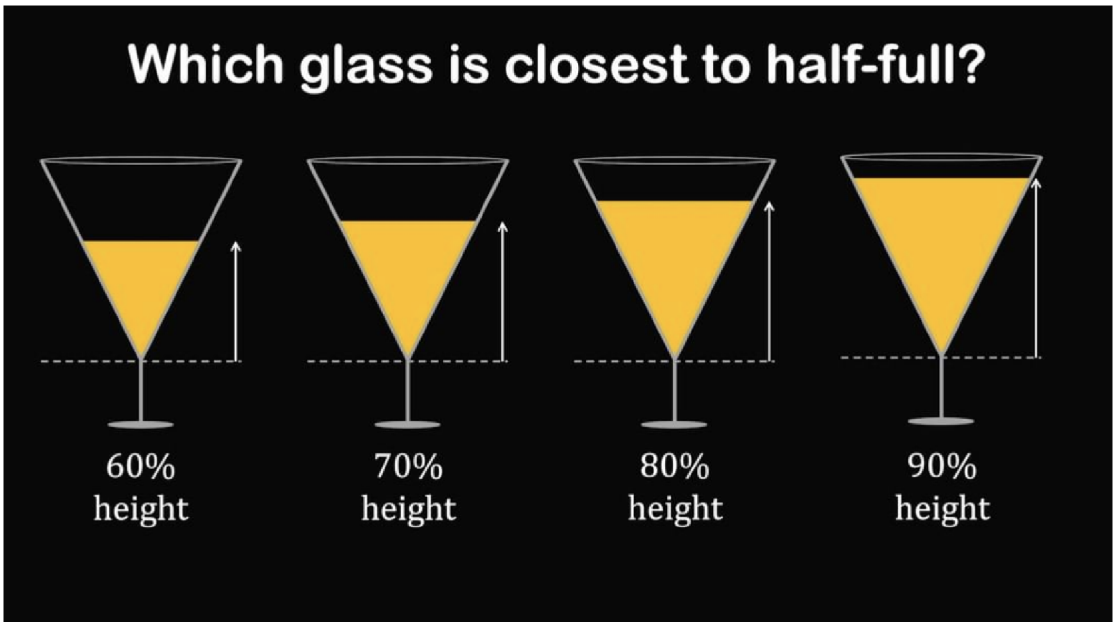
Random Questions #1

Which glass is closest to half-full?

Question from Presh Talwalkar on Instagram, linked here: https://www.instagram.com/p/C7X-UdNkuxjU/?igsh=OGZwYm85bWhtaTVy



Solution:

To solve this, we need to use the following equation(s):

$$V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

where r is the radius and h is the height of the cone.

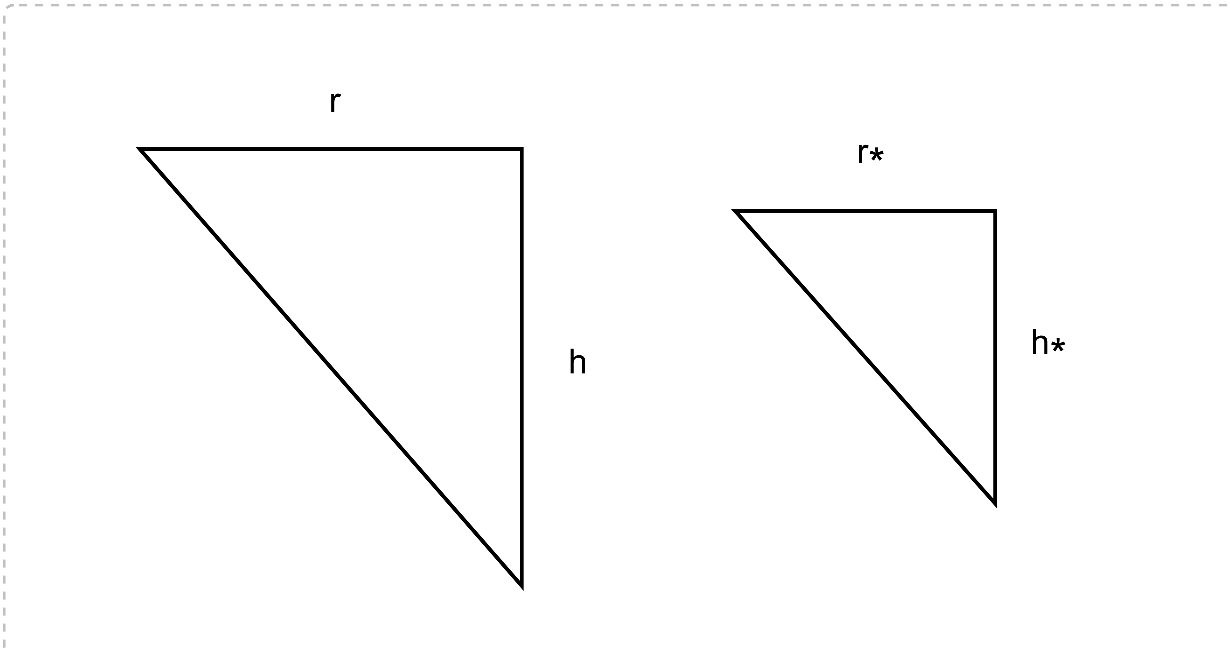
Let’s let the entire cone-shaped glass have a radius r and height h. Then the fractional cone comprised of the liquid has a radius $r^*$and height h^*. We want to know which glass is closest to half-full, or rather, which fractional cone’s volume is equal to half of the entire cone’s volume. Which means, we wish to solve the following equation:

