Magic

Saturday, June 11, 2022 12:40 PM

Opening question: magic of quantum advantages?

Entanglement ? Not really.

Outline:

1. Stabilizer state and Clifford group (What is not magic?).
2. Quantum magic.
3. Some theorems.
4. In many body physics $\begin{cases} \text{SPT.} \\ \text{OTOC.} \end{cases}$

---

1. Stabilizer states and Clifford group

$|\psi\rangle \in \mathcal{H}(2^n)$

$P|\psi\rangle = +|\psi\rangle$

$\{P_1, P_2, \ldots, P_n\}$

s.t. $P_i|\psi\rangle = +|\psi\rangle$

$U|\psi\rangle$

$\{U P_1 U^\dagger, U P_2 U^\dagger, \ldots, U P_n U^\dagger\}$

U: Clifford group

$|\psi\rangle = |0 \ldots 0\rangle \quad P_1 = +Z \otimes I \cdots$

$$P_2 = +I \otimes Z \otimes \cdots$$

$$\vdots$$

$$P_n = +I \otimes \cdots \otimes Z$$

$$\begin{pmatrix} Z & I & \cdots & I \\ I & Z & \cdots & - \\ & \vdots & & \\ I & I & \cdots & Z \end{pmatrix} \longrightarrow \begin{pmatrix} X & Z & \cdots & I \\ \cdots & - & - & - \\ - & & - & - \\ \cdot & & \cdot & - \end{pmatrix}$$

$n \times n$

(QuantumClifford.jl)

- Graph states $\longrightarrow$ n

2. Quantum magic

(Lin 2022)

(a) Min-relative entropy of magic:

$$\mathfrak{D}_{\min}(\rho) = \min_{\sigma \in \mathrm{STAB}} D_{\min}(\rho\|\sigma)$$

with the min-relative entropy $D_{\min}(\rho\|\sigma) := -\log \operatorname{Tr} \Pi_\rho \sigma$, where Π_ρ is the projector onto the support of ρ. For a pure state $|\psi\rangle$, $\mathfrak{D}_{\min}(\psi) = -\log \max_{\phi \in \mathrm{STAB}} |\langle\psi|\phi\rangle|^2$.

(b) Max-relative entropy of magic:

$$\mathfrak{D}_{\max}(\rho) = \min_{\sigma \in \mathrm{STAB}} D_{\max}(\rho\|\sigma)$$

with the max-relative entropy $D_{\max}(\rho\|\sigma) := \log \min\{\lambda : \rho \leq \lambda\sigma\}$, where the matrix inequality $\rho \leq \lambda\sigma$ means that $\lambda\sigma - \rho$ is positive semidefinite. This measure is also known as log-generalized robustness, $\mathfrak{D}_{\max}(\rho) = \log(1 + R_g(\rho))$, where

$$R_g(\rho) = \min s \geq 0 \quad \text{such that} \quad \rho \in (1+s)\,\mathrm{STAB} - s\,\mathcal{S}.$$

Set of all states

Here the subscript "g" is a label for "generalized robustness," signifying its difference with the free-robustness measure that will also be discussed.

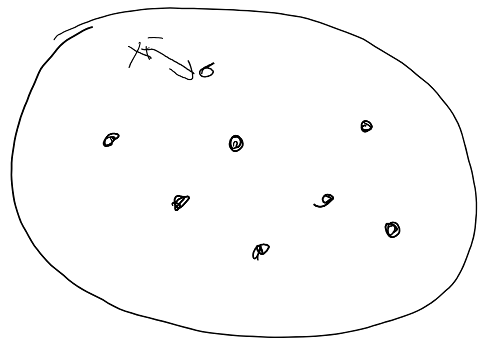
(c) Free robustness of magic:

$$R(\rho) = \min s \geq 0 \quad \text{such that} \quad \rho \in (1+s)\,\mathrm{STAB} - s\,\mathrm{STAB}.$$

The log-free robustness is $\mathrm{LR}(\rho) = \log[1 + R(\rho)]$.

Note that, for any state ρ, it holds that

$$\mathfrak{D}_{\min}(\rho) \leq \mathfrak{D}_{\max}(\rho) \leq \mathrm{LR}(\rho). \tag{1}$$



3. Some theorems

① Upper bounds

$$\mathfrak{D}_{\max}(\mathrm{SEP}) = \mathfrak{D}_{\min}(\mathrm{SEP}) \overset{\leq}{=} [\log(3-\sqrt{3})]n \approx 0.34n,$$

attained on the tensor product of the "golden state" $G = \frac{1}{2}[I + (X+Y+Z)/\sqrt{3}]$ [17], due to weak additivity. Note that these measures carry fundamental operational interpretations in terms of value in transformations. How large can they get when we consider general states?

