Opera Seminar I: Basics of C^*-Algebra

Motivation: Why operator alg.?
(What's wrong with Hilb?)

Consider (1+1) Infinite spin chain

$i \in \mathbb{Z}$

Naively $\boxed{\mathcal{H} = \bigotimes_{i\in\mathbb{Z}} \mathcal{H}_i}$ $\quad$ $\mathcal{H}_i$ spin $\frac{1}{2}$ DOF.

Not well-defined inner prod. $\rightarrow$ Hilb $\left\{ \begin{matrix} \text{Complete} \\ \text{normed} \end{matrix} \right\}$ vec.

$$|\psi_1\rangle = \bigotimes_{i\in\mathbb{Z}} |\uparrow\rangle_i$$

$$|\psi_2\rangle = \bigotimes_{i\in\mathbb{Z}} e^{i\theta} |\uparrow\rangle_i$$

$$\langle\psi_1|\psi_2\rangle = \prod_{i\in\mathbb{Z}} e^{i\theta} = e^{i\Sigma\theta}$$

Generically: No well-defed inner prod.

Possible solution: All states

$\leftarrow$ $|\uparrow\rangle$ $\qquad$ $\rightarrow$ $|\uparrow\rangle$

finite $\boxed{\dfrac{(|\uparrow\cdots\uparrow\rangle + |\downarrow\cdots\downarrow\rangle)}{\sqrt{2}}}$ ✓

$\downarrow$ inf.

? such limit $\nexists$

(thermal dyn. limit?).

Alternatively, we choose to describe by operators.

· Basics on Operator Algebra.

抽象化 $B(\mathcal{H})$ Bounded op. on $\mathcal{H}$. $\left\{ \begin{matrix} \text{Linear space} \\ \underline{\text{multiplication}} \Rightarrow \text{Algebra.} \end{matrix} \right.$

$M_{\mathbb{C}}(\mathcal{H})$

Def. An alg. A is a vector space $\mathbb{C}$ equipped with multiplication $(\cdot,\cdot): A\times A \to A$.

① $\mathbb{C}$-linear

② Associativity $A\cdot(B\cdot C) = (A\cdot B)\cdot C$

③ Distribution $A\cdot(B+C) = A\cdot B + A\cdot C$

$B(\mathcal{H})$ + $*$-structure $(\psi, A\omega) = (A^*\psi, \omega)$

$*$-str. $\begin{cases} *: \mathcal{A} \to \mathcal{A} \\ (AB)^* = B^*A^* \\ (\lambda A+B)^* = \bar{\lambda}A^* + B^* \end{cases}$ $*$-algebra

$B(\mathcal{H})$. + norm structure

matrix norm $\|\cdot\|: \mathcal{A} \to \mathbb{R}_{\geq 0}$ Banach $*$-alg.

compatible with $*$ $\|AB\| \leq \|A\|\cdot\|B\|$

$\|A^*\| = \|A\|$

+ C^*-property C^*-algebra.

$\|A^*A\| = \|A\|^2$

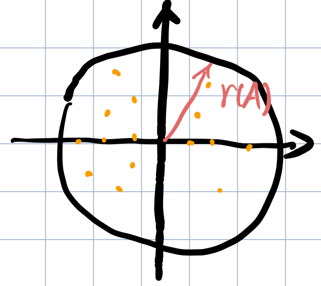
Eg. $B(\mathcal{H})$ Bounded op. on a Hilbert space $\mathcal{H}$. is a C^*-algebra.

$*$: adjoint. $\|A\| = \sup\limits_{\substack{\psi\in\mathcal{H} \\ |\psi|=1}} |A\psi|_{\mathcal{H}}$

· Spectra u. Positive Op.
and.

Spectra $\longleftrightarrow$ Eigenvalue.

Def. $A\in\mathcal{A}$ $\mathrm{Spec}(A) = \{\lambda\in\mathbb{C} \mid (A-\lambda I) \text{ not invertible}\}$.



$$r(A) = \sup_{\lambda \in \mathrm{Spec}(A)} |\lambda|$$

$\subset \mathbb{C}$

Prop. $\forall A \in \mathcal{A}$, $r(A) \leq \|A\|$, $\mathrm{Spec}(A)$ is compact. ~有界闭.