$$\frac{1}{6}\pi r^2 h = \frac{1}{3}\pi (r^*)^2 h^*$$

Using similar triangles from the cross-section of the cone, we can find the following relationship:

$$\frac{r^*}{r} = \frac{h^*}{h}$$

In[41]:=
```
fullConeCrossSection = Triangle[{{0, 0}, {0, 1}, {-1, 1}}];
fractionalConeCrossSection = Triangle[{{1, 0}, {1, 0.5}, {0.5, 0.5}}];
Show[Graphics[{FaceForm[White], EdgeForm[Directive[Thick, Black]],
   EdgeLabels → {1 ↔ 2 → "Hey"}, fullConeCrossSection}], Graphics[{FaceForm[White],
   EdgeForm[Directive[Thick, Black]], fractionalConeCrossSection}]]
```



Now we can use Mathematica to solve this problem:

In[44]:= $\text{VCone} = \frac{1}{3}\pi\, r^2\, h;$

$\text{eq} = \frac{1}{2}\text{VCone} == \frac{1}{3}\pi\, \text{rs}^2\, \text{hs};$

In[46]:= $\text{rs} = \frac{\text{hs}\ r}{h};$

eq

Out[47]=

$$\frac{1}{6} h\,\pi\, r^2 == \frac{\text{hs}^3\,\pi\, r^2}{3\, h^2}$$

Now we have an equation in terms of h, r, h^*, and physical constants, where technically h and r should already be known. However, we know from the question stem the following equation:

$h^* = f\, h$

where f is the fraction of the original height in decimal form (for example, 60% height as shown in the question/image would equate to an f value of 0.60).

In[48]:=
```
hs = f h;
eq
```

Out[49]=

$$\frac{1}{6} h \pi r^2 == \frac{1}{3} f^3 h \pi r^2$$

In[52]:=
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eq = FullSimplify[eq, {r > 0, h > 0}]
```

Out[52]=

$$2 f^3 == 1$$

In[54]:=
```
NSolve[eq, f, Reals]
```

Out[54]=

```
{{f → 0.793701}}
```

From this, we can see that the exact fractional height that makes the glass half-full would be $f \approx 0.794$, or for the purposes of this question, $f \approx 0.80$, meaning that the solution to this question is “80% height”. We can confirm this by testing the volumes using the values $r = h = 1$.

In[55]:=

$$N\left[\frac{1}{6} \pi (1)^2 (1)\right]$$

$$N\left[\frac{1}{3} \pi (0.8)^2 (0.8)\right]$$

Out[55]=

```
0.523599
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Out[56]=

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0.536165
```

Or if we use the exact value of f:

In[57]:=

$$N\left[\frac{1}{6} \pi (1)^2 (1)\right]$$

$$N\left[\frac{1}{3} \pi (0.793701)^2 (0.793701)\right]$$

Out[57]=

```
0.523599
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Out[58]=

```
0.5236
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Solving the exact equation above yields the exact solution for f of $f = \frac{1}{\sqrt[3]{2}}$