*

$$f(A \cup B) - f(A) = \sum_i f(A \cup \{b_1, \dots, b_{i-1}\} \cup \{b_i\}) - f(A \cup \{b_1, \dots, b_{i-1}\}) \leq \sum_i f(A \cup \{b_i\}) - f(A) \tag{3.4}$$

□

**Proposition 3.2** (Nemhauser et al. [1978]). Algorithm 3.1 is  $1 - \frac{1}{e}$  optimal.

*Proof.* For optimal set  $S^*$

$$f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) \geq \max_{u \in S^*} \{f(A_{i-1} \cup \{u\}) - f(A_{i-1})\} \geq \frac{1}{k} \sum_{u \in S^*} [f(A_{i-1} \cup \{u\}) - f(A_{i-1})] \geq \quad (3.5)$$

$$\geq \frac{1}{k} \left( f(A_{i-1} \cup S^*) - f(A_{i-1}) \right) \geq \frac{1}{k} \left[ f(S^*) - f(A_{i-1}) \right] \quad (3.6)$$

We got a recursion equation:

$$f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) \geq \frac{1}{k} \left[ f(S^*) - f(A_{i-1}) \right] \quad (3.7)$$

We can solve the recursion and acquire

$$f(A_k) \geq \left( 1 - \left( 1 - \frac{1}{k} \right)^k \right) f(S^*) + \left( 1 - \frac{1}{k} \right)^k f(A_0) \geq \left( 1 - \frac{1}{e} \right) f(S^*) \quad (3.8)$$

□

**Theorem 3.3** (Nemhauser and Wolsey [1978]). For all constant  $\epsilon > 0$  each algorithm acquiring  $1 - \frac{1}{e} + \epsilon$  requires exponential number of requests to value oracle.

**Theorem 3.4** (Feige [1998]). For MAX-K-COVER all constant  $\epsilon > 0$  each algorithm acquiring  $1 - \frac{1}{e} + \epsilon$  requires exponential number of requests to value oracle unless  $P = NP$ .

**Note** Runtime of Algorithm 3.1 on the preceding page is  $O(nk)$ . It is possible to acquire  $O(n \lg(\frac{1}{\epsilon}))$  runtime and  $1 - \frac{1}{e} - \epsilon$  optimality by looking on some subset of  $N$  at each step instead of the whole set.

### 3.1 Non-monotonic functions

What happens if  $f$  is not monotonic? First of all, does greed algorithm work? Not only it is not optimal approximation, it can be as bad as  $\frac{2}{N}$ . However, it can be fixed. The idea is to randomize algorithm to prevent it from “bad” choices.

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#### Algorithm 3.2

---

```

1: procedure RANDOMIZED GREEDY( $N$ )
2:    $A \leftarrow \emptyset$ 
3:   for  $i = 1$  to  $k$  do
4:      $M_i \leftarrow \arg \max_{B \subseteq N : |B| \leq k} \sum_{u \in B} f(A_{i-1} \cup \{u\}) - f(A_{i-1})$ 
5:      $A_i \leftarrow \begin{cases} A_{i-1} \cup \{u\} & \forall u \in M_i \text{ with } P = \frac{1}{k} \\ A_{i-1} & \text{with } P = 1 - \frac{|M_i|}{k} \end{cases}$ 
6:   end for
7:   return  $A_k$ 
8: end procedure

```

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#### Randomized greedy algorithm

**Theorem 3.5** (Buchbinder et al. [2014]). In monotonic case, Algorithm 3.2 is  $1 - \frac{1}{e}$  optimal in expectation.

*Proof.* Take a look at  $i^{th}$  iteration and condition on previous iterations, denote a chosen element from  $M_i$  as  $u_i$ :

$$\mathbb{E} \left[ f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) | A_{i-1} \right] = \frac{1}{k} \sum_{u_i \in M_i} f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) \geq \quad (3.9)$$

$$\geq \frac{1}{k} (f(A_{i-1} \cup S^*) - f(A_{i-1})) \geq \quad (3.10)$$

$$\geq \frac{1}{k} (f(S^*) - f(A_{i-1})) \geq \frac{1}{k} (f(S^*) - f(A_{i-1})) \quad (3.11)$$

If the inequality is right for any  $A_{i-1}$  it is right, from tower property, in expectation over  $A_{i-1}$ :

$$\mathbb{E} \left[ f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) \right] \geq \frac{1}{k} (f(S^*) - \mathbb{E}[f(A_{i-1})]) \quad (3.12)$$

And thus we can once again solve the recurrence and acquire same result as in Proposition 3.2 on the previous page.  $\square$

**Lemma 3.6.** Given set  $B \subseteq N$  such that

$$\forall u \in N \quad P(u \in B) \leq p \quad (3.13)$$

then

$$\mathbb{E}[f(B)] \geq (1-p)f(\emptyset) \quad (3.14)$$

*Proof.* WLoG  $p(u_1 \in B) \geq p(u_2 \in B) \geq \dots \geq p(u_n \in B)$ . Denote

$$X_i = \mathbb{1}_{u_i \in B} N_i = \bigcup_{j=1}^i u_j \quad (3.15)$$

We can then rewrite

$$f(B) = f(N_0) + \sum_{i=1}^n X_i \left( f(B \cap N_i) - f(B \cap N_{i-1}) \right) \quad (3.16)$$

$$\mathbb{E}[f(B)] = f(N_0) + \sum_{i=1}^n \mathbb{E} \left[ X_i \left( f(B \cap N_i) - f(B \cap N_{i-1}) \right) \right] \geq \quad (3.17)$$

$$\geq f(N_0) + \sum_{i=1}^n \left( f(N_i) - f(N_{i-1}) \right) \mathbb{E}[X_i] = f(N_0) + \sum_{i=1}^n \left( f(N_i) - f(N_{i-1}) \right) p_i = \quad (3.18)$$

$$= f(N_0)(1-p_1) + \sum_{i=1}^n \underbrace{f(N_i)(p_i - p_{i+1})}_{\leq 0} \geq f(N_0)(1-p_1) \geq f(\emptyset)(1-p) \quad (3.19)$$

$\square$

**Lemma 3.7.** Given set  $A \subseteq N$  and set  $B \subseteq N$  such that

$$\forall u \in N \quad P(u \in B) \leq p \quad (3.20)$$

$$\mathbb{E}[f(A \cup B)] \geq (1-p)f(A) \quad (3.21)$$

*Proof.* Define

$$g_A(S) = f(A \cup S) \quad (3.22)$$

Obviously,  $g_A$  is also submodular (from diminishing returns). Then, from Lemma 3.6

$$\mathbb{E}[f(A \cup B)] = \mathbb{E}[g_A(B)] \geq (1-p)g_A(\emptyset) = (1-p)f(A) \quad (3.23)$$

$\square$

**Theorem 3.8** (Buchbinder et al. [2014]). In non-monotonic case, Algorithm 3.2 on page 4 is  $\frac{1}{e}$  optimal in expectation.

*Proof.* Similarly to monotonic case, take a look at  $i^{th}$  iteration and condition on previous iterations, denote a chosen element from  $M_i$  as  $u_i$ :

$$\mathbb{E}\left[f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) | A_{i-1}\right] = \frac{1}{k} \sum_{u_i \in M_i} f(A_{i-1} \cup \{u_i\}) - f(A_{i-1}) \geq \frac{1}{k}(f(A_{i-1} \cup S^*) - f(A_{i-1})) \quad (3.24)$$

Since

$$P(u \in A_{i-1}) \leq 1 - \left(1 - \frac{1}{k}\right)^{i-1} \quad (3.25)$$

from Lemma 3.7 on the preceding page

$$\mathbb{E}[f(A_{i-1} \cup S^*)] \geq \left(1 - \frac{1}{k}\right)^{i-1} f(S^*) \quad (3.26)$$

Thus, taking expectation

$$\mathbb{E}\left[f(A_{i-1} \cup \{u_i\}) - f(A_{i-1})\right] \geq \frac{1}{k}(f(A_{i-1} \cup S^*) - f(A_{i-1})) \geq \frac{1}{k}\left[\left(1 - \frac{1}{k}\right)^{i-1} f(S^*) - \mathbb{E}\left[f(A_{i-1})\right]\right] \quad (3.27)$$

$$\mathbb{E}\left[f(A_i)\right] \geq \frac{1}{k}(f(A_{i-1} \cup S^*) - f(A_{i-1})) \geq \frac{1}{k}\left[\left(1 - \frac{1}{k}\right)^{i-1} f(S^*) - \mathbb{E}\left[f(A_{i-1})\right]\right] \quad (3.28)$$

Solving the recurrence we get

$$\mathbb{E}[f(A_i)] \geq \frac{i}{k} \left(1 - \frac{1}{k}\right)^{k-1} f(S^*) \geq \frac{1}{e} f(S^*) \quad (3.29)$$

i.e.,

$$\mathbb{E}[f(A_k)] \geq \left(1 - \frac{1}{k}\right)^{k-1} f(S^*) \geq \frac{1}{e} f(S^*) \quad (3.30)$$

□

**Note** Algorithm 3.2 on page 4 is not optimal. In addition, the upper bound of the best approximation is  $\sim 0.49$ .

