Chapter 7
Sampling

Every time we process data, we actually process *sampled* data. The theory underlying *sampling*, which has relevant consequences on practical implementations, makes up the topic of this note.

Crucial terms

Before starting with our discussion, it is suitable to define and name some important quantities:

T	sampling period;	
f_s	sampling frequency or rate,	$f_s = 1/T$;
ω_s	sampling (angular) frequency,	$\omega_s = 2\pi f_s = 2\pi/T$;
$f_{\mathtt{Ny}}$	Nyquist frequency,	$f_{\mathtt{Ny}} = f_s/2$;
$\omega_{\mathtt{Ny}}$	Nyquist (angular) frequency,	$\omega_{\mathtt{Ny}} = \omega_s/2$;
$(-f_{\mathtt{Ny}}, f_{\mathtt{Ny}})$	Nyquist band (in terms of frequency)	does not include $\pm f_{\mathtt{Ny}}$;
$(-\omega_{\mathtt{Ny}}, \omega_{\mathtt{Ny}})$	Nyquist band (in terms of angular frequency)	does not include $\pm\omega_{\mathtt{Ny}}$.

Parentheses about "angular" refer to the fact that most physicists use the same word to identify f or ω without falling in disarray.

Sampling a signal

Let $f(t)$ be a function of time, i.e. a signal. For example, $f(t)$ could be a voltage. Given a sampling period T, we measure $f(t)$ at integer multiples of

T: $t = nT$. In this way, we obtain a so-called sequence $\{f(nT)\}$ of values (see Fig. 7.1).

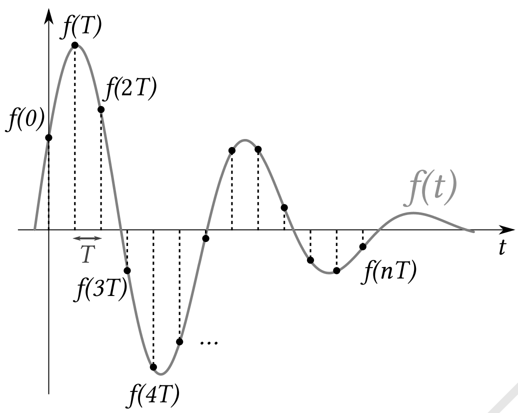

Figure 7.1: Sampling a signal $f(t)$ with a sampling period T.

7.1 Poisson sum formula and related identities

We now consider the Fourier transform of $f(t)$: $\mathcal{F}[f(t)] = \tilde{F}(\omega)$.

Given $\omega_s = 2\pi/T$, let us define a *folding* $\overline{F}(\omega)$ as

$$\overline{F}(\omega) = \sum_{n=-\infty}^{\infty} \tilde{F}(\omega + n\omega_s)\,. \tag{7.1}$$

The folding $\overline{F}(\omega)$ is by construction periodic in the frequency space, with period ω_s. Consequently, $\overline{F}(\omega)$ can be expressed as a Fourier series, whose frequencies are given by $2\pi k/\omega_s$, i.e. by kT.

$$\begin{cases} \overline{F}(\omega) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{ikT\omega} \quad \text{where} \\ a_k = \displaystyle\frac{1}{\omega_s}\int_{-\omega_s/2}^{\omega_s/2} \overline{F}(\omega) e^{-ikT\omega} \mathrm{d}\omega\,. \end{cases}$$

By using the definition of $\overline{F}$ given by Eq. (7.1), and swapping the series and the integral, the coefficients a_k can be computed as

$$\begin{aligned} a_k &= \frac{1}{\omega_s}\sum_{n=-\infty}^{\infty}\int_{-\omega_s/2}^{\omega_s/2}\tilde{F}(\omega+n\omega_s)e^{-ikT\omega}\mathrm{d}\omega = \\ &= \frac{1}{\omega_s}\sum_{n=-\infty}^{\infty}\int_{-\omega_s/2+n\omega_s}^{\omega_s/2+n\omega_s}\tilde{F}(\omega')e^{-ikT\omega'}e^{ikTn\omega_s}\mathrm{d}\omega' = \\ &= \frac{2\pi}{\omega_s}\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{F}(\omega')e^{-ikT\omega'}\mathrm{d}\omega' = \\ &= \frac{2\pi}{\omega_s}f(kT)\,. \end{aligned}$$