First, observe that the value of $\mathfrak{D}_{\max}$ or log-generalized robustness (and so of all entropic measures) is capped at n.

Theorem 1. *On an n-qubit system,* $\max_\rho \mathfrak{D}_{\max}(\rho) \leq n$.

② Magic of random pure states

Theorem 3. *Let $|\psi\rangle$ be a random n-qubit state drawn from the Haar measure. Then, for any $n \geq 6$,*

$$\Pr\{\mathfrak{D}_{\min}(\psi) \leq \gamma\} < \exp(0.54n^2 - 2^{n-\gamma}). \qquad (11)$$

③ Pauli MBQC (Too much is bad.)
(Measurement Based)

Theorem 5. *Pauli MBQC with any n-qubit resource state $|\Psi\rangle$ with $\mathfrak{D}_{\min}(\Psi) \geq n - O(\log n)$ cannot achieve superpolynomial speedups over* BPP *machines (classical randomized algorithms) for problems in* NP.

④ t-design $\frac{1}{|G|}\sum_{g \in G} U(g)\, \rho^{\otimes t}\, U^{\dagger}(g) = \int_{\text{Haar}} dU\, U \rho^{\otimes t} U^{\dagger \otimes t}$

Clifford group : 2-design , ✗ 3-design

Achieve t-design, how much magic?

- When do (approximate) *t*-designs achieve maximal magic?
  - $\tilde{O}[t^4 \log(1/\epsilon)]$ non-Clifford gates → *t*-design ⇒ $t = \tilde{\Omega}\left(n^{1/4}\right)$ [Haferkamp *et al.* '20]
  - Conjecture in light of entanglement theory $t = O(n)$ [ZWL *et al.* PRL'18, JHEP'18]
  - Good probe for **circuit complexity growth**? Saturation expected to happen in poly time regime [Brown/Susskind '14] [Haferkamp *et al.* NP'22]... Complexity: exp time regime

15

4. In many-body physics

① SPT

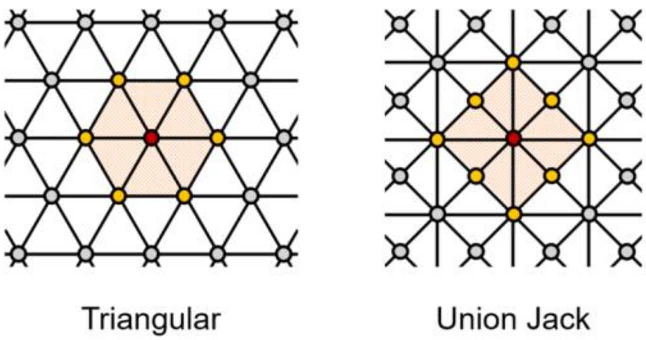
1.

(B) SPT

4

concreteness, think about the well-known Levin-Gu state $|\Psi_{\mathrm{LG}}\rangle$ [41] defined on the 2D triangular lattice (see Fig. 1), which takes the form

$$|\Psi_{\mathrm{LG}}\rangle = U_{CCZ} U_{CZ} U_Z H^{\otimes n} |0\rangle^{\otimes n}, \tag{38}$$



(a) Triangular lattice: $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) < 0.56n$.
(b) Union Jack lattice: $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) < 0.46n$.

of the derivation. Note that we expect the above bounds to be loose, and it can likely be shown that $\mathfrak{D}_{\max}(\hat{\Psi}) = \mathfrak{D}_{\min}(\hat{\Psi}) \leq [2 - (2/3)\log 6]n \lesssim 0.28n$ for all regular triangulated lattices (also see Appendix C for

SEP $< 0.34n$

- Result (informal): A large class of SPT phases in ≥2D must have symmetry-protected magic

One case:

Theorem [Ellison/Kato/ZWL/Hsieh, Quantum'21]

Alt+M或按住空格键开启麦克风

Any state in group cohomology SPT phases in ≥2D protected by $G = \mathbb{Z}_q^m$ symmetry (represented by Pauli strings) is magical, assuming it is defined on q-dimensional qudits.

Typical example: Z_2 Levin-Gu [Levin/Gu, PRB'12]

Magic in SPT phases

Theorem [Ellison/Kato/ZWL/Hsieh, Quantum'21]

Any state in group cohomology SPT phases in ≥2D protected by $G = \mathbb{Z}_q^m$ symmetry (represented by Pauli strings) is magical, assuming it is defined on *q*-dimensional qudits.