**Runtime** Runtime of Algorithm 3.2 on page 4 is  $O(nk)$ .

## 4 Maximization of the submodular function without constraints

$$\max f(S) \quad (4.1)$$

### Examples

- MAX-CUT
- MAX-DIRECTED-CUT
- Max Facility Location
- MAX-SAT (with all literals in a clause having same sign).

**Proposition 4.1** ([Feige et al., 2011]). Algorithm which choose random solution as following:  $u \in S$  with probability  $\frac{1}{2}$  independently, is  $\frac{1}{4}$  approximation in expectation:

$$\mathbb{E}[f(S)] \geq \frac{1}{4} f(S^*) \quad (4.2)$$

**Proposition 4.2** ([Feige et al., 2011]). If  $f$  is symmetric, the same algorithm is  $\frac{1}{2}$  approximation in expectation:

$$\mathbb{E}[f(S)] \geq \frac{1}{2} f(S^*) \quad (4.3)$$

**Proposition 4.3** ([Feige et al., 2011]). For any constant  $\epsilon > 0$  it is impossible to acquire  $(\frac{1}{2} + \epsilon)$  approximation in polynomial time, even in symmetric case.

Note that for  $\bar{f}(S) = f(\bar{S})$ , we can use the same oracle. So a "conjugate" algorithm would be start from  $N$  and drop elements from it.

---

#### Algorithm 4.1

---

```

1: procedure DOUBLE GREEDY( $N$ )
2:    $X \leftarrow \emptyset, Y \leftarrow N$ 
3:   for  $i = 1$  to  $n$  do
4:      $a_i = f(X_{i-1} \cup \{u_i\}) - f(X_i)$ 
5:      $b_i = f(Y_{i-1} \setminus \{u_i\}) - f(Y_i)$ 
6:     if  $a_i > b_i$  then
7:        $X_i \leftarrow X_{i-1} \cup \{u_i\}$ 
8:        $Y_i \leftarrow Y_{i-1}$ 
9:     else
10:       $X_i \leftarrow X_{i-1}$ 
11:       $Y_i \leftarrow Y_{i-1} \setminus \{u_i\}$ 
12:    end if
13:  end for
14:  return  $X_N$ 
15: end procedure

```

---



---

#### Algorithm 4.2

---

```

1: procedure RANDOMIZED DOUBLE GREEDY( $N$ )
2:    $X \leftarrow \emptyset, Y \leftarrow N$ 
3:   for  $i = 1$  to  $n$  do
4:      $a_i = \max \{0, f(X_{i-1} \cup \{u_i\}) - f(X_i)\}$ 
5:      $b_i = \max \{0, f(Y_{i-1} \setminus \{u_i\}) - f(Y_i)\}$ 
6:      $(X_i, Y_i) \leftarrow \begin{cases} (X_{i-1} \cup \{u_i\}, Y_i) & \text{with } P = \frac{a_i}{a_i + b_i} \\ (X_{i-1}, Y_{i-1} \setminus \{u_i\}) & \text{with } P = \frac{b_i}{a_i + b_i} \end{cases}$ 
7:   end for
8:   return  $X_N$ 
9: end procedure

```

---

**Proposition 4.4.** It's impossible that both  $f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) < 0$  and  $f(Y_{i-1} \setminus \{u_i\}) - f(Y_i) < 0$ .

*Proof.* From diminishing returns:

$$f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) \geq f(Y_i) - f(Y_{i-1} \setminus \{u_i\}) \quad (4.4)$$

$$f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1} \setminus \{u_i\}) \geq 0 \quad (4.5)$$

Thus at least one of  $f(X_{i-1} \cup \{u_i\}) - f(X_{i-1})$  and  $f(Y_{i-1} \setminus \{u_i\}) - f(Y_i)$  is greater than 0.  $\square$

**Lemma 4.5.** Let  $S^*$  be an optimal solution and

$$S_i^* = S^* \cup X_i \cap Y_i \quad (4.6)$$

i.e., optimal solution to which we add everything Algorithm 4.1 added and drop everything it dropped.  
For all  $i$ :

$$f(S_{i-1}^*) - f(S_i^*) \leq f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}) \quad (4.7)$$

**Lemma 4.6.** Let  $S^*$  be an optimal solution and

$$S_i^* = S^* \cup X_i \cap Y_i \quad (4.8)$$

i.e., optimal solution to which we add everything Algorithm 4.2 added and drop everything it dropped.  
For all  $i$ :

$$\mathbb{E} \left[ f(S_{i-1}^*) - f(S_i^*) \right] \leq \frac{1}{2} \mathbb{E} \left[ f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}) \right] \quad (4.9)$$

*Proof.* Take a look at  $i^{th}$  iteration and condition on previous iterations:

$$\mathbb{E} \left[ f(X_i) - f(X_{i-1}) + f(Y_i) - f(Y_{i-1}) \middle| X_{i-1}, Y_{i-1} \right] = \quad (4.10)$$

$$= \frac{a_i}{a_i + b_i} \underbrace{(f(X_{i-1} \cup \{u_i\}) - f(X_{i-1}))}_{=a_i \text{ if } a_i \neq 0} + \frac{b_i}{a_i + b_i} \underbrace{(f(Y_{i-1} \cup \{u_i\}) - f(Y_{i-1}))}_{=b_i \text{ if } b_i \neq 0} = \frac{a_i^2 + b_i^2}{a_i + b_i} \quad (4.11)$$

Now divide into two cases:  $u_i \in S^*$  and  $u_i \notin S^*$ .

- If  $u_i \notin S^*$ , in particular,  $u_i \notin S_{i-1}^*$ :

$$\mathbb{E} [f(S_{i-1}^*) - f(S_i^*)] = \frac{a_i}{a_i + b_i} (f(S_{i-1}^*) - f(S_{i-1}^* \cup \{u_i\})) \stackrel{S_{i-1}^* \subseteq Y_{i-1} \setminus \{u_i\}}{\leq} \quad (4.12)$$

$$\leq \frac{a_i}{a_i + b_i} (f(Y_{i-1}^* \setminus \{u_i\}) - f(Y_{i-1}^*)) \leq \frac{a_i b_i}{a_i + b_i} \quad (4.13)$$

- If  $u_i \in S^*$ , in particular,  $u_i \in S_{i-1}^*$ :

$$\mathbb{E} [f(S_{i-1}^*) - f(S_i^*)] = \frac{b_i}{a_i + b_i} (f(S_{i-1}^*) - f(S_{i-1}^* \setminus \{u_i\})) \stackrel{X_{i-1} \subseteq S_{i-1}^* \setminus \{u_i\}}{\leq} \quad (4.14)$$

$$\leq \frac{b_i}{a_i + b_i} (f(X_{i-1}^* \cup \{u_i\}) - f(X_{i-1}^*)) \leq \frac{a_i b_i}{a_i + b_i} \quad (4.15)$$

And since  $a_i^2 - 2a_i b_i + b_i^2 = (a_i - b_i)^2 \geq 0$  (and by tower property), we get the required.  $\square$

**Theorem 4.7** (Buchbinder et al. [2015]). Algorithm 4.2 on the previous page is  $\frac{1}{2}$  approximation in expectation.

*Proof.* Denote

$$S_{alg} = S_n^* = X_n = Y_n \quad (4.16)$$

Then

$$\mathbb{E} [f(S_0^*) - f(S_n^*)] \leq \frac{1}{2} \mathbb{E} [f(X_n) - f(X_0) + f(Y_n) - f(Y_0)] \quad (4.17)$$

$$\mathbb{E} [f(S^*) - f(S_{alg})] \leq \frac{1}{2} \mathbb{E} [2S_{alg} - f(X_0) - f(Y_0)] \stackrel{f(S) \geq 0}{\leq} \mathbb{E} [S_{alg}] \quad (4.18)$$

Thus

$$\mathbb{E} [S_{alg}] \geq \frac{1}{2} \mathbb{E} [f(S^*)] \quad (4.19)$$

$\square$

**Collary 4.7.1.** Algorithm 4.1 on the preceding page is  $\frac{1}{3}$  approximation.



**Note** Algorithms 4.1 and 4.2 on page 7 run in  $O(N)$  time.

## 5 Knapsack constraints

Let each element of set have price  $c_i$  and budget  $B$ , then

$$\max f(S) \tag{5.1}$$

$$\text{s.t. } \sum_{i \in S} c_i \leq B \tag{5.2}$$

---

### Algorithm 5.1

---

```

1: procedure DENSITY GREEDY( $N$ )
2:    $S \leftarrow \emptyset$ 
3:   while  $N \neq \emptyset$  do
4:      $x^* \leftarrow \arg \max \left\{ \frac{f(S \cup \{x\}) - f(S)}{c_i} \right\}$ 
5:     if  $c(S) + c_{x^*} \leq B$  then
6:        $S \leftarrow S \cup \{x^*\}$ 
7:     end if
8:      $N \leftarrow N \setminus \{x^*\}$ 
9:   end while
10:  return  $S$ 
11: end procedure

```

---

Note that this is generalization of cardinality constraint.

