# Chapter 6
# Oscillators





Oscillators are a main topic in electronics. The reason is that they are a main topic in everyone's life: how could we live without clocks, computers, GPS, and so on...? One might argue that, apart from the last century, people were living pretty well without all this stuff. However, it is undeniable that everyone holds a nice oscillator, located within the thorax, oscillating at approximately 1 Hz and, hopefully, working for (at least) three billion seconds...

This chapter deals with the physics of self-sustaining oscillators, i.e. systems that keep on oscillating without damping, provided that they are fed with energy. We start from the beautiful example provided by the quadrature oscillator that, by relying on three op-amps, generates two sinusoidal signals in quadrature. Thereupon, we derive the Van der Pol equation, which underlies most oscillators, and we finally address the two most famous electronic oscillators: the Wien bridge oscillator and the relaxation oscillator.

## 6.1 Quadrature oscillator

Let us consider the circuit in Fig. 6.1, and let us assume that the three op-amps are ideal. By neglecting the parts colored in blue and red--and assuming for a while the non-inverting input of the rightmost op-amp to be connected to ground--we have a circuit made of two integrators in series, where the inverted output of the second one is fed back into the input of the first one. It is straightforward to see that

$$V_1 = -\frac{1}{\tau}\int V_0 \mathrm{d}t \qquad \texttt{i.e.} \quad V_0 = -\tau\dot{V}_1 \tag{6.1a}$$

$$V_2 = -\frac{1}{\tau}\int V_1 \mathrm{d}t \qquad \texttt{i.e.} \quad V_1 = -\tau\dot{V}_2\,, \tag{6.1b}$$

$$V_{\text{out}} = -V_2\,. \tag{6.1c}$$

By taking into account that, due to the connecting wire, $V_0 = V_{\text{out}}$, one gets the following equation for, for example, $V_1$:

$$\ddot{V}_1 + \omega_s^2 V_1 = 0\,,$$

where $\omega_s = 1/\tau$. This last equation corresponds to the well-known differential equation describing a harmonic oscillator. Consequently, the integrators' outputs $V_1$ and $V_2$ are expected to oscillate harmonically with a relative $\pi/2$ phase shift, i.e. *"in quadrature"*.

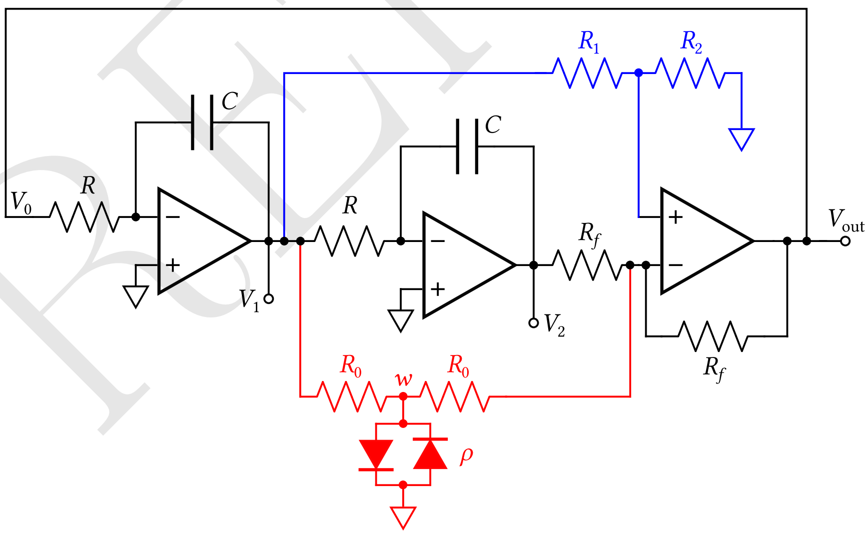


Figure 6.1: Quadrature oscillator circuit built by means of two integrators.

Unfortunately, in the real world, due to the system's intrinsic losses, an oscillation would eventually fade out[1]. To describe the related damping term,

[1] A main source of this intrinsic losses can be shown to be due to the frequency response of the op-amps.

let us consider the blue part of the circuit (remember to disconnect the non-inverting input of the rightmost op-amp from ground!). Choosing two finite values for $R_1$ and $R_2$ allows to explicitly add a controllable damping term that can be parametrized by $r' = R_2/(R_1 + R_2)$; typically, $r' \ll 1$. The damping contributes to the output voltage $V_{\text{out}}$ with a term given by $2r'V_1$, so that Eq. (6.1c) above is replaced by the following one:

$$V_{\text{out}} = 2r'V_1 - V_2 \,. \tag{6.2}$$

Consequently, the equation for $V_1$ becomes

$$\ddot{V}_1 + 2r\omega_s\dot{V}_1 + \omega_s^2 V_1 = 0 \,.$$

The intrinsic losses described above can be parametrized by adding a suitable positive constant value $r_0$ to $r'$ and setting $r = r_0 + r'$; the last equation, which contains $r$ instead of $r'$, already takes into account this fact.

