This is a worksheet that finds the recurrence relation in *Random meander model for links,* Lemma 4.2.

Maple worksheet
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See the paper:
Random meander model for links,

We first will define the Catalan numbers.

$$C := n \rightarrow \frac{\binom{2\,n}{n}}{n+1}$$

$$C := n \mapsto \frac{\binom{2\cdot n}{n}}{n+1} \tag{1}$$

Now we will define the overcount $\mathrm{O}_s(k)$ and the number of meander graphs with zero pierced circles. $E_s(0)$.

$$Oh := (s, k) \rightarrow \binom{2\,s-k-1}{k}\,C(s-k)^2$$

$$Oh := (s, k) \mapsto \binom{2\cdot s-k-1}{k}\cdot C(s-k)^2 \tag{2}$$

$$E := s \rightarrow \sum_{m=0}^{s} \left((-1)^m Oh(s, m) \right)$$

$$E := s \mapsto \sum_{m=0}^{s} (-1)^m \cdot Oh(s, m) \tag{3}$$

Now we will use the built in package to call Zeilberger's Algorithm.
$with(SumTools[Hypergeometric])$:

We call the function Zeilberger, with the term of $E_s(0)$ as first input, s, m as the second and third which just are the variables. And last is the variable x which denotes the shift operator.

$$Zz := Zeilberger\left((-1)^m Oh(s, m), s, m, x\right)$$

$$Zz := \left[(2 s^3 + 17 s^2 + 40 s + 16) x^3 + (-26 s^3 - 141 s^2 - 226 s - 81) x^2 + (-26 s^3 - 93 s^2 - 82 s - 30) x + 2 s^3 + s^2 - 8 s + 5, \left(2 \left(\frac{(130 s^3 + 513 s^2 + 464 s + 129) m^6}{(s^2 + 8 s + 16) (2 s + 1)} - \frac{(1148 s^4 + 6536 s^3 + 11797 s^2 + 7978 s + 1911) m^5}{(s^2 + 8 s + 16) (2 s + 1)} + \frac{(4190 s^5 + 31281 s^4 + 84087 s^3 + 101238 s^2 + 54607 s + 11385) m^4}{(s^2 + 8 s + 16) (2 s + 1)} - \frac{1}{(s^2 + 8 s + 16) (2 s + 1)} ((8072 s^6 + 74816 s^5 + 267346 s^4 + 470140 s^3 + 426777 s^2 + 190440 s + 34785) m^3) + \frac{1}{(s^2 + 8 s + 16) (2 s + 1)} ((8638 s^7 + 95947 s^6 + 428750 s^5 + 998077 s^4 + 1300923 s^3 + 944248 s^2 + 356331 s + 57246) m^2) - \frac{1}{(s^2 + 8 s + 16) (2 s + 1)} ((4860 s^8 + 63096 s^7 + 339337 s^6 + 987490 s^5 + 1694280 s^4 + 1743444 s^3 + 1045895 s^2 + 338444 s + 47904) m) + \frac{1}{(2 s + 1) (s + 4)} (1122 s^8 + 12219 s^7 + 56423 s^6 + 143292 s^5 + 216794 s^4 + 197991 s^3 + 105965 s^2 + 30834 s + 3960)\right) (-2 s + 2 m - 1) (-2 s + m) m (-1)^m \binom{2 s - m - 1}{m} \binom{2 s - 2 m}{s - m}^2 (s^2 + 8 s + 16) (2 s + 1)\right) \Big/ \left((-s - 2 + m)^3 (-s - 4 + m)^2 (-s + m - 1) (-s - 3 + m)^2 (-s + m) (s - m + 1)^2\right) \right] \quad (4)$$

The first item in this output is L and contains the polynomails $P_i(s)$.

$L := Zz[1]$

$$L := \left(2s^3+17s^2+40s+16\right)x^3+\left(-26s^3-141s^2-226s-81\right)x^2+\left(-26s^3-93s^2-82s-30\right)x+2s^3+s^2-8s+5 \quad \textbf{(5)}$$

And the second is the function $G(s,m)$.