If $A^* = A$, $\mathrm{Spec}(A) \subset \mathbb{R}$.

Rem. $\mathcal{A} \sim$ Bounded op.s

$\Rightarrow$ finite norm $\Rightarrow$ $\mathrm{Spec}(A)$ bounded.

(and closed)

Positive op. $\leftrightarrow$ Positive op.

Def. (Thm.) The three following statements are equiv.

$\mathcal{A}$. $A \in \mathcal{A}$.

A is positive if

① $\mathrm{Spec}(A) \subset [0, +\infty)$ and $A^* = A$.

② $\exists H \in \mathcal{A}$ s.t. $A = H^* H$

③ $\exists K^* = K \in \mathcal{A}$ s.t. $A = K^2$

Rem. For determining normalization.

- States and GNS construction.

A state ω $\longleftrightarrow$ Give all possible A's $\langle \psi | A | \psi \rangle$

Def. A state ω of a C^*-alg. $\mathcal{A}$ is a functional

$$\omega : \mathcal{A} \to \mathbb{C}$$

① $\omega(\mathbb{1}) = 1$ $\quad\quad$ $\omega(H^* H) \geq 0$

② $\omega(\lambda A + B) = \lambda \omega(A) + \omega(B)$

(A state $\omega \in \mathcal{A}^*$).

Def. (Pure/Mixed). If the decomposition

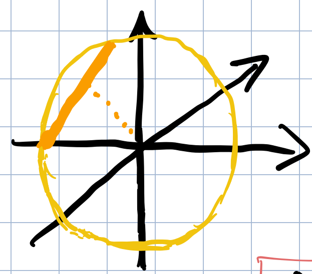
$$\omega = \lambda \omega_1 + (1-\lambda)\omega_2 \qquad \lambda \in (0,1)$$

must be trivial (i.e. $\omega_1 = \omega_2 = \omega$)

then ω is pure.

otherwise ω is mixed.

Rem.



space of states is convex. with pure states being the boundary.

For pure state, how to $|\psi\rangle \mapsto |\psi\rangle\langle\psi|$ in C^*-alg.?

Q: Such assignment of state is heavy.

Any better way to probe info of the state?

A: Gel'fand-Naimark-Segal construction.

GNS construction.

Given a $\mathcal{A}$, ω. pure

$$N_\omega = \{A \in \mathcal{A} \mid \omega(A^*A) = 0\}$$

$$\langle\psi|A^*A|\psi\rangle = \| A|\psi\rangle \| = 0.$$

① $\mathcal{H}_\omega = \mathcal{A}/N_\omega \rightsquigarrow$ A Hilbert space.

Need to prove N_ω is an ideal

inner prod. $\langle [A], [B] \rangle = \omega(A^*B) \qquad A, B \in \mathcal{A}$

$$(\omega(A^*A) \geq 0)$$

How does $\mathcal{A}$ represent on $\mathcal{H}_\omega$.

② $\pi_\omega(A)[B] \equiv [AB] \in \mathcal{H}_\omega$ Verify ① is a rep. ② w.r.t. mod N_ω

act on $\mathcal{H}_\omega$ ③ norm related.

③ $\Omega \in \mathcal{H}_\omega$ $\langle \Omega, \pi_\omega(A)\Omega\rangle = \omega(A)$.

[I]

$(\mathcal{H}_\omega, \pi_\omega, \Omega)$ - GNS construction of $(\mathcal{A}, \omega)$

Rem. ① GNS construction for spin $\frac{1}{2}$

$\mathcal{H} = \mathbb{C}^2$ $B(\mathcal{H}) = \mathcal{A} = \mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$\omega \sim |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $|\downarrow\rangle$

$N_{|\uparrow\rangle} = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \middle| \, |a|^2+|c|^2=0 \Leftrightarrow a=c=0\right\}$

$\mathcal{H}_{|\uparrow\rangle} = \mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$\begin{pmatrix}a&0\\b&0\end{pmatrix}$

GNS construction is rarely physical.

