Ref: arxiv 1904.04051

Algebraic QFT $\leftarrow$ Axiom

① Wightman Axioms

$\mathcal{H}$. ϕ: field is operator-valued distribution is spacetime.

$f(x): \mathbb{R}^4 \to \mathbb{R} / \mathbb{C}$ $\phi: f \mapsto$ operator on $\mathcal{H}$.

Intuitively: $\phi(f) = \int d^4x \, f(x) \underset{\sim}{\phi(x)}$.

$(\Box^2 + m)\phi = 0$

Properties:

a) Rep. of Poincaré Group.

b) Spectral condition

Spectrum of $\underline{P}$ $\subset$ forward lightcone.

generator of rep. in (a)

① $E > 0$

② $|E| \leq |\vec{P}|$.

c) $\exists!$ Vacuum $|0\rangle \in \mathcal{H}$ Invariant under Poin...

d) Causality if O_1, O_2 spatially-separated regions.

$\mathrm{supp}\, f_1 \subset O_1$, $\mathrm{supp}\, f_2 \subset O_2$, $[\phi(f_1), \phi(f_2)]_\pm = 0$

e) $\langle \phi(f_1)\phi(f_2)\cdots\phi(f_n)|0\rangle\rangle$ dense in $\mathcal{H}$.

Difficulty: ϕ not necessarily bounded.

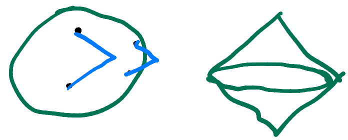
② Haag-Kastler Axioms

Do not assume $\mathcal{H}$!

algebra of observables

a) $\forall$ causally closed O, $\exists$ C^*-algebra $A(O)$

convex w.r.t. ~~light~~ time-like path.



b) Net structure: $O_1 \subset O_2$, $A(O_1) \hookrightarrow A(O_2)$.

c) Locality: O_1, O_2 spatially sep., $[A(O_1), A(O_2)] = 0$

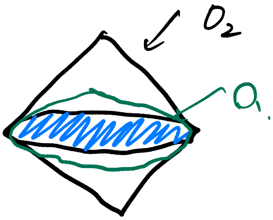
d) Poincaré. $\forall \rho \in \mathcal{P}$ ← P. group,

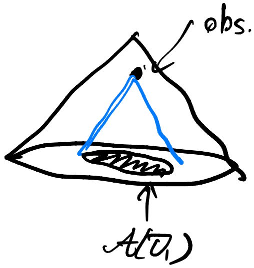
$\exists$ isomorphism $\alpha(\rho) : A(O) \to A(\rho O)$

e) If $O_1 \subset O_2$, O_1 contains Cauchy surface of O_2

any worldline in O_2 must intersect

then $A(O_1) = A(O_2)$





Ground State / Vacuum.

$\exists$ state w. vacuum. $\quad$ $\mathbb{M}$: spacetime.

1) w is Poincaré invariant.

$\hookrightarrow$ $w(\alpha(\rho)A) \equiv w(A)$ $\quad \forall \rho \in \mathcal{P}$

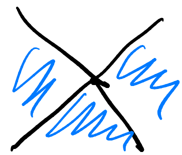
$A \in \mathcal{A}(\mathbb{M})$

2) $\forall$ $Q = \int f(x)\, \underbrace{\beta_x}_{\text{translation}}(A)\, d^4x$

Fourier trans. $\underline{f(k)}$ support outside lightcone

arbitrary obs.

$w(Q^* Q) = 0.$?



k

Take GNS construction w.r.t. w.

1) $\Rightarrow$ Poincaré group unitarily represented on $\mathcal{H}_w$.

SNAG Theorem $\quad \underset{\text{unitary}}{U}(\vec{t})$. $\quad \vec{t} \in \mathbb{R}^n$, strongly continuous.

$\exists$ projection-valued measure $dP(\vec{\lambda})$

s.t. $U(\vec{t}) = \int_{\mathbb{R}^n} e^{i\vec{t}\cdot\vec{\lambda}}\, \underline{\underline{dP(\vec{\lambda})}}$

$\sim \int d^4p\; e^{ip\cdot t}\, |p\rangle\langle p|.$

Poincaré rep. $U(x) \sim e^{iP_\mu x^\mu}$.

supp $P(\vec{x})$. $\Longrightarrow$ spectrum of P^μ.

Further: 2) spectrum of $P^\mu \subset$ forward lightcone.

(1311 arxiv sec 5.2).

Superselection sectors.