In order to sustain the oscillatory behavior, an "anti-damping" (enhancing) term is necessary. This term can be implemented by means of an additional network, which in Fig. 6.1 is colored in red. This network contains a diode pair that, as shown below, provides a voltage-dependent resistance: we will soon see that these nonlinear elements are crucial in order to get a self-oscillating circuit. The contribution provided by this network to $V_{\text{out}}$ can be computed as follows.

Let $\rho$ be the resistance of the diode pair and $w$ the voltage across it. The Thévenin equivalent of the network given by $\rho$ and the leftmost $R_0$ is a generator of voltage $V_1\rho/(R_0+\rho)$ in series with an impedance given by the parallel $R_0 \parallel \rho$. Because the inverting input of the third op-amp is almost at virtual ground due to $r' \ll 1$, and taking into account the rightmost resistance $R_0$, the contribution to $V_{\text{out}}$ due to the $R_0$-$\rho$-$R_0$ network is:

$$-R_f \frac{V_1\rho}{R_0 + \rho} \frac{1}{R_0 + R_0 \parallel \rho} = -\frac{R_f}{R_0} \frac{V_1}{2 + R_0/\rho} = -\frac{R_f}{R_0} w \,,$$

where the voltage across the diode pair $w$ is

$$w = \frac{V_1 \rho}{R_0 + \rho} \frac{R_0}{R_0 + R_0 \parallel \rho} = \frac{V_1}{2 + R_0/\rho} \, . \tag{6.3}$$

It is now necessary to find an expression of the resistance $\rho$ as a function of the voltage $w$. According to the Shockley diode equation, the current through the diode that sinks the current to ground is $I_s \left(e^{\beta w/n} - 1\right)$, where $I_s$ is the saturation current, $w$ the applied voltage, $\beta$ the temperature factor[2] and $n$ the ideality factor ($n \sim 1 \div 2$). The current through the diode that sources the current from ground is instead $-I_s \left(e^{-\beta w/n} - 1\right)$. Adding the two currents leads to

$$I = I_s \left(e^{\beta w/n} - e^{-\beta w/n}\right) = 2 I_s \sinh\left(\frac{\beta w}{n}\right) ,$$

so that the diode pair resistance is

$$\rho = \frac{w}{2 I_s \sinh\left(\frac{\beta w}{n}\right)} \, .$$

Inserting this last expression in Eq. (6.3) gives

$$w = \frac{V_1}{2 + \frac{2 I_s R_0}{w} \sinh\left(\frac{\beta w}{n}\right)} ,$$

which can be rewritten as

$$2w + 2 I_s R_0 \sinh\left(\frac{\beta w}{n}\right) = V_1 \, .$$

Expanding $\sinh$ to the third order provides

$$2w + 2 I_s R_0 \left(\frac{\beta w}{n} + \frac{\beta^3 w^3}{6 n^3}\right) + \mathcal{O}(w^5) = V_1 \, . \tag{6.4}$$

It can be easily shown that[3], if $y = ax + bx^3$ and $x \ll 1$, the inverse function

[2] $\beta = e/(k_B T) \approx 40\,\mathrm{V}^{-1}$ at $T = 300\,\mathrm{K}$

[3] It is sufficient to try with $x = cy + dy^3 + \mathcal{O}(y^5)$ by replacing it in the equation $y = ax + bx^3$. Identity holds provided that $c = 1/a$ and $d = -b/a^4$.

is also approximately a cubic function, i.e. $x = y/a - by^3/a^4 + \mathcal{O}(y^5)$.