Remarks:

- Alternative: "Symmetry-protected" magic is an intrinsic property of such SPT phases (cannot be removed by symmetric finite-depth circuits)
- Previous talk: circumvents the obstruction by considering higher dimensional qudits [Ellison *et al.*, PRXQ'22]
- A stronger result for "decorated domain wall" SPTs: the restriction on qudit dimension can be removed
- Being considered here are 0-form symmetries. Would be interesting to consider subsystem and higher-form symmetries.

23

② OTOC

Consider the Pauli group on n qubits $\mathbb{P}(n)$ with elements P. Any state ρ in the $d \equiv 2^n$-dimensional Hilbert space can be decomposed in the Pauli basis as $\rho = d^{-1}\sum_{P\in\mathbb{P}} \mathrm{tr}(P\rho)P$. Define the purity functional $\mathrm{Pur}(x) \equiv \mathrm{tr}\, x^2$, and a probability distribution over the coefficients of such expansion by $\Xi_\rho := \{\mathrm{Pur}^{-1}(\rho)d^{-1}\,\mathrm{tr}^2(P\rho)\}_P$. Note that $\Xi_\rho(P) \geqslant 0$ and sum to one. The α-Stabilizer Rényi entropy is defined as[1]:

$$M_\alpha(\rho) := S_\alpha(\Xi_\rho) + S_2(\rho) - \log d \qquad (1)$$

magic

- When do (approximate) *t*-designs achieve maximal magic?

We can naturally associate a stabilizer Rényi entropy to unitary operators U through the Choi state $|U\rangle := (\mathbb{1}\otimes U)|I\rangle$, where $|I\rangle := d^{-1/2}\sum_i |i\rangle\otimes|i\rangle$. Let

- When do (approximate) *t*-designs achieve maximal magic?

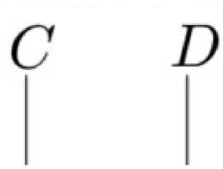
$\tilde{O}[t^4 \log(1/\epsilon)]$ non-Clifford gates → *t*-design ⇒ $t = \tilde{\Omega}(n^{1/4})$ [Haferkamp *et al.* '20]

$t = O(n)$ [ZWL *et al.* PRL'18, JHEP'18]

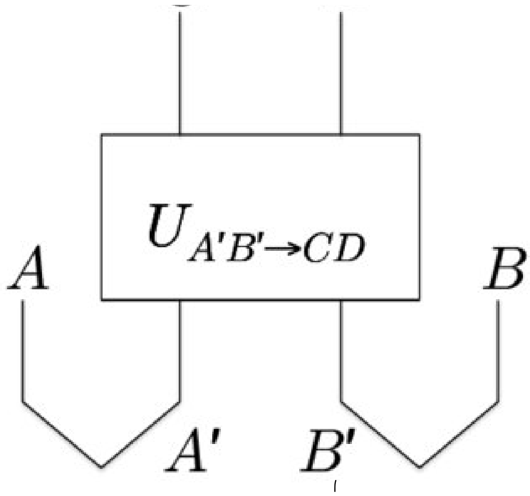
pected to happen in poly time regime

Complexity: exp time regime

15



$U\ A'B' \to CD$



$U_{A'B'\to CD}$

$\Big\Downarrow$ Choi

$|\psi\rangle_{ACBD}$

Lemma 2. *The α-stabilizer Rényi entropy of* $|\mathbb{U}\rangle$, *for* $\alpha > 1$, *equals the* 2α*-points out-of-time order correlator*

choi state

channel

$$M_\alpha(|\mathbb{U}\rangle) = \frac{1}{1-\alpha} \log \mathrm{OTOC}_{2\alpha}(\mathbb{U}) \qquad (3)$$

where $\mathrm{OTOC}_{2\alpha} := d^{-2} \sum_{P,P'} \mathrm{otoc}_{2\alpha}(P, P')$, *and* $d \times \mathrm{otoc}_{2\alpha}(P, P') := \mathrm{tr}[\langle P_{2\alpha} \prod_{i=1}^{2\alpha} P^{(\mathbb{U})} P' P_{i-1} P_i \rangle]$ *with* $P_0 \equiv \mathbb{1}$ *and* $\langle \cdot \rangle$ *is the average over* $P_1, \ldots, P_{2\alpha}$.

$P_{2\alpha}\, P^{(u)} P' P_0 P_1 P^{(u)} P' P_2 P_1 \ldots$

$P^{(u)} = U P U^{\dagger}$