$G := Zz[2]$

$$G := \left(2\left(\frac{\left(130s^3+513s^2+464s+129\right)m^6}{\left(s^2+8s+16\right)(2s+1)} - \frac{\left(1148s^4+6536s^3+11797s^2+7978s+1911\right)m^5}{\left(s^2+8s+16\right)(2s+1)} + \frac{\left(4190s^5+31281s^4+84087s^3+101238s^2+54607s+11385\right)m^4}{\left(s^2+8s+16\right)(2s+1)} - \frac{1}{\left(s^2+8s+16\right)(2s+1)}\left(\left(8072s^6+74816s^5+267346s^4+470140s^3+426777s^2+190440s+34785\right)m^3\right) + \frac{1}{\left(s^2+8s+16\right)(2s+1)}\left(\left(8638s^7+95947s^6+428750s^5+998077s^4+1300923s^3+944248s^2+356331s+57246\right)m^2\right) - \frac{1}{\left(s^2+8s+16\right)(2s+1)}\left(\left(4860s^8+63096s^7+339337s^6+987490s^5+1694280s^4+1743444s^3+1045895s^2+338444s+47904\right)m\right) + \frac{1}{(2s+1)(s+4)}\left(1122s^8+12219s^7+56423s^6+143292s^5+216794s^4+197991s^3+105965s^2+30834s+3960\right)\right)(-2s+2m-1)(-2s+m)\,m\,(-1)^m\binom{2s-m-1}{m}\binom{2s-2m}{s-m}^2\left(s^2+8s+16\right)(2s+1)\right) \Big/ \left((-s-2+m)^3(-s-4+m)^2(-s+m-1)(-s-3+m)^2(-s+m)\left(s-m+1\right)^2\right) \quad \textbf{(6)}$$

Maple has a built in function to verify this is correct.

$Verify\left((-1)^m Oh(s,m), 'Zz', s, m, x\right)$

$$true \quad \textbf{(7)}$$

This verifies Lemma 4.2, but we can be explicit and see that the sum is zero, as claimed.

First, we build the polynomails listed in $Zz[1]$:

$$P0 := s \to 2\,s^3 + s^2 - 8\,s + 5$$

$$P0 := s \mapsto 2 \cdot s^3 + s^2 - 8 \cdot s + 5 \tag{8}$$

$$P1 := s \to -26\,s^3 - 93\,s^2 - 82\,s - 30$$

$$P1 := s \mapsto -26 \cdot s^3 - 93 \cdot s^2 - 82 \cdot s - 30 \tag{9}$$

$$P2 := s \to -26\,s^3 - 141\,s^2 - 226\,s - 81$$

$$P2 := s \mapsto -26 \cdot s^3 - 141 \cdot s^2 - 226 \cdot s - 81 \tag{10}$$

$$P3 := s \to 2\,s^3 + 17\,s^2 + 40\,s + 16$$

$$P3 := s \mapsto 2 \cdot s^3 + 17 \cdot s^2 + 40 \cdot s + 16 \tag{11}$$

Next, we manually check that the recurrence sums to zero.

$$P0(s)E(s) + P1(s)E(s+1) + P2(s)E(s+2) + P3(s)E(s+3)$$

$$\frac{1}{\pi\,\Gamma(s+2)^2}\left(\left(2\,s^3 + s^2 - 8\,s + 5\right)\Gamma\left(s + \frac{1}{2}\right)^2 2^{4\,s}\,\mathrm{hypergeom}\left(\left[-s+1, -s-1, -s-1\right], \left[-2\,s+1, -s+\frac{1}{2}\right], \frac{1}{4}\right)\right) + \frac{1}{\pi\,\Gamma(s+3)^2}\left(16\left(-26\,s^3 - 93\,s^2 - 82\,s - 30\right)\Gamma\left(s + \frac{3}{2}\right)^2 \mathrm{hypergeom}\left(\left[-s, -s-2, -s-2\right], \left[-1-2\,s, -s-\frac{1}{2}\right], \frac{1}{4}\right) 2^{4\,s}\right) + \frac{1}{\pi\,\Gamma(s+4)^2}\left(256\left(-26\,s^3 - 141\,s^2 - 226\,s - 81\right)\Gamma\left(s + \frac{5}{2}\right)^2 2^{4\,s}\,\mathrm{hypergeom}\left(\left[-s-3, -s-3, -s-1\right], \left[-3-2\,s, -\frac{3}{2}-s\right], \frac{1}{4}\right)\right) + \frac{1}{\pi\,\Gamma(s+5)^2}\left(4096\left(2\,s^3 + 17\,s^2 + 40\,s + 16\right)\Gamma\left(s + \frac{7}{2}\right)^2 2^{4\,s}\,\mathrm{hypergeom}\left(\left[-s-4, -s-4, -s-2\right], \left[-2\,s-5, -s-\frac{5}{2}\right], \frac{1}{4}\right)\right) \tag{12}$$

$$\xrightarrow{\text{assuming integer}}$$

$$0 \tag{13}$$

This completes the Maple file showing the recurrence is true.