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HIGHLY ACCURATE FINITE DIFFERENCE METHOD FOR COUPLED NONLINEAR SCHRÖDINGER EQUATION

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The coupled nonlinear Schrödinger equation models several interesting physical phenomena. It presents a model equation for optical fiber with linear birefringence. In this article, we write a finite difference scheme to solve this equation. The method is fourth-order in space and second-order in time. It is unconditionally stable and extrapolation is used in the temporal direction and this makes the method fourth-order in the two directions, space and time. Many numerical tests have been conducted to display the robustness of the scheme.

Keywords: Coupled nonlinear Schrödinger equation; Finite difference method; Vector solitons

C.R. Category: G.1.7; G.1.8

1 INTRODUCTION

The coupled nonlinear Schrödinger equations (CNLSE) arise in a great variety of physical situations. In fiber communication systems, such equations have been shown to govern pulse propagation along orthogonal polarization axes in nonlinear optical fibers and in wavelength-division-multiplexed systems. These equations also model beam propagation inside crystals or photorefractives as well as water wave interactions. Solitary waves in these equations are often called vector solitons in the literature as they generally contain two components. In all the above physical situations, collision of vector solitons is an important issue. This system has been studied intensively in the past ten years. It has been shown that, in addition to passing-through collision, vector solitons can also bounce off each other or trap each other. However, what is still unknown is that this collision can be much more complex and regular at the same time [7, 10]. In this article, we consider CNLSE:

$$i\frac{\partial \Psi_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + (|\Psi_1|^2 + e|\Psi_2|^2)\Psi_1 = 0, \quad -\infty < x < \infty, \tag{1}$$

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$$i\frac{\partial \Psi_2}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_2}{\partial x^2} + (e|\Psi_1|^2 + |\Psi_2|^2)\Psi_2 = 0, \quad -\infty < x < \infty, \tag{2}$$

where Ψ_1 and Ψ_2 are the wave amplitudes in two polarizations and e is the cross-phase modulational coefficient, with the initial conditions

$$\Psi_1(x,0) = g_1(x), \quad \Psi_2(x,0) = g_2(x) \tag{3}$$

and boundary conditions

$$\frac{\partial \Psi_1(x,t)}{\partial x} = \frac{\partial \Psi_2(x,t)}{\partial x} = 0 \quad \text{as } |x| \longrightarrow \infty. \tag{4}$$

For linearly birefringent fibers, $e = 2/3$ and for elliptically birefringent fibers, e can take other positive values. When $e = 0$, this system becomes two decoupled nonlinear Schrödinger (NLS) equations and when $e = 1$, the system is called Manakov equations. In both cases, the system is integrable. In particular, solitons of one polarization should pass through pulses of the opposite polarization without creating shadows and so the collision is elastic. For other values of e, the system is nonintegrable. Solitons in optical fibers can be defined as nonlinear pulses that propagate nearly distortion-free for long distances and that undergo elastic collision.

Following the discussion of Wadati [10], the exact solutions of Eqs. (1) and (2) are

$$\Psi_1(x,t) = \sqrt{\frac{2\alpha}{1+e}}\,\mathrm{sech}(\sqrt{2\alpha}(x - vt))\exp i\left\{vx - \left[\frac{v^2}{2} - \alpha\right]t\right\}, \tag{5}$$

$$\Psi_2(x,t) = \pm\sqrt{\frac{2\alpha}{1+e}}\,\mathrm{sech}(\sqrt{2\alpha}(x - vt))\exp i\left\{vx - \left[\frac{v^2}{2} - \alpha\right]t\right\}, \tag{6}$$

with the conserved quantities

(i) Mass conservations:

$$I_1 = \int_{-\infty}^{\infty} |\Psi_1|^2\,\mathrm{d}x, \quad I_2 = \int_{-\infty}^{\infty} |\Psi_2|^2\,\mathrm{d}x. \tag{7}$$

(ii) Momentum conservation:

$$I_3 = \int_{-\infty}^{\infty} i\sum_{j=1}^{2}\left(\bar{\Psi}_j\frac{\partial \Psi_j}{\partial x} - \frac{\partial \bar{\Psi}_j}{\partial x}\Psi_j\right)\mathrm{d}x. \tag{8}$$

(iii) Energy conservation:

$$I_4 = \int_{-\infty}^{\infty}\left\{\sum_{j=1}^{2}\frac{1}{2}\left|\frac{\partial \Psi_j}{\partial x}\right|^2 - \frac{1}{2}\sum_{j=1}^{2}|\Psi_j|^4 - e|\Psi_1|^2|\Psi_2|^2\right\}\mathrm{d}x. \tag{9}$$

In this article, we derive a highly accurate finite difference method for solving the system in Eqs. (1) and (2), and this method is analyzed for accuracy and stability. The method is fourth-order in space and second-order in time. An extrapolation has been used to increase the accuracy to fourth-order in both directions, space and time. Many numerical tests are given and the exact solution for a single soliton is used to measure the accuracy of the method. Newton's

and a predictor–corrector technique are used to solve the nonlinear system obtained from the full discertization of the given problem.

The article is organized as follows. In Section 2, the derivation of the method and the analysis of the properties of the scheme, like stability, accuracy and the solution of the nonlinear system are given. Numerical tests with various initial conditions of the CNLS equations like a single soliton, interaction of two solitons and interaction of two orthogonal solitons are reported in Section 3. Conclusions are given in Section 4.

2 NUMERICAL METHOD

2.1 Space Discertization

We assume that, in the time interval $0 \le t \le T_{\max}$ under consideration, the solution of the system given in Eqs. (1) and (2) is negligibly small outside the interval $x_L \le x \le x_R$. Therefore in our numerical study we consider the following system

$$\begin{aligned} i\frac{\partial \Psi_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + (|\Psi_1|^2 + e|\Psi_2|^2)\Psi_1 = 0, \quad x_L < x < x_R, \\ i\frac{\partial \Psi_2}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_2}{\partial x^2} + (e|\Psi_1|^2 + |\Psi_2|^2)\Psi_2 = 0, \quad x_L < x < x_R, \end{aligned} \tag{10}$$

with the initial conditions

$$\Psi_1(x,0) = g_1(x), \quad \Psi_2(x,0) = g_2(x),$$

and the boundary conditions

$$\frac{\partial \Psi_1(x,t)}{\partial x} = \frac{\partial \Psi_2(x,t)}{\partial x} = 0 \quad \text{at } x = x_L, x_R.$$