In the step from the first to the second row the substitution $\omega = \omega' - n\omega_s$ was carried out, while in the step from the second to the third row we took into account the fact that $e^{ikTn\omega_s} = e^{2\pi ik} = 1, \forall k$. Replacing the last result within the expression of $\overline{F}(\omega)$ as a Fourier series yields

$$\overline{F}(\omega) = \frac{2\pi}{\omega_s}\sum_{k=-\infty}^{\infty}f(kT)e^{ikT\omega} = T\sum_{k=-\infty}^{\infty}f(kT)e^{ikT\omega}\,. \tag{7.2}$$

The last equation is known as *Poisson sum formula*.

An interesting corollary of the Poisson sum formula is provided by choosing $f(t) = 1$. In this case, $\tilde{F}(\omega) = 2\pi\delta(\omega)$, so that

$$\sum_{n=-\infty}^{\infty}\delta(\omega+n\omega_s) = \frac{1}{\omega_s}\sum_{k=-\infty}^{\infty}e^{ikT\omega}\,.$$

Taking the inverse Fourier transform of this last expression yields

$$\sum_{k=-\infty}^{\infty}\delta(t-kT) = \frac{1}{T}\sum_{n=-\infty}^{\infty}e^{in\omega_s t}\,.$$

These last two expressions show how infinite series of delta functions are equivalent to infinite sums of rotating phasors. One could almost say that an infinite sum of bumps is actually equal to an infinite sum of wheels.

A final, important expression that will be soon very useful is derived by considering the inverse Fourier transform of the folding $\overline{F}(\omega)$: by exploiting

Eq. (7.2) one gets

$$\mathcal{F}^{-1}\left[\overline{F}(\omega)\right] = T \sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT)\,. \tag{7.3}$$

7.2 Nyquist band

The Nyquist band defined above has the following mathematical representation, which we refer to as the *Nyquist window*:

$$W_{\pi/T}(\omega) = \begin{cases} 1 & \text{if } |\omega| < \dfrac{\pi}{T} \\ 0 & \text{if } |\omega| \geqslant \dfrac{\pi}{T} \end{cases}. \tag{7.4}$$

It is interesting to consider the inverse Fourier transform of this frequency interval. One gets

$$\mathcal{F}^{-1}\left[W_{\pi/T}(\omega)\right] = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{-i\omega t}\mathrm{d}\omega = \frac{1}{2\pi}\frac{e^{-i\omega t}}{-it}\bigg|_{-\pi/T}^{\pi/T} = \frac{\sin\left(\frac{\pi t}{T}\right)}{\pi t} = \frac{1}{T}\mathtt{sinc}\left(\frac{\pi t}{T}\right),$$

where sinc(x) is defined as $\sin(x)/x$ for $x \neq 0$ and 1 for $x = 0$.

7.3 Sampling theorem

We now suppose to have a signal $f(t)$ which is $\frac{\pi}{T}$-bandlimited ($\frac{\pi}{T}$-BL), i.e. its Fourier transform $\tilde{F}(\omega)$ is zero outside the frequency band defined in Eq. (7.4).

For such a signal, we can write

$$\tilde{F}(\omega) = W(\omega)\overline{F}(\omega)\,, \tag{7.5}$$

i.e. the Fourier transform $\tilde{F}(\omega)$ can be reconstructed by the folding $\overline{F}(\omega)$ by multiplying it by the Nyquist window. Please note that this operation is allowed only because $f(t)$ is π/T-BL.

By taking the inverse Fourier transform of Eq. (7.5) we get

$$f(t) = \mathcal{F}^{-1}\left[W(\omega)\overline{F}(\omega)\right] = \left[\frac{1}{T}\texttt{sinc}\left(\frac{\pi t}{T}\right)\right] * \left[T\sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT)\right],$$

where the well-known theorem on the Fourier transform of a convolution is used. By finally computing the convolution between the sinc and the delta functions we get

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT)\,\texttt{sinc}\left(\pi\frac{t-kT}{T}\right). \tag{7.6}$$

This last expression corresponds to the *sampling theorem*: to sum up, provided that a signal is $\frac{\pi}{T}$-BL, the samples $f(kT)$ obtained by sampling $f(t)$ at regular intervals T allow to perfectly reconstruct the original signal $f(t)$ by applying Eq. (7.6).

