Slow-Fast Decomposition of Motion of the Kapitza System

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The purpose of this document is to do the slow-fast decomposition of motion for the Kapitza pendulum system, for the high-driving, low-amplitude regime. In turn, we derive the effective potential U_{eff} and show that the vertical pendulum position $\phi = \pi$ is indeed stable in the long-term motion. We also find the condition for stability and show it holds for the example system. This is the continuation of setup done in a numerical Kapitza pendulum project for Physics 5300 (Theoretical Mechanics).

1 Slow-Fast Decomposition of Motion of the Kapitza System

From our derivation, we have the given Lagrangian equation of motion,

$$\ddot{\phi} = -\frac{g}{l}\sin(\phi) - \frac{a\nu^2}{l}\cos(\nu t)\sin(\phi).$$

We are going to investigate the regime called the high-driving, low-amplitude regime. For this, we assume a small driving amplitude a such that $a << l$ and a large driving frequency ν such that $\nu >> \omega_0$. In doing so, we propose a solution for $\phi(t)$ that *decomposes* the motion, being a superposition of a slowly varying motion $\Phi(t)$ (the *average* motion) and a rapidly oscillating component $\xi(t)$ (the *vibration*). Hence,

$$\phi(t) = \Phi(t) + \xi(t).$$

In this, the slow time scale is governed by the natural frequency of the pendulum $\omega_0 = \sqrt{g/l}$, while the fast time scale is governed by the driving frequency ν.

1.1 Fast Component ξ

We assume that the fast component $\xi(t)$ is small, mainly driven by the $\cos(\nu t)$ term. From our assumed form of the motion, we can see that

$$\ddot{\phi} = \ddot{\Phi} + \ddot{\xi} = -\frac{g}{l}\sin(\Phi + \xi) - \frac{a\nu^2}{l}\cos(\nu t)\sin(\Phi + \xi).$$

On the fast time scale, since Φ is slow, we can assume $\ddot{\Phi}$ is small. Likewise, we can say that the first term of $\ddot{\phi}$, giving us the approximation

$$\ddot{\xi} \approx -\frac{a\nu^2}{l}\cos(\nu t)\sin(\phi).$$

Also, on the fast time scale, ϕ is not varying as much, and thus we can approximate $\phi \approx \Phi$, yielding

$$\ddot{\xi} \approx -\frac{a\nu^2}{l}\cos(\nu t)\sin(\Phi).$$

We integrate this twice, but we ignore the integration constants, as we will absorb them into the slow motion Φ. This gives us

$$\boxed{\xi(t) \approx \frac{a}{l}\cos(\nu t)\sin(\Phi(t)).}$$

This is the **fast component** $\xi(t)$, an oscillation at frequency ν with amplitude proportional to $\frac{a}{l}\sin(\Phi)$.

1.2 Slow Component Φ

Recall,

$$\ddot{\phi} = \ddot{\Phi} + \ddot{\xi} = -\frac{g}{l}\sin(\Phi+\xi) - \frac{a\nu^2}{l}\cos(\nu t)\sin(\Phi+\xi).$$

We use the sine addition formula,

$$\sin(\Phi+\xi) = \sin(\Phi)\cos(\xi) + \cos(\Phi)\sin(\xi).$$

We use the fact that ξ is small, and use the Taylor approximations,

$$\begin{aligned}\sin(\xi) &\approx \xi\\ \cos(\xi) &= 1 - \frac{1}{2}\xi^2.\end{aligned}$$

This gives us,

$$\sin(\Phi+\xi) \approx \sin(\Phi) + \xi\cos(\Phi) - \frac{1}{2}\xi^2\sin(\Phi).$$

We substitute this back into our ODE, and average the entire equation over the fast time period $T = 2\pi/\nu$. The averaging operation we use is simply

$$\langle x(t)\rangle = \frac{1}{T}\int_0^T x(t)\ dt.$$

We recall that single $\sin(\nu t)$ and $\cos(\nu t)$ terms average to zero, or

$$\langle \sin(\nu t)\rangle = 0 \text{ and } \langle \cos(\nu t)\rangle = 0.$$

Squared terms average to 1/2, or

$$\langle \cos^2(\nu t)\rangle = \frac{1}{2}.$$

The averaged equation of motion will be

$$\langle\ddot{\Phi}\rangle + \langle\ddot{\xi}\rangle = -\frac{g}{l}\langle\sin(\Phi+\xi)\rangle - \frac{a\nu^2}{l}\langle\cos(\nu t)\sin(\Phi+\xi)\rangle.$$

