Chap 3.

part 1 : C^*- opera construction on infinite spin systems

quasi-local algebra, reps and translational symmetry

infinite Lattice

naively. $\bigotimes_{i\in\mathbb{Z}} \mathcal{H}_i$ as the Hilbert Space.

$\hookrightarrow$ $\langle \psi, \xi \rangle = \prod_{i\in\mathbb{Z}} \langle \psi_i, \xi_i \rangle_{\mathcal{H}_i}$ does not converge

way out. 1. choose reference vector $\Omega_i \in \mathcal{H}_i$ $\forall i$.

and consider only these sequences ψ s.t. $\psi_i \neq \Omega_i$ for finitely many i

$\cdots \uparrow\uparrow \cdots \uparrow\uparrow$, $\Omega = |\uparrow \cdots \uparrow\rangle$ $\underline{\psi = |\text{finitely many } \downarrow\text{'s}\rangle}$

$\hookrightarrow$ completion $\mathcal{H}_\Omega \longrightarrow \boxed{\mathcal{A} = \mathcal{B}(\mathcal{H}_\Omega)}$

$\downarrow$

$\underline{\text{no local structure}}$ $\rightarrow$ support on infinite space ?

remark: different Ω's often corresponds to inequiv. reps of opera

$[\sigma_n^i, \sigma_m^j] = 2\delta_{nm}\, i \sum_{k=1}^{3} \varepsilon_{ijk}\, \sigma_n^k$

$\Omega_1 = |\uparrow \cdots \uparrow\rangle \rightarrow$ rep. 1

$\Omega_2 = |\downarrow \cdots \downarrow\rangle \rightarrow$ rep 2

$\updownarrow \not\exists\, U$

2. quasi-local opera from local opera

What is the tensor product of two operas ?

C^*- algebra $=$ $*$-algebra $+$ completion w.r.t some norm

Def: Suppose $\mathcal{A}, \mathcal{B}$ are $*$-alg.

then. their algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ consists of linear combinations

of elements $A \otimes B$. $A, B \in \mathcal{A}, \mathcal{B}$.

$\mathcal{A} \odot \mathcal{B}$ is a $*$-alg. by setting

$(A_1 \otimes B_1)(A_2 \otimes B_2) := A_1 A_2 \otimes B_1 B_2$

$(A \otimes B)^* := A^* \otimes B^*$

$\mathcal{A} \odot \mathcal{B}$ is a C^*-alg. by some completion

Γ labels the sites of the system

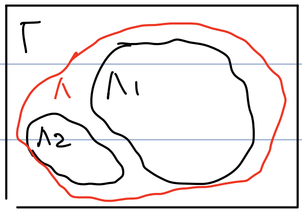
$\mathcal{P}(\Gamma)$ is the set of all subsets of Γ

$\mathcal{P}_f(\Gamma)$ is the subset of $\mathcal{P}(\Gamma)$. where all its elements are finite subsets

assume that $\forall x \in \Gamma$. there is an opera $\mathcal{A}(\{x\}) := M_d(\mathbb{C})$

if $\Lambda \in \mathcal{P}_f(\Gamma)$. $\mathcal{A}(\Lambda) = \bigotimes_{x\in\Lambda} \mathcal{A}(\{x\})$

which gives an assignment $\Lambda \mapsto \mathcal{A}(\Lambda)$ $\forall \Lambda \in \mathcal{P}_f(\Gamma)$



prop: this assignment preserves a local net structure

if $\Lambda_1 . \Lambda_2 \in \mathcal{P}_f(\Gamma)$. and $\Lambda_1 \cap \Lambda_2 = \phi$.

then $[\mathcal{A}(\Lambda_1). \mathcal{A}(\Lambda_2)] = \{0\}$

here $\mathcal{A}(\Lambda_i)$ $i=1.2$ is embedded in to a sufficiently large Λ

local observables . $\mathcal{A}_{loc} = \bigcup_{\Lambda\in\mathcal{P}_f(\Gamma)} \mathcal{A}(\Lambda)$

$\rightarrow$ completion w.r.t norms on each $\mathcal{A}(\Lambda) \rightarrow$ C^* algebra called $\mathcal{A}$

(quasi-local algebra)

additional construction

if $A \in \mathcal{A}(\Lambda)$ for some $\Lambda \in \mathcal{P}_f(\Gamma)$ A is localized in Λ

the smallest Λ is called $\mathrm{supp}(A)$ we require $\mathcal{A}(\phi) = \mathbb{C}\mathbb{1}$

