> *restart*; *libname*;

"D:\Maple 2017\lib" **(1)**

In order for Maple to load the GRTensor library, add the directory it was installed in to the Maple libpath variable.

> *libname* := "D:/grtensor/released/grt3/lib", *libname*;

libname := "D:/grtensor/released/grt3/lib", "D:\Maple 2017\lib" **(2)**

Now the GRTensorIII module can be loaded using the with() command.

> *with*(*grtensor*);

"GRTensor III v2.1.10 Oct 3, 2017"

"Copyright 2017, Peter Musgrave, Denis Pollney, Kayll Lake"

"Latest version is at http://github.com/grtensor/grtensor"

"For help ?grtensor"

"Support/contact grtensor3@gmail.com"

[*Asym*, *KillingCoords*, *PetrovReport*, *Sym*, *autoAlias*, *cmcompare*, *difftool*, *grDalias*, *grF_strToDef*, *gralter*, *gralterd*, *grapply*, *grarray*, *grcalc*, *grcalc1*, *grcalcalter*, *grcalcd*, *grclear*, *grcomponent*, *grconstraint*, *grdata*, *grdebug*, *grdef*, *grdisplay*, *grdump*, *greqn2set*, *grinit*, *grload*, *grload_maplet*, *grmap*, *grmetric*, *grnewmetric*, *grnormalize*, *groptions*, *grsaveg*, *grt2DG*, *grtestinput*, *grtransform*, *grundef*, *hypersurf*, *join*, *kdelta*, *makeg*, *nprotate*, *nptetrad*, *qload*, *spacetime*] **(3)**

Help for the GRTensorIII commands is now available by entering ?<command name>. For a list of commands use `?grt_commands`

The `spacetime` command is used to specify the input for GRTensor. The most common use is to enter the line element of the spacetime you want to calculate objects in.

In this example we will follow the development in Poisson and Will for a general spherically symmetric spacetime with co-ordinates r, theta, phi and t. The intent is to solve Einstein's equation and "discover" the Schwarzschild solution so the metric begins with general functions. With reference to (10.7) the spacetime command is used to enter this general spherical metric. Note that the dependence of the functions alpha, beta and gamma on r is explicitly indicated.

As is noted in the text, the specific form of the metric is somewhat bizarre, but is general. The form used is chosen to make the steps in the derivation simpler. (Knowing what the answer is can be helpful!)

>

> $spacetime\left(spherical, coord = [r, \theta, \phi, t], ds = -\exp\left(\frac{-2\cdot \Phi(r)}{c^2}\right)\cdot d[t]^2 + \left(1 - \frac{2\cdot G\cdot m(t,r)}{c^2 r}\right)^{-1}\cdot d[r]^2 + r^2\cdot\left(d[\theta]^2 + \sin(\theta)^2\cdot d[\phi]^2\right)\right);$

```
Calculated ds for spherical (0.000000 sec.)
```

CPU Time = 0.015

For the spherical spacetime:

Line element

(4)

$$ds^2 = \frac{dr^2}{1 - \frac{2\,G\,m(t,r)}{c^2\,r}} + r^2\,d\theta^2 + r^2\sin(\theta)^2\,d\phi^2 - \frac{dt^2}{\left(e^{\frac{\Phi(r)}{c^2}}\right)^2} \tag{4}$$

The metric spherical is now available for use.

The most common operations in GRTensorIII are the definition, calculation and simplification of the components of tensors in the spacetime. Definitions are provided for all of the commonly used tensors. A full list can be found on the `?grt_objects` help screen.

To indicate a tensor object in GRTensor a Maple function expression is used. The name of the function is the tensor name and the arguments specify the number and type of indices. For example the covariant (indices down) metric tensor is `g(dn,dn)`; the contravariant version of the same tensor is `g(up,up)`.

To calculate one of the pre-defined tensor objects the command `grcalc()` is used. The result of the calculation is not automatically displayed (because in some case the expression may be very large and require simplification). It can be displayed with `grdisplay()`. In cases where the output is expected to be small a tensor can be calulated and displayed using `grcalcd()`.