GNS construction $\sim$ purification of states?

② For mixed states.

Lemma. $\forall \rho$ mixed.

$$\rho = \sum_i \lambda_i \omega_i \qquad \begin{cases} \sum_i \lambda_i = 1 \\ \lambda_i \in (0,1) \\ \omega_i \text{ pure} \end{cases}$$

GNS construction

$$\bigoplus_i (\mathcal{H}_{\omega_i}, \pi_{\omega_i}, \Omega_i)$$

$\oplus\Omega_i$ not necessarily a pure state?

Thm. States ω_1, ω_2 without equiv. GNS construction cannot purely superpose.

③ Prop. The followings are equiv., for a given $\mathcal{A}$ (on a lattice system).

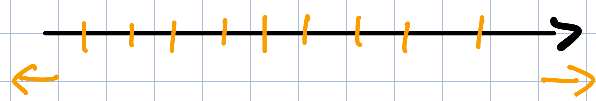
$s \in \mathbb{Z}^d$. Γ finite set $\subset \mathbb{Z}^d$. $\mathcal{H}_\Gamma = \bigotimes_{i\in\Gamma} \mathcal{H}_i$

(i) ω_1, ω_2 are states. with equiv. GNS constructions

(ii) $\forall \varepsilon > 0$. $\exists$ finite Γ_ε s.t. $\forall A \in \mathcal{A}_{\mathbb{Z}^d \setminus \Gamma_\varepsilon}$

$$\frac{|\omega_1(A) - \omega_2(A)|}{\|A\|} < \varepsilon .$$

For lattice system, $\omega_1 \sim \omega_2$. then they can be different for only finite regions.



Eg. $|\uparrow \cdots \uparrow \cdots \rangle$ / $|\downarrow \cdots \downarrow \cdots \rangle$ are not equiv.

④ Thm. States ω_1, ω_2 without equiv. GNS construction cannot purely superpose.

⑤ GNS construction of $(\mathcal{A}, \omega)$ is unique up to unitary equivalence.

$$\pi_\omega \to \mathrm{Ad}_{\mathcal{U}} \pi_\omega$$
$$\mathcal{U} : \mathcal{H}_\omega \to \mathcal{H}_{\omega'} \qquad \mathcal{U}^* \mathcal{U} = \mathrm{id}_{\mathcal{H}_\omega}$$
$$\Omega_\omega \to \mathcal{U}\Omega \qquad \mathcal{U}\mathcal{U}^* = \mathrm{id}_{\mathcal{H}_{\omega'}}$$

- Dynamics and Symmetry.

  $\alpha \in \mathrm{Aut}(\mathcal{A})$.

① Time evolution

② Sym: as automorphisms commutative with time evolution.

Time evolution is also generated by automorphism.

Def. A C^*-dyn. system is $(\mathcal{A}, \alpha_t)$

$$\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A}).\quad 单参子群.$$

Def. A sym. is $\beta \in \mathrm{Aut}(\mathcal{A})$. s.t.

$$\beta \circ \alpha_t = \alpha_t \circ \beta \qquad \forall t \in \mathbb{R}.$$

· Quantum Mechanics

$\mathcal{H}$. $\quad B(\mathcal{H})$. $\quad$ Observables $A = A^* \in B(\mathcal{H})$

$$\omega'(A) = \frac{\omega(PAP^*)}{\omega(P^*P)}$$

P: projection operators w.r.t. an eigenspace of $A^* = A$.

Schrödinger's equation.

$(\mathcal{A}, \alpha_t) \qquad \alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$.

$\omega. \Rightarrow (\mathcal{H}_\omega, \pi_\omega, \Omega)$ unique up to unitary transformation

$$\pi_\omega(\alpha_t(A)) = \mathcal{U}^*_{(t)} \pi_\omega(A)\, \mathcal{U}(t)$$

$$= \exp(iH_\omega t)\, \pi_\omega(A) \exp(-iH_\omega t)$$

$$= \pi_{\mathcal{U}\omega}(A)$$