HW1

1) $6x + 10y = 2 = \gcd(6, 10)$

~~Let $2x = p$, $2y = q$~~

~~$\Rightarrow 3p + 5q = 1 = \gcd(3, 5)$~~

Using Euclid's Algorithm

$m = 3 \quad n = 5 \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$q = \left\lfloor \frac{n}{m} \right\rfloor = 1$

$\Rightarrow \quad m = n \% m = 2 \qquad n = 3$

$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$q = 1$

$\Rightarrow m = 1 \qquad n = 2$

$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$q = 2$

$\Rightarrow$

$m = 0 \qquad n = 1$

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$$

$\Rightarrow$ ~~$x = 2 \quad y = -1$~~

$$\Rightarrow \boxed{x = 2 \qquad y = -1}$$

2) $6x + 10y + 15z = 1$

Let $2y + 3z = p$

$\Rightarrow 6x + 5p = 1 = \gcd(6, 5)$

Using Euclid's algorithm,

$m = 5 \qquad n = 6 \qquad q = \left\lfloor \frac{n}{m} \right\rfloor = 1$

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow m = n \% m = 1 \qquad n = 5 \qquad q = 5$

$$Q = \begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ -1 & 1 \end{bmatrix}$$

$\Rightarrow m = n \% m = 0 \qquad n = 1$

$\Rightarrow p = -1 \quad x = 1$

Now, to solve $2y + 3z = p = -1$

Consider $2a + 3b = 1$

Then, $m = 2 \quad n = 3 \quad q = 1$

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$m = n \% m = 1 \quad n = 2 \quad q = 2$

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$m = n \% m = 0 \quad n = 1$

$Q \Rightarrow a = -1 \quad b = 1$

$\Rightarrow y = -a = 1 \quad z = -b = -1$

$$\Rightarrow \boxed{x = 1 \quad y = 1 \quad z = -1}$$

3) Let $g = \gcd(a^m - 1, a^n - 1)$

Let $n > m$, then using

$\gcd(a, b) = \gcd(a, b - a)$

$g = \gcd(a^m - 1, a^n - a^m)$

$= \gcd(a^m - 1, a^m(a^{n-m} - 1))$

a^m & $a^m - 1$ can not have a common factor. Hence,

$g = \gcd(a^m - 1, a^{n-m} - 1)$

Performing this step until $n - km < m$

$g = \gcd(a^m - 1, a^{n \% m} - 1)$

Similar to Euclid's algorithm, ~~we~~ we will get

$\cancel{g = \gcd(g^{0} - 1, g^{\gcd})}$

$g = \gcd(a^{0} - 1, a^{\gcd(m, n)} - 1)$

$\Rightarrow g = \gcd(0, a^{\gcd(m,n)} - 1)$

$\Rightarrow \boxed{g = a^{\gcd(m,n)} - 1}$

4b) $2x + 3y + 5z = 0$

Let $2x + 3y = a$

Then $a + 5z = 0$

$m = 1 \quad n = 5 \quad q = 5$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 1 & 0 \end{bmatrix}$

$m = n \% m = 0 \quad n = 1$

$\Rightarrow a = -5K \quad z = K$

Now, $2x + 3y = a = -5K$

Consider $2x + 3y = 1$

$\Rightarrow m = 2 \quad n = 3 \quad q = 1$

$Q = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow m = n \% m = 1 \quad n = 2 \quad q = 2$

$$Q = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$m = 0 \qquad n = 1$

$\Rightarrow \cancel{r = \quad -t \quad s = \quad t}$

$\Rightarrow r = -1 \qquad s = 1$

$\Rightarrow x = 5K \qquad y = -5K$

$\cancel{\Rightarrow x, y}$

$\Rightarrow \boxed{x = 5K,\ y = -5K,\ z = K \quad \text{where } K \in \mathbb{Z}}$