We first demonstrate `grcalc()` by calculating the Christoffel symbols.

```
> grcalcd(Chr(dn, dn, up));
Calculated g(dn,dn,pdn) for spherical (0.000000 sec.)
Calculated Chr(dn,dn,dn) for spherical (0.000000 sec.)
Calculated detg for spherical (0.000000 sec.)
Calculated g(up,up) for spherical (0.000000 sec.)
Calculated Chr(dn,dn,up) for spherical (0.000000 sec.)
```

CPU Time $= 0.$

For the spherical spacetime:

Christoffel symbol of the second kind (symmetric in first two indices)

$$\Gamma_{r\,r}{}^{r} = \frac{G\left(-\left(\frac{\partial}{\partial r}\,m(t,r)\right)r + m(t,r)\right)}{r\left(-c^2\,r + 2\,G\,m(t,r)\right)}$$

$$\Gamma_{r\,r}{}^{t} = \frac{\left(e^{\frac{\Phi(r)}{c^2}}\right)^2 c^2\,r\,G\left(\frac{\partial}{\partial t}\,m(t,r)\right)}{\left(-c^2\,r + 2\,G\,m(t,r)\right)^2}$$

$$\Gamma_{r\,\theta}{}^{\theta} = \frac{1}{r}$$

$$\Gamma_{r\,\phi}{}^{\phi} = \frac{1}{r}$$

$$\Gamma_{r\,t}{}^{r} = -\frac{G\left(\frac{\partial}{\partial t}\,m(t,r)\right)}{-c^2\,r + 2\,G\,m(t,r)}$$

$$\Gamma_{r\,t}{}^{t} = -\frac{\frac{d}{dr}\,\Phi(r)}{c^2}$$

$$\Gamma_{\theta\theta}{}^{r} = \frac{-c^2 r + 2\, G\, m(t, r)}{c^2}$$

$$\Gamma_{\theta\phi}{}^{\phi} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\Gamma_{\phi\phi}{}^{r} = \frac{\left(-c^2 r + 2\, G\, m(t, r)\right) \sin(\theta)^2}{c^2}$$

$$\Gamma_{\phi\phi}{}^{\theta} = -\sin(\theta) \cos(\theta)$$

$$\Gamma_{tt}{}^{r} = \frac{\left(-c^2 r + 2\, G\, m(t, r)\right) \left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right)}{r\, c^4 \left(\mathrm{e}^{\frac{\Phi(r)}{c^2}}\right)^2} \tag{5}$$

Any components not shown are zero. We go on to calculate the Einstein tensor

```
> grcalcd(G(up, dn));
Created definition for G(up,dn)
Calculated R(dn,dn) for spherical (0.000000 sec.)
Calculated Ricciscalar for spherical (0.000000 sec.)
Calculated G(dn,dn) for spherical (0.000000 sec.)
Calculated G(up,dn) for spherical (0.000000 sec.)
```

CPU Time $= 0.$

For the spherical spacetime:

G(up,dn)

$G(up, dn)$

$$G^{r}{}_{r} = \frac{2\left(-\left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right) c^2 r^2 + 2\, G\, m(t, r) \left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right) r - G\, m(t, r)\, c^2\right)}{r^3 c^4}$$

$$G^{t}{}_{r} = \frac{2\left(\mathrm{e}^{\frac{\Phi(r)}{c^2}}\right)^2 G \left(\frac{\partial}{\partial t}\, m(t, r)\right)}{\left(-c^2 r + 2\, G\, m(t, r)\right) r}$$

$$G^{\theta}{}_{\theta} = \frac{1}{r^3 c^6 \left(-c^2 r + 2\, G\, m(t, r)\right)^2} \Bigg(-G \left(\mathrm{e}^{\frac{\Phi(r)}{c^2}}\right)^2 \left(\frac{\partial^2}{\partial t^2}\, m(t, r)\right) c^8 r^4 + 2\, G^2 m(t,$$
$$r) \left(\mathrm{e}^{\frac{\Phi(r)}{c^2}}\right)^2 \left(\frac{\partial^2}{\partial t^2}\, m(t, r)\right) c^6 r^3 - 3 \left(\mathrm{e}^{\frac{\Phi(r)}{c^2}}\right)^2 c^6 r^3 G^2 \left(\frac{\partial}{\partial t}\, m(t, r)\right)^2 - \Bigg(\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$
$$\Phi(r)\Bigg) c^8 r^5 + 6\, G\, m(t, r) \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}\, \Phi(r)\right) c^6 r^4 + G \left(\frac{\partial}{\partial r}\, m(t, r)\right) \left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right) c^6 r^4$$
$$- G \left(\frac{\partial}{\partial r}\, m(t, r)\right) c^8 r^3 + \left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right)^2 c^6 r^5 - \left(\frac{\mathrm{d}}{\mathrm{d}r}\, \Phi(r)\right) c^8 r^4 - 12\, G^2 m(t, r)^2 \Bigg(\frac{\mathrm{d}^2}{\mathrm{d}r^2}$$