Symmetry, conserved charge. Q.

$$\langle Q=0 \,|\, \underline{\text{any observable}} \,|\, Q=1 \rangle = 0.$$

$$\text{rep. of } \mathcal{A} = \begin{array}{cc} Q=0 & Q=1 \\ \end{array}$$
$$\underline{\begin{pmatrix} & 0 \\ 0 & \end{pmatrix}} \Longrightarrow \text{reducible.}$$

"Charge sector" $\Longleftrightarrow$ inequivalent irreps of $\mathcal{A}$.

$\boxed{\text{DHR condition}}$

A rep π of $\mathcal{A}(M)$ is DHR acceptable if

$\underline{\forall \text{ caus. closed } D,}$ $\quad \pi|_{\underline{\mathcal{A}(D')}} \overset{U}{\simeq} \underline{\pi_0}|_{\mathcal{A}(D')}$

complement of D. vacuum GNS rep.

intuitively: unitarily, excitation transport into O.

U: transport excitations in π into O.

Limitation: theory cannot be long-ranged.

e.g. EMF. Gauss Law.

Excitation as Endomorphisms.

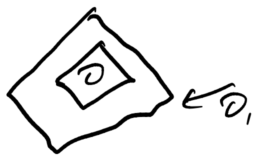
Fix O. Consider a DHR rep, π.

$$\pi|_{A(O')} \overset{V}{\simeq} \pi_0|_{A(O')}.$$

$$\tilde{\pi}(A) = V^{-1}\pi(A)V. \qquad \tilde{\pi}(A(O')) = \pi_0.$$

rep on $\mathcal{H}_0$

Choose $O_1 \supset O$.



$\forall A \in A(O_1)\ \forall B \in A(O_1') \subset A(O')$.

$$\tilde{\pi}(B) = \pi_0(B).$$

$$[\tilde{\pi}(A), \pi_0(B)] = [\tilde{\pi}(A), \tilde{\pi}(B)] = \tilde{\pi}([A,B]) = 0.$$

$$[\tilde{\pi}(A(O_1)), \pi_0(A(O_1'))] = 0.$$

$$\Rightarrow \tilde{\pi}(A(O_1)) \subset \pi_0(A(O_1'))' \overset{\text{Haag Duality}}{=} \pi_0(A(O_1))$$

$X' = \{ Y \in \text{operators} \mid [Y, X] = 0 \}$

$\tilde{\pi}(\mathcal{A}(\mathcal{O}_1)) \subset \pi_0(\mathcal{A}(\mathcal{O}_1))$.

Endomorphism $\rho_\pi := \pi_0^{-1} \circ \tilde{\pi} : \mathcal{A} \to \mathcal{A}$.

depend on choice of $\mathcal{O}$.

$\rho_1 \sim \rho_2$ if $\rho_1 = \iota \circ \rho_2$

ι inner automorphism, $\iota(A) = UAU^{-1}$.

Property (Transportability).

$\forall$ ρ defined on $\mathcal{O}$, $\forall$ $\mathcal{O}$,

$\exists$ U, localized in $\mathcal{O}_2 \supset \mathcal{O} \cup \mathcal{O}$,

s.t. $\iota_U \circ \rho$ defined on $\mathcal{O}$.

Prop. if ρ is defined on $\mathcal{O}$, then

$\rho|_{\mathcal{O}'} = \mathrm{id}$.

Particle Statistics

Prop. if $\mathcal{O}_1, \mathcal{O}_2$ satisfy $\mathcal{O}_1 \subset \mathcal{O}_2'$.

$\rho_1 \in \Delta(\mathcal{O}_1)$ all endomorphism localized in $\mathcal{O}_1$

$\rho_2 \in \Delta(\mathcal{O}_2)$.

then $\rho_1\rho_2 = \rho_2\rho_1$.

What if

$\rho_1, \rho_2 \in \Delta(\mathcal{O})$

Prop: $\rho_1\rho_2 \sim \rho_2\rho_1$

Proof: Choose U_1 s.t. $l_{U_1} \circ \rho_1 \in \Delta(\mathcal{O}_1)$

similarly choose $\mathcal{O}_2$.

$\mathcal{O}_1, \mathcal{O}_2$ separated.

let $A \in \mathcal{A}(\mathcal{O})$. let $\mathcal{O}_1, \mathcal{O}_2$ away from $\mathcal{O}$.

$\rho_1\rho_2(A)$

$l_{U_i} \circ \rho_i(A) = A$.

$= \rho_1 \circ l_{U_2^*}(A) = \rho_1 U_2^* A U_2$ $\quad \rho_i(A) = l_{U_i^*}(A)$

$= l_{\rho_1(U_2^*)} \circ l_{U_2^*}(A) = \rho_1(U_2^*) U_2^* A U_2 \rho_1(U_2)$?

$= l_U \circ \rho_2\rho_1(A)$

<u>Lv</u> intertwiner.