---

### Algorithm 5.2

---

```

1: procedure OPTIMIZED DENSITY GREEDY( $N$ )
2:    $S_1 \leftarrow$  output of Algorithm 5.1
3:    $S_2 \leftarrow \left\{ \arg \max_{\substack{i \in N \\ c_i \leq B}} f(i) \right\}$ 
4:   return  $\arg \max_{S \in \{S_1, S_2\}} f(S)$ 
5: end procedure

```

---

**Proposition 5.1** (Khuller et al. [1999]). Algorithm 5.2 is  $\frac{1}{2}(1 - \frac{1}{e})$ -optimal.

**Proposition 5.2.** Algorithm 5.2 is  $(1 - \frac{1}{\sqrt{e}})$ -optimal.

**Theorem 5.3** (Khuller et al. [1999], Sviridenko [2004]). If a set of  $l$  most dense items in optimal solution  $S^*$ , it is possible to get good approximation to the optimal solution.

Enumerating all sets of up to 3 most dense items in optimal solution  $S^*$ , we can acquire  $1 - \frac{1}{e}$ -approximation of optimal solution. Since cardinality constraint is a particular case of knapsack constraint, this is best polynomial approximation.

## 6 Introduction to matroids

Matroid is a basic concept in combinatorial optimization. It was first defined by Whitney [1935].

**Definition 6.1 (matroid).** Matroid  $\mathcal{M}$  is a pair  $(E, \mathcal{I})$ .  $E$  is a finite set (called the ground set) and  $\mathcal{I} \neq \emptyset$  is a family of subsets of  $E$  (called the independent sets) with the following properties:

1. If  $Y \in \mathcal{I}$  then for all  $X \subseteq Y$ ,  $X \in \mathcal{I}$ .
2. If  $X, Y \in \mathcal{I}$  and  $|Y| > |X|$ , then exists  $e \in Y \setminus X$ ,  $X \cup \{e\} \in \mathcal{I}$ .

**Notes** All maximal independent sets have same size. Those sets are called basis.

## Examples

### Uniform manifold

$$\mathcal{M}_k = \left( E, \left\{ X \subseteq E \mid |X| \leq k \right\} \right) \quad (6.1)$$

**Linear manifold** Let  $A \in \mathbb{R}^{m \times n}$  be a matrix. Let  $E$  be a set of columns of  $A$ . The set  $X \subseteq E$  is independent if its elements are independent. Alternatively, for sub-matrix  $A_X$  consisting of columns of  $A$ :

$$\mathcal{I} = \{X \subseteq E \mid \text{rank}(A_X) = |X|\} \quad (6.2)$$

**Graphic matroids** Let  $G = (V_G, E_G)$  be a graph,  $E = E_G$  and

$$\mathcal{I} = \{X \subseteq E_G \mid X \text{ is forest}\} \quad (6.3)$$

**Proposition 6.1.**  $M = (E_G, \mathcal{I})$  is matroid.

The basis is then spanning trees (or forests if graph is not connected).

**Partition matroid** For a set  $E$  let  $E_1, \dots, E_k$  be some partition of  $E$ . Then

$$\mathcal{I} = \{X \subseteq E \mid \forall i = 1..k \ |X| \cap E_i \leq 1\} \quad (6.4)$$

**Proposition 6.2.**  $M = (E, \mathcal{I})$  is matroid.

Note that partition matroid encodes constraints of submodular welfare problem.

Constraint of matching in the bipartite graph can be defined as intersection of two partition matroids.

**Definition 6.2 (Circuit).** Circuit in matroid  $M = (E, \mathcal{I})$  is a dependent set  $X$  ( $X \notin \mathcal{I}$ ) and for all  $x \in X$ ,  $X \setminus \{x\} \in \mathcal{I}$ .

**Definition 6.3 (Rank function).** For matroid  $M = (E, \mathcal{I})$  rank function  $r : 2^E \rightarrow \mathbb{N}$  is defined as

$$r(x) = \max \{|Y| \mid Y \subseteq X, Y \in \mathcal{I}\} \quad (6.5)$$

**Definition 6.4 (Rank of matroid).** For matroid  $M = (E, \mathcal{I})$  rank of matroid is  $\text{rank}(E)$ .

**Proposition 6.3.** Rank of matroid is submodular function.

---

### Algorithm 6.1

---

```

1: procedure GREEDY( $E, \mathcal{I}$ )
2:    $S \leftarrow \emptyset$ 
3:   for  $e \in E$  do
4:     if  $S \cup \{e\} \in \mathcal{I}$  then
5:        $S \leftarrow S \cup \{e\}$ 
6:     end if
7:   end for
8:   return  $S$ 
9: end procedure

```

---

**Proposition 6.4.** Algorithm 6.1 returns basis of  $E$ .

*Proof.* Assume  $S$  is not a basis and let  $B$  be a basis. Exists  $x \in B \setminus S$  such that  $S \cup \{x\}$  is independent. However, since we have not added  $x$  to  $S$ , it got to be dependent with  $S$ .  $\square$

---

**Algorithm 6.2**

---

```
1: procedure GREEDY( $E, I$ )
2:    $S \leftarrow \emptyset$ 
3:   for  $e \in E$  from heaviest to lightest do
4:     if  $S \cup \{e\} \in I$  then
5:        $S \leftarrow S \cup \{e\}$ 
6:     end if
7:   end for
8:   return  $S$ 
9: end procedure
```

---

**Question** Given matroid over  $E$  (via independence oracle), let weight function  $w : E \rightarrow \mathbb{R}$  and weight of set be  $w(X) = \sum_{x \in X} w(x)$ . We want to find independent set (pr basis) of maximal weight.

**Proposition 6.5.** Algorithm 6.2 solves the problem of maximal weight basis.

*Proof.* We know that for  $k = \text{rank}(M)$ , the size of the output of algorithm is  $k$  and so is size of optimal solution  $S^*$ . Lets assume  $S$  is not optimal, thus exists  $i$  such that  $w(e_i^*) > w(e_i)$ . At iteration at which we added  $e_i$  to  $S$ . At this iteration  $|S| = i - 1$ . take a look at first  $i$  elements of  $S^*$ : this is independent set, and from definition of matroid, exists  $e'$  such that  $S$  is independent with  $e'$ . However,  $w(e') \geq w(e^* - i) > w(e_i)$ , thus we should have added it beforehand.  $\square$

## 7 Continuous extensions of submodular functions

**Definition 7.1 (Continuous extensions of function).** Denote by  $\mathbb{1}_S \in \{0, 1\}^N$  indicator of  $S$ . For  $f : 2^N \rightarrow \mathbb{R}$  extension of  $f$  is  $F : [0, 1]^N \rightarrow \mathbb{R}$  such that for all  $S \in N$   $F(\mathbb{1}_S) = f(S)$ .

There exist many extensions of submodular functions. In particular, there exist convex and concave extensions of submodular functions.

**Main idea** For point  $x \in [0, 1]^N$  define distribution on subsets of  $N$ ,  $D_x$  such that for  $R \sim D_x$ :

$$P(i \in R) = x_i \quad (7.1)$$

Then we define

$$F(x) = \mathbb{E}_{R \sim D_x}[f(R)] \quad (7.2)$$

**Example** For all  $x$  choose  $D_x$  such that  $F(x)$  is maximized:

$$f^+(x) = \max_{D_x} \mathbb{E}_{R \sim D_x}[f(R)] \quad (7.3)$$

Similarly, we can choose  $D_x$  such that  $F(x)$  is minimized:

$$f^-(x) = \min_{D_x} \mathbb{E}_{R \sim D_x}[f(R)] \quad (7.4)$$

**Proposition 7.1.**  $f^+$  is concave and  $f^-$  is convex.

*Proof.* Let  $x, y \in [0, 1]^N$  lets show that for

$$z = \lambda x + (1 - \lambda)y \quad (7.5)$$

concave property is fulfilled:

$$f^-(z) \geq \lambda f^-(x) + (1 - \lambda)f^-(y) \quad (7.6)$$

Let  $\{\alpha_S\}_{S \subseteq N}$  be a distribution which defined  $f(x)$ :  $P(R = S) = \alpha_S$ , which fulfills:

$$f^+(x) = \mathbb{E}_{R \sim \alpha_S}[f(R)] \quad (7.7)$$

$$\sum_{S: i \in S} \alpha_S = x_i \quad (7.8)$$

Similarly, let  $\{\beta_S\}_{S \subseteq N}$  be a distribution defining  $f(y)$ .