$\Lambda \in \mathcal{P}(\Gamma)$'s complement is denoted as Λ^c

Q: can A be localized in infinite regions? . ($\Lambda \in \mathcal{P}(\Gamma)$ $\Lambda \notin \mathcal{P}_f(\Gamma)$)

A: yes. $\mathcal{A}(\Lambda) = \overline{\bigcup_{\Lambda_f\in\mathcal{P}_f(\Lambda)} \mathcal{A}(\Lambda_f)}^{\|\cdot\|}$ $\nwarrow$ completion same as quasi-local algebra

$\downarrow$

is $\mathcal{A}(\Gamma)$ quasi local algebra ?

Representation .

prop: $\mathcal{A}(\mathbb{Z}^d)$ (quasi-local algebra on $\mathbb{Z}^d$) is simple. $\nearrow$ a C^*-alg $\mathcal{A}$ is simple iff only closed two-sided ideals are $\{0\}$ and $\mathcal{A}$

cor: every non-zero rep of $\mathcal{A}(\mathbb{Z}^d)$ is faithful and hence isometric

$\mathcal{A}(\mathbb{Z}^d) \subset \mathcal{B}(\mathcal{H}_\Omega)$

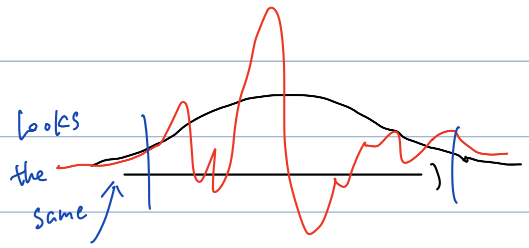
we don't need to do the entire GNS construction

equivalence under GNS rep. ?

prop: $\mathcal{A} := \mathcal{A}(\Gamma)$ is a quasi-local opera

and $\omega_1 . \omega_2$ are pure states on $\mathcal{A}$.

the following criteria are equivalent:



1. corresponding GNS reps of $\omega_1 . \omega_2$ are equivalent

2. $\forall \varepsilon > 0$ $\exists \Lambda_\varepsilon \in \mathcal{P}_f(\Gamma)$ s.t.

$$\underline{|\omega_1(A) - \omega_2(A)| < \varepsilon \|A\|}$$

$$\forall A \in \mathcal{A}(\Lambda) \text{ with } \Lambda \in \mathcal{P}_f(\boxed{\Lambda_\varepsilon^c})$$

→ if we restrict to observables "far away" the two states look the same

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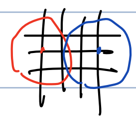
Translation symmetry

if translation symmetry exists on $\Gamma = \mathbb{Z}^d$

we can define a map $\tau_y : \mathcal{A}_{loc} \to \mathcal{A}_{loc}$ $\forall y \in \mathbb{Z}^d$



by defining $\tau_y(A(x)) = A(x+y)$

this map can be extended to an automorphism of $\mathcal{A}(\mathbb{Z}^d)$

$$\tau_y(\mathcal{A}(\Lambda)) = \mathcal{A}(\Lambda + y) \quad \forall \Lambda \in \mathcal{P}(\mathbb{Z}^d)$$

$\Rightarrow \tau : \mathbb{Z}^d \to \mathrm{Aut}(\mathcal{A}(\mathbb{Z}^d))$ is a group homomorphism

translation sym $\begin{cases} \text{site.} \\ \text{opera} \\ \text{state} \end{cases}$

translation invariant state can be defined as a state ω.

such that $\omega(\tau_x(A)) = \omega(A)$ $\forall x \in \mathbb{Z}^d$ and $A \in \mathcal{A}$

ω has the property that. $\exists$ a unitary representation

$x \mapsto U(x)$. $U(x)$ is a unitary in GNS rep w.r.t ω

implementing the translation

$$\pi(\tau_x(A)) = U(x)\, \pi(A)\, U(x)^*$$

asymptotic Abelianess of quasi-local algebra,

if we move a local operator far enough.

it will commute with any other local operator

Thm: for a quasi-local $\mathcal{A}(\mathbb{Z}^d)$ with $x \to \tau_x$, as the action of translation group. then $\forall A, B \in \mathcal{A}(\mathbb{Z}^d)$,

$$\lim_{|x| \to \infty} \| [\tau_x(A), B] \| = 0$$

part 2. time translation, dynamics. $\to$ Hamiltonian

$\downarrow$

GS, & KMS state