INVITATIONALS SAMPLE PROBLEMS

- This is a set of sample problems selected to represent the difficulty range of the actual test.
- Ordering is based on difficulty, which increases as the question number increases.
- No aids are permitted other than pencils, scratch papers, graph papers, rulers, compasses, and erasers. No calculators, smartwatches, or other computing devices are allowed. No problems on the test will require the use of a calculator.
- Figures are not necessarily drawn to scale.

PROBLEMS

Problem 1

Compute $2021 \times 2021 - 2040 \times 2002$.

Problem 2

Suppose there exists an n-gon with k angles that measure $180 - n$ degrees. What is the maximum value of k?

Problem 3