$$
\Phi(r)\Big) c^4 r^3 - 4 G^2 m(t,r) \left(\frac{\partial}{\partial r} m(t,r)\right) \left(\frac{d}{dr} \Phi(r)\right) c^4 r^3 + 4 G^2 m(t,r) \left(\frac{\partial}{\partial r} m(t,r)\right) c^6 r^2 - 6 G m(t,r) \left(\frac{d}{dr} \Phi(r)\right)^2 c^4 r^4 + 5 G m(t,r) \left(\frac{d}{dr} \Phi(r)\right) c^6 r^3 + G m(t,r) c^8 r^2
$$

$$
+ 8 G^3 m(t,r)^3 \left(\frac{d^2}{dr^2} \Phi(r)\right) c^2 r^2 + 4 G^3 m(t,r)^2 \left(\frac{\partial}{\partial r} m(t,r)\right) \left(\frac{d}{dr} \Phi(r)\right) c^2 r^2
$$

$$
- 4 G^3 m(t,r)^2 \left(\frac{\partial}{\partial r} m(t,r)\right) c^4 r + 12 G^2 m(t,r)^2 \left(\frac{d}{dr} \Phi(r)\right)^2 c^2 r^3 - 8 G^2 m(t,r)^2 \left(\frac{d}{dr} \Phi(r)\right) c^4 r^2 - 4 G^2 m(t,r)^2 c^6 r - 8 G^3 m(t,r)^3 \left(\frac{d}{dr} \Phi(r)\right)^2 r^2 + 4 G^3 m(t,r)^3 \left(\frac{d}{dr} \Phi(r)\right) c^2 r + 4 G^3 m(t,r)^3 c^4\Big)
$$

$$
G^{\phi}{}_{\phi} = \frac{1}{r^3 c^6 \left(-c^2 r + 2 G m(t,r)\right)^2} \Bigg(-G \left(e^{\frac{\Phi(r)}{c^2}}\right)^2 \left(\frac{\partial^2}{\partial t^2} m(t,r)\right) c^8 r^4 + 2 G^2 m(t,r) \left(e^{\frac{\Phi(r)}{c^2}}\right)^2 \left(\frac{\partial^2}{\partial t^2} m(t,r)\right) c^6 r^3 - 3 \left(e^{\frac{\Phi(r)}{c^2}}\right)^2 c^6 r^3 G^2 \left(\frac{\partial}{\partial t} m(t,r)\right)^2 - \left(\frac{d^2}{dr^2} \Phi(r)\right) c^8 r^5 + 6 G m(t,r) \left(\frac{d^2}{dr^2} \Phi(r)\right) c^6 r^4 + G \left(\frac{\partial}{\partial r} m(t,r)\right) \left(\frac{d}{dr} \Phi(r)\right) c^6 r^4
$$

$$
- G \left(\frac{\partial}{\partial r} m(t,r)\right) c^8 r^3 + \left(\frac{d}{dr} \Phi(r)\right)^2 c^6 r^5 - \left(\frac{d}{dr} \Phi(r)\right) c^8 r^4 - 12 G^2 m(t,r)^2 \left(\frac{d^2}{dr^2} \Phi(r)\right) c^4 r^3 - 4 G^2 m(t,r) \left(\frac{\partial}{\partial r} m(t,r)\right) \left(\frac{d}{dr} \Phi(r)\right) c^4 r^3 + 4 G^2 m(t,r) \left(\frac{\partial}{\partial r} m(t,r)\right) c^6 r^2 - 6 G m(t,r) \left(\frac{d}{dr} \Phi(r)\right)^2 c^4 r^4 + 5 G m(t,r) \left(\frac{d}{dr} \Phi(r)\right) c^6 r^3 + G m(t,r) c^8 r^2
$$