For our numerical work, we decompose the complex functions Ψ_1, Ψ_2, g_1 and g_2 into their real and imaginary parts by writing

$$\begin{aligned} \Psi_1(x,t) = u_1(x,t) + iu_2(x,t), \quad \Psi_2(x,t) = u_3(x,t) + iu_4(x,t) \\ g_1(x) = g_{1\mathrm{R}} + ig_{1\mathrm{I}}, \quad g_2(x) = g_{2\mathrm{R}} + ig_{2\mathrm{I}}, \end{aligned} \tag{11}$$

where $u_i (i = 1, \ldots, 4)$, $g_{1\mathrm{R}}$, $g_{1\mathrm{I}}$, $g_{2\mathrm{R}}$ and $g_{2\mathrm{I}}$ are real functions, see Refs. [3, 4, 5]. By substituting Eq. (11) into Eq. (10), we obtain the following system.

$$\begin{aligned} \frac{\partial u_1}{\partial t} + \frac{1}{2}\frac{\partial^2 u_2}{\partial x^2} + z_1 u_2 = 0, \\ \frac{\partial u_2}{\partial t} - \frac{1}{2}\frac{\partial^2 u_1}{\partial x^2} - z_1 u_1 = 0, \\ \frac{\partial u_3}{\partial t} + \frac{1}{2}\frac{\partial^2 u_4}{\partial x^2} + z_2 u_4 = 0, \\ \frac{\partial u_4}{\partial t} - \frac{1}{2}\frac{\partial^2 u_3}{\partial x^2} - z_2 u_3 = 0, \end{aligned} \tag{12}$$

where

$$z_2 = (u_1^2 + u_2^2) + e(u_3^2 + u_4^2) z_2 = e(u_1^2 + u_2^2) + (u_3^2 + u_4^2) \tag{13}$$

and

$$u_1(x,0) = g_{1\mathrm{R}}(x), \quad u_2(x,0) = g_{1\mathrm{I}}(x)$$
$$u_3(x,0) = g_{2\mathrm{R}}(x), \quad u_4(x,0) = g_{2\mathrm{I}}(x)$$

and the homogenous Neumann boundary conditions

$$\frac{\partial u_1}{\partial x} = \frac{\partial u_2}{\partial x} = \frac{\partial u_3}{\partial x} = \frac{\partial u_4}{\partial x} = 0 \quad \text{at } x = x_L, x_R.$$

The system (12) can be written in a matrix-vector form as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} A \frac{\partial^2 \mathbf{u}}{\partial x^2} + B(\mathbf{u})\mathbf{u} = \mathbf{0}, \tag{14}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

and

$$B(\mathbf{u}) = \begin{bmatrix} 0 & z_1 & 0 & 0 \\ -z_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2 \\ 0 & 0 & -z_2 & 0 \end{bmatrix}.$$

To develop a numerical method for solving the system (13), the region $R = [x_L < x < x_R] \times [t > 0]$ with its boundary ∂R consisting of the ordinates $x = x_L, x = x_R$ and the axis $t = 0$ is covered with a rectangular mesh of points with coordinates

$$x_m = x_L + mh, \quad m = 0, 1, \ldots, N$$
$$t_n = nk, \quad n = 0, 1, 2, \ldots,$$

where h and k are the mesh space step size and time step size, respectively.

Now, we approximate the exact solution $u(x_m, t)$ by $U_m(t)$ and the second-order space derivative $\partial^2 \mathbf{u}/\partial x^2$, by

$$\frac{1}{h^2}\delta_x^2 \mathbf{u}_m = \left(1 + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\right)\frac{\partial^2 \mathbf{u}_m}{\partial x^2} + \mathrm{o}(h^4), \tag{15}$$

where $\delta_x^2 \mathbf{u}_m = \mathbf{u}_{m+1} - 2\mathbf{u}_m + \mathbf{u}_{m-1}$.

By approximating the second-order operator in Eq. (14) by

$$\frac{\partial^2}{\partial x^2} = \frac{1}{h^2}\delta_x^2 + \mathrm{o}(h^2), \tag{16}$$

and substituting Eq. (15) into Eq. (14) we get the fourth-order approximation

$$\frac{\partial^2 \mathbf{u}}{\partial x^2} = \frac{1}{h^2}\left(1 + \frac{1}{12}\delta_x^2\right)^{-1} \delta_x^2 \mathbf{u}_m + \mathrm{o}(h^4), \tag{17}$$

which is known as a Numorov approximation or Douglas approximation.

Substituting Eq. (16) into Eq. (13) and dropping the error terms we get the semidiscrete system

$$\dot{\mathbf{U}}_m + \frac{A}{2h^2}\left(1 + \frac{1}{12}\delta_x^2\right)^{-1} \delta_x^2 \mathbf{U}_m + B(\mathbf{U}_m)\mathbf{U}_m = \mathbf{0}, \tag{18}$$

where the dot $(\cdot)$ denotes differentiation with respect to time.

The semidiscrete system in Eq. (17) can be presented as

$$\dot{\mathbf{U}}_m = G(\mathbf{U}_m), \tag{19}$$

where

$$G(\mathbf{U}_m) = -\frac{A}{2h^2}\left(1 + \frac{1}{12}\delta_x^2\right)^{-1} \delta_x^2 \mathbf{U}_m - B(\mathbf{U}_m)\mathbf{U}_m.$$

By making use of the following definitions

$$\mathbf{u} = [\mathbf{u}_1^t, \mathbf{u}_2^t, \ldots, \mathbf{u}_N^t]^t,$$

where

$$\mathbf{u}_i = [(U_{1,i}, U_{2,i}, U_{3,i}, U_{4,i}]^t, \quad i = 1, 2, \ldots, N.$$

The resulting system (17) after imposing the homogeneous Neumann boundary conditions can be written as

$$P\dot{\mathbf{u}} = [S + PD(\mathbf{u})]\mathbf{u}, \tag{20}$$

where each of $D(\mathbf{u})$, P and S are a block-diagonal skew symmetric, a block-tridiagonal symmetric and a block-tridiagonal skew symmetric matrices, respectively and the structures of these matrices are

$$P = \frac{1}{12}\begin{bmatrix} 5I & I & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ I & 10I & I & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & I & 10I & I \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & I & 5I \end{bmatrix},$$

$$S = \frac{-1}{2h^2}\begin{bmatrix} -A & A & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ A & -2A & A & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \ddots & \mathbf{0} & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & A & -2A & A \\ \mathbf{0} & \cdots & \cdots & \mathbf{0} & A & -A \end{bmatrix},$$

$$D(\mathbf{u}) = -\mathrm{diag}[B_1(\mathbf{u}_1), B_2(\mathbf{u}_2), \ldots, B_N(\mathbf{u}_N)],$$

where $\mathbf{0}$ and I are (4×4) zero and identity matrices, respectively and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B_i(\mathbf{u}_i) = \begin{bmatrix} 0 & (z_1)_i & 0 & 0 \\ -(z_1)_i & 0 & 0 & 0 \\ 0 & 0 & 0 & (z_2)_i \\ 0 & 0 & -(z_2)_i & 0 \end{bmatrix},$$

$$z_1 = U_1^2 + U_2^2 + e(U_3^2 + U_4^2)$$
$$z_2 = e(U_1^2 + U_2^2) + U_3^2 + U_4^2.$$

