Jul. 11 Opera Seminar

Lattice Γ metric d interaction Φ

$\cdots \bullet \bullet \overset{x}{\bullet} \bullet \bullet \bullet \bullet \bullet \bullet \cdots$

$\mathcal{H}_x$ $\max_{x\in\Gamma} \dim(\mathcal{H}_x) \leq N$

Local Hamiltonian $H_\Lambda = \sum_{X\subset\Lambda} \Phi(X)$

$\Lambda \subset \mathcal{P}_f(\Gamma)$

Def λ-norm (a family of norms with spacial decay rate λ)

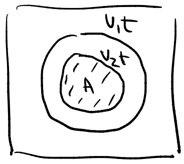
$$\|\Phi\|_\lambda := \sup_{x\in\Gamma} \sum_{\Lambda\ni x} |\Lambda|\cdot\|\Phi(\Lambda)\|\cdot N^{2|\Lambda|} e^{\lambda\cdot \mathrm{diam}(\Lambda)} \qquad (\lambda>0)$$

$\lambda_1 \quad \lambda_2$
$\downarrow \quad \downarrow$
$v_1 \quad v_2$

Def λ-regularity $\mathbb{Z}^d$ ✓ $d_{min} \sim \dfrac{2|t|\cdot\|\Phi\|_\lambda}{\lambda}$

We say Γ is λ-regular if $\sum_{n=1}^{\infty} f(n)\, e^{-\lambda n} < \infty$, where

$$f(n) = |B_n(x_0)\setminus B_{n-1}(x_0)|$$



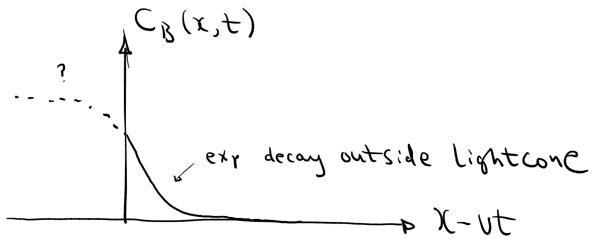
counts the number of sites in a shell of radius n for some fixed x_0.

Def $x\in\Gamma$

$$C_{A,B}(x,t) := [\alpha_t(A), B] \quad\Rightarrow\quad C_B(x,t) := \sup_{A\in\mathcal{A}(\{x\})\setminus\{0\}} \frac{\|C_{A,B}(x,t)\|}{\|A\|}.$$

where $A\in\mathcal{A}(\{x\})$, $B\in\mathcal{A}$

strictly local algebra   quasi-local algebra

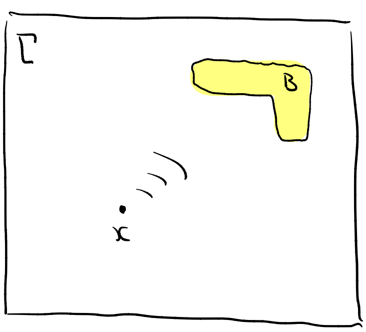


WTS a bound on $C_B(x,t)$

Thm (Lieb-Robinson) λ-quasi-local

Let $\lambda>0$ and Γ be λ-regular. Then for all interactions Φ with $\|\Phi\|_\lambda<\infty$, sites $x\in\Gamma$ and $t\in\mathbb{R}$ and $B\in\mathcal{A}$, there is a bound

$$C_B(x,t) \leq e^{2|t|\cdot\|\Phi\|_\lambda}\cdot C_B(x,0) + \sum_{y\in\Gamma\setminus\{x\}} e^{-\lambda d(x,y)}\left(e^{2|t|\cdot\|\Phi\|_\lambda}-1\right) C_B(y,0).$$



for any $A\in\mathcal{A}(\{x\})$

$$\|[\alpha_t(A), B]\| = \|C_{A,B}(x,t)\| \leq \|A\|\cdot C_B(x,t)$$

Observation

- For A not hyperlocal. (still strictly local) $|\mathrm{Supp}(A)| = k > 1$

$$A = \sum_{i_1,\cdots,i_k} \lambda_{i_1,\cdots,i_k} E_{i_1} \cdots E_{i_k} \qquad \text{with} \qquad \|E_{i_\ell}\| = 1$$

note the Commutator bound

$$\begin{aligned} \|[A_1 A_2, B]\| &= \|[A_1, B] A_2 + A_1 [A_2, B]\| \\ &\leq \|A_1 [A_2, B]\| + \|[A_1, B] A_2\| \\ &\leq \|A_1\| \cdot \|[A_2, B]\| + \|A_2\| \cdot \|[A_1, B]\| \end{aligned} \qquad \text{(useful)}$$

