CO21BTECH11002
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Assignment 5

1)

a)

Mass gain in r-direction $= \frac{\partial(\rho V_r \, dr \, r \, d\theta)}{\partial r} dr$

Mass gain in θ-direction $= \frac{\partial(\rho V_\theta \, dr \, r \, d\theta)}{\partial \theta} d\theta$

Net mass gain $= \left(\frac{\partial(\rho V_r r)}{r \, \partial r} + \frac{\partial(\rho V_\theta)}{r \, \partial \theta} \right) r \, dr \, d\theta$

For steady state, incompressible flow,
Net mass gain $= 0$, $\rho =$ constant

$$\Rightarrow \left(\frac{\partial(\rho V_r r)}{r \, \partial r} + \frac{\partial(\rho V_\theta)}{r \, \partial \theta} \right) r \, dr \, d\theta = 0$$

$$\Rightarrow \frac{\partial(V_r r)}{r \, \partial r} + \frac{\partial V_\theta}{r \, \partial \theta} = 0$$

$$\Rightarrow \boxed{\frac{\partial V_r}{\partial r} + \frac{V_r}{r} + \frac{\partial V_\theta}{r \, \partial \theta} = 0}$$

6b In cartesian coordinates :-

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

In cylindrical coordinates;

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\Rightarrow r = \sqrt{x^2+y^2} \qquad \cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta$$

Also, $$\frac{\partial T}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial T}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial T}{\partial \theta}$$

& $$\frac{\partial T}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial T}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial T}{\partial \theta}$$

Also, $$-\sin\theta \frac{d\theta}{dx} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{\sin^2\theta}{r}$$

$$\Rightarrow \frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\cos\theta \frac{d\theta}{dy} = \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{\cos^2\theta}{r}$$

$$\Rightarrow \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$$

Now, $$\frac{\partial^2 T}{\partial x^2} = \left(\cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial\theta}\right)\left(\cos\theta\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\frac{\partial T}{\partial\theta}\right)$$

$$\frac{\partial^2 T}{\partial y^2} = \left(\sin\theta \frac{\partial T}{\partial r} + \frac{\cos\theta}{r}\frac{\partial T}{\partial \theta}\right)\left(\sin\theta \frac{\partial T}{\partial r} + \frac{\cos\theta}{r}\frac{\partial T}{\partial \theta}\right)$$

From the above two equations,

$$\boxed{\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}}$$

2)

b) $$\frac{\partial T}{\partial x} = K\frac{\partial^2 T}{\partial x^2}$$

$$\Rightarrow \frac{T_i^{n+1} - T_i^n}{\Delta t} = \frac{K}{(\Delta x)^2}\left[\frac{4}{3}\left(T_{i+1}^n + T_{i-1}^n\right) - \frac{1}{12}\left(T_{i+2}^n + T_{i-2}^n\right) - \frac{5}{2}T_i^n\right]$$

$$\Rightarrow T_i^{n+1} = T_i^n + \lambda\left[\frac{4}{3}\left(T_{i+1}^n + T_{i-1}^n\right) - \frac{1}{12}\left(T_{i+2}^n + T_{i-2}^n\right) - \frac{5}{2}T_i^n\right]$$

$$\Rightarrow T_i^{n+1} = T_i^n\left(1 - \frac{5\lambda}{2}\right) + \frac{4\lambda}{3}\left(T_{i+1}^n + T_{i-1}^n\right) - \frac{\lambda}{12}\left(T_{i+2}^n + T_{i-2}^n\right)$$

$$\Rightarrow \varepsilon_j^{n+1} = \varepsilon_j^n\left(1 - \frac{5\lambda}{2}\right) + \frac{4\lambda}{3}\left(\varepsilon_{j+1}^n + \varepsilon_{j-1}^n\right) - \frac{\lambda}{12}\left(\varepsilon_{j+2}^n + \varepsilon_{j-2}^n\right)$$

Putting $\varepsilon_j^n = e^{\sigma_r t^n} e^{iKx_j}$

where $x_j = jh$, $t^n = n\Delta t$

$$\Rightarrow e^{\sigma_r \Delta t(n+1)} e^{iKjh} = e^{\sigma_r n\Delta t} e^{iKjh}\left(1 - \frac{5\lambda}{2}\right) + \frac{4\lambda}{3}\left(e^{\sigma_r n\Delta t} e^{iK(j+1)h} + e^{\sigma_r n\Delta t} e^{iK(j-1)h}\right) - \frac{\lambda}{12}\left(e^{\sigma_r n\Delta t} e^{iK(j+2)h} + e^{\sigma_r n\Delta t} e^{iK(j-2)h}\right)$$

$$\Rightarrow e^{\sigma \Delta t} = 1 - \frac{5\lambda}{2} + \frac{4\lambda}{3}\left(e^{ikh} + e^{-ikh}\right) - \frac{\lambda}{12}\left(e^{2ikh} + e^{-2ikh}\right)$$

$$\left| \frac{\varepsilon_j^{n+1}}{\varepsilon_j^{n}} \right| = \left| e^{\sigma \Delta t} \right| < 1$$

$$\Rightarrow \left| 1 - \frac{5\lambda}{2} + \frac{4\lambda}{3}\left(e^{ikh} + e^{-ikh}\right) - \frac{\lambda}{12}\left(e^{2ikh} + e^{-2ikh}\right) \right| < 1$$