Consequently, from Eq. (6.4) it follows that the voltage $w$ across the diode pair is itself a cubic function of the voltage at the first op-amp's output $V_1$:

$$w \cong \chi V_1 - \xi V_1^3 \,,$$

where $\chi = \frac{1}{2}(1+I_s R_0 \beta/n)^{-1}$ and $\xi = I_s R_0 \beta^3/(3n^3\chi^4)$. The additional anti-damping term is thus $-\frac{R_f}{R_0} V_1 \left(\chi - \xi V_1^2\right)$ so that Eq. (6.2) becomes

$$V_{\mathtt{out}} \cong -V_2 - V_1 \left[2r + \frac{R_f}{R_0}\left(\chi - \xi V_1^2\right)\right] .$$

By evaluating the time derivative of this last expression and remembering that, according to Eqs. (6.1a, 6.1b), $V_{\mathtt{out}} = V_0 = -\tau \dot{V}_1$ and $-\dot{V}_2 = V_1/\tau$, we get:

$$-\tau \ddot{V}_1 \cong \frac{V_1}{\tau} - \dot{V}_1 \left[2r + \frac{R_f}{R_0}\left(\chi - 3\xi V_1^2\right)\right] ,$$

i.e.

$$\ddot{V}_1 - \omega_s \dot{V}_1 \left[\left(\frac{R_f}{R_0}\chi - 2r\right) - 3\frac{R_f}{R_0}\xi V_1^2\right] + \omega_s^2 V_1 \cong 0 \,.$$

Let us suppose $\frac{R_f}{R_0}\chi > 2r$. In addition, let

$$\begin{aligned} \mu &\equiv \omega_s \left(\frac{R_f}{R_0}\chi - 2r\right) . \\ V_{\mathtt{ref}} &\equiv \left[\left(\frac{R_f}{R_0}\chi - 2r\right) / \left(3\frac{R_f}{R_0}\xi\right)\right]^{1/2} , \end{aligned}$$

The last differential equation can be finally written as

$$\ddot{V}_1 - \frac{\mu}{V_{\mathtt{ref}}^2} \dot{V}_1 \left(V_{\mathtt{ref}}^2 - V_1^2\right) + \omega_s^2 V_1 \cong 0 \,. \tag{6.5}$$

Now, if the amplitude $V_1$ gets a starting value $V_1(t=0)$ small enough so that the factor that multiplies $\dot{V}_1$ is still positive, Eq. (6.5) describes an oscillating system with anti-damping and thus an increasing amplitude. However, as soon as $V_1^2$ becomes so large that the bracket takes on, at least occasionally, negative values, the term containing $\dot{V}_1$ becomes a damping one. Could such a mechanism lead to self-oscillation? We answer this question in the following section, but before discussing the solution of Eq. (6.5), where does the system takes its energy from? That's easy: from the power supply feeding the op-amps.

## 6.2 Van der Pol oscillator

For the sake of simplicity, let us set $x \equiv V_1/V_{\mathtt{ref}}$ and replace $\cong$ with $=$. Equation (6.5) is then equivalent to

$$\ddot{x} - \mu\left(1 - x^2\right)\dot{x} + \omega_s^2 x = 0\,. \tag{6.6}$$

This last differential equation is known as Van der Pol equation. The parameter $\mu$, having dimension s$^{-1}$, is typically much smaller than $\omega_s$. The evolution of the Van der Pol oscillator can be shown to converges to a *limit cycle*: whatever the initial conditions, the system eventually reaches a steady-state oscillation regime, as shown in Fig. 6.5. The physics underlying this behavior is relatively easy. If one starts with a small, nonzero amplitude ($x = 0$ is a stationary solution of the equation!) the term proportional to $\dot{x}$ is positive, thus leading to an amplitude increase. It is only when $x$ overcomes 1 that the same term acts as a drag force.

In order to assess the solution of the Van der Pol equation we proceed according to a *perturbative* approach. We assume that the solution can be expressed as a sum of sinusoidal terms multiplied by increasing powers of $\mu$:

$$x(t) = \sum_{n=0}^{\infty} \mu^n a_n \cos(\omega_n t + \phi_n)\,.$$

Due to the freedom in the selection of the starting time, the phase $\phi_0$ can be set to zero. We consider here only the terms that are linear in $\mu$, corresponding to the first order of the perturbation:

$$x(t) = a_0 \cos(\omega_0 t) + a_1 \mu \cos(\omega_1 t + \phi_1) + \mathcal{O}(\mu^2)\,.$$

Inserting this expression in the Van der Pol equation yields

$$a_0(\omega_s^2-\omega_0^2)\cos(\omega_0 t)+a_1\mu(\omega_s^2-\omega_1^2)\cos(\omega_1 t+\phi_1)+\mu\left[1-a_0^2\cos^2(\omega_0 t)\right]a_0\omega_0\sin(\omega_0 t)+\mathcal{O}(\mu^2)=0\,.$$

This last equation must hold for any time $t$ and for each different order in $\mu$. Therefore, the coefficients of the different components must vanish. We then have:

- order $\mu^0$:

$$a_0(\omega_s^2-\omega_0^2)\cos(\omega_0 t)=0 \qquad \Rightarrow \qquad \omega_0=\omega_s\,.$$

- order $\mu^1$:

$$\begin{aligned} a_1(\omega_s^2-\omega_1^2)\cos(\omega_1 t+\phi_1)+a_0\omega_0\left[1-a_0^2\cos^2(\omega_0 t)\right]\sin(\omega_0 t) &= 0\,, \quad \texttt{i.e.} \\ a_1(\omega_s^2-\omega_1^2)\cos(\omega_1 t+\phi_1)+a_0\omega_0\left(1-\frac{a_0^2}{4}\right)\sin(\omega_0 t)-\frac{a_0^3\omega_0}{4}\sin(3\omega_0 t) &= 0\,. \end{aligned}$$

We now have to set $\omega_1$ and $\phi_1$. We are given two possibilities: either to compensate the term proportional to $\sin(\omega_0 t)$ or the term proportional to $\sin(3\omega_0 t)$. In the former case, we would remain with the term proportional to $\sin(3\omega_0 t)$, whose suppression would imply setting $a_0=0$ and, thereupon, also $a_1=0$...a trivial solution. Therefore, we must have:

$$a_0\omega_0\left(1-\frac{a_0^2}{4}\right)\sin(\omega_0 t)=0 \qquad \Rightarrow \qquad a_0=2\,,$$

and

$$a_1(\omega_s^2-\omega_1^2)\cos(\omega_1 t+\phi_1)-\frac{a_0^3\omega_0}{4}\sin(3\omega_0 t)=0 \quad \Rightarrow$$

$$\Rightarrow \quad \begin{cases} \cos(\omega_1 t+\phi_1)=\sin(3\omega_0 t) \quad \Rightarrow \quad \omega_1=3\omega_0\,,\ \phi_1=-\pi/2 \\ a_1=-\frac{1}{4\omega_s}\,. \end{cases}$$

The overall solution at the first perturbative level is then

$$x(t) = 2\cos(\omega_s t) - \frac{\mu}{4\omega_s}\sin(3\omega_s t) + \mathcal{O}(\mu^2)\,. \tag{6.7}$$

The phase-space trajectory $(x(t), \dot{x}(t))$ of the approximate solution is also shown in the Fig. 6.5.

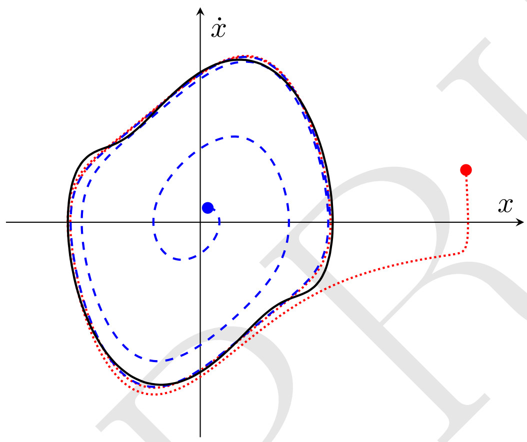


Figure 6.2: Approximate solution of Eq. (6.7) (black solid line) and numerical integrations (dashed and dotted lines) of the Van der Pol oscillator with $\omega_s = 1$ and $\mu = 0.6$; starting points for the numerical integrations are marked with dots.

## 6.3 Rayleigh oscillator

Another nonlinear relation that leads to a limit cycle is the Rayleigh oscillator equation,

$$\ddot{x} - \mu\left(1 - \dot{x}^2\right)\dot{x} + \omega_s^2 x = 0\,, \tag{6.8}$$

in which the nonlinear term depends on the squared velocity rather than the squared amplitude. This equation can be easily shown to correspond to a Van der Pol equation: it just takes to take its derivative and set $y = \dot{x}\sqrt{3}$. In other words, a Rayleigh equation is a Van der Pol one if we consider the velocity of the former as the amplitude of the latter.

## 6.4 The Wien bridge oscillator

Let us now study a celebrated circuit, namely the Wien bridge oscillator, which provides a sinusoidal wave while requiring no input. The oscillator, shown in Fig. 6.3, consists of an op-amp with both a positive and a negative feedback loop on which a resistive network with a power-dependent gain is present. For this purpose, an instructive implementation relies on a small incandescent lamp. Alternatively, networks using pairs of diodes can be used.

### 6.4.1 Stability

Let $R_0(1-\varepsilon)$ be the resistance of the lamp. Henceforth, we assume that $|\varepsilon| \ll 1$; besides that, $\varepsilon$ can be both negative and positive. Moreover, we first assume

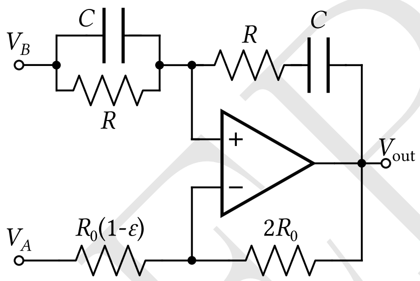


Figure 6.3: Wien bridge oscillator.

to work in the frequency domain, more specifically in the $s$-space. All quantities are therefore assumed to be Laplace transforms of standard, time-dependent variables, as highlighted by tildes.

To analyze stability, let us apply the generalized expression for the voltage difference $\Delta\tilde{V}$ at the op-amp's inputs, provided in a previous note. We have

$$\Delta\tilde{V} = \tilde{V}_{\text{out}}\left(\frac{Z_P}{Z_P+Z_S} - \frac{1-\varepsilon}{3-\varepsilon}\right) + \frac{Z_S}{Z_P+Z_S}\tilde{V}_B - \frac{2}{3-\varepsilon}\tilde{V}_A\,,$$

where $Z_P$ and $Z_S$ are the impedances of the $RC$ parallel and $RC$ series on the positive feedback loop, respectively. By replacing $\tilde{V}_{\text{out}} = \tilde{A}\Delta\tilde{V}$, it follows

$$\Delta\tilde{V} = \frac{\frac{Z_S}{Z_P+Z_S}\tilde{V}_B - \frac{2}{3-\varepsilon}\tilde{V}_A}{1+\tilde{A}\left(\frac{1-\varepsilon}{3-\varepsilon} - \frac{1}{1+Z_S/Z_P}\right)}\,. \tag{6.9}$$

By explicitly writing $Z_S/Z_P = 2 + s\tau + 1/s\tau$, the denominator $\mathtt{den}(\tilde{G}_{\mathtt{cl}})$ of the last expression, which coincides with the denominator of the closed loop gain $\tilde{G}_{\mathtt{cl}}$, becomes

$$\mathtt{den}(\tilde{G}_{\mathtt{cl}}) = 1 + \tilde{\tilde{A}}\,\frac{1-\varepsilon}{3-\varepsilon}\,\frac{s^2\tau^2 - 2s\tau\frac{\varepsilon}{1-\varepsilon} + 1}{s^2\tau^2 + 3s\tau + 1}\,.$$

For the sake of notation, it is worth defining $\delta \equiv \varepsilon/(1-\varepsilon)$, so that $\varepsilon = \delta/(\delta + 1)$. The last expression then becomes

$$\mathtt{den}(\tilde{G}_{\mathtt{cl}}) = 1 + \frac{\tilde{A}}{3+2\delta}\,\frac{s^2\tau^2 - 2s\tau\delta + 1}{s^2\tau^2 + 3s\tau + 1}\,.$$

We now take into account the frequency response of the op-amp by expliciting $\tilde{A} = A_0\omega_0/(s+\omega_0)$:

$$\mathtt{den}(\tilde{G}_{\mathtt{cl}}) = \frac{(s+\omega_0)(s^2\tau^2 + 3s\tau + 1) + \frac{A_0\omega_0}{3+2\delta}\left(s^2\tau^2 - 2s\tau\delta + 1\right)}{\left(s^2\tau^2 + 3s\tau + 1\right)(s+\omega_0)}\,.$$

In order to assess the stability of the Wien bridge oscillator, we look for the zeroes of $\mathtt{den}(\tilde{G}_{\mathtt{cl}})$. To this purpose, we single out its numerator $\eta$. By defining $z \equiv s\tau$, $\nu \equiv \omega_0\tau$ and $A' \equiv A_0\omega_0\tau/(3+2\delta)$, we get

$$\eta = (z+\nu)\left(z^2 + 3z + 1\right) + A'\left(z^2 - 2z\delta + 1\right)\,.$$

In this expression, the first, cubic term is small compared to the second one due to the $A'$ factor. Consequently, the numerator $\eta$, normalized by the constant $A'$, can be expressed as

$$\begin{aligned} \frac{\eta}{A'} &= z^2 - 2z\delta + 1 + \frac{1}{A'}\varphi(z) \qquad \mathtt{where} \\ \varphi(z) &\equiv (z+\nu)\left(z^2 + 3z + 1\right)\,. \end{aligned}$$

The term $\varphi(z)/A'$ is a "perturbation" to the quadratic term. If $A_0 \to \infty$, the perturbation vanishes and the zeroes of the numerator are simply[4]

$$z_{1,2} = \delta \pm \sqrt{\delta^2 - 1} = \delta \pm i\sqrt{1 - \delta^2}\,. \tag{6.10}$$

The two roots have an imaginary part close to unity and a real part equal to $\delta$ ($|\delta| \ll 1$). In this approximation, stability is determined by the sign of $\delta$, and thus by the sign of $\varepsilon$. Marginal stability holds for $\delta = 0$, corresponding to $z_{1,2} = \pm i$ and, consequently, to $s = \pm i/\tau$. Under this last condition, the system exhibits an oscillation with frequency $\omega_s = \tau^{-1}$.

We now consider the general case in which $A'$ is finite, though still large with respect to unity. We have:

$$\begin{array}{lllll} \eta = & 1 \cdot z^3 + & (\nu + 3 + A')z^2 + & (3\nu + 1 - 2\delta A')z + & (\nu + A') = 0 \\ & \downarrow & \downarrow & \downarrow & \downarrow \\ & a & b & c & d \end{array}$$

The last equation is a cubic equation with real coefficients $a$, $b$, $c$, $d$ (respectively of $z^3 \ldots z^0$). Therefore, there must be either three real roots, or one real root and two complex-conjugate ones. The former case can be ruled out by taking into account the behavior at $A' \to \infty$ and assuming a *continuous migration* of the roots once $A'$ is made finite (and large). The roots can then be written as

$$z_1 = p + iq \qquad z_2 = p - iq \qquad z_3 = r\,,$$

where $p$, $q$ and $r$ are real numbers. Vieta's formulas provide the relationship

[4] The third root $r_3$, which is clearly real, has gone to $-\infty$.

between the equation coefficients and roots:

$$
\begin{aligned}
&\text{(I)} \quad r_1 + r_2 + r_3 = -\frac{b}{a} && \Rightarrow \quad 2p + r = -(\nu + 3 + A') \\
&\text{(II)} \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{c}{a} && \Rightarrow \quad p^2 + q^2 + 2pr = 3\nu + 1 - 2\delta A' \\
&\text{(III)} \quad r_1 r_2 r_3 = -\frac{d}{a} && \Rightarrow \quad (p^2 + q^2) r = -(\nu + A')
\end{aligned}
$$

By virtue of (III), the real root $r$ must be negative. Moreover, in the limit $A' \to \infty$ we know from Eq. (6.10) that $p^2 + q^2 = 1$. Consequently, for finite but large values of $A'$, $r$ must be large in absolute value and approximately equal to $-A'$.

Combining together (II) and (III) yields

$$
2pr = 3\nu + 1 - 2\delta A' + \frac{\nu + A'}{r} = 3\nu + 1 + \frac{\nu}{r} - A' \left( 2\delta - \frac{1}{r} \right) .
$$

Because (quite typically) $-\frac{A'}{r} \sim 1$, $-r \sim A' \gg 1$, $\nu \lesssim 1$, and excluding critical as well as unlikely cases in which $1/(2r) < \delta < 0$, the dominant term in the last expression is $-2A'\delta$. Therefore, $pr \approx -A'\delta$, and because $r < 0$, it turns out that the sign of $p$ is equal to the sign of $\delta$. In other words, as in Eq. (6.10), stability is determined by the sign of $\varepsilon$.

### 6.4.2 Oscillations

After discussing stability, we are now interested in studying the differential equation that describes the system oscillations. The two input terminals can be connected to ground, so that $\tilde{V}_A = \tilde{V}_B = 0$. Equation (6.9) can be recast as follows:

$$
\Delta\tilde{V}\,\tilde{A}\,\frac{1-\varepsilon}{3-\varepsilon}\left(s^2\tau^2 - 2s\tau\frac{\varepsilon}{1-\varepsilon} + 1\right) = \left(-\Delta\tilde{V}\right)\left(s^2\tau^2 + 3s\tau + 1\right) . \tag{6.11}
$$

As in the previous stability analysis, quantities such as $\tilde{A}$, $\tilde{V}_{\text{out}}$, etc. are defined in the $s$-space.

In the limit of very large $\tilde{A}$, only the left-hand term matters, and because $\tilde{A}\Delta\tilde{V} = \tilde{V}_{\mathrm{out}}$, Eq. (6.11) becomes

$$\tilde{V}_{\mathrm{out}} \frac{1-\varepsilon}{3-\varepsilon} \left( s^2 - 2s\tau \frac{\varepsilon}{1-\varepsilon} + 1 \right) \cong 0 \,,$$

This $s$-space equation can be straightforwardly transformed back into the time domain:

$$\ddot{V}_{\mathrm{out}} \tau^2 - 2\tau \frac{\varepsilon}{1-\varepsilon} \dot{V}_{\mathrm{out}} + V_{\mathrm{out}} \cong 0 \,.$$

Defining $\omega_s^2 \equiv 1/\tau^2$, and because $|\varepsilon| \ll 1$, finally yields

$$\ddot{V}_{\mathrm{out}} - \frac{2\varepsilon}{\tau} \dot{V}_{\mathrm{out}} + \omega_s^2 V_{\mathrm{out}} \cong 0 \,. \tag{6.12}$$

We now look for an expression for $\varepsilon$. Any nonzero inverting input voltage $V_1 \equiv V_-$ yields a current across the lamp, thus heating it up and increasing its resistance according to

$$R_1 = R_L \left( 1 + \alpha V_1^2 \right) \,. \tag{6.13}$$

The voltage $V_{10}$ at which the lamp resistance $R_1$ becomes exactly equal to $R_0 = R_2/2$ can be promptly derived:

$$R_L \left( 1 + \alpha V_{10}^2 \right) = R_0 \quad \Rightarrow \quad V_{10} = \sqrt{\frac{R_0 - R_L}{\alpha R_L}} \,.$$

Consequently,

$$\begin{aligned} R_1 &= R_L \left[ 1 + \alpha \left( V_{10}^2 + V_1^2 - V_{10}^2 \right) \right] = R_0 + \alpha R_L (V_1^2 - V_{10}^2) = \\ &= R_0 \left[ 1 - \alpha \frac{R_L}{R_0} \left( V_{10}^2 - V_1^2 \right) \right] = R_0 (1 - \varepsilon) \,. \end{aligned}$$

At the equilibrium point, namely if $\varepsilon = 0$, we have $V_{10} \equiv V_1(\varepsilon = 0) = V_{\mathrm{out}}(\varepsilon = 0)/3$. If $|\varepsilon|$ is small enough so that higher-order corrections can be ignored the inverting input voltage $V_1$ is still approximately given by $V_1 \cong V_{\mathrm{out}}/3$. Therefore, by defining $V_0 \equiv 3V_{10} = V_{\text{out}}(\varepsilon = 0)$, the expression for $\varepsilon$ becomes

$$\varepsilon \cong \frac{\alpha}{9}\frac{R_L}{R_0}\left(V_0^2 - V_{\text{out}}^2\right) .$$

We now insert the expression for $\varepsilon$ back into the differential equation, Eq. (6.12), thus producing

$$\ddot{V}_{\text{out}} - \frac{2}{\tau}\frac{\alpha R_L}{9R_0}\left(V_0^2 - V_{\text{out}}^2\right)\dot{V}_{\text{out}} + \omega_s^2 V_{\text{out}} \cong 0 .$$

If we define $x \equiv V_{\text{out}}/V_0$, $\mu \equiv 2\alpha V_0^2 R_L/(9\tau R_0)$, the last differential equation can be written in a very fascinating and general way...

## 6.5 Relaxation oscillator

A beautiful circuit that exhibits oscillations is the so-called *relaxation oscillator*, shown in Fig. 6.4. In this case, the output oscillates by switching between

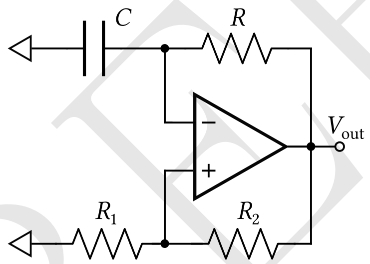


Figure 6.4: Relaxation oscillator.

the two saturation voltages, namely between $\pm V_S$ for a symmetrically-powered op-amp. The capacitor is charged and discharged by the two saturation voltages and the switch between them occurs when the drop on the capacitor overcomes (in absolute value) the thresholds alternatively set by the voltage divider making up the positive feedback[5].

Goal of the present section is to show that the relaxation oscillator can be eventually led back to an equation in which the dissipative/enhancing term is equal to that described by the Van der Pol equation.

[5]The configuration relying on a voltage divider on the positive feedback like that one described here is called a "Schmitt trigger".

We first note that the inverting input voltage $V_-$ is given by

$$\dot{V}_- + \frac{V_-}{\tau} = \frac{V_{\text{out}}}{\tau}\,. \tag{6.14}$$

Moreover, due to the voltage divider on the positive feedback network, the non-inverting input voltage $V_+$ is given by

$$V_+ = \frac{V_{\text{out}}}{k}\,, \tag{6.15}$$

where $k = (R_2 + R_1)/R_1 > 1$.

To parametrize the finite bandwidth and also the nonlinear behavior of a real op-amp, let us model it by separating the frequency-dependent part from the nonlinear one. Let $W$ be the op-amp output voltage if saturation would not occur. Due to the well-known frequency-dependent behavior of the op-amp ($\tilde{A}(s) = A_0\omega_0/(s+\omega_0)$), the equation governing $W$ can be written as

$$\dot{W} + \omega_0 W = A_0\omega_0(V_+ - V_-)\,. \tag{6.16}$$

The saturation behavior can than be described by means of a nonlinear, sigmoid function $f(x)$ such that $f(x) \to \pm 1$ when $x$ overcomes 1:

$$V_{\text{out}} = V_S\, f\left(\frac{W}{V_S}\right)\,, \tag{6.17}$$

where $V_S$ is the saturation voltage. A possible analytical form for $f(z)$ is

$$f\,(z) = \frac{z}{(1+z^2)^{\frac{1}{2}}}\,.$$

We note that

$$f'\,(z) = \frac{1}{(1+z^2)^{\frac{3}{2}}}\,.$$

It is suitable to set $x \equiv t/\tau$, $y \equiv V_-/V_S$, $z \equiv W/V_S$ and $\nu \equiv \omega_0\tau$. Consequently, by combining Eqs. (6.14-6.17), we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(z) \tag{6.18a}$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} + \nu z = \nu A_0 \left[\frac{1}{k} f(z) - y\right] . \tag{6.18b}$$

Taking the derivative of Eq. (6.18b),

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \nu \frac{\mathrm{d}z}{\mathrm{d}x} = \nu A_0 \left[\frac{1}{k} f'(z) \frac{\mathrm{d}z}{\mathrm{d}x} - \frac{\mathrm{d}y}{\mathrm{d}x}\right] ,$$

summing the result with Eq. (6.18b) itself, and using Eq. (6.18a) yields

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \left[\nu + 1 - \frac{\nu A_0}{k} f'(z)\right] \frac{\mathrm{d}z}{\mathrm{d}x} + \left[\nu + \frac{\nu A_0}{k} \frac{f(z)}{z} (k-1)\right] z = 0 .$$

Considering the expansion up to the second order of $f(z)/z$ and $f'(z)$, one gets

$$\frac{\mathrm{d}^2 z}{\mathrm{d}x^2} + \left[\nu + 1 - \frac{\nu A_0}{k} \left(1 - \frac{3}{2} z^2\right)\right] \frac{\mathrm{d}z}{\mathrm{d}x} + \left[\nu + \frac{\nu A_0}{k} \left(1 - \frac{z^2}{2}\right) (k-1)\right] z \simeq 0 .$$

The resulting equation contains a nonlinear correction to the amplitude and, more importantly, a Van-der-Pol-like term with regard to the dissipative/enhancing term (and with the right sign!). Figure 6.5 shows a numerical integration of

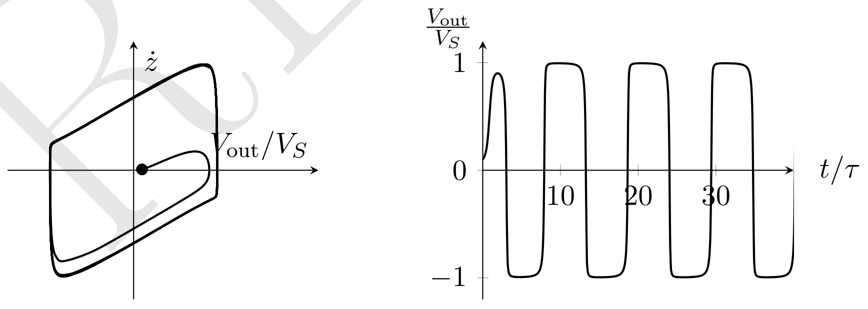


Figure 6.5: Numerical integration of Eqs. 6.18a, 6.18b.

the last equation where the parameters are set to $A_0 = 10^5$, $\nu = 10^{-4}$, $k = 1.5$

One might argue that the result is due to the particular choice of $f(z)$. However, any symmetric sigmoid-like curve is given by a linear term multiplied a function that, for symmetry reasons, must depend on $z^2$.