We will have

$$\begin{aligned} \|[\alpha_t(A), B]\| &= \Big\| \sum_{i_1 \cdots i_k} \lambda_{i_1 \cdots i_k} [\alpha_t(E_{i_1}) \cdots \alpha_t(E_{i_k}), B] \Big\| \\ &\leq \|A\| \cdot \sum_{i_1 \cdots i_k} \|[\alpha_t(E_{i_1}) \cdots \alpha_t(E_{i_k}), B]\| \\ &\leq \|A\| \cdot \sum_{i_1 \cdots i_k} \sum_{\lambda=1}^{k} \underbrace{\|\alpha_t(E_{i_1})\|}_{1} \cdots \|[\alpha_t(E_{i_\lambda}), B]\| \cdots \underbrace{\|\alpha_t(E_{i_k})\|}_{1} \\ &= \|A\| \cdot \sum_{i_1 \cdots i_k} \Big(\sum_{\lambda=1}^{k} \|[\alpha_t(E_{i_\lambda}), B]\| \Big) \\ &\leq \|A\| \cdot N^{2k} \cdot \sum_{\lambda \in \mathrm{Supp}(A)} C_B(\lambda, t) \end{aligned}$$

can then apply Lieb-Robinson for each $C_B(\lambda, t)$

- For B strictly local, further estimate

$$C_B(y, 0) = 0 \qquad \text{if} \quad y \notin \mathrm{Supp}(B)$$

and

$$C_B(y, 0) = \sup_{\substack{A \in \mathcal{A}(\{y\}) \\ \|A\| = 1}} \|C_{A,B}(y, 0)\| = \sup_{A} \|[A, B]\| \leq 2 \overset{1}{\cancel{\|A\|}} \cdot \|B\|$$

And rewrite

$$C_B(x, t) \leq e^{2|t| \cdot \|\Phi\|_\lambda} \cdot C_B(x, 0) + \sum_{y \in \Gamma \setminus \{x\}} e^{-\lambda d(x,y)} \left(e^{2|t| \cdot \|\Phi\|_\lambda} - 1 \right) C_B(y, 0).$$

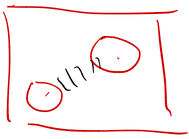
$$\leq e^{2|t| \cdot \|\Phi\|_\lambda} C_B(x, 0) + 2 |\mathrm{Supp}(B)| \cdot \|B\| \cdot e^{-\lambda d[x, \mathrm{Supp}(B)]} \left(e^{2|t| \cdot \|\Phi\|_\lambda} - 1 \right)$$

Cor (loose) With $A \in \mathcal{A}(\Lambda_1)$, $B \in \mathcal{A}(\Lambda_2)$ local ops

$v \sim \frac{2\|\Phi\|_\lambda}{\lambda}$

$$\|[\alpha_t(A), B]\| \leq 4 \|A\| \cdot \|B\| \cdot |\Lambda_1| \cdot |\Lambda_2| \cdot N^{2|\Lambda_1|} \boxed{e^{2|t| \cdot \|\Phi\|_\lambda - \lambda d[\Lambda_1, \Lambda_2]}}$$

Rmk With assumptions $\Lambda_1 \cap \Lambda_2 = \phi$ and geometry, this can be improved.



Caveats and Proof Strategy

Q Does α_t exist ?

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

Strategy Consider $\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$ Local dynamics

1° Show a local dynamics version of Lieb-Robinson Thm.