Since we are averaging on the fast time scale for comparison, we can see that if ξ is purely oscillatory, then $\langle\ddot{\xi}\rangle = 0$, and since Φ is effectively constant on the fast scale, $\langle\ddot{\Phi}\rangle = \ddot{\Phi}$. The first averaged term on the RHS is,

$$-\frac{g}{l}\langle\sin(\Phi+\xi)\rangle \approx -\frac{g}{l}\langle\sin(\Phi) + \xi\cos(\Phi) - \frac{1}{2}\xi^2\sin(\Phi)\rangle.$$

From earlier, we see that $\xi \propto \cos(\nu t)$ and $\langle\xi\rangle = 0$, the average is simply

$$-\frac{g}{l}\langle\sin(\Phi+\xi)\rangle \approx -\frac{g}{l}\left(\sin(\Phi) - \frac{1}{2}\sin(\Phi)\langle\xi^2\rangle\right).$$

The second term on the RHS is

$$-\frac{a\nu^2}{l}\langle\cos(\nu t)\sin(\Phi+\xi)\rangle = -\frac{a\nu^2}{l}\langle\cos(\nu t)\left[\sin(\Phi) + \xi\cos(\Phi) - \frac{1}{2}\xi^2\sin(\Phi)\right]\rangle.$$

Since Φ is slow on the fast time scale and doesn't vary much, we can say that

$$\langle \cos(\nu t) \sin(\Phi) \rangle \approx \sin(\Phi) \langle \cos(\nu t) \rangle = 0.$$

Now, we substitute $\xi = \frac{a}{l} \cos(\nu t) \sin(\Phi)$, and see that

$$-\frac{a\nu^2}{l} \langle \cos(\nu t) \left[\sin(\Phi) + \xi \cos(\Phi) - \frac{1}{2} \xi^2 \sin(\Phi) \right] \rangle = -\frac{a\nu^2}{l} \langle \frac{a}{l} \cos^2(\nu t) \sin(\Phi) \cos(\Phi) \rangle$$
$$- \frac{a\nu^2}{l} \langle \frac{-a^2}{2l^2} \cos^3(\nu t) \sin^3(\Phi) \rangle.$$

The first term, we can pull out the Φ parts and average the cosine, getting $1/2$. The second term would have an average of a cosine cubed, which conveniently equals 0. Hence, we can say that

$$-\frac{a\nu^2}{l} \langle \cos(\nu t) \sin(\Phi + \xi) \rangle = -\frac{a\nu^2}{l} \frac{a}{l} \frac{\sin(2\Phi)}{2} \frac{1}{2} = -\frac{a^2\nu^2}{4l^2} \sin(2\Phi).$$

Hence, putting it all together, we claim that

$$\ddot{\Phi} = -\frac{g}{l} \sin(\Phi) - \frac{a^2\nu^2}{4l^2} \sin(2\Phi) + \frac{g}{2l} \sin(\Phi) \langle \xi^2 \rangle.$$

We ignore the higher order term to get that

$$\boxed{\ddot{\Phi} \approx -\frac{g}{l} \sin(\Phi) - \frac{a^2\nu^2}{4l^2} \sin(2\Phi).}$$

1.3 Deriving U_{eff}

To derive the effective potential to show that we have a stable equilibrium in the vertical position, we first recall that

$$\tau_{\text{eff}} = \text{Effective Force/Torque} = -\frac{dU_{\text{eff}}}{d\Phi}.$$

This equation only holds for a *conservative system*, which ours technically isn't, but the *averaged motion* can be approximated as a conservative system, seeing how our $\ddot{\Phi}$ does not depend explicitly on time. In general as well,

$$\tau_{\text{eff}} = I\ddot{\Phi} = ml^2\ddot{\Phi}.$$

Hence,

$$\ddot{\Phi} = \frac{1}{ml^2} \tau_{\text{eff}} = -\frac{1}{ml^2} \frac{dU}{d\Phi}.$$

This implies

$$U_{\text{eff}}(\Phi) = -ml^2 \int \ddot{\Phi} \, d\Phi = -ml^2 \int \left[-\frac{g}{l} \sin(\Phi) - \frac{a^2\nu^2}{4l^2} \sin(2\Phi) \right] \, d\Phi.$$