Now, take a look at linear combination of  $\alpha_S$  and  $\beta_S$ :

$$P(S) = \lambda \alpha_S + (1 - \lambda) \beta_S \quad (7.9)$$

Note that this distribution conserves marginal values of  $z$ , since:

$$\sum_{S: i \in S} \lambda \alpha_S + (1 - \lambda) \beta_S = \lambda \sum_{S: i \in S} \alpha_S + (1 - \lambda) \sum_{S: i \in S} \beta_S = \lambda x_i + (1 - \lambda) y_i = z_i \quad (7.10)$$

By definition,

$$f^+(z) \geq \mathbb{E}_{R \sim \lambda \alpha_S + (1 - \lambda) \beta_S}[f(R)] = \sum_{S \subseteq N} P(R = S) f(S) = \quad (7.11)$$

$$= \sum_{S \subseteq N} [\lambda \alpha_S + (1 - \lambda) \beta_S] f(S) = \lambda f^+(x) + (1 - \lambda) f^+(y) \quad (7.12)$$

□

**Proposition 7.2.** Evaluating concave extensions of submodular function in some point is NP-hard.

**Definition 7.2 (Lovasz extension).**

$$f_L(x) = \mathbb{E}_{\theta \sim [0,1]}[f(\{i : x_i \geq \theta\})] \quad (7.13)$$

**Theorem 7.3 (Lovasz).**  $f_L(x) = f^-(x)$  iff  $f$  is submodular.

*Proof.*  $\Leftarrow$ : Denote  $\{\alpha_S\}_{S \subseteq N}$  a distribution defining  $f^-(x)$ , and out of those the one that maximizes  $\sum_{S \subseteq N} \alpha_S |S|^2$ . We'll show that for such  $\alpha$ , the sets for which  $\alpha_S > 0$  are chain (i.e., a set of sets such that for  $A, B$  in the set either  $A \subseteq B$  or  $B \subseteq A$ ).

Note that there is unique distribution that conserves marginal values and its support is chain: the Lovasz distribution.

Lets show how we can "fix" the distribution  $\alpha_S$  which support is not a chain (uncrossing). Suppose there are  $A, B$  such that  $\alpha_A \geq \alpha_B > 0$  and  $A \not\subseteq B$  and  $B \not\subseteq A$ . For that, lets reduce the probability of  $A$  and  $B$  by  $\alpha_B$ , and increase probability of  $A \cap B$  and  $A \cup B$  by  $\alpha_B$ .

Does the new distribution conserve marginal values? For all of cases  $x \in A \cap B$ ,  $x \in A \setminus B$  and  $x \in B \setminus A$ , the probability did not change.

What happened to  $\mathbb{E}[f(R)]$ ? From submodularity,

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \quad (7.14)$$

and since we removed LHS and added RHS multiplied same constant, the expectation can not grow.

What happens to  $\sum_{S \subseteq N} \alpha_S |S|^2$ ?

$$|A \cup B|^2 + |A \cap B|^2 = (|A| + |B \setminus A|)^2 + (|B| - |B \setminus A|)^2 = \quad (7.15)$$

$$= |A|^2 + |B|^2 + 2|B \setminus A| \cdot (|A| - |B| + |B \setminus A|) = \quad (7.16)$$

$$= |A|^2 + |B|^2 + 2|B \setminus A| \underbrace{(|A \cup B| - |B|)}_{>0} > |A|^2 + |B|^2 \quad (7.17)$$

Thus, if we choose the set which maximizes  $\sum_{S \subseteq N} \alpha_S |S|^2$ , there are no two sets  $A, B$  such that  $A \not\subseteq B$  and  $B \not\subseteq A$ , i.e., support is the chain.

$\Rightarrow$ :

□

**Definition 7.3** (Multilinear extension).

$$F(x) = \sum_{S \subseteq N} f(S) \prod_{i \in S} x_i \prod_{i \notin S} (1 - x_i) \quad (7.18)$$

i.e., each element is chosen independently.

**Proposition 7.4.** Let  $f$  be monotonous function. Then for  $\mathbf{x} \in [0, 1]^N$ ,  $\mathbb{R}^n \ni \mathbf{y} > 0$  (coordinate-wise), and  $g(t) = F(\mathbf{x} + t\mathbf{y})$ ,  $g$  is monotonous, i.e.,

$$\frac{\partial F}{\partial \mathbf{y}} \geq 0 \quad (7.19)$$

In other words, for  $i \in N$

$$\frac{\partial F}{\partial x_i} \geq 0 \quad (7.20)$$

**Proposition 7.5.** Let  $f$  be submodular function. Then for  $\mathbf{x} \in [0, 1]^N$ ,  $\mathbb{R}^n \ni \mathbf{y} > 0$  (coordinate-wise), and  $g(t) = F(\mathbf{x} + t\mathbf{y})$ ,  $g$  is concave, i.e.,

$$\frac{\partial^2 F}{\partial \mathbf{y}^2} \leq 0 \quad (7.21)$$

In other words, for  $i, j \in N$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} \leq 0 \quad (7.22)$$

## 8 Matroid constraints

Let  $\mathcal{M} = (E, \mathcal{I})$  be a matroid.

$$\max f(S) \quad (8.1)$$

$$\text{s.t. } S \in \mathcal{I} \quad (8.2)$$

Let

$$\mathcal{P}_{\mathcal{M}} = \left\{ z \in [0, 1]^N \mid \forall S \subseteq N \sum_{i \in S} z_i \leq \text{rank}(S) \right\} \quad (8.3)$$

---

### Algorithm 8.1

---

```

1: procedure CONTINUOUS GREEDY( $N$ )
2:    $\mathbf{y}(0) \leftarrow \mathbf{0}$ 
3:   for  $t' \in (0, 1)$  do
4:      $\mathbf{x}(t') \leftarrow \arg \max_{\mathbf{x} \in \mathcal{P}_{\mathcal{M}}} \{\mathbf{x} \cdot \nabla F(\mathbf{y}(t'))\}$ 
5:      $\frac{\partial \mathbf{y}}{\partial t}(t') \leftarrow \mathbf{x}(t')$ 
6:   end for
7:   return  $\mathbf{y}(1)$ 
8: end procedure

```

---

**Lemma 8.1.** For  $\mathbf{x} \in [0, 1]^N$

$$\sum_{i \in S} F(\max\{\mathbf{x}, \mathbf{1}_i\}) - F(\mathbf{x}) \geq F(\max\{\mathbf{x}, \mathbf{1}_S\}) - F(\mathbf{x}) \quad (8.4)$$

*Proof.* Denote by  $D_{\mathbf{x}}$  random distribution of taking each element independently with probability  $x_i$ , i.e.,

$$F(\mathbf{x}) = \mathbb{E}_{R \sim D_{\mathbf{x}}}[f(R)] \quad (8.5)$$

Then

$$\sum_{i \in S} F(\max\{\mathbf{x}, \mathbf{1}_i\}) - F(\mathbf{x}) = \sum_{i \in S} \mathbb{E}_{R \sim D_{\max\{\mathbf{x}, \mathbf{1}_i\}}}[f(R)] - \mathbb{E}_{R \sim D_{\mathbf{x}}}[f(R)] = \sum_{i \in S} \mathbb{E}_{R \sim D_{\mathbf{x}}}[f(R \cup \{i\}) - f(R)] = \quad (8.6)$$

$$= \mathbb{E}_{R \sim D_{\mathbf{x}}} \left[ \sum_{i \in S} f(R \cup \{i\}) - f(R) \right] \geq \mathbb{E}_{R \sim D_{\mathbf{x}}}[f(R \cup S) - f(R)] = \quad (8.7)$$

$$= F(\max\{\mathbf{x}, \mathbf{1}_S\}) - F(\mathbf{x}) \quad (8.8)$$

□

**Theorem 8.2** (Calinescu et al. [2011]). For monotonous submodular  $f$ ,

$$F(\mathbf{y}(1)) \geq \left(1 - \frac{1}{\epsilon}\right) f(S^*) \quad (8.9)$$

where  $\mathbf{y}(t)$  is output of Algorithm 8.1 on the previous page:

$$\mathbf{y}(t) = \int_0^t \mathbf{x}(t') dt' \quad (8.10)$$

and  $S^*$  is optimal solution of

$$\max f(S) \quad (8.11)$$

$$\text{s.t. } S \in \mathcal{I} \quad (8.12)$$

*Proof.* Lets bound  $\frac{\partial F}{\partial t}$ :