2° and show $\alpha_t^\Lambda \longrightarrow \alpha_t$ as Λ grows $\to \infty$

Derivation $\delta(A) = \sum_{X \cap \Lambda \neq \phi} [\Phi(X), A]$ (*)

$$\|\delta(A)\| \leq \sum_{x \in \Lambda} \sum_{X \ni x} \|[\Phi(X), A]\| \leq 2\|A\| \sum_{x \in \Lambda} \underbrace{\sum_{X \ni x} \|\Phi(X)\|}$$

$$\leq 2\|A\| \cdot |\Lambda| \cdot \|\Phi\|_\lambda < \infty$$

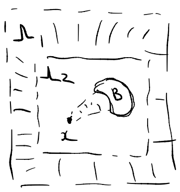
$$\|\Phi\|_\lambda := \sup_{x \in \Gamma} \sum_{\Lambda \ni x} |\Lambda| \cdot \|\Phi(\Lambda)\| \cdot N^{2|\Lambda|} e^{\lambda \cdot \mathrm{diam}(\Lambda)}$$

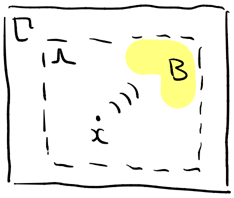
Proof sketch

Lemma Same as Thm, but with Λ being finite subsets of Γ, then

$$C_B^\Lambda(x,t) \leq e^{2|t| \cdot \|\Phi\|_\lambda} C_B(x,0) + \sum_{y \in \Gamma \setminus \{x\}} e^{-\lambda d(x,y)} \left(e^{2|t| \cdot \|\Phi\|_\lambda} - 1\right) C_B(y,0)$$

where $\alpha_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$, and $C_{A,B}^\Lambda$ are similarly altered

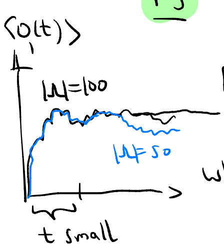




$$C_{A,B}^\Lambda(x,t) = [\alpha_t^\Lambda(A), B]$$

$$C_B^\Lambda(x,t) = \sup_{\substack{A \in \mathcal{A}(\{x\}) \\ \|A\| = 1}} \|C_{A,B}^\Lambda(x,t)\|$$

Pf Let $A \in \mathcal{A}(\{x\})$, $B \in \mathcal{A}$



$$[\alpha_t^\Lambda(A), B] - [A, B] = \int_0^t \frac{d}{ds}[\alpha_s^\Lambda(A), B]\, ds = \int_0^t [\alpha_s^\Lambda(\delta_\Lambda(A)), B]\, ds$$

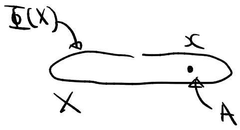
where $\delta_\Lambda(A) := i[H_\Lambda, A]$, $H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$

$$C_{A,B}^\Lambda(x,t) = C_{A,B}^\Lambda(x,0) + i \sum_{X \subset \Lambda, X \ni x} \int_0^t \left[\alpha_s^\Lambda([\Phi(X), A]), B\right] ds$$

get bound

$$C_B^\Lambda(x,t) \leq C_B^\Lambda(x,0) + \sum_{X \subset \Lambda, X \ni x} \int_0^t \sup_{\substack{A \in \mathcal{A}(\{x\}) \\ \|A\| = 1}} \underbrace{\left\| [\alpha_s^\Lambda([\Phi(X), A]), B] \right\|}\, ds$$

Note that $[\Phi(X), A] \in \mathcal{A}(X)$. Write as $\sum_{i_1 \cdots i_{|X|}} \lambda_{i_1 \cdots i_{|X|}} E_{i_1} \cdots E_{i_{|X|}}$



Thus

$$\left\| \left[\alpha_s^\Lambda\big([\Phi(X), A]\big), B \right] \right\| \le \underbrace{\|[\Phi(X), A]\|}_{\le 2\|A\| \cdot \|\Phi(X)\|} \cdot N^{2|X|} \sum_{y \in X} C_B^\Lambda(y, s)$$

$$\Rightarrow \quad C_B^\Lambda(x,t) \le C_B^\Lambda(x,0) + 2 \sum_{\substack{X \subset \Lambda \\ X \ni x}} \|\Phi(X)\| \cdot N^{2|X|} \sum_{y \in X} \underbrace{\int_0^t C_B^\Lambda(y,s)\, ds}$$