7.4 Aliasing

Let us now move backwards and suppose that we have sampled a signal $f(t)$ without asking ourselves whether it was $\frac{\pi}{T}$-BL. If we try to reconstruct the signal starting from the samples $\{f(kT)\}$ we get

$$r(t) = \sum_{k=-\infty}^{\infty} f(kT)\texttt{sinc}\left(\pi\frac{t-kT}{T}\right).$$

We first note that, if we sample $r(t)$ at times given by $t = nT$, we get:

$$r(nT) = \sum_{k=-\infty}^{\infty} f(kT)\texttt{sinc}\left(\pi\frac{nT-kT}{T}\right) = \sum_{k=-\infty}^{\infty} f(kT)\delta_{n,k} = f(nT),$$

where we have exploited the fact that $\texttt{sinc}(n\pi) = 0$ if n is a nonzero integer number and $\texttt{sinc}(0) = 1$. So the reconstruction seems to work at least at the sampling times. However, ...

The Fourier transform of $r(t)$ is:

$$\tilde{r}(\omega) = W(\omega)\mathcal{F}\left[T\sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT)\right] = W(\omega)\overline{F}(\omega)\,.$$

Now, if $f(t)$ were indeed $\frac{\pi}{T}$-BL, we would get

$$\begin{aligned} \tilde{r}(\omega) &= W(\omega)\overline{F}(\omega) = \tilde{F}(\omega) \quad \text{and thus} \\ r(t) &= f(t)\,. \end{aligned}$$

The other possibility is that $f(t)$ is not $\frac{\pi}{T}$-BL. In this case,

$$\begin{aligned} \tilde{r}(\omega) &= W(\omega)\overline{F}(\omega) \neq \tilde{F}(\omega) \quad \text{so that} \\ r(t) &\neq f(t)\,. \end{aligned} \tag{7.7}$$

In other words, $r(t)$ is not $f(t)$! The reconstruction does not work the way we wished, it works in a different way or, in Latin, *alias*!

A graphical representation of a correct reconstruction and a wrong one - also known as *aliasing* - is shown in Fig. 7.2. In the case of a signal that is not π/T-BL (Fig. 7.2 left), the operations of folding and of multiplication times the Nyquist window yield a spectrum $W(\omega)\overline{F}(\omega)$ which is different from the original one, $\tilde{F}(\omega)$. Indeed, when building the folding, higher-frequency contributions end up corrupting the Fourier spectrum within the Nyquist window, so that the multiplication times the Nyquist window does not restore the original spectrum. This inconvenient does not take place if the signal is π/T-BL.

It is worth noting that, in the case of a non-π/T-BL $f(t)$, the reconstructed function $r(t)$ is π/T-BL.

We also finally note that the sinc function makes up the "perfect" interpolating *kernel* function, simply because its Fourier transform exactly coincides with the Nyquist window. One might be tempted to use a different $k(t)$ such that $k(0) = 1$ and $k(nT) = 0\;\forall n \neq 0$. For example, one might consider the linear interpolation