$$\frac{\partial F}{\partial t} = \nabla F \cdot \frac{\partial \mathbf{y}}{\partial t} = \nabla F(\mathbf{y}(t')) \cdot \mathbf{x}(t') \geq \nabla F(\mathbf{y}(t')) \cdot \mathbf{1}_{S^*} = \sum_{i \in S^*} (\nabla F(\mathbf{y}(t')))_{\mathbf{i}} \stackrel{F \text{ is multilinear}}{\geq} \quad (8.13)$$

$$\geq \sum_{i \in S^*} \frac{F(\max\{\mathbf{y}, \mathbf{1}_i\}) - F(\mathbf{y})}{1 - y_i} \geq \sum_{i \in S^*} F(\max\{\mathbf{y}, \mathbf{1}_i\}) - F(\mathbf{y}) \geq \quad (8.14)$$

$$\geq F(\max\{\mathbf{y}, \mathbf{1}_{S^*}\}) - F(\mathbf{y}) \geq f(S^*) - F(\mathbf{y}) \quad (8.15)$$

where max is coordinate-wise maximum. We got

$$F(\mathbf{y}(0)) \geq 0 \quad (8.16)$$

$$\frac{\partial F}{\partial t} \geq f(S^*) - F(\mathbf{y}) \quad (8.17)$$

with solution  $F(\mathbf{y}(t)) \geq (1 - e^{-t}) f(S^*)$ , i.e.,

$$F(1) \geq \left(1 - \frac{1}{\epsilon}\right) f(S^*) \quad (8.18)$$

□

**Note**  $\nabla F$  can be estimated efficiently with random sampling.

**Lemma 8.3.** For set  $S \subseteq N$  and vector  $\mathbf{z}$  such that  $\|\mathbf{z}\|_{\infty} \leq p$

$$F(\max\{\mathbf{1}_S, \mathbf{z}\}) \geq (1 - p)f(S) \quad (8.19)$$

*Proof.*

$$F(\max\{\mathbf{1}_S, \mathbf{z}\}) \geq f_L(\max\{\mathbf{1}_S, \mathbf{z}\}) \geq (1 - p)f(S) \quad (8.20)$$

□

---

**Algorithm 8.2**

---

```
1: procedure MEASURED CONTINUOUS GREEDY( $N$ )
2:    $\mathbf{y}(0) \leftarrow \mathbf{0}$ 
3:   for  $t' \in (0, 1)$  do
4:      $\mathbf{x}(t') \leftarrow \arg \max_{\mathbf{z} \in \mathcal{P}_M} \{\sum_{i \in N} z_i \cdot (F(\max\{\mathbf{y}, \mathbb{1}_i\}) - F(\mathbf{y}))\}$ 
5:      $\left(\frac{\partial \mathbf{y}}{\partial t}\right)_i(t') \leftarrow \mathbf{x}_i(t')(1 - \mathbf{y}_i(t'))$ 
6:   end for
7:   return  $\mathbf{y}(1)$ 
8: end procedure
```

---

**Theorem 8.4** (Feldman et al. [2011]). For submodular  $f$ ,

$$F(\mathbf{y}(1)) \geq \left(1 - \frac{1}{e}\right) f(S^*) \quad (8.21)$$

where  $\mathbf{y}(t)$  is output of Algorithm 8.2:

$$\mathbf{y}(t) = \int_0^t \mathbf{x}(t') dt' \quad (8.22)$$

and  $S^*$  is optimal solution of

$$\max f(S) \quad (8.23)$$

$$\text{s.t. } S \in \mathcal{I} \quad (8.24)$$

*Proof.* Lets bound  $\frac{\partial F}{\partial t}$ :

$$\frac{\partial F}{\partial t} = \nabla F \cdot \frac{\partial \mathbf{y}}{\partial t} = \nabla F(\mathbf{y}(t')) \cdot \mathbf{x}(t') \stackrel{\text{Fis multilinear}}{=} \sum_{i \in N} \frac{F(\max\{\mathbf{y}, \mathbb{1}_i\}) - F(\mathbf{y})}{1 - \mathbf{y}_i} \cdot \mathbf{x}(t') \cdot (1 - \mathbf{y}_i(t')) = \quad (8.25)$$

$$= [F(\max\{\mathbf{y}, \mathbb{1}_i\}) - F(\mathbf{y})] \cdot \mathbf{x}(t') \geq \sum_{i \in S^*} F(\max\{\mathbf{y}, \mathbb{1}_i\}) - F(\mathbf{y}) \geq F(\max\{\mathbf{y}, \mathbb{1}_{S^*}\}) - F(\mathbf{y}) \quad (8.26)$$

where max is coordinate-wise maximum.

Since

$$\frac{\partial y_i}{\partial t} \leq 1 - y_i \quad (8.27)$$

we get

$$y_i \leq 1 - e^{-t} \quad (8.28)$$

From 8.3 we got

$$\frac{\partial F}{\partial t} \geq F(\max\{\mathbf{y}, \mathbb{1}_{S^*}\}) - F(\mathbf{y}) \geq e^{-t} f(S^*) - F(\mathbf{y}) \quad (8.29)$$

Solving differential inequality

$$F(\mathbf{y}(0)) \geq 0 \quad (8.30)$$

$$\frac{\partial F}{\partial t} \geq f(S^*) - F(\mathbf{y}) \quad (8.31)$$

we acquire the solution  $F(\mathbf{y}(t)) \geq te^{-t} f(S^*)$  with maximum in  $t = 1$ :

$$F(1) \geq \frac{1}{e} f(S^*) \quad (8.32)$$

Note that now  $\mathbf{y}$  is not a linear combination of elements of polytope, but rather something which is at most linear combination.  $\square$

**Note** It is known that  $\frac{1}{e}$  is not optimal, but rather at least 0.372 [Ene and Nguyen, 2016].

**Rounding of the fractional solution** Remember the submodular welfare problem. We define polytope

$$\mathcal{P}_M = \left\{ \mathbf{z} \in [0, 1]^N \mid \sum_{j=1}^k z_{i,j} \leq 1 \right\} \quad (8.33)$$

Then an intuitive way to round the vector in polytope is just sample from random variable  $Z_i$ :

$$P(Z_i = j) = z_{i,j} \quad (8.34)$$

with  $Z_i$  independent. Denote by  $S$  the set acquired by this rounding.

**Proposition 8.5.**

$$\mathbb{E}[f(S)] \geq F(\mathbf{z}) \quad (8.35)$$

**Note** For submodular welfare with  $k$  players we can get  $1 - (1 - \frac{1}{k})^k$  approximation, by running Algorithm 8.2 on the previous page and stopping after  $t = 1$ .

**Pipage rounding** We choose two components of vector  $\mathbf{z}$ ,  $z_i$  and  $z_j$  and move mass from  $z_i$  to  $z_j$  such that

- At least one of  $z_i$  and  $z_j$  becomes integer
- Total value isn't reduced (due to concavity in two variables)
- New vector is still in  $\mathcal{P}_M$ .

**Note** There exists effective algorithm to find, given  $\mathbf{x}$ , a convex combination of basis  $\{B_1, B_2, \dots, B_n\}$ .

---

#### Algorithm 8.3

---

```

1: procedure MERGE BASIS( $\beta_1, B_1, \beta_2, B_2$ )
2:   while  $B_1 \neq B_2$  do
3:      $i \in B_1, j \in B_2$  such that  $i \in B_1 \setminus B_2, j \in B_2 \setminus B_1$ 
4:      $(B_1, B_2) = \begin{cases} (B_1, B_2 \setminus \{j\} \cup \{i\}) & \text{w.p. } \frac{\beta_2}{\beta_1 + \beta_2} \\ (B_1 \setminus \{i\} \cup \{j\}, B_2) & \text{w.p. } \frac{\beta_1}{\beta_1 + \beta_2} \end{cases}$ 
5:   end while
6:   return  $B_1$ 
7: end procedure

```

---



---

#### Algorithm 8.4

---

```

1: procedure SWAP ROUND( $x = \sum_{l=1}^n \beta_l \mathbb{1}_{B_l}$ )
2:    $C_1 = B_1$ 
3:   for  $k = 1..n - 1$  do
4:      $C_{k+1} = \text{MergeBasis}(\sum_{l=1}^k \beta_l, C_k, \beta_{k+1}, B_{k+1})$ 
5:   end for
6:   return  $C_n$ 
7: end procedure

```

---



**Swap rounding** Decompose  $\mathbf{z}$  as a convex combination over basis. From two vectors in combination we create a new one: there exists  $x \in B_i \setminus B_j$  and  $y \in B_j \setminus B_i$  such that  $B_i \setminus \{x\} \cup \{y\}$  and  $B_j \setminus \{y\} \cup \{x\}$  is a basis.

