236621 - Algorithms for Submodular Optimization

Roy Schwartz

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Abstract

1 Introduction

We are looking on $f : 2^N \to \mathbb{R}$ for some set $N = \{1, \dots n\}$

Definition 1.1. f is submodular if

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B) \tag{1}$$

Definition 1.2. Return of u wrt A is $f(A \cup \{u\}) - f(A)$

Definition 1.3 (Diminishing returns)**.** f has diminishing returns if for $A \subseteq B$

$$f(A \cup \{u\}) - f(A) \geq f(B \cup \{u\}) - f(B) \tag{2}$$

Proposition 1.1. f is submodular iff f has diminishing returns

Proof. $\Rightarrow$:
Let $A \subseteq B \subseteq N$ and $u \notin B$. Lets use submodularity property on $A \cup \{u\}$ and B:

$$f(A \cup \{u\}) + f(B) \geq f(A \cup \{u\} \cup B) + f(9A \cup \{u\}0 \cap B) = f(B \cup \{u\}) + f(B) \tag{3}$$

Thus

$$f(A \cup \{u\}) \geq f(B \cup \{u\}) \tag{4}$$

□

$\Leftarrow$:
We'll proof by induction over $|A \cup B| - |A \cap B|$, i.e., size of symmetric difference.
Basis: $|A \cup B| - |A \cap B| = 0$, then $A = B$, and then submodular property is fulfilled.
Step: assume $|A \cup B| - |A \cap B| = k$. WLOG let $u \in A$ such that $u \notin B$.

$$f(A) + f(B) = f(A) - f(A \setminus \{u\}) + f(A \setminus \{u\}) + f(B) \geq \tag{5}$$
$$\geq f(A) - f(A \setminus \{u\}) + f(A \setminus \{u\} \cup B) + f(A \setminus \{u\} \cap B) \geq \tag{6}$$
$$\geq f(A \cup B) - f(A \cup B \setminus \{u\}) + f(A \cup B \setminus \{u\}) + f(A \cap B) = f(A \cup B) + f(A \cap B) \tag{7}$$

Definition 1.4 (Monotonous function)**.** f is non-decreasing monotonous if $\forall A \subseteq B \subseteq N$, $f(A) \leq f(B)$.

Definition 1.5 (Symmetric function)**.** f is symmetric if $\forall S \subseteq N$, $f(S) \leq f(N \setminus S)$.

Definition 1.6 (Normalized function)**.** f is normalized if $f(\emptyset) = 0$.

Examples

Linear function $\forall n \in N$ exists weight w_n and

$$f(S) = \sum_{u \in S} w_u + b \tag{8}$$

Such f is submodular.

Budget additive function (clipped linear function) $\forall n \in N$ exists weight w_n and

$$f(S) = \min \left\{ \sum_{u \in S} w_u, b \right\} \tag{9}$$

Such f is submodular.

Coverage function Given set X and n subsets $S_1, S_2, \ldots, S_n \subset X$ define

$$f(S) = \left| \bigcup_{i \in S} S_i \right| \tag{10}$$

This f is obviously submodular.

Graph cuts Let $G + (V, E)$ be a graph and $w : E \to \mathbb{R}^+$ weights of edges. Given a cut $S \subseteq V$ define $\delta(S)$ to be sum of weights of all edges going through the cut. $\delta : 2^V \to \mathbb{R}^+$ is submodular, normalized, and symmetric.

Rank function Let $v_1, \ldots, v_n \in \mathbb{R}^d$ vectors, and

$$f(S) = \operatorname{rank}(S) = \dim \operatorname{span}\big(\{v_i | i \in S\}\big) \tag{11}$$

2 Submodular optimization

Given world N, submodular function $f : 2^N \to \mathbb{R}^+$, and a family of feasible solutions $\mathcal{I} \subseteq 2^N$

$$\max f(S) \tag{12}$$
$$\text{s.t. } S \in \mathcal{I} \tag{13}$$

Note Most of submodular functions (except for logarithm of determinant of submatrix) are nonnegative. We use the condition to have properly defined multiplicative approximation.

Note How f is given in input? Obviously, not as a list of values, since it's exponential in $|N|$. Thus we represent f with black box, and same applies for constraints. Usually, constraints are simple.

2.1 Examples of submodular optimization problems

Example f is submodular and there are no constraints. It generalizes MAX-CUT, MAX-DICUT

Example f is submodular and there is size constraint:

$$\max f(S) \tag{14}$$
$$\text{s.t. } |S| \leq k \tag{15}$$

. It generalizes MAX-K-COVER.

Submodular welfare