# Random Questions #2

Question from mmathemagics on Instagram, linked here:
https://www.instagram.com/p/C-F2-uLNiny/?igsh=bzY0bnNsOHo5eWtn

Let $k \in \mathbb{R}$, $M(k) := \max_{x \in [-1,1]} \{ | 4x^2 - 4x + k | \}$, $\min_{k \in \mathbb{R}} M(k) = ?$

We know that $\forall\, k \in \mathbb{R}$ that $f(x) = 4x^2 - 4x + k$ is continuous on the interval $x \in [-1, 1]$, so by the Extreme Value Theorem we know that $\exists\, a, b \in [-1, 1]$ such that $f(a) \le f(x) \le f(b)\ \forall\, x \in [-1, 1]$.

We can use the first derivative test:

$f(x, k) = | 4x^2 - 4x + k |$

$\frac{\partial f}{\partial x} = \frac{4x^2 - 4x + k}{| 4x^2 - 4x + k |}(8x - 4) = 0$

$\Rightarrow (4x^2 - 4x + k)(8x - 4) = 0$

$\Rightarrow 4x^2 - 4x + k = 0 \text{ or } 8x - 4 = 0$

$\Rightarrow x = \frac{1}{2}\left(1 \pm \sqrt{1-k}\right) \text{ or } x = \frac{1}{2}$

So, we know by the Extreme Value Theorem, that the maximum value of $f(x)$ for some k is at $x = -1$ or $x = 1$ or $x = \frac{1}{2}$ or $x = \frac{1}{2}\left(1 \pm \sqrt{1-k}\right)$.

$f(-1, k) = | 4(-1)^2 - 4(-1) + k | = | 8 + k |$

$f\left(\frac{1}{2}, k\right) = \left| 4\left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right) + k \right| = \left| -1 + k \right|$

$f(1, k) = | 4(1)^2 - 4(1) + k | = | k |$

The other two options we will use Mathematica to help us evaluate.

In[ ]:=
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f[x_] := Abs[4 x^2 - 4 x + k];
FullSimplify[f[1/2 (1 + Sqrt[1 - k])]]
FullSimplify[f[1/2 (1 - Sqrt[1 - k])]]
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Out[ ]=
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0
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Out[ ]=
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0
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Both of these points end up with the function being equal to 0. Therefore we can also elect to ignore that sometimes those two points are extraneous, and do not fall in $x \in [-1, 1]$. We can ignore these since the function value must have an global maximum greater than 0 due to the absolute value, as seen by the three values above. Even if $k = 0$, then the maximum is $|8 + k| = 8$.

Now, $|8 + k| = |-1 + k|$ when $k = -3.5$. When $k > -3.5$, $|8 + k|$ yields the largest value, while $|-1 + k|$ yields the largest value for $k < -3.5$. The value $|k|$ will never yield the largest value as $\forall\, k \in \mathrm{R}$ one of the other two values must yield a larger value.

Therefore, we can define $M(k)$ piecewise as follows:
$M(k) = |-1 + k|, k \leq -3.5$
$M(k) = |8 + k|, k > -3.5$

As k decreases from -3.5, $M(k)$ increases, and likewise as k increases from -3.5, $M(k)$ increases. Therefore, the minimum for $M(k)$ must be exactly at $k = -3.5$.

$\therefore \min_{k \in \mathrm{R}} M(k) = |-1 - 3.5|$
$\Rightarrow \min_{k \in \mathrm{R}} M(k) = 4.5$ when $k = -3.5$