**Theorem 8.6** (Chekuri et al. [2010]). *Proof.* What is value of  $\mathbf{x}$  after one iteration of basis merge? To simplify, assume  $\beta_1 = \beta_2 = \frac{1}{m}$ . The expectation is

$$\mathbb{E}[F(\mathbf{x}')] = \frac{1}{2}F(\mathbf{x} + \beta_1(\mathbb{1}_i) + \frac{1}{m}(\mathbb{1}_i - \mathbb{1}_j)) + \frac{1}{2}F(\mathbf{x} + \beta_1(\mathbb{1}_i) + \frac{1}{m}(\mathbb{1}_j - \mathbb{1}_i)) \quad (8.36)$$

Define

$$g(s) = F(\mathbf{x} + s(\mathbb{1}_i - \mathbb{1}_j)) \quad (8.37)$$

Then, from convexity

$$\mathbb{E}[F(\mathbf{x}')] = \frac{1}{2}g\left(\frac{1}{m}\right) + \frac{1}{2}g\left(-\frac{1}{m}\right) \geq g(0) = F(\mathbf{x}) \quad (8.38)$$

□

## 9 Complexity of unconstrained submodular optimization

**Theorem 9.1.** For all  $\epsilon > 0$  exists submodular function such that random algorithm that achieves  $(\frac{1}{2} + \epsilon)$  approximation in expectation requires at least  $e^{\frac{1}{8}\epsilon^2 N}$  queries to value oracle.

*Proof.* We'll show two submodular functions such that

- Ratio between values is approximately 2.
- The functions can be distinguished only with exponential number of queries to value oracles.

Let  $g(S) = |S|(n - |S|)$  be a cut size of clique.

To define  $f$  suppose there is partition of  $N$  into two sets of equal size:  $C \cup D = N$ ,  $|C| = |D| = \frac{n}{2}$ .

Denote for  $S' \subseteq N$ , denote  $k = |S' \cap C|$  and  $l = |S' \cap D|$ . Then define

$$f(S) = \begin{cases} |S|(n - |S|) & |k - l| \leq \epsilon n \\ |k|(n - 2l) + l(n - 2k) - \underbrace{(\epsilon^2 n^2 - 2\epsilon n |k - l|)}_{O(\epsilon n^2)} & |k - l| \geq \epsilon n \end{cases} \quad (9.1)$$

where we added  $O(\epsilon n^2)$  term to make  $f$  submodular. □

The optimal value of  $g$  is  $\frac{n^2}{4}$ . For  $f$ , if  $S = C$ ,  $f(S) = (\frac{1}{2} - \epsilon)n^2$ .

The choice of sets  $C, D$  is random and unknown to algorithm. Define

$$Y_i = \begin{cases} 1 & i \in C \\ -1 & i \in D \end{cases} \quad (9.2)$$

Then  $k - l = \sum_{i \in S} Y_i$ , and  $P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2}$ . We know that for independent  $-1 \leq Y_i \leq 1$  such that  $\mathbb{E}[Y_i] = 0$  then

$$P\left(\sum_{i=1}^n Y_i > \lambda\right) \leq e^{-\frac{1}{2} \frac{\lambda^2}{n}} \quad (9.3)$$

In our case, we want to bound  $P(|\sum_{i \in S} Y_i| > \epsilon n)$ . To remove the dependence we define  $Z_i$  to be independent variables. Then

$$P\left(\left|\sum_{i \in S} Y_i\right| > \epsilon n\right) = P\left(\left|\sum_{i \in S} Z_i\right| > \epsilon n \mid \sum_{i \in N} Z_i = 0\right) = \frac{P(|\sum_{i \in S} Z_i| > \epsilon n, \sum_{i \in N} Z_i = 0)}{P(\sum_{i \in N} Z_i = 0)} \leq \quad (9.4)$$

$$\leq \frac{P(|\sum_{i \in S} Z_i| > \epsilon n)}{P(\sum_{i \in N} Z_i = 0)} \leq \frac{2e^{-\frac{1}{2} \frac{\epsilon^2 n^2}{|S|}}}{\frac{1}{2^{N+1}}} \leq 2(2N + 1)e^{-\frac{1}{2} \epsilon^2 N} \leq e^{-\frac{1}{4} \epsilon^2 N} \quad (9.5)$$

To bound the probability to get at least one non-balanced query, we'll use union bound. Even for exponential number of queries, the probability to get one such query is exponentially small. We can use Yao's lemma to show that exists input for any algorithm that requires exponential number of queries.

**Note** There exists a general method called “symmetry gap” [Vondrák, 2013] to proof complexity results on submodular optimization.

## 10 Submodular minimization

Simplest minimization problems are unconstrained ones

$$\min_{S \subseteq N} \{f(S)\} \quad (10.1)$$

which generalizes MIN-CUT, MIN-S-T-CUT. Iwata et al. [2001] have extended the dual maximal flow problem to general unconstrained submodular minimization.

### 10.0.1 Continuous relaxation

The Lovasz extension together with convex minimization can be used to minimize submodular function [Grötschel et al., 1981]. Since Lovasz extension is convex, we can easily minimize it. Now we only need to round it. Choose  $\theta \sim U[0, 1]$  and output  $\{i : x_i^* \geq \theta\}$ . By definition

$$\mathbb{E}_\theta [f(\{i : x_i^* \geq \theta\})] = f_L(\bar{X}^*) \quad (10.2)$$

**Note** For any  $\theta$ ,  $\{i : x_i^* \geq \theta\}$  got to be optimal, since any solution can not be better than optimum.

### 10.0.2 Cardinality constraint

Even for monotone  $f$ , the best known algorithm has  $\tilde{O}(\sqrt{n})$ -approximation [Svitkina and Fleischer, 2011].

## 10.1 Multiway partition

Given the graph  $G = (V, E)$ , set of terminals  $V \supseteq T = \{t_1, t_2, \dots, t_k\}$ , and weight function  $w : E \rightarrow \mathbb{R}_+$ . We want to find  $F \subseteq E$  such that in  $G_F = (V, E \setminus F)$  there is no path from  $t_i$  to  $t_j$ , minimizing  $\sum_{e \in F} w_e$ . On the other hand, we want to partition graph into  $k$  parts, each of which contains one terminal with minimal weight of connecting weights.

---

#### Algorithm 10.1

---

```

1: procedure PARTITION( $\{t_1, \dots, t_k\}$ )
2:   for  $i = 1..k$  do
3:      $F_i \leftarrow$  minimal cut between  $t_1$  and  $\{t_2, \dots, t_k\}$ .
4:   end for
5:   return  $\bigcup_{i=1}^k F_i$ 
6: end procedure
```

---

**Lemma 10.1.** Output of Algorithm 10.1 is always legal.

**Lemma 10.2.** Algorithm 10.1 provides 2-approximation of multicut.

*Proof.* For optimal solution  $S^*$ ,

$$\sum_{e \in F_i} w_e \leq \sum_{e \in \delta(S_i^*)} w_e \quad (10.3)$$

Thus, for  $S$ , the output of Algorithm 10.1 on the preceding page

$$f(S) \leq \sum_i \sum_{e \in \delta(S_i^*)} w_e \leq 2f(S^*) \quad (10.4)$$

□

## 10.2 Submodular multiway partition

Given the world  $N$  and  $N \supset T = \{t_1, t_2, \dots, t_k\}$  and submodular  $f : 2^N \rightarrow \mathbb{R}_+$ , we want to find partition of  $N$  into  $\{S_1, \dots, S_k\}$  such that  $t_i \in S_i$  and minimize

$$\sum_{i=1}^k f(S_i) \quad (10.5)$$

### Examples

- Multiway cut:  $f$  if cut function of undirected graph.
- Hypergraph multiway cut
  - Each hyperedge cost is number of groups it is present in:

$$f(S) = \sum_{\substack{e \in E \\ 0 \leq |e \cap S| \leq |e|}} w_e \quad (10.6)$$

- Each hyperedge cost is  $w_e$  if it is cut and 0 otherwise. In this case, for each edge we choose a vertex  $u_e \in e$ :

$$f(S) = \sum_{\substack{e \in E \\ u_e \in S \\ |e \cap S| \leq |e|}} w_e \quad (10.7)$$

For symmetric  $f$  there exists 1.5-approximation and for general case there exists 2-approximation.

**Relaxation of the problem** Define for each vertex  $u$  variable  $x(u, i)$ :

$$x(u, i) \geq 0 \quad (10.8)$$

$$\sum_{i=1}^k x(u, i) = 1 \quad (10.9)$$

Our goal is

$$\min \sum_{i=1}^k f_L(\mathbf{x}_i) \quad (10.10)$$

$$\text{s.t. } \sum_{i=1}^k x(u, i) = 1 \quad (10.11)$$

$$x(t_i, i) = 1 \quad (10.12)$$

$$x(u, i) \geq 0 \quad (10.13)$$

Denote by  $\mathbf{x}_i \in [0, 1]^N$  such that  $(\mathbf{x}_i)_u = x(u, i)$ . We can solve the relaxed problem (since it is convex problem). But how do we round the solution?

