Ravninske krivulje s pitagorejskim hodogramom

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Kubične PH krivulje

- Primitivni PH: $w = 1$, $\max\{\deg(u), \deg(v)\} = 1$
- Zapišimo u in v v Bernsteinovi bazi:

$$u(t) = u_0 B_0^1(t) + u_1 B_1^1(t)$$

$$v(t) = v_0 B_0^1(t) + v_1 B_1^1(t)$$

  Pri tem predpostavimo, da velja $u_0 : u_1 \neq v_0 : v_1$.
- Po zgornjem izreku tako dobimo hodograf

$$x'(t) = (u_0^2 - v_0^2) B_0^2(t) + (u_0 u_1 - v_0 v_1) B_1^2(t) + (u_1^2 - v_1^2) B_2^2(t),$$

$$y'(t) = 2 u_0 v_0 B_0^2(t) + (u_0 v_1 + u_1 v_0) B_1^2(t) + 2 u_1 v_1 B_2^2(t).$$

- Kubično PH krivljo tako zapišemo

$$r(t) = (x(t), y(t))^T = \sum_{i=0}^{n} b_k B_k^n(t)$$

kjer sta x in y izražena kot

$$x(t) = \int_0^t (u^2(t) - v^2(t))dt,$$

$$y(t) = \int_0^t (2u(t)v(t))dt$$

- Pravilo za integriranje Bernstainovih baznih polinomov:

$$\int B_k^n(t)dt = \frac{1}{n+1} \sum_{j=k+1}^{n+1} b_k^{n+1}(t)$$

- Integracija nam tako poda kontrolne točke kubične Bézierjeve krivulje

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{b}_0 + \frac{1}{3}(u_0^2 - v_0^2, 2u_0v_0)^T, \\
\mathbf{b}_2 &= \mathbf{b}_1 + \frac{1}{3}(u_0u_1 - v_0v_1, u_0u_1 + v_0v_1)^T, \\
\mathbf{b}_3 &= \mathbf{b}_2 + \frac{1}{3}(u_1^2 - v_1^2, 2u_1v_1)^T,
\end{aligned}$$

Pri čemer je $\mathbf{b}_0$ poljubna kontrolna točke, ki ustreza konstantam pri integraciji.

$$
\begin{aligned}
\mathbf{p}_1 &= \mathbf{p}_0 + \frac{1}{5}(u_0^2 - v_0^2, 2u_0 v_0), \\
\mathbf{p}_2 &= \mathbf{p}_1 + \frac{1}{5}(u_0 u_1 - v_0 v_1, u_0 v_1 + u_1 v_0), \\
\mathbf{p}_3 &= \mathbf{p}_2 + \frac{2}{15}(u_1^2 - v_1^2, 2u_1 v_1) + \\
&\quad \frac{1}{15}(u_0 u_2 - v_0 v_2, u_0 v_2 + u_2 v_0), \\
\mathbf{p}_4 &= \mathbf{p}_3 + \frac{1}{5}(u_1 u_2 - v_1 v_2, u_1 v_2 + u_2 v_1), \\
\mathbf{p}_5 &= \mathbf{p}_4 + \frac{1}{5}(u_2^2 - v_2^2, 2u_2 v_2),
\end{aligned}
$$

Parametrična hitrost in dolžina loka

$$\sigma(t) = |r\prime(t)| = \sqrt{x\prime^2(t) + y\prime^2(t)} = u^2(t) + v^2(t)$$

$$\sigma(t) = \sum_{k=0}^{n-1} \sigma_k B_k^{n-1}(t),$$

kjer so

$$\sigma_k = \sum_{j=max(0,k-m)}^{min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{n-1}{k}} (u_j u_{k-j} + v_j v_{k-j}),$$

$$k = 0, \ldots, n-1.$$

$$\begin{aligned}
\sigma_0 &= u_0^2 + v_0^2, \\
\sigma_1 &= u_0 u_1 + v_0 v_1, \\
\sigma_2 &= u_1^2 + v_1^2.
\end{aligned}$$

$$\begin{aligned}
\sigma_0 &= u_0^2 + v_0^2, \\
\sigma_1 &= u_0 u_1 + v_0 v_1, \\
\sigma_2 &= \frac{2}{3}(u_1^2 + v_1^2) + \frac{1}{3}(u_0 u_2 + v_0 v_2), \\
\sigma_3 &= u_1 u_2 + v_1 v_2, \\
\sigma_4 &= u_2^2 + v_2^2.
\end{aligned}$$

$$s(t) = \sum_{k=0}^{n} s_k B_k^n(t),$$

kjer je $s_0 = 0$ in $s_k = \frac{1}{n}\sum_{j=0}^{k-1} \sigma_j, k = 1, \ldots, n.$

$S = s(1) = \frac{\sigma_0+\sigma_1+\ldots+\sigma_{n-1}}{n}.$

$\Delta s = S/N$
Začetni približek:

$$t_k^{(0)} = t_{k-1} + \frac{\Delta s}{\sigma(t_{k-1})}$$

Newton-Raphson:

$$t_k^{(r)} = t_k^{(r-1)} - \frac{s(t_k^{(r-1)}) - k\Delta s}{\sigma(t_k^{(r-1)})}, r = 1, 2, \ldots.$$

Lastnosti odvoda krivulje

$$\mathbf{t} = \frac{(u^2 - v^2, 2uv)}{\sigma}$$

$$\mathbf{n} = \frac{(2uv, v^2 - u^2)}{\sigma}$$

$$\kappa = 2\frac{uv' - u'v}{\sigma^2}.$$

Racionalni odmiki krivulj s PH

$$r_d(t) = r(t) + d\mathbf{n}(t),$$

kjer je $\mathbf{n}(t)$ enotska normala .

Racionalni odmiki krivulj s PH

$$\mathbf{P}_k = (W_k, X_k, Y_k) = (1, x_k, y_k), \qquad k = 0, \ldots, n.$$

$$\Delta\mathbf{P}_k = \mathbf{P}_{k+1} - \mathbf{P}_k = (0, \Delta x_k, \Delta y_k), \qquad k = 0, \ldots, n-1$$

$$\Delta\mathbf{P}_k^{\perp} = (0, \Delta y_k, -\Delta x_k).$$

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Racionalni odmik

$$r_d(t) = \left(\frac{X(t)}{W(t)}, \frac{Y(t)}{W(t)}\right),$$

kjer je $\mathbf{O}_k = (W_k, X_k, Y_k)$, $\quad k = 0, \dots, 2n-1$, določeno z

$$\mathbf{O}_k = \sum_{j=max(0,k-n)}^{min(n-1,k)} \frac{\binom{n-1}{j}\binom{n}{k-j}}{\binom{2n-1}{k}} (\sigma_j \mathbf{P}_{k-j} + dn\Delta\mathbf{P}_j^{\perp}).$$