2.2 Full Discertization

We assume that $\mathbf{u}^n$ is the full discrete approximation to the exact solution $\mathbf{u}(t_n)$. Equation (20) represents a system of ordinary differential equations. Many numerical methods can be used to solve this system and some of these methods with a suitable stability region include [1]

(i) Explicit midpoint rule (Leap-Frog scheme):

$$\mathbf{U}_m^{n+1} - \mathbf{U}_m^{n-1} = 2kG(\mathbf{U}_m^n). \tag{21}$$

(ii) Trapezoidal rule:

$$\mathbf{U}_m^{n-1} - \mathbf{U}_m^n = \frac{k}{2}[G(\mathbf{U}_m^{n+1}) + G(\mathbf{U}_m^n)]. \tag{22}$$

(iii) Implicit midpoint rule:

$$\mathbf{U}_m^{n-1} - \mathbf{U}_m^n = kG\left(\frac{\mathbf{U}_m^{n+1} + \mathbf{U}_m^n}{2}\right). \tag{23}$$

These three second-order methods are in wide use. The explicit method (21) is extremely easy to use and has a very low cost per step. On the other hand, it can only operate if k is limited by a stability restriction. The implicit methods (22) and (23) can usually work with larger time steps and this is due to the stability property of both. For nonlinear problems we usually need to solve a nonlinear system at each time step and this may or may not offset the disadvantage of their higher cost per step.

In this article, we adopt the implicit midpoint rule (23). To describe this method, we assume that $\mathbf{U}_m^n$ denotes the discrete approximation to the exact solution $\mathbf{u}(x_m, t_n)$. The full discretization of Eq. (20) after applying the implicit midpoint rule is

$$P(\mathbf{u}^{n+1} - \mathbf{u}^n) = k\left[S + PD\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right)\right]\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right). \tag{24}$$

The resulting system (24) is a block nonlinear tridiagonal system. This system of $4N$ nonlinear algebraic equations can be solved by using Newton's method or by a predictor–corrector method and this will be given in detail in the next subsections.

2.3 Newton's Method

To describe this method, the nonlinear system obtained in Eq. (24) can be written as

$$\mathbf{Q}(\mathbf{w}) = P(\mathbf{u}^{n+1} - \mathbf{u}^n) - k\left[S + PD\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right)\right]\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right) = \mathbf{0}, \tag{25}$$

where $\mathbf{w} = \mathbf{u}^{n+1}$.

Newton's method for this system can be given as

$$\mathbf{w}^{(l+1)} = \mathbf{w}^{(l)} - J^{-1}(\mathbf{w}^{(l)})\mathbf{Q}(\mathbf{w}^{(l)}), \quad l = 0, 1, \ldots, \tag{26}$$

where $J(\mathbf{w})$ is the Jacobian matrix which contains the partial derivatives of $\mathbf{Q}(\mathbf{w})$ and is an $N \times N$ square matrix whose elements are blocks of size 4×4. These derivatives are calculated analytically since $\mathbf{Q}(\mathbf{w})$ is not complicated. The initial guess vector $\mathbf{w}^{(0)}$ is assumed to be equal to $\mathbf{u}^n$.

The iterative scheme in Eq. (26) is usually solved in two steps.

(i) Solve the linear block-tridiagonal system

$$J(\mathbf{w}^{(l)})\mathbf{y} = \mathbf{Q}(\mathbf{w}^{(l)})$$

by Crout's method for the intermediate vector y.

(ii) Update your solution by using

$$\mathbf{w}^{(l+1)} = \mathbf{w}^{(l)} - \mathbf{y}.$$

So steps (i) and (ii) are applied iteratively till the following condition is satisfied:

$$\|\mathbf{w}^{(l+1)} - \mathbf{w}^{(l)}\| \leq \text{tol},$$

where tol is a prescribed tolerance.

2.4 Predictor–Corrector Method

To avoid the solution of the nonlinear system (24) using Newton's method, a predictor–corrector pair is assumed

(i) Predictor:

$$P\mathbf{u}^* = P\mathbf{u}^n + k[S + PD(\mathbf{u}^n)]\mathbf{u}^n. \tag{27}$$

(ii) Corrector:

$$P(\mathbf{u}^{n+1} - \mathbf{u}^n) = kS\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right) + kPD\left(\frac{\mathbf{u}^* + \mathbf{u}^n}{2}\right)\left(\frac{\mathbf{u}^* + \mathbf{u}^n}{2}\right).$$

We call this method the Douglas predictor–corrector method.

The vectors $\mathbf{u}^*$ and $\mathbf{u}^{n+1}$ are obtained by solving a system of linear equations, whose matrices P and $(P - (k/2)S)$ do not change with n and therefore can be factorized once, and for all, as LU decomposition. Thus at each time level, only forward and backward solvers are required.

We have noticed that this method is not conservative [8]. To circumvent this difficulty we modify the predictor–corrector pair as follows.

(i) Predictor:

$$P\mathbf{u}^{[0]} = P\mathbf{u}^n + k[S + PQ(\mathbf{u}^n)]\mathbf{u}^n. \tag{28}$$

(ii) Corrector:

$$\left[P - \frac{k}{2}S\right]\mathbf{u}^{[r-1]} = \left[P + \frac{k}{2}S\right]\mathbf{u}^n + kPQ\left(\frac{\mathbf{u}^{[r]} + \mathbf{u}^n}{2}\right)\left(\frac{\mathbf{u}^{[r]} + \mathbf{u}^n}{2}\right), \quad r = 0, 1, \ldots. \tag{29}$$

To execute this pair, we apply the predictor Eq. (28) first to find an initial approximation $\mathbf{u}^{[0]}$, and then we apply the corrector formula iteratively until two consecutive iterants $\mathbf{u}^{[r]}$and $\mathbf{u}^{[r+1]}$ satisfy the condition $\|\mathbf{u}^{[r+1]} - \mathbf{u}^{[r]}\| \leq \text{tol}$, where tol is a prescribed tolerance; then $\mathbf{u}^{n+1}$ is taken to be $\mathbf{u}^{(r+1)}$. In this method the factorization of the matrices P and $(P - (k/2)S)$ is done at the beginning of the computation, then an unspecified number of backward and forward solvers per time step are needed to achieve the required accuracy. As we see from the numerical results, this method conserves the mass and almost conserves the other quantities.

