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A bite of random matrices

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Abstract: A little bite of random matrices, including some not standard contents. We will discuss motivation, research status and properties of random matrices. At the end, we propose a very toy example to show how properties of random matrices are related to entanglement dynamics.

Contents

1 Why random

Let's start with an opening question: why we are using random models (Hamiltonians) in studying many-body dynamics?

We have lots of example: Anderson localization model with random disorder terms, SYK model with random coupling, quantum circuit model with random gates, etc. It is usually OK if we fix the "seed" to de-randomize the models and still get the dynamical properties we want. But I think there are two good reasons to keep randomness:[1]

1. Random models are easy to deal with. For the case of SYK and quantum circuits, random parameters ensure a high symmetry, which make it possible for us to get some analytical results. Numerically, fixed seed will lead to fluctuations, which are canceled by averaging.

2. In the ensemble of random models, almost every model shares some similar properties. So random models can faithfully reflect those properties. They are also referred as *universal* properties. A consequence is that universal properties should be insensitive to change of the distribution that random parameters obey. We may expect a mild dependence on lower moments.

An extreme way to make things random is by assuming a completely random Hamiltonian, dropping all details of interactions. Gaussian random matrices are good examples. We can analytically calculate many properties of them. We will have more discussion in 3.

2 Studies on random matrices

Speaking of random matrices, we can mean in a general sense or a specific sense:

- In a general sense, all physical models with random Hamiltonians can be regarded as random matrices. (Well, they are matrices and they are random.)

[1] Not so true in quantum gravity, where the ensemble plays a more essential role.

- In a special sense, we focus on Gaussian random matrices. And because of universality, Gaussian random matrices are expected to have similar properties as almost all random matrices.

When people say something about random matrices, they are usually referring to the special sense.

Many research fields are related to random matrix theory.

- Physics.
  - Nuclear physics: Wigner's early work.
  - Quantum gravity: [CGAH+17].
  - Quantum many-body dynamics.
- Math: works by Erdos et al. [EY12] and by Tao et al. And Tao wrote a good textbook [Tao]. They are studying properties of random matrices, including universality.
- Statistical machine learning theory.

3 Properties of Gaussian random matrices

3.1 Definitions

Gaussian random matrices are matrices whose entries obey an i.i.d. Gaussian distribution. The ensemble of Gaussian random matrices is called β-Gaussian ensemble, where β denotes symmetry. If $\beta = 1$, the entries are real number; if $\beta = 2$, the entries are complex number; if $\beta = 4$, the entries are quaternion number.

3.2 Eigenstates

To study the properties of Gaussian random matrices, we apply eigendecomposition to them:

$$H = U \Lambda U^{-1}, \tag{3.1}$$

where Λ contains eigenvalues, U contains eigenstates. And we will discuss the properties of eigenstates in this section and properties of eigenvalues later.

Note that Gaussian random matrices have a high symmetry. By applying an arbitrary unitary matrix U_0, we will not change Gaussian random matrices. So by definition, U follows Haar distribution.

A property of Haar random unitary matrices at large N is that if we apply it on tensor states like $|0\rangle^{\otimes N}$, we will almost surely get a maximal entangled states. So Haar random unitary matrices have a strong entangling ability.

Note that usually for physical models, not all eigenstates have strong entanglement. In most cases, only eigenstates at middle energy spectrum have volume-law entanglement. And eigenstates at the ends have area-law entanglement. Random matrix theory cannot describe the phenomena.

3.3 Eigenvalues

Level repulsion See [EKR18] or [Meh04] for details. Note that we only need the assumption of Haar measure of U, so it can be extended to random matrices other than Gaussian.

For Gaussian random matrices, level spacing distribution is referred to Wigner's surmise

$$P(s) \sim s^{\beta} e^{-\alpha_{\beta} s^2}. \tag{3.2}$$

An exact derivation of (3.2) can be found in [Meh04].

Spectral form factor. Spectral form factor is a quantity closely related to level statistics. It is defined as $\langle \frac{Z(\beta+it)Z(\beta-it)}{|Z(\beta)|^2} \rangle$, where Z is thermal partition function. Details can be found in [CGAH+17].

References

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