$$\Rightarrow \left| 1 - \frac{5\lambda}{2} + \frac{4\lambda}{3}\, 2\cos kh - \frac{\lambda}{12}\, 2\cos 2kh \right| < 1$$

$$\Rightarrow \left| 1 - \frac{5\lambda}{2} + \frac{8\lambda}{3}\cos kh - \frac{\lambda}{6}\cos 2kh \right| < 1$$

$$\Rightarrow \left| 1 - \frac{5\lambda}{2} + \frac{8\lambda}{3}\cos kh - \frac{\lambda}{6}\left(2\cos^2 kh - 1\right) \right| < 1$$

$$\Rightarrow \left| 1 - \frac{5\lambda}{2} + \frac{8\lambda}{3}\cos kh - \frac{\lambda}{3}\cos^2 kh + \frac{\lambda}{6} \right| < 1$$

$$\Rightarrow \left| 1 - \frac{7\lambda}{3} + \frac{\lambda}{3}\left(8\cos kh - \cos^2 kh\right) \right| < 1$$

$$\Rightarrow -2 < -\frac{7\lambda}{3} + \frac{\lambda}{3}\left(8\cos kh - \cos^2 kh\right) < 0$$

$$\Rightarrow \frac{7\lambda}{3} - \frac{\lambda}{3}\left(8\cos kh - \cos^2 kh\right) < 2$$

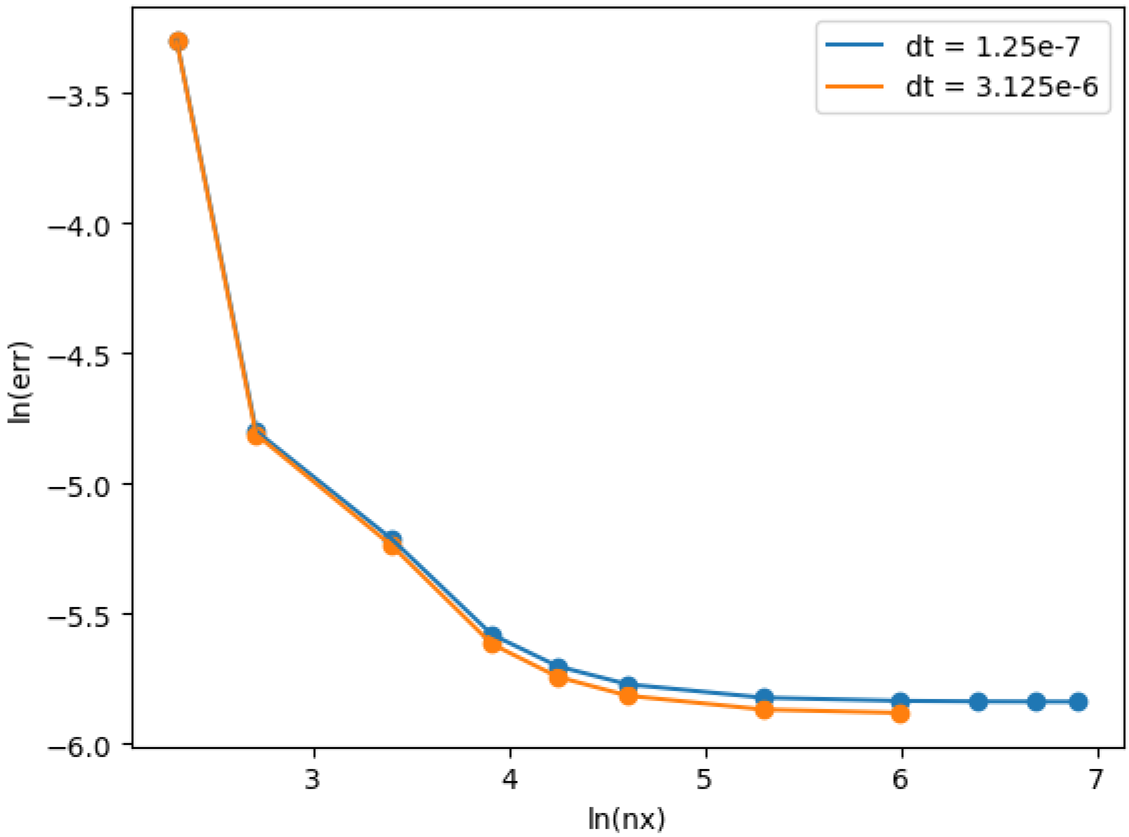
Max value of $-\left(8\cos kh - \cos^2 kh\right) = 9$
when $\cos kh = -1$

$$\Rightarrow \frac{7\lambda}{3} + \frac{9\lambda}{3} < 2 \qquad \Rightarrow \frac{16\lambda}{3} < 2$$

$$\Rightarrow \boxed{\lambda < \frac{3}{8}}$$

where $\lambda = \dfrac{\kappa \Delta t}{(\Delta x)^2}$

c)



Observations:

- Near the start, value of slope = -3.86 which shows that this scheme is 4th order.
- After nx becomes greater than about 100, the decrease in error almost saturates.
- For dt = 3.125e-6, max value of nx is 565 (from the relation λ < ⅜). After that the solution blows up and we get inf as the error value.
- For dt = 1.25e-7, max value of nx is 2828. Hence, we get a complete curve till nx = 1000 without the solution blowing up.
- For larger value of dt, the solution has a faster convergence rate.