2.5 Conserved Quantities

In this section, we study the conservation properties of the system (1–2) and analogous property of the numerical method [2, 3].

In this work, we derive the conservation laws

$$\int_{-\infty}^{\infty} |\Psi_1|^2 \, \mathrm{d}x \quad \text{and} \quad \int_{-\infty}^{\infty} |\Psi_2|^2 \, \mathrm{d}x \tag{30}$$

and the other quantities are calculated using a numerical quadrature.

To show that the system in Eq. (1) satisfies Eq. (30), we multiply Eq. (1) by $\bar{\Psi}_1$(the complex conjugate of Ψ_1), and its complex conjugate by Ψ_1 and subtract the latter from the former to obtain

$$\begin{aligned} \bar{\Psi}_1 \left(i\frac{\partial \Psi_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \Psi_1}{\partial x^2} + (|\Psi_1|^2 + e|\Psi_2|^2)\Psi_1 \right) \\ - \Psi_1 \left(i\frac{\partial \bar{\Psi}_1}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{\Psi}_1}{\partial x^2} + (|\Psi_1|^2 + e|\Psi_2|^2)\bar{\Psi}_1 \right) = 0, \end{aligned} \tag{31}$$

and this gives

$$i\left[\bar{\Psi}_1 \frac{\partial \Psi_1}{\partial t} + \Psi_1 \frac{\partial \bar{\Psi}_1}{\partial t}\right] + \frac{1}{2}\left[\bar{\Psi}_1 \frac{\partial^2 \Psi_1}{\partial x^2} - \Psi_1 \frac{\partial^2 \bar{\Psi}_1}{\partial x^2}\right] = 0,$$

or, equivalently

$$i\frac{\partial}{\partial t}(\Psi_1 \bar{\Psi}_1) + \frac{1}{2}\frac{\partial}{\partial x}\left(\bar{\Psi}_1 \frac{\partial \Psi_1}{\partial x} - \Psi_1 \frac{\partial \bar{\Psi}_1}{\partial x}\right) = 0. \tag{32}$$

Integration of both sides of Eq. (32) with respect to x in $-\infty < x < \infty$ gives

$$i\frac{\partial}{\partial t}\int_{-\infty}^{\infty} |\Psi_1|^2 \, \mathrm{d}x + \frac{1}{2}\left[\bar{\Psi}_1 \frac{\partial \Psi_1}{\partial x} - \Psi_1 \frac{\partial \bar{\Psi}_1}{\partial x}\right]_{-\infty}^{\infty} = 0.$$

By utilizing the homogenous Neumann boundary conditions $\partial\Psi_1/\partial x = 0$ as $|x| \to \infty$ we find that

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} |\Psi_1|^2 \, \mathrm{d}x = 0,$$

and hence the first conserved quantity

$$I_1 = \int_{-\infty}^{\infty} |\Psi_1|^2 \, \mathrm{d}x = \text{constant} \tag{33}$$

is obtained.

By adopting the same criteria, the second conserved quantity

$$I_2 = \int_{-\infty}^{\infty} |\Psi_2|^2 \, \mathrm{d}x = \text{constant} \tag{34}$$

can be obtained.

By using the exact solution Eqs. (5) and (6), we obtain the exact values of these quantities as

$$I_1 = I_2 = \frac{2}{1+e}\sqrt{2\alpha}. \tag{35}$$

Conservation of I_1 and I_2implies L^2 boundedness of the solution and no blow up is expected during the simulations of the method.

To study the analogous property of the discrete system (24), we rewrite this system as

$$(\mathbf{u}^{n+1} - \mathbf{u}^n) = \frac{1}{2}kP^{-1}S(\mathbf{u}^{n+1} + \mathbf{u}^n) + \frac{1}{2}kD\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right)(\mathbf{u}^{n+1} + \mathbf{u}^n). \tag{36}$$

Premultiplying Eq. (36) by $(\mathbf{u}^{n+1} + \mathbf{u}^n)^t$ we get

$$\begin{aligned}(\mathbf{u}^{n+1} + \mathbf{u}^n)^t(\mathbf{u}^{n+1} - \mathbf{u}^n) = {} & \frac{1}{2}k(\mathbf{u}^{n+1} + \mathbf{u}^n)^t P^{-1}S(\mathbf{u}^{n+1} + \mathbf{u}^n) \\ & + \frac{1}{2}k(\mathbf{u}^{n+1} + \mathbf{u}^n)^t D\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right)(\mathbf{u}^{n+1} + \mathbf{u}^n).\end{aligned} \tag{37}$$

By using the symmetric property of P and the skew symmetric property of A the following relations hold [9].

$$(\mathbf{u}^{n+1} + \mathbf{u}^n)^t P^{-1}S(\mathbf{u}^{n+1} + \mathbf{u}^n) = 0 \tag{38}$$

$$(\mathbf{u}^{n+1} + \mathbf{u}^n)^t D\left(\frac{\mathbf{u}^{n+1} + \mathbf{u}^n}{2}\right)(\mathbf{u}^{n+1} + \mathbf{u}^n) = 0. \tag{39}$$

By using Eqs. (38) and (39) in Eq. (37), we get

$$(\mathbf{u}^{n+1} + \mathbf{u}^n)^t(\mathbf{u}^{n+1} - \mathbf{u}^n) = 0$$

and from this we deduce that

$$(\mathbf{u}^{n+1})^t(\mathbf{u}^{n+1}) = (\mathbf{u}^n)^t\mathbf{u}^n, \quad n = 0, 1, \ldots. \tag{40}$$

This proves that the derived scheme possesses the conservation property which ensures the boundedness of the approximations $\mathbf{u}_m^n$ as $n \to \infty$, ruling out the occurrence of the nonlinear blow up [8].

2.6 Stability

To study the stability of the numerical method, that is, the sensitivity of the numerical solution to perturbations in the initial data, the von Neumann stability analysis is used. This method is applicable only for linear problems. To apply this method, we assume that

$$\mathbf{U}_m^n = G^n \phi \mathrm{e}^{i\beta mh} \tag{41}$$

is the test function, where $i = \sqrt{-1}$, $\beta \in \mathbb{R}$, $\phi \in \mathbb{R}^4$ and $G \in \mathbb{R}^{4\times 4}$ is the amplification matrix. The necessary condition for stability of the difference system is

$$\max_j |\lambda_j| \leq 1, \quad j = 1, 2, 3, 4, \tag{42}$$

where λ_j, $j = 1, 2, 3, 4$ are the eigenvalues of the amplification matrix G. To apply von Neumann stability analysis, we consider the linearized form of the system in Eq. (14) and this can be assumed of the form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} A \frac{\partial^2 \mathbf{u}}{\partial x^2} + \alpha A \mathbf{u} = \mathbf{0}, \tag{43}$$

where $\alpha = \max\{z_1, z_2\}$.