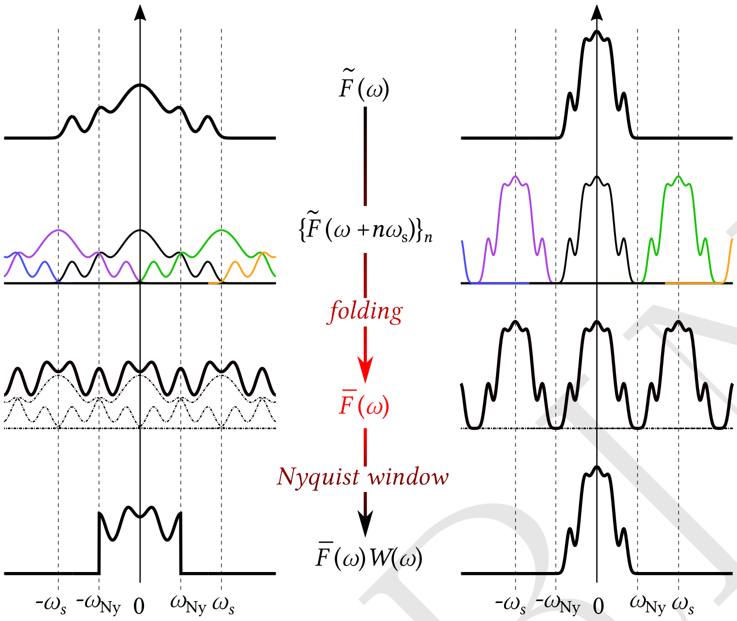

Figure 7.2: Folding and windowing of a non-$\frac{\pi}{T}$-BL signal (left) and a $\frac{\pi}{T}$-BL one.

kernel, which is given by

$$k(t) = \begin{cases} \dfrac{T - |t|}{T} & \text{for } |t| \leqslant T \\ 0 & \text{for } |t| > T\,, \end{cases}$$

and whose Fourier transform is

$$\begin{aligned} \tilde{k}(\omega) &= \frac{1}{T}\int_{-T}^{0}(T+t)e^{i\omega t}\mathrm{d}t + \frac{1}{T}\int_{0}^{T}(T-t)e^{i\omega t}\mathrm{d}t = \\ &= \frac{2}{T}\int_{0}^{T}(T-t)\cos(\omega t)\mathrm{d}t = \frac{2}{\omega^2 T}\int_{0}^{T}(\omega T - \omega t)\cos(\omega t)\,\mathrm{d}(\omega t) = \\ &= \frac{2}{\omega^2 T}\Big[\omega T\sin(x) - \cos(x) - x\sin(x)\Big]_0^{\omega T} = T\frac{\sin^2(\omega T/2)}{(\omega T/2)^2} = T\,\mathtt{sinc}^2\left(\frac{\omega T}{2}\right). \end{aligned}$$

Clearly, $\tilde{k}(\omega) \neq W(\omega)$.

7.5 Aliasing of a sinusoidal function

An interesting question concerns what happens in the special case that a sinusoidal signal violates the $\frac{\pi}{T}$-bandlimitedness. We consider a signal $f(t)$ of frequency

$\omega_0 > 0$:

$$f(t) = \frac{1}{\pi} \cos\left(\omega_0 t - \varphi\right)$$

that has a Fourier spectrum $\tilde{F}(\omega)$ equal to

$$\tilde{F}(\omega) = e^{i\varphi} \delta\left(\omega - \omega_0\right) + e^{-i\varphi} \delta\left(\omega + \omega_0\right) .$$

The signal is then sampled with a sampling (angular) frequency ω_s. The folding of the spectrum is given by

$$\overline{F}(\omega) = \sum_{n=-\infty}^{\infty} \left[e^{i\varphi} \delta\left(\omega - \omega_0 + n\omega_s\right) + e^{-i\varphi} \delta\left(\omega + \omega_0 + n\omega_s\right) \right] .$$

It is worth expressing the signal's frequency ω_0 in terms of the sampling frequency ω_s as follows: starting from

$$\left(\omega_0 + \frac{\omega_s}{2}\right) \frac{1}{\omega_s} = m + r \,,$$

where $m \in \mathbb{N}$ and r is a real number, $r \in [0, 1[$, ω_0 is given by

$$\omega_0 = \omega_s \left(m + r - \frac{1}{2}\right) .$$

The folding of the spectrum is thus

$$\overline{F}(\omega) = \sum_{n=-\infty}^{\infty} \left\{ e^{i\varphi} \delta \left[\omega - \omega_s \left(m + r - \frac{1}{2} \right) + n\omega_s \right] + e^{-i\varphi} \delta \left[\omega + \omega_s \left(m + r - \frac{1}{2} \right) + n\omega_s \right] \right\} ,$$