Choose threshold  $\theta \in [0, 1]$  and define  $A_i(\theta) = \{u | x(u, i) \geq \theta\}$ . Then, for example, if  $\theta \sim U[0, 1]$

$$\mathbb{E}[f(A_i(\theta))] = f_L(\mathbf{x}_i) \quad (10.14)$$

There are two “problems” we want to fix:  $A_i(\theta) \cap A_j(\theta) \neq \emptyset$  and  $\bigcup_i A_i(\theta) \neq N$ .  
Let  $A, B$  such that  $A \cap B \neq \emptyset$ . In symmetric case

$$f(A) + f(B) = f(N \setminus A) + f(B) \geq f((N \setminus A) \cup B) + f((N \setminus A) \cap B) = f(A \cap B) + f(B \setminus A) \quad (10.15)$$

Since

$$\frac{1}{2}[f(A \setminus B) + f(B)] + \frac{1}{2}[f(B \setminus A) + f(A)] \leq f(A) + f(B) \quad (10.16)$$

at least one of choices  $\{A \setminus B, B\}$  and  $\{B \setminus A, A\}$  is at least as good as  $\{A, B\}$ . This action is called uncrossing. By repeating the process we will get disjoint sets.

Now, denote  $A(\theta) = \bigcup_i A_i(\theta)$  and  $V(\theta) = N \setminus A(\theta)$ . What would be value of output:

$$\sum_{i=1}^{k-1} f(A_i(\theta)) + f(A_k(\theta) \cup f(V(\theta))) \leq \sum_{i=1}^k f(A_i(\theta)) + f(V(\theta)) \quad (10.17)$$

Thus we only need to bound  $f(V(\theta))$ :

$$\mathbb{E}_\theta[f(V(\theta))] \leq \alpha \sum_{i=1}^k f_L(\mathbf{x}_i) \quad (10.18)$$

Naively, in symmetric case, we could bound by

$$f(V(\theta)) = f(A(\theta)) \leq \sum_{i=1}^k f(A_i(\theta)) \quad (10.19)$$

which provides 2-approximation.

**Lemma 10.3.** For all  $\frac{1}{2} \leq \delta \leq 1$

$$\int_0^1 f(V(\theta)) d\theta \leq \int_0^\delta f(A_1(\theta) \cap A_2(\theta)) d\theta + \int_0^\delta f\left(\left[A_1(\theta) \cup A_2(\theta)\right] \cap A_3(\theta)\right) d\theta + \quad (10.20)$$

$$+ \dots + \int_0^\delta f\left(\left[A_1(\theta) \cup A_2(\theta) \cup \dots \cup A_{k-1}(\theta)\right] \cap A_k(\theta)\right) d\theta = \quad (10.21)$$

$$= \sum_{i=1}^{k-1} \int_0^\delta f\left(\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta)\right) d\theta \quad (10.22)$$

*Proof.* First assume  $\delta = 1$ , then

$$\int_0^1 f\left(\left[A_1(\theta) \cup \dots \cup A_i(\theta)\right] \cap A_{i+1}(\theta)\right) d\theta = f_L(\mathbf{y}_i) \quad (10.23)$$

where

$$\mathbf{y}_i = \min \left\{ \mathbf{x}_{i+1}, \max_{1 \leq j \leq i} \mathbf{x}_j \right\} \quad (10.24)$$

for elementwise maximum and minimum. Since

$$\min \{x_1, x_2\} + \min \{\max \{x_1, x_2\}, x_3\} + \dots + \quad (10.25)$$

$$+ \min \{\max \{x_1, x_2, \dots, x_{n-1}\}, x_n\} + \max \{x_1, x_2, \dots, x_n\} = \sum x_i \quad (10.26)$$

we get

$$\sum_{i=1}^{k-1} \mathbf{y}_i + \max_{1 \leq j \leq k} \mathbf{x}_j = \mathbf{1} \quad (10.27)$$

On the other hand, since  $(1 - \theta)$  is also uniform

$$\int_0^1 f(V(\theta)) d\theta = f_L\left(\mathbf{1} - \max_{1 \leq j \leq k} \mathbf{x}_j\right) = f_L\left(\sum_{i=1}^{k-1} \mathbf{y}_i\right) = (k-1)f_L\left(\frac{1}{k-1} \sum_{i=1}^{k-1} \mathbf{y}_i\right) \stackrel{\text{convex}}{\leq} \sum_{i=1}^{k-1} f_L(\mathbf{y}_i) \quad (10.28)$$

Now, assuming  $f(\emptyset) = 0$ , we can rewrite

$$\int_0^1 f\left(\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta)\right) d\theta = \int_0^\delta f\left(\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta)\right) d\theta + \quad (10.29)$$

$$+ \int_\delta^1 f\left(\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta)\right) d\theta \quad (10.30)$$

However, since we are on simplex, for  $\delta > \frac{1}{2}$ ,  $\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta) = \emptyset$ .  $\square$

**Lemma 10.4.**

$$\sum_{i=1}^k f(A_i(\theta)) \geq f(A_1(\theta) \cap A_2(\theta)) + f\left(\left[A_1(\theta) \cup A_2(\theta)\right] \cap A_3(\theta)\right) + \quad (10.31)$$

$$+ \cdots + f\left(\left[A_1(\theta) \cup A_2(\theta) \cup \cdots \cup A_{k-1}(\theta)\right] \cap A_k(\theta)\right) + f\left(\bigcup_{i=1}^k A_i(\theta)\right) \quad (10.32)$$

*Proof.* By induction. Basis. For  $k = 2$ :

$$f(A_1(\theta)) + f(A_2(\theta)) = f(A_1(\theta) \cap A_2(\theta)) + f(A_1(\theta) \cup A_2(\theta)) \quad (10.33)$$

Step

$$\sum_{i=1}^k f(A_i(\theta)) \geq \sum_{i=1}^{k-2} f\left(\bigcup_{j=1}^i A_j(\theta) \cap A_{i+1}(\theta)\right) + f\left(\bigcup_{j=1}^{k-1} A_j(\theta)\right) + f(A_k(\theta)) \geq \quad (10.34)$$

$$\geq \sum_{i=1}^{k-1} f\left(\bigcup_{j=1}^i A_j(\theta) \cap A_{i+1}(\theta)\right) + f\left(\bigcup_{j=1}^k A_j(\theta)\right) \quad (10.35)$$

$\square$

**Theorem 10.5.** For all  $\frac{1}{2} \leq \delta \leq 1$

$$\sum_{i=1}^k \int_0^\delta f(A_i(\theta)) d\theta \geq \int_0^1 f(V(\theta)) d\theta + \int_0^\delta f(A(\theta)) d\theta \quad (10.36)$$

*Proof.* Thus, substituting Lemma 10.3 on the previous page into Lemma 10.4

$$\sum_{i=1}^k \int_0^\delta f(A_i(\theta)) d\theta \geq \sum_{i=1}^{k-1} \int_0^\delta f\left(\left[\bigcup_{j=1}^i A_j(\theta)\right] \cap A_{i+1}(\theta)\right) + f\left(\bigcup_{i=1}^k A_i(\theta)\right) d\theta \geq \quad (10.37)$$

$$\geq \int_0^1 f(V(\theta)) d\theta + \int_0^\delta f(A(\theta)) d\theta \quad (10.38)$$

$\square$

**Theorem 10.6** (Chekuri and Ene [2011]). There exist 1.5-approximation for symmetric submodular multiway partition.

*Proof.* For symmetric  $f$ , we choose  $\delta = 1$  in Theorem 10.5, and get:

$$\sum_{i=1}^k \int_0^1 f(A_i(\theta)) d\theta \geq \int_0^1 f(V(\theta)) d\theta + \int_0^1 f(A(\theta)) d\theta = 2 \int_0^1 f(V(\theta)) d\theta \quad (10.39)$$

$$2\mathbb{E}_\theta f(V(\theta)) \leq \mathbb{E}_\theta \sum_{i=1}^k f(A_i(\theta)) = \sum_{i=1}^k f(A_i(\theta)) \quad (10.40)$$

i.e.,  $\alpha = \frac{1}{2}$  and we got 1.5-approximation.  $\square$

**Theorem 10.7** (Chekuri and Ene [2011]). There exist 2-approximation for submodular multiway partition.

*Proof.* If  $f$  is not symmetric, we can choose  $\theta \sim U[0.5, 1]$ ,  $A_i$  will be guaranteed to be disjoint. In this case

$$\mathbb{E}_\theta \left[ \sum_{i=1}^{k-1} f(A_i(\theta)) + f(A_k(\theta) \cup f(V(\theta))) \right] \leq \mathbb{E}_\theta \left[ \sum_{i=1}^k f(A_i(\theta)) f(V(\theta)) \right] = \quad (10.41)$$

$$= \sum_{i=1}^k \int_{\frac{1}{2}}^1 2f(A_i(\theta)) d\theta + \int_{\frac{1}{2}}^1 2f(V(\theta)) d\theta = \quad (10.42)$$

$$= 2 \left[ \sum_{i=1}^k f_L(\mathbf{x}_i) - \sum_{i=1}^k \int_0^{\frac{1}{2}} f(A_i(\theta)) d\theta + \int_{\frac{1}{2}}^1 f(V(\theta)) d\theta \right] \stackrel{\text{Th. 10.5}}{\leq} \quad (10.43)$$

$$\leq 2 \sum_{i=1}^k f_L(\mathbf{x}_i) \quad (10.44)$$

$\square$

**Lower bounds** For symmetric problem, it is unknown whether  $\frac{3}{2} - \frac{1}{k}$  is optimal. The known lower bound is 1.26 [Ene et al., 2013].