$$
+ 8 G^3 m(t,r)^3 \left(\frac{d^2}{dr^2} \Phi(r)\right) c^2 r^2 + 4 G^3 m(t,r)^2 \left(\frac{\partial}{\partial r} m(t,r)\right) \left(\frac{d}{dr} \Phi(r)\right) c^2 r^2
$$

$$
- 4 G^3 m(t,r)^2 \left(\frac{\partial}{\partial r} m(t,r)\right) c^4 r + 12 G^2 m(t,r)^2 \left(\frac{d}{dr} \Phi(r)\right)^2 c^2 r^3 - 8 G^2 m(t,r)^2 \left(\frac{d}{dr} \Phi(r)\right) c^4 r^2 - 4 G^2 m(t,r)^2 c^6 r - 8 G^3 m(t,r)^3 \left(\frac{d}{dr} \Phi(r)\right)^2 r^2 + 4 G^3 m(t,r)^3 \left(\frac{d}{dr} \Phi(r)\right) c^2 r + 4 G^3 m(t,r)^3 c^4\Bigg)
$$

$$
G^{r}{}_{t} = \frac{2 G \left(\frac{\partial}{\partial t} m(t,r)\right)}{r^2 c^2}
$$

$$G^{t}{}_{t} = -\frac{2\,G\left(\frac{\partial}{\partial r}\,m(t,r)\right)}{c^2\,r^2} \tag{6}$$

The angular terms of the Einstein tensor are formidable even in the case of spherical symmetry. Fortunately we can proceed without using them (since due to the Bianchi identity the components of the Einstein tensor are inter-dependent and we can choose to work with the simpler expressions involving coordinates r and t.)

In the case of a vacuum solution, Einstein's equation gives us G(up,dn) = 0.

Examining the G(t,t) component we can use Maple dsolve to solve for m(r,t). [In general it is not typical that we get a differential equation that is so co-operative!]

```
> Gtt := grcomponent(G(up, dn), [t, t]) = 0;
```

$$Gtt := -\frac{2\,G\left(\frac{\partial}{\partial r}\,m(t,r)\right)}{c^2\,r^2} = 0 \tag{7}$$

```
> dsolve(Gtt, m(t, r));
```

$$m(t,r) = \_F1(t) \tag{8}$$

If we impose the assumption of a time invariant solutions, then we can simplify this to m(r,t) = M, M some constant. We can now define this function forMaple and re-evaluate the Einstein tensor

```
> m(r, t) := M;
```

$$m := (r,t) \mapsto M \tag{9}$$

```
> grdisplay(G(up, dn));
```

For the spherical spacetime:

G(up,dn)

$G(up, dn)$

$$G^{r}{}_{r} = \frac{2\left(-\left(\frac{d}{dr}\,\Phi(r)\right)c^2\,r^2 + 2\,G\,M\left(\frac{d}{dr}\,\Phi(r)\right)r - G\,M\,c^2\right)}{r^3\,c^4}$$

$$G^{\theta}{}_{\theta} = \frac{1}{r^3\,c^6\,(-c^2\,r + 2\,G\,M)^2}\left(-\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^8\,r^5 + 6\,G\,M\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^6\,r^4 + \left(\frac{d}{dr}\,\Phi(r)\right)^2 c^6\,r^5 - \left(\frac{d}{dr}\,\Phi(r)\right)c^8\,r^4 - 12\,G^2\,M^2\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^4\,r^3 - 6\,G\,M\left(\frac{d}{dr}\,\Phi(r)\right)^2 c^4\,r^4 + 5\,G\,M\left(\frac{d}{dr}\,\Phi(r)\right)c^6\,r^3 + G\,M\,c^8\,r^2 + 8\,G^3\,M^3\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^2\,r^2 + 12\,G^2\,M^2\left(\frac{d}{dr}\,\Phi(r)\right)^2 c^2\,r^3 - 8\,G^2\,M^2\left(\frac{d}{dr}\,\Phi(r)\right)c^4\,r^2 - 4\,G^2\,M^2\,c^6\,r - 8\,G^3\,M^3\left(\frac{d}{dr}\,\Phi(r)\right)^2 r^2 + 4\,G^3\,M^3\left(\frac{d}{dr}\,\Phi(r)\right)c^2\,r + 4\,G^3\,M^3\,c^4\right)$$