and the windowing with the Nyquist window yields

$$
\begin{aligned}
\overline{F}(\omega)W(\omega) &= \sum_{n=-\infty}^{\infty}\left\{e^{i\varphi}\delta\left[\omega-\omega_s\left(r-\frac{1}{2}\right)+\omega_s(n-m)\right]+\right.\\
&\qquad\left.+e^{-i\varphi}\delta\left[\omega+\omega_s\left(r-\frac{1}{2}\right)+\omega_s(n+m)\right]\right\}W(\omega)=\\
&= \sum_{n=-\infty}^{\infty}\left\{e^{i\varphi}\delta\left[\omega-\omega_s\left(r-\frac{1}{2}\right)+\omega_s(n-m)\right]\delta_{n,m}+\right.\\
&\qquad\left.+e^{-i\varphi}\delta\left[\omega+\omega_s\left(r-\frac{1}{2}\right)+\omega_s(n+m)\right]\delta_{n,-m}\right\}W(\omega)=\\
&= e^{i\varphi}\delta\left[\omega-\omega_s\left(r-\frac{1}{2}\right)\right]+e^{-i\varphi}\delta\left[\omega+\omega_s\left(r-\frac{1}{2}\right)\right]W(\omega)\,.
\end{aligned}
$$

In other words, the sinusoidal signal reconstructed out of the sampled sequence turns out to be a sinusoidal signal given by

$$
a(t)=\frac{1}{2\pi}e^{i\varphi}e^{-i\omega_a t}+\frac{1}{2\pi}e^{-i\varphi}e^{i\omega_a t}=\frac{1}{\pi}\cos\left(\omega_a t-\varphi\right)\,,
$$

whose frequency is

$$
\omega_a=\omega_s\left(r-\frac{1}{2}\right)=\omega_0-\left\lfloor\frac{1}{2}+\frac{\omega_0}{\omega_s}\right\rfloor\omega_s\,,
$$

where $\lfloor x\rfloor$ represents the greatest integer less than or equal to x. This *aliasing* frequency ω_a can also be negative despite ω_0 being positive.

We now consider three relevant cases:

- $\omega_0<\frac{\omega_s}{2}$; then, $\lfloor\frac{\omega_0}{\omega_s}+\frac{1}{2}\rfloor=0$ and thus $\omega_a=\omega_0$: nothing is changed by the sampling process;

- $\frac{\omega_s}{2}\leqslant\omega_0<\omega_s$; then, $\lfloor\frac{\omega_0}{\omega_s}+\frac{1}{2}\rfloor=1$ and thus $\omega_a=\omega_0-\omega_s<0$. Consequently, the reconstruction fails, yielding a sinusoidal signal with a different (and opposite in sign) frequency. Now suppose that we are taking snapshot of a dot moving along a circumference, so that

$$\begin{aligned} x(t) &= \cos(2\pi f_0 t)\,, \\ y(t) &= \sin(2\pi f_0 t)\,. \end{aligned}$$

If the rate f_s at which the snapshots are taken (the sampling rate) is such that $\frac{f_s}{2} \leqslant f_0 < f_s$, then the reconstructed motion of the dot is described by

$$\begin{aligned} x'(t) &= \cos[2\pi(f_s - f_0)t]\,; \\ y'(t) &= -\sin[2\pi(f_s - f_0)t]\,. \end{aligned}$$

This is the notorious *back-rotating wheel* effect;

- $\omega_0 = \omega_s$; then, $\lfloor \frac{\omega_0}{\omega_s} + \frac{1}{2} \rfloor = 1$ and $\omega_a = 0$: the reconstructed signal is constant.

7.6 Does the Nyquist band include the Nyquist frequency?

One might argue that the Nyquist frequency could belong to the Nyquist band, so that the $\frac{\pi}{T}$-bandlimitedness concerns spectral components that are, in absolute value, strictly greater than $\frac{\pi}{T}$. It is not difficul to prove, via a *reductio ad absurdum*, that the Nyquist band does not include the Nyquist frequency.

Let us consider an input signal given by

$$f(t) = A\cos\left(\frac{\pi}{T}t - \varphi\right)\,.$$

The sequence of sampled values is then given by

$$f(nT) = A\cos(n\pi - \varphi) = (-1)^n A\cos\varphi\,.$$

Two different signals having the same product $A\cos\varphi$ produce the same sequence and any reconstruction cannot therefore reproduce both of them. QED.