## 10.3 Submodular covering problems

### 10.3.1 Submodular vertex cover

For graph  $G = (V, E)$  and submodular  $f : 2^V \rightarrow \mathbb{R}_+$ . The goal is to find  $A \subset V$ , such that  $G$  is vertex cover (i.e., for  $(u, v) \in E$ ,  $u \in A$  or  $v \in A$ ) and  $f(A)$ .

For each vertex define variable  $x_i$ , and the relaxation is

$$\min f_L(\mathbf{x}) \quad (10.45)$$

$$\text{s.t. } \forall (u, v) \in E \quad x_u + x_v \geq 1 \quad (10.46)$$

$$\forall u \in V \quad 0 \leq x_u \leq 1 \quad (10.47)$$

**Proposition 10.8** (Iwata and Nagano [2009]). There exists half-integer optimum of Eq. (10.45), i.e.,  $\forall v \in V$   $x_v^k \in \{0, \frac{1}{2}, 1\}$ . It is possible to find it efficiently.

*Proof.* Let  $\mathbf{x}^*$  be the optimum of Eq. (10.45). Assume

$$x_{u_1}^* \geq x_{u_2}^* \geq \dots \geq x_{u_n}^* \quad (10.48)$$

We can rewrite

$$f_L(\mathbf{x}^*) = (1 - x_{u_1}^*) \cdot f(\emptyset) + (x_{u_1}^* - x_{u_2}^*) f(\{u_1\}) + \dots + x_{u_n}^* f(N) \quad (10.49)$$

We can now write optimization problem as a linear programming problem:

$$\min (1 - x_{u_1}) \cdot f(\emptyset) + (x_{u_1} - x_{u_2})f(\{u_1\}) + \cdots + x_{u_n}f(N) \quad (10.50)$$

$$\text{s.t. } x_{u_1} \geq x_{u_2} \geq \cdots \geq x_{u_n} \quad (10.51)$$

$$\forall (u, v) \in E \quad x_u + x_v \geq 1 \quad (10.52)$$

$$\forall u \in V \quad 0 \leq x_u \leq 1 \quad (10.53)$$

Obviously the optimal value of Eq. (10.50) is less or equal than the optimal value of Eq. (10.45) on the preceding page.

Let  $\mathbf{x}^*$  be the optimal solution of Eq. (10.50). Define

$$V^- = \left\{ u : 0 < x_u^* < \frac{1}{2} \right\} \quad (10.54)$$

$$V^+ = \left\{ u : \frac{1}{2} < x_u^* < 1 \right\} \quad (10.55)$$

$$(10.56)$$

Now define two new solutions:

$$y_u = \begin{cases} x_u^* + \epsilon & u \in V^+ \\ x_u^* - \epsilon & u \in V^- \\ x_u^* & \text{otherwise} \end{cases} \quad (10.57)$$

$$z_u = \begin{cases} x_u^* - \epsilon & u \in V^+ \\ x_u^* + \epsilon & u \in V^- \\ x_u^* & \text{otherwise} \end{cases} \quad (10.58)$$

Since  $\mathbf{x}^* = \frac{\mathbf{y} + \mathbf{z}}{2}$ , the value of at least one of  $\mathbf{y}$ ,  $\mathbf{z}$  is not worse than value of  $\mathbf{x}^*$  (from linearity). In particular, both  $\mathbf{y}$ ,  $\mathbf{z}$  have same value as  $\mathbf{x}^*$ , since  $\mathbf{x}^*$  is optimal.

We can choose  $\epsilon$  such that  $y$ ,  $z$  are non-negative and less than 1.

For edge constrain, it is impossible that  $u, v \in V^-$ . If  $x_u + x_v = 1$ , both  $y$  and  $z$  doesn't change value of  $x_u + x_v$ .

If  $x_u + x_v > 1$ , there exists  $\epsilon$ , such that edge constrain is still fulfilled.

Similarly, order constraints are fulfilled for small enough  $\epsilon$ .

Since  $\mathbf{y}$  moves values towards 0 and 1 and  $\mathbf{z}$ , we choose feasible  $\epsilon$  such that one of values in  $\mathbf{y}$  or  $\mathbf{z}$  becomes half-integer, thus increasing number of half-integer elements in optimal solution, until we arrive to half-integer solution.  $\square$

**Proposition 10.9** (Iwata and Nagano [2009]). Given half-integer optimal solution,

$$A = \left\{ u \mid x_u^* \geq \frac{1}{2} \right\} \quad (10.59)$$

is 2-approximation of submodular vertex cover.

*Proof.*

$$f_L(\mathbf{x}^*) = \frac{1}{2}f(A) + \frac{1}{2}f(\{u \mid x_u^* = 1\}) \leq \frac{1}{2}f(A) \quad (10.60)$$

$\square$

**Proposition 10.10** (Khot and Regev [2008]). Under unique games conjecture, 2-approximation is optimal.

### 10.3.2 Submodular set cover

Given  $n$  elements  $\{1, \dots, n\}$  and  $m$  subsets  $S_1, \dots, S_m \subseteq \{1, \dots, n\}$  with submodular function  $f : 2^{1, \dots, m} \rightarrow \mathbb{R}_+$ . The goal is to find cover  $X \subseteq \{S_1, \dots, S_m\}$  which minimizes  $f(X)$ .

**Greedy algorithm** At each step we choose set which minimizes

$$V_i(S_j) = \frac{f(X_{i-1} \cup \{S_j\}) - f(X_{i-1})}{\left| \bigcup_{k \in X_{i-1} \cup \{S_j\}} S_k \right| - \left| \bigcup_{k \in X_{i-1}} S_k \right|} \quad (10.61)$$

This algorithm gives  $\log(n)$  approximation in case of weighted or regular set cover.

**Example** Let

$$S_j = \{j\} \quad (10.62)$$

$$S_{n-1} = \{1, 2, \dots, n\} \quad (10.63)$$

$$f(X) = \min \{|X|, n - |X|, 1\} \quad (10.64)$$

In this case the optimal solution is  $X = \{S_1, \dots, S_n\}$  while the algorithm will choose  $X = \{S_{n+1}\}$

**Frequency** We define frequency of the elements as

$$\eta_i = |\{j : i \in S_j\}| \quad (10.65)$$

and frequency of input is  $\max_i \eta_i$ .

Similarly to vertex cover, we'll define relaxed problem

$$\min f_L(\mathbf{x}) \quad (10.66)$$

$$\text{s.t. } \forall i \in \{1, 2, \dots, n\} \quad \sum_{j: i \in S_j} x_j \geq 1 \quad (10.67)$$

$$\forall i \in \{1, 2, \dots, n\} \quad 0 \leq x_i \leq 1 \quad (10.68)$$

and we can find optimal solution  $\mathbf{x}^*$ . We define solution in the following way:

$$X = \left\{ S_j : x_j^* \geq \frac{1}{\eta} \right\} \quad (10.69)$$

Obviously the solution is feasible.

**Proposition 10.11** (Iwata and Nagano [2009]). The solution is  $\eta$ -approximation

*Proof.* If  $f$  is symmetric,

$$f(X) = f_L(\mathbb{1}_X) \leq f_L(\min \{\eta \cdot \mathbf{x}^*, \mathbb{1}\}) \leq \eta \cdot f_L(\mathbf{x}^*) \quad (10.70)$$

If  $f$  is not monotonic we can “monotonize”:

$$\tilde{f}(A) = \min \{f(B) : A \subseteq B \subseteq \{1, \dots, m\}\} \quad (10.71)$$

Take a look at

$$\tilde{X} = \arg \min_T \{f(T) : X \subseteq T\} \quad (10.72)$$

we can use the proof of monotonic case with  $\tilde{f}$  and show that the algorithm is  $\eta$ -approximation even if  $f$  is not monotonic.  $\square$

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