We split this into two parts. The first is

$$-ml^2 \int \left(-\frac{g}{l}\sin(\Phi)\right) \, d\Phi = -mgl\cos(\Phi) + C_1,$$

for an integration constant C_1. The second part is

$$-ml^2 \int \left(-\frac{a^2\nu^2}{4l^2}\sin(2\Phi)\right) \, d\Phi = -\frac{ma^2\nu^2}{8}\cos(2\Phi) + C_2,$$

for another integration constant C_2. Hence, the total effective potential is

$$U_{\text{eff}}(\Phi) = -mgl\cos(\Phi) - \frac{ma^2\nu^2}{8}\cos(2\Phi) + C,$$

where C is some combined constant. Using the identity that $\cos(2\Phi) = 1 - 2\sin^2(\Phi)$, and absorbing all the constants, we get the form

$$U_{\text{eff}}(\Phi) = -mgl\cos(\Phi) + \frac{ma^2\nu^2}{4}\sin^2(\Phi) + C',$$

where C' is the combined constant. We can drop this constant, it just shifts our potential, and we reserve the right as physicists to define our equilibrium potential. If we make sure the $\xi(t)$ is indeed incredibly small, than we can roughly say that $\phi(t) \approx \Phi(t)$, and hence we get our expected effective potential,

$$\boxed{U_{\text{eff}}(\phi) = -mgl\cos(\phi) + \frac{ma^2\nu^2}{4}\sin^2(\phi).}$$

1.4 Plots

Below is a figure for a system with $m = l = g = 1$ (ignoring units) and $a = 0.1$ with $\nu = 30$. This shows the U_{eff} plot vs. ϕ, and we see the expected trough near $\phi = \pi$, or the vertical position.

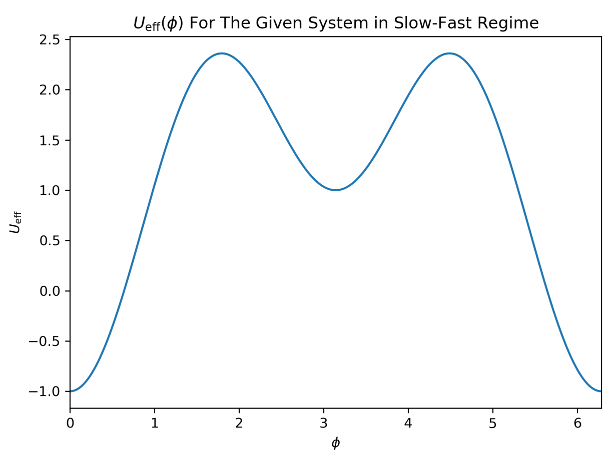

Figure 1: U_{eff} vs. ϕ Plot, Showing Vertical Stability.

1.5 Condition for Stability

To find the stability condition, we first set the effective potential derivative to 0.

$$\frac{dU_{\text{eff}}}{d\phi} = mgl\sin(\phi) + \frac{ma^2\nu^2}{4}\sin(2\phi) = \sin(\phi)\left[mgl + \frac{ma^2\nu^2}{2}\cos(\phi)\right] = 0.$$

There are clearly two solutions, where $\phi = 0$ (vertically downward) or $\phi = \pi$ (vertically upward/inverted). These two make $\sin(\phi) = 0$, satisfying the condition. These equilibria ϕ_{eq} are stable if the potential energy is at a local minimum, or

$$\left.\frac{d^2U_{\text{eff}}}{d\phi^2}\right|_{\phi=\phi_{\text{eq}}} > 0.$$

Here,

$$\frac{d^2U_{\text{eff}}}{d\phi^2} = \frac{d}{d\phi}\frac{dU_{\text{eff}}}{d\phi} = mgl\cos(\phi) + \frac{ma^2\nu^2}{2}\cos(2\phi).$$

We can test and see that $\phi_{\text{eq}} = 0$ yields that this result is always greater than 0 in this regime, but we leave that aside. For the inverted position, we see that

$$\left.\frac{d^2U_{\text{eff}}}{d\phi^2}\right|_{\phi=\pi} = -mgl + \frac{ma^2\nu^2}{2} = m\left[\frac{a^2\nu^2}{2} - gl\right] > 0.$$

This gets us our stability condition,

$$\boxed{a^2\nu^2 > 2gl.}$$

For our plotted system, we see that $a^2\nu^2 = 9$ and $2gl = 2$, hence the condition holds, which further shows that the inverted position is stable.

1.6 Addendum

An LLM (Gemini) was used in the making of this paper. The derivation is by-hand (and could be subject to errors). The LLM was used only to refine explanations and make explanations clear and concise.