The linearized version of the proposed scheme applied to Eq. (43) is

$$\begin{aligned} \left(1 + \frac{1}{12}\delta_x^2\right)(\mathbf{U}_m^{n+1} - \mathbf{U}_m^n) + \frac{1}{4} r A \delta_x^2 (\mathbf{U}_m^{n+1} + \mathbf{U}_m^n) \\ + \frac{1}{2} k \alpha A \left(1 + \frac{1}{12}\delta_x^2\right)(\mathbf{U}_m^{n+1} - \mathbf{U}_m^n) = \mathbf{0}, \end{aligned} \tag{44}$$

where $r = k/h^2$.

By substituting Eq. (41) into Eq. (44), we get after some manipulation

$$\gamma[G - I] - \left(r\mu A - \frac{k}{2}\gamma\alpha A\right)[G + I] = \mathbf{0}, \tag{45}$$

where

$$\mu = \sin^2 \frac{\beta h}{2} \quad \text{and} \quad \gamma = 1 - \frac{\mu}{3}.$$

Equation (45) can be written as

$$[\gamma I - \omega A]G - [\gamma I + \omega A] = \mathbf{0}, \tag{46}$$

where

$$\omega = \mu - \frac{k}{2}\gamma\alpha.$$

Recall the skew symmetric property of A. It is easy to see that matrix $[\gamma I - \omega A]$ is nonsingular and shares the same set of eigenvalues $\{\gamma + \omega i, \gamma - \omega i, \gamma + \omega i, \gamma - \omega i\}$ with the matrix $[\gamma I + \omega A]$. Thus, the maximal module of the eigenvalues of G is one. The eigenvalues of G can also be obtained by writing the matrix G in Eq. (46) explicitly as

$$G = [\gamma I - \omega A]^{-1}[\gamma I + \omega A]. \tag{47}$$

By using Matlab, the eigenvalues of the matrix G are

$$\lambda_1 = \frac{\gamma^2 - \omega^2 + 2i\gamma\omega}{\gamma^2 + \omega^2}, \quad \lambda_2 = \frac{\gamma^2 - \omega^2 - 2i\gamma\omega}{\gamma^2 + \omega^2}$$
$$\lambda_3 = \frac{\gamma^2 - \omega^2 + 2i\gamma\omega}{\gamma^2 + \omega^2}, \quad \lambda_4 = \frac{\gamma^2 - \omega^2 + 2i\gamma\omega}{\gamma^2 + \omega^2}.$$

The modulus of all of these eigenvalues is equal to 1. This means that our scheme is unconditionally stable in the linear sense according to von Neumann stability analysis, which means that no restriction on the time step size k can be imposed, but we must choose it in such a way that we get accurate results.

2.7 Accuracy of the Scheme

To study the accuracy of the derived scheme (24), the proposed scheme at a typical node $(x, t) = (mh, nk)$ is

$$\left(1 + \frac{1}{12}\delta_x^2\right)\left(\frac{\mathbf{U}_m^{n+1} - \mathbf{U}_m^n}{k}\right) + \frac{1}{4h^2}A\delta_x^2(\mathbf{U}_m^{n-1} + \mathbf{U}_m^n) + \left(1 + \frac{1}{12}\delta_x^2\right)\mathbf{g}\left(\frac{\mathbf{U}_m^{n+1} + \mathbf{U}_m^n}{2}\right) = \mathbf{0}, \tag{48}$$

where

$$\mathbf{g}(\mathbf{u}) = B(\mathbf{u})\mathbf{u}.$$

The Taylor's series expansion of all terms in Eq. (48) about $\mathbf{u}_m^n$, where $\mathbf{u}_m^n$ is the exact solution vector, have the following expressions

$$\left(1 + \frac{1}{12}\delta_x^2\right)\left(\frac{\mathbf{u}_m^{n+1} - \mathbf{u}_m^n}{k}\right) = \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{k}{2}\frac{\partial^2 \mathbf{u}}{\partial t^2} + \frac{h^2}{12}\frac{\partial^3 \mathbf{u}}{\partial x^2 \partial t} + \frac{kh^2}{24}\frac{\partial^4 \mathbf{u}}{\partial x^2 \partial t^2} + \mathrm{o}(k^2 + k^2h^2 + h^4)\right)_m^n \tag{49}$$

$$\frac{1}{4h^2}\delta_x^2(\mathbf{u}_m^{n+1} + \mathbf{u}_m^n) = \left(\frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{k}{4}\frac{\partial^3 \mathbf{u}}{\partial x^2 \partial t} + \frac{h^2}{24}\frac{\partial^4 \mathbf{u}}{\partial x^4} + \frac{kh^2}{48}\frac{\partial^5 \mathbf{u}}{\partial t \partial x^4} + \mathrm{o}(k^2 + k^2h^2 + h^4)\right)_m^n \tag{50}$$

$$\left(1 + \frac{1}{12}\delta_x^2\right)\mathbf{g}\left(\frac{\mathbf{u}_m^{n+1} + \mathbf{u}_m^n}{2}\right) = \left(\mathbf{g} + \frac{k}{2}\frac{\partial \mathbf{g}}{\partial t} + \frac{h^2}{12}\frac{\partial^2 \mathbf{g}}{\partial x^2} + \frac{kh^2}{24}\frac{\partial^3 \mathbf{g}}{\partial t\, \partial x^2} + \mathrm{o}(k^2 + k^2h^2 + h^4)\right)_m^n. \tag{51}$$

Substituting these expansions into Eq. (48), we get

$$T_m^n = \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}(\mathbf{u})\right)_m^n + \frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}(\mathbf{u})\right)_m^n + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\left\{\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}(\mathbf{u})\right)_m^n\right\} \times \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\left\{\frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{g}(\mathbf{u})\right)_m^n\right\} + \mathrm{o}(k^2 + k^2h^2 + h^4). \tag{52}$$

By using the differential system (14) all terms inside the brackets are equal to zero and so we get

$$T_m^n = \mathrm{o}(k^2 + k^2h^2 + h^4),$$

and from this we deduce that the proposed scheme is second-order in time and fourth-order in space. The scheme is consistent since $T_m^n \to 0$ as $h, k \to 0$.

3 NUMERICAL RESULTS

To get a reliable solution, discrete conservation laws are important features in computing smooth soliton solution of the CNLS equations. The accuracy is measured by using the L_∞ and L_2 error norms defined by

$$\|ER\|_\infty = \max_{1<m<N}\{|(\|\Psi_1(x_m, t_n)\| - \|U_{1,m}^n + iU_{2,m}^n\|)|\}$$

and

$$\|ER\|_2 = \left[\sum_{i=1}^{N}(\|\Psi_1(x_m, t_n)\| - \|U_{1,m}^n + iU_{2,m}^n\|)^2\right]^{1/2}.$$

We calculate the conserved quantities by using the trapezoidal rule. We consider the following problems to highlight the properties of the derived scheme.

3.1 Single Soliton

In this test, we take the initial condition

$$\begin{aligned}\Psi_1(x, 0) &= \sqrt{\frac{2\alpha}{1+e}}\,\mathrm{sec}\,h(\sqrt{2\alpha}x)\exp i\{vx\}\\ \Psi_2(x, 0) &= \sqrt{\frac{2\alpha}{1+e}}\,\mathrm{sec}\,h(\sqrt{2\alpha}x)\exp i\{vx\},\end{aligned} \tag{53}$$

where α, e and v are constants.

To compute the numerical solution, we impose the natural boundary conditions

$$\frac{\partial \Psi_1}{\partial x} = \frac{\partial \Psi_2}{\partial x} = 0 \quad \text{at } x = x_L, x_R \text{ for all } t \geq 0.$$

The following parameters are used in this test

$$x_L = -20, \quad x_R = 80, \quad h = 0.1, \quad k = 0.01, \quad v = 1.0, \quad \alpha = 1.0, \quad e = \frac{2}{3}.$$

We solve the nonlinear system (24) using Newton's method, three iterations at most are needed for convergence with tolerance 10^{-6}. This method conserves I_1 and I_2 exactly and nearly conserves I_3 and I_4. The results are presented in Table I. Perspective views of the travelling soliton are presented in Figure 1. By using the same parameters and the predictor–corrector method we get the results which are displayed in Table II. We can easily see from Tables I and II that the two methods give almost the same results regarding the accuracy and the conserved quantities.

The rate of convergence of the proposed method can be calculated from the formula

$$p = \log\left(\frac{\|\Psi_1(x_m, t_n) - (U^n_{1,m} + iU^n_{2,m})_{h_1}\|}{\|\Psi_1(x_m, t_n) - (U^n_{1,m} + iU^n_{2,m})_{h_2}\|}\right) \Bigg/ \log\left(\frac{h_1}{h_2}\right). \tag{54}$$

The value of p is called the rate of convergence.

In Table III we present the rate of convergence of our method which is almost equal to 4, and this justifies that our method is fourth-order in space.

To improve the temporal accuracy of the proposed method, we use extrapolation on the computed solution to eliminate the lower-order term in the truncation error. Since the implicit

TABLE I Single soliton using Newton's method.

T	I_1	I_3	I_4	L_∞	L_2
0.0	1.189207	−5.628650	0.439351	0.000000	0.000000
10	1.189207	−5.628648	0.439346	0.000211	0.000858
20	1.189207	−5.628648	0.439346	0.000406	0.001702
30	1.189207	−5.628648	0.439346	0.000600	0.002547
40	1.189207	−5.628648	0.439346	0.000797	0.003395

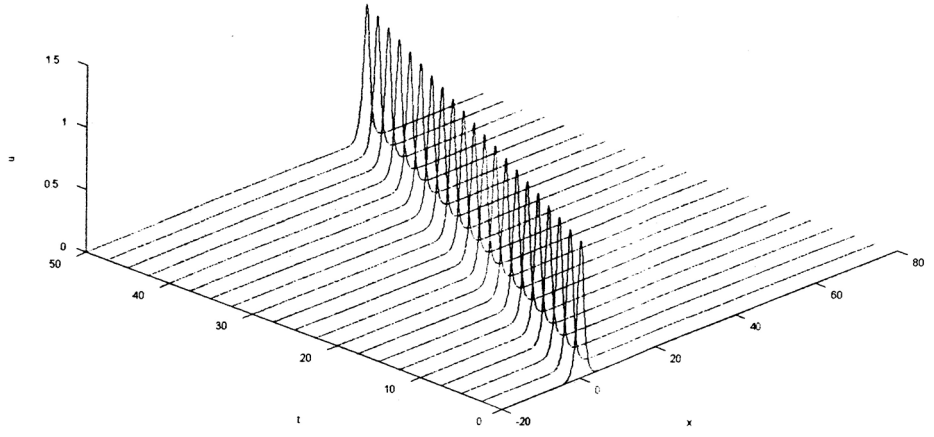

FIGURE 1 Single soliton at different time levels.

TABLE II Single soliton using predictor–corrector method.

T	I_1	I_3	I_4	L_∞	L_2
0.0	1.189207	−5.628650	0.439351	0.000000	0.000000
10	1.189205	−5.628629	0.439351	0.000211	0.000856
20	1.189203	−5.628613	0.439355	0.000407	0.001700
30	1.189202	−5.628595	0.439360	0.000604	0.002549
40	1.189200	−5.628579	0.439363	0.000803	0.003403

TABLE III Single soliton and Newton's method with $e = 2/3$, $\alpha = 1$, $v = 1.0$, $k = 0.005$.

Time	$\|ER(h = 0.2)\|_\infty$	$\|ER(h = 0.1)\|_\infty$	p
4	0.001401	0.000089	3.98
8	0.002474	0.000161	3.94
12	0.003495	0.000230	3.93
16	0.004572	0.000301	3.93
20	0.005672	0.000371	3.93

midpoint rule applied to the scheme is in the form $O(k^2) + O(k^4)$, we use

$$\mathbf{u}_m^n \approx \frac{4\mathbf{U}_m^n(k/2) - \mathbf{U}_m^n(k)}{3} \tag{55}$$

to eliminate the term $O(k^2)$, which makes the final solution fourth-order accurate in both the temporal and spatial dimension. Although the extrapolation requires two times as much computation as the original scheme plus the application of the formula (55), the resulting high-order accuracy allows the use of much larger time steps in the computation [6].

By using the parameters $k = 0.1, 0.05$, $h = 0.2$, $e = 2/3$, $v = 1$ and $\alpha = 1$, with the extrapolation formula (55), we obtain results which are displayed in Table IV.

To compare the computational efficiency between Newton's method and the predictor–corrector method, we calculate the computing time needed for simulation of our proposed method using different values of k and the final time is $t = 20$. The results are given in Table V. We found that the predictor–corrector method is faster than Newton's by a ratio of approximately 3.5.

TABLE IV Richardson extrapolation using L_∞ norm.

t	$k = 0.1$	$k = 0.05$	*Extrapolation*
4	0.003512	0.001847	0.001321
8	0.006546	0.003417	0.002408
12	0.009599	0.005001	0.003750
16	0.012661	0.006543	0.005429
20	0.015653	0.008124	0.007226

TABLE V Computing time using Newton's method and predictor–corrector method with parameters $h = 0.1, \alpha = 1, e = 2/3, v = 1$.

k	*CPU (Newton's)*	*CPU (PC)*
0.1	21.04	6.65
0.05	41.8	11.43
0.01	137.59	38.44

3.2 Collision of Two Solitons

To study the interaction of two solitons, we take the initial conditions

$$\Psi_1(x,0) = \sum_{j=1}^{2} \sqrt{\frac{2\alpha_j}{1+e}} \sec h(\sqrt{2\alpha_j} x_j) \exp i\{(v_j - x_j)\} \tag{56}$$

and

$$\Psi_2(x,0) = \sum_{j=1}^{2} \sqrt{\frac{2\alpha_j}{1+e}} \sec h(\sqrt{2\alpha_j} x_j) \exp i\{(v_j + x_j)\} \tag{57}$$

and the homogenous Neumann boundary conditions

$$\frac{\partial \Psi_1}{\partial x} = \frac{\partial \Psi_2}{\partial x} = 0 \quad \text{at } x = x_L, x_R.$$

The exact total sum of the masses can be given asymptotically by

$$I_1 = \frac{2}{1+e} \sum_{j=1}^{2} \sqrt{2\alpha_j}.$$

In this test we choose the parameters $x_1 = 0, x_2 = 25, x_L = -20, x_R = 80, v_1 = 1.0, v_2 = 0.1, \alpha_1 = 1, \alpha_2 = 0.5,\ e = 2/3$.

The initial conditions represent two wave forms separated by a distance equal to 25 units. The faster wave is situated initially at $x = 0$ and the slower one at $x = 25$.

By using the proposed scheme and Newton's method, we found that three iterations at most are needed for convergence with tolerance 10^{-6}. This method exactly conserves I_1 and nearly conserves I_3 and I_4. The simulation of interaction is depicted at different times in Figure 2. It is observed that the two waves travelling at different velocities collide and separate after the interaction is unchanged in shape and velocities. In Table VI we calculate velocity and the conserved quantities. It is very clear that I_1 is exactly conserved while I_3 and I_4 are almost conserved. The velocity regains its initial value after the interaction. By using the same parameters and the predictor–corrector method we have found that seven iterations at most are needed for convergence with tolerance 10^{-6}. The results are presented in Table VII, where we have displayed the velocity and the conserved quantities during the interaction scenario.

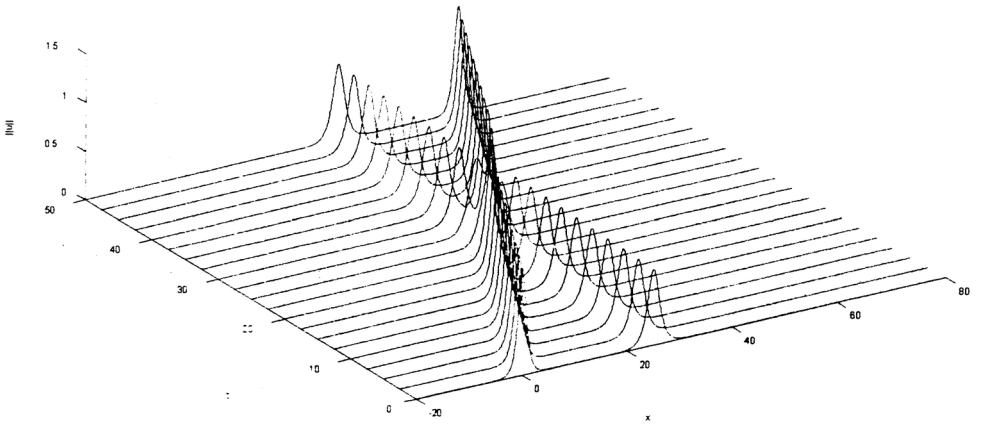

FIGURE 2 Interaction of two solitons.

3.3 Soliton Collisions of Different Initial Conditions

In this section, we solve Eq. (24) together with different initial condition

$$\begin{aligned}\Psi_1(x,0) &= \sqrt{2\alpha_1}\,\mathrm{sec}\,h(\sqrt{2\alpha_1}x+\zeta_{10})\exp(i\,v_1 x),\\ \Psi_2(x,0) &= \sqrt{2\alpha_2}\,\mathrm{sec}\,h(\sqrt{2\alpha_2}x+\zeta_{20})\exp(i\,v_2 x),\end{aligned} \tag{58}$$

where amplitudes $\sqrt{2\alpha_j}$, velocities v_j and initial phase constants ζ_{j0} $(j=1,2)$ with $v=-v=v/2(>0)$. So the approaching velocity of the two solitons is v. We also fix $\alpha_1=1.2$ and $\alpha_2=1$. The choices for the initial position parameters ζ_{10} and ζ_{20} can be totally arbitrary. They do not affect the collision outcome as long as $\zeta_{10}>0$ and $\zeta_{10}<0$ are large enough. We

TABLE VI Interaction of two solitons using Newton's method.

t	v_1	I_1	I_3	I_4
0	1.0	1.702074	−3.803570	−0.494114
10	1.0	1.702074	−3.803595	−0.493484
20	1.0	1.702074	−3.803534	−0.495458
30	1.14	1.702074	−3.803677	−0.496713
40	0.99	1.702074	−3.803727	−0.490079
50	1.0	1.702074	−3.803625	−0.492114

TABLE VII Interaction of two solitons using predictor–corrector method.

t	v_1	I_1	I_3	I_4
0	1.0	1.702074	−3.803570	−0.494114
10	1.0	1.702072	−3.803588	−0.493475
20	1.0	1.702069	−3.803522	−0.495441
30	1.14	1.702068	−3.803661	−0.496688
40	0.99	1.702065	−3.803703	−0.490046
50	1.0	1.702061	−3.803593	−0.492070

fix $\zeta_{10} = 10$ and $\zeta_{20} = -10$ in our calculations. The only free parameters left are the cross-phase modulational (XPM) coefficient e and the collision, velocity v, which are used as control parameters. In all calculations, we choose $h = 0.1$ and $k = 0.1$.

In the first case, we choose $e = 1$ and $v = 0.4$. This is a Manakov model which is completely integrable and hence we expect the interaction of the two solitons to be elastic and this is indeed the case. Figure 3 displays the contours of the interaction scenario where we can see that the two waves retain their shapes and velocity after the interaction. The velocity and the conserved quantities are given in Table VIII.

In the second test, we select $e = 2/3$ which corresponds to linearly birefringent fibers, and $v = 0.4$. We observe that collision takes place at $t \approx 40$. During collision, we observe that the velocity of the right moving soliton steadily decreases and becomes negative when it emerges from the collision. This means that this soliton is reflected back by the collision. The same thing happens to the other soliton, it initially moves to the left, but turns around after the collision. This reflection scenario has been reported in Ref. [11]. The amplitudes of the two solitons also changed after the collision, the larger soliton gets even larger, and the smaller one gets even smaller. The polarization of the two solitons are also shifted. This means that the energy inside a vector soliton has been partially transferred from one polarization axis to the other due to the collision. This is called a daughter wave. These daughter waves are small pulses that split off from the solitary wave and propagate beside it, but in the other mode. This scenario is displayed in Figure 4 for $\|\Psi_1\|$, and this has been reported in Refs. [11, 12].

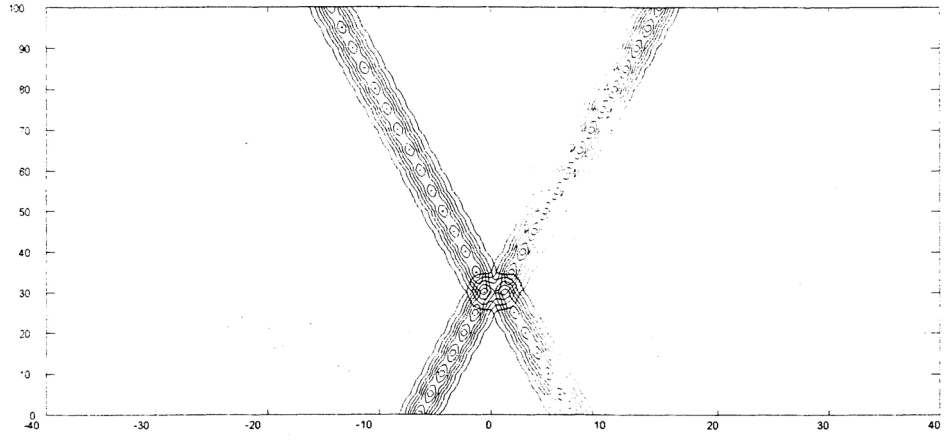

FIGURE 3 Manakov model: contours of $\|\Psi_1\|$ and $\|\Psi_2\|$, with parameters $h = 0.1$, $k = 0.1$, $e = 1$, $v = 0.4$.

TABLE VIII Interaction of two solitons (Manakov model) $e = 1$, $h = 0.1$, $k = 0.1$, $v = 0.4$.

t	$max(u)$	v	I_1	I_4
0	1.549193	0.2	1.760223	−2.087838
10	1.5430	0.2	1.760223	−2.087547
20	1.5424	0.19	1.760223	−2.087529
30	1.2113	0.34	1.760223	−2.087473
40	1.5452	0.2	1.760223	−2.087473
50	1.5450	0.2	1.760223	−2.087485
60	1.5449	0.2	1.760223	−2.087471

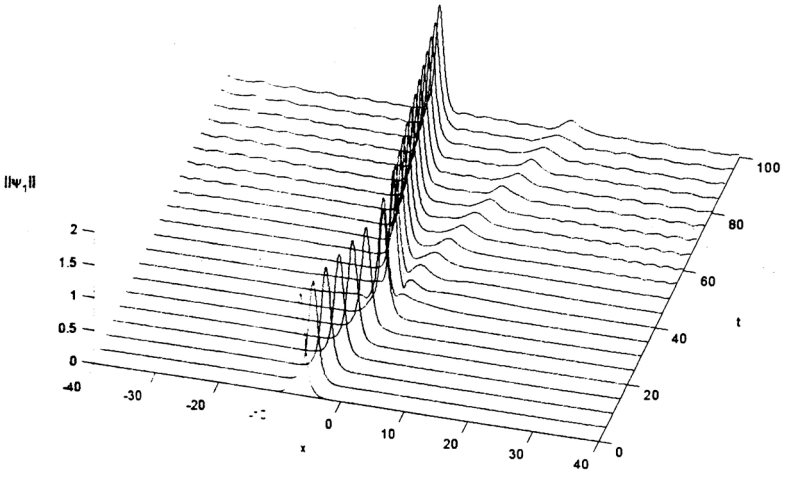

FIGURE 4 Reflection $\|\Psi_1\|$, $h = 0.1$, $k = 0.1$, $e = 2/3$, $v = 0.4$.

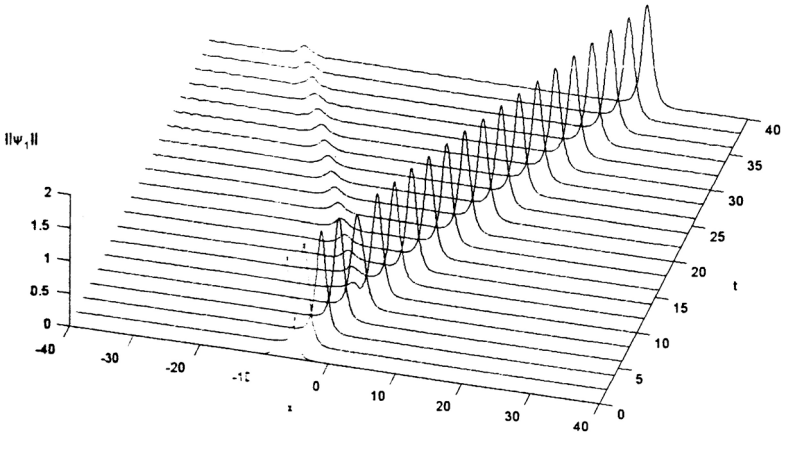

FIGURE 5 Transmission $\|\Psi_1\|$, with parameters $h = 0.1$, $k = 0.1$, $e = 2/3$, $v = 1.6$.

We repeat the test by increasing the velocity to $v = 1.6$. We have noticed that the two waves pass through each other with some reshaping and radiation shedding and daughter waves are generated. This transmission scenario is displayed in Figure 5.

4 CONCLUSIONS

We have developed a numerical method for solving the CNLSE, which is fourth-order in space and second-order in time, unconditionally stable. We have studied the interaction of two solitons of different amplitudes. The accuracy of the numerical results highlighted the efficiency of the method and displayed the interaction picture clearly in the case of a single soliton, two solitons and interaction of two orthogonal solitons and we simulate some of the scenarios reported in Ref. [11]. This method can be easily generalized to solve N coupled NLS equation and this will be considered next.

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