The following formulae were provided for this iteration of the MATH 152 final exam

Formulas for Finite Sums:

$$\sum_{i=1}^{n} c = cn, \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Double Angle Formulas:

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Table of Trigonometric Substitutions:

Expression	Substitution	Identity	Range of θ
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$1-\sin^2\theta = \cos^2\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\quad(\cos\theta \ge 0)$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$1+\tan^2\theta = \sec^2\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ $\quad(\sec\theta > 0)$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$ $\quad(\tan\theta \ge 0)$

Error Bounds in the Trapezoidal and Midpoint Rules:

Suppose that $|f''(x)| \le K$ for all $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12\,n^2} \qquad \text{and} \qquad |E_M| \le \frac{K(b-a)^3}{24\,n^2}.$$

Error Bound in Simpson's Rule:

Suppose that $|f^{(4)}(x)| \le K$ for all $a \le x \le b$. If E_S is the error in using Simpson's Rule, then

$$|E_S| \le \frac{K(b-a)^5}{180\,n^4}.$$

Taylor's Inequality:

If

$$\left|f^{(n+1)}(x)\right| \le M \quad \text{for} \quad |x-a| \le d,$$

then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for} \quad |x-a| \le d.$$

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1. Compute the following integrals.

[4] (a) $\displaystyle\int \frac{e^x}{e^{2x} - 4e^x + 5}\,dx$

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[4] (b) $\displaystyle\int \frac{1}{x^2\sqrt{4 - x^2}}\,dx$, using the substitution $x = 2\cos\theta$.

2. Compute the following integrals.

[4] (a) $\displaystyle\int_0^{\pi} |\sin x - 1| dx$

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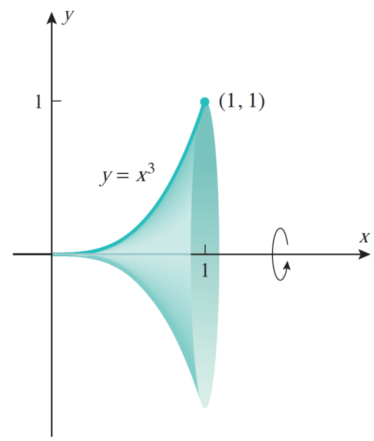
[4] (b) $\displaystyle\int_{-1}^{1} \ln(x+1)\ dx$

[5] **3.** (a) Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.



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[3] (b) Find the area inside the polar curve $r = \sqrt{\theta}$, $0 \leq \theta \leq 2\pi$.

4. The parts of this problem are not related.

[4] (a) Find the derivatives $f'(x)$ and $f''(x)$ if $f(x) = \int_1^x \frac{e^t}{t}\, dt$.

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[4] (b) Suppose you know that $\{b_n\}$ is a decreasing sequence and all its terms lie between the numbers 4 and 6. Explain why the sequence has a limit. What can you say about the value of the limit?

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[2] (c) Write down the definite integral which can be expressed by the limit $\lim_{n\to\infty} \left(\frac{2}{n} \sum_{i=1}^{n} e^{1+2i/n} \right)$

5. A tank contains 100 litres of pure water. Brine that contains 0.1 kg of salt per litre enters the tank at a rate of 10 litres per minute. The solution is kept thoroughly mixed and drains from the tank at the same rate.

[5] (a) Find an expression in terms of for the amount of salt in the tank at any time t. Give your answer as a function of t.

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[2] (b) How much salt is in the tank after 6 minutes?

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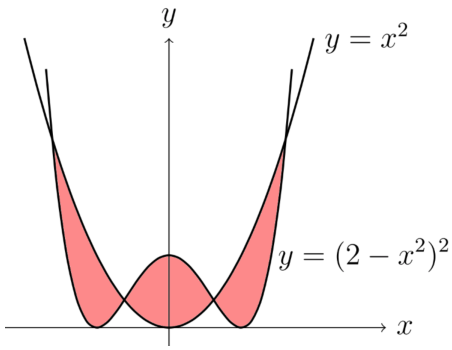
[1] (c) How much salt is in the tank after a very long time; that is, as $t \to \infty$.

[5] **6.** Find the area of the region bounded by the curves $y = x^2$ and $y = (2 - x^2)^2$.



7. All parts of this question concern the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

[1] (a) What is the $\lim_{n\to\infty}$ of a_n for this series?

[1] (b) Does this series converge or diverge?

[4] (c) **Justify** your answer to part (b) above. You may **NOT** simply state that it is p-series with $p = \frac{1}{2}$. Clearly state the test(s) you are using and justify the steps in using the test.

[3] (d) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. Does this series **converge** or **diverge**? Justify your answer.

[1] (e) Is the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ said to be **absolutely convergent** or **conditionally convergent**?

8. Let R be the region bounded by the graph of $y = e^{-\sqrt{x}}$ and the x-axis, for $x \geq 0$.

[5] (a) Find the volume of the solid generated when R is revolved about the x-axis.

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[3] (b) Let C denote the curve with parametric equations

$$x = t - 2\sin t, \quad y = 1 - 2\cos t, \quad 0 \leq t \leq 2\pi.$$

Set up BUT DO NOT EVALUATE the integral for the length of the curve C.

9. For each series determine whether it is **convergent** or **divergent.** In each case, state the test(s) you are using and justify the steps in using the test.

[4] (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k^{3/2} - 1}$

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[4] (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

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[4] (c) $\displaystyle\sum_{n=3}^{\infty} \frac{e^{-2n}}{n^2 + 2n}$

[3] **10.** (a) Define the **Taylor Series** of a function f at a.

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[4] (b) Compute the Taylor series for the function $f(x) = e^{-2x}$ at $a = 1$. State your answer using the summation notation.

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[4] (c) What is the bound on the error when using the T_3 polynomial to approximate e^{-2x} over the interval $x \in [0, 2]$?

[8] **11.** (a) For the following series compute the radius R of convergence and the interval I of convergence. Justify your answer. Credit will only be given in so far as the reasons are adequate.

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{2n+1}.$$

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[4] (b) Find the power series representation and the radius of convergence for the function:

$$f(x) = \frac{x^3}{x^3+27}.$$