$$G^{\phi}{}_{\phi} = \frac{1}{r^3\,c^6\,(-c^2\,r + 2\,G\,M)^2}\left(-\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^8\,r^5 + 6\,G\,M\left(\frac{d^2}{dr^2}\,\Phi(r)\right)c^6\,r^4 + \left(\frac{d}{dr}\right.\right. \tag{10}$$

$$\Phi(r)\Big)^2 c^6 r^5 - \left(\frac{d}{dr}\Phi(r)\right) c^8 r^4 - 12\, G^2 M^2 \left(\frac{d^2}{dr^2}\Phi(r)\right) c^4 r^3 - 6\, G M \Big(\frac{d}{dr}$$

$$\Phi(r)\Big)^2 c^4 r^4 + 5\, G M \left(\frac{d}{dr}\Phi(r)\right) c^6 r^3 + G M c^8 r^2 + 8\, G^3 M^3 \left(\frac{d^2}{dr^2}\Phi(r)\right) c^2 r^2$$

$$+ 12\, G^2 M^2 \left(\frac{d}{dr}\Phi(r)\right)^2 c^2 r^3 - 8\, G^2 M^2 \left(\frac{d}{dr}\Phi(r)\right) c^4 r^2 - 4\, G^2 M^2 c^6 r$$

$$- 8\, G^3 M^3 \left(\frac{d}{dr}\Phi(r)\right)^2 r^2 + 4\, G^3 M^3 \left(\frac{d}{dr}\Phi(r)\right) c^2 r + 4\, G^3 M^3 c^4\Big)$$

>

First,note that the G(t,t) term is not shown because it is zero.

We can now use the G(r,r) component to determine Phi(r). Again the equation is simple enough that dsolve is viable.

> *Grreqn* := *grcomponent*(*G*(*up*, *dn*), [*r*, *r*]) = 0;

$$Grreqn := \frac{2\left(-\left(\frac{d}{dr}\Phi(r)\right) c^2 r^2 + 2\, G M \left(\frac{d}{dr}\Phi(r)\right) r - G M c^2\right)}{r^3 c^4} = 0 \quad \textbf{(11)}$$

> *phiSolution* := *dsolve*(*Grreqn*, Phi(*r*));

$$phiSolution := \Phi(r) = \frac{c^2 \ln(r)}{2} - \frac{c^2 \ln(-c^2 r + 2\, G M)}{2} + \_C1 \quad \textbf{(12)}$$

This can now be substituted into the metric or globally defined. We can use the Maple rhs() function to extract the RHS. We then cut and paste this into a definition for Phi, omitting the _C1 which we can take to be zero.

> *rhs*(*phiSolution*);

$$\frac{c^2 \ln(r)}{2} - \frac{c^2 \ln(-c^2 r + 2\, G M)}{2} + \_C1 \quad \textbf{(13)}$$

> Phi(*r*) := $\frac{c^2 \ln(r)}{2} - \frac{c^2 \ln(-c^2 r + 2\, G M)}{2}$;

$$\Phi := r \mapsto \frac{c^2 \ln(r)}{2} - \frac{c^2 \ln(-c^2 r + 2\, G M)}{2} \quad \textbf{(14)}$$

> *grdisplay*(*g*(*dn*, *dn*));

For the spherical spacetime:

Covariant metric tensor

g(*dn*, *dn*)

(15)

$$g_{ab} = \begin{bmatrix} \frac{1}{1 - \frac{2\,G\,M}{c^2\,r}} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & -\frac{1}{\left(\mathrm{e}^{\frac{\frac{c^2 \ln(r)}{2} - \frac{c^2 \ln(-c^2 r + 2\,G\,M)}{2}}{c^2}}\right)^2} \end{bmatrix} \tag{15}$$

Making use of the generic Maple simplification routine we can collapse the exp(ln()) type terms:

> $gralterd(g(dn, dn), simplify)$

```
Component simplification of a GRTensorIII object:

Applying routine simplify to object g(dn,dn)
```

CPU Time $= 0.016$

For the spherical spacetime:

Covariant metric tensor

$g(dn, dn)$

$$g_{ab} = \begin{bmatrix} -\frac{c^2\,r}{-c^2\,r + 2\,G\,M} & 0 & 0 & 0 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2 \sin(\theta)^2 & 0 \\ 0 & 0 & 0 & \frac{c^2\,r - 2\,G\,M}{r} \end{bmatrix} \tag{16}$$

We have recovered the conventional form of the Schwarzshild metric in curvature coordinates.

>