Set $\quad \varphi(x,y) := \sum_{X \ni x,y} \|\Phi(X)\| \cdot N^{2|X|}$

$$C_B^\Lambda(x,t) \le C_B^\Lambda(x,0) + 2 \sum_{y \in \Gamma} \varphi(x,y) \cdot \int_0^t C_B^\Lambda(y,s)\, ds$$

Iterate

$$C_B^\Lambda(x,t) \le C_B^\Lambda(x,0) + 2 \sum_{y \in \Gamma} \varphi(x,y) \int_0^t \left[C_B^\Lambda(y,0) + 2 \sum_{y'} \varphi(y,y') \int_0^s C_B^\Lambda(y', s')\, ds' \right] ds$$

$$\le \cdots$$

$$\le C_B^\Lambda(x,0) + 2t \sum_y \varphi(x,y)\, C_B^\Lambda(y,0) + \frac{(2t)^2}{2!} \sum_{y y'} \varphi(x,y')\, \varphi(y',y)\, C_B^\Lambda(y,0) + \frac{(2t)^3}{3!} \sum_{y y' y''} \varphi(x,y'')\, \varphi(y'',y')\, \varphi(y',y)\, C_B^\Lambda(y,0) + \cdots$$

$$d(x,y) \le d(x,y') + d(y',y'') + \cdots + d(y^{(n)}, y)$$

$$\Rightarrow \quad C_B^\Lambda(x,t) \le \sum_y e^{-\lambda d(x,y)} f(x,y)\, C_B^\Lambda(y,0)$$

where $f(x,y) = \delta_{xy} + 2t\, \underbrace{\varphi_\lambda(x,y)} + \frac{(2t)^2}{2!} \sum_{y'} \varphi_\lambda(x,y')\, \varphi_\lambda(y',y) + \frac{(2t)^3}{3!} \sum_{y' y''} \cdots$

$$\boxed{\varphi_\lambda(x,y) := e^{\lambda d(x,y)}\, \varphi(x,y)}$$

Since

$$\sum_y \varphi_\lambda(x,y) = \sum_y \sum_{X \ni x,y} \|\Phi(X)\| \cdot N^{2|X|} e^{\lambda d(x,y)}$$

$$\leq \sum_{X \ni x} \sum_{y \in X} \|\Phi(X)\| \cdot N^{2|X|} e^{\lambda \operatorname{diam}(X)}$$

$$= \sum_{X \ni x} |X| \cdot \|\Phi(X)\| \cdot N^{2|X|} e^{\lambda \operatorname{diam}(X)} \leq \|\Phi\|_\lambda$$

$$\Rightarrow f(x,y) \leq \delta_{xy} + 2t\|\Phi\|_\lambda + \frac{(2t)^2}{2!}\|\Phi\|_\lambda^2 + \cdots$$

$$= \delta_{xy} + \left(e^{2t\|\Phi\|_\lambda} - 1\right)$$

$$\Rightarrow C_B^\Lambda(x,t) \leq C_B^\Lambda(x,0) + \sum_{y \in \Gamma \setminus \{x\}} e^{-\lambda d(x,y)} \left(e^{2|t| \cdot \|\Phi\|_\lambda} - 1\right) C_B^\Lambda(y,0)$$

□ (Lemma)

sub Thm (α_t) Suppose Γ is λ-regular and $\|\Phi\|_\lambda < \infty$,

Then local dynamics converge to a strongly continuous 1-parameter group of automorphisms α_t, i.e.,

$$\lim_{\Lambda \to \infty} \|\alpha_t^\Lambda(A) - \alpha_t(A)\| = 0 \qquad \forall A \in \mathcal{A}$$

This convergence is uniform in t on a given compact $[-T, T]$

Pf sketch $\Lambda_n \subset \Lambda_m$

$t \in [-T, T]$

$$\|\alpha_t^{\Lambda_m}(A) - \alpha_t^{\Lambda_n}(A)\| \leq \sum_{x \in \Lambda_m \setminus \Lambda_n} \sum_{X \ni x} \int_0^t \|[\Phi(X), \alpha_{t-s}^{\Lambda_n}(A)]\| ds$$

$$\leq \sum_{x \in \Lambda_m \setminus \Lambda_n} \sum_{X \ni x} \|\Phi(X)\| N^{2|X|} \sum_{y \in X} \int_0^t C_A^{\Lambda_n}(y, t-s) ds$$

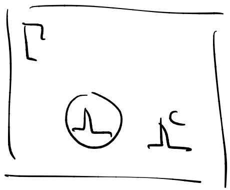
$$\leq 2 C_T \|\Phi\|_\lambda \cdot \|A\| \sum_{x \in \Lambda_m \setminus \Lambda_n} \sum_{z \in \Lambda} e^{-\lambda d(x,z)}$$

□ (sub-Thm)

Sub-Thm shows global dynamics exist. And claim on α_t follows from Lemma

□ (Lieb-Robinson)

Implications



Fact $\Lambda \in \mathcal{P}_f(\Gamma)$, $\varepsilon > 0$, $A \in \mathcal{A}$ and

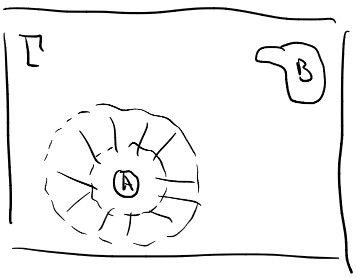
$$\|[A,B]\| \leq \varepsilon \|A\| \cdot \|B\| \qquad \forall B \in \mathcal{A}(\Lambda^c)$$

then $\exists A_\Lambda \in \mathcal{A}(\Lambda)$ s.t. $\|A_\Lambda - A\| \leq \varepsilon \|A\|$

Example $A \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ $\quad \|[A, I \otimes B]\| \leq \varepsilon \|A\| \cdot \|B\| \quad \forall B \in \mathcal{B}(\mathcal{H}_2)$

finite $\Rightarrow A' := \int_{U(\mathcal{H}_2)} (I \otimes U) A (I \otimes U^*) \, dU_{\text{Haar}} \in \mathcal{B}(\mathcal{H}_1) \otimes I$

$\Rightarrow A' \simeq A$.



4.3.2

Cor Same as Thm, but with $\Gamma = \mathbb{Z}^d$. $A \in \mathcal{A}(\Lambda)$, $B \in \mathcal{A}(\Lambda')$

Then $\exists C > 0$ s.t.

$$\|[\alpha_t(A), B]\| \leq C \cdot \|A\| \cdot \|B\| \cdot |\Lambda| \cdot N^{2|\Lambda|} e^{2|t| \cdot \|\Phi\|_\lambda - \lambda d(\Lambda, \text{supp}(B))}$$

Rmk Pick d_{min} s.t. $\varepsilon > C|\Lambda| \cdot N^{2|\Lambda|} e^{2|t| \cdot \|\Phi\|_\lambda - \lambda d_{min}}$

$\Rightarrow$ If $d(\text{supp}(B), \text{supp}(A)) \geq d_{min}$, $\quad \|[\alpha_t(A), B]\| < \varepsilon \|A\| \cdot \|B\|$ $\forall B$

$$d_{min} \sim \left. \frac{2|t| \cdot \|\Phi\|_\lambda}{\lambda} \right\} \leftarrow \lambda \in (0, \lambda_0]$$

Thm $\|\Phi\|_\lambda < \infty$, unique GS ω, Hamiltonian H in GS representation has a gap $\gamma > 0$. Then $\exists \mu > 0$

$$|\omega(AB) - \omega(A)\omega(B)| \leq C(A,B,\gamma) e^{-\mu d(\Lambda_1, \Lambda_2)}$$

for all disjoint $\Lambda_1, \Lambda_2 \in \mathcal{P}_f(\Gamma)$ and $A \in \mathcal{A}(\Lambda_1)$, $B \in \mathcal{A}(\Lambda_2)$

Rmk GS entanglement: gapped 1D $S_A(\rho_{GS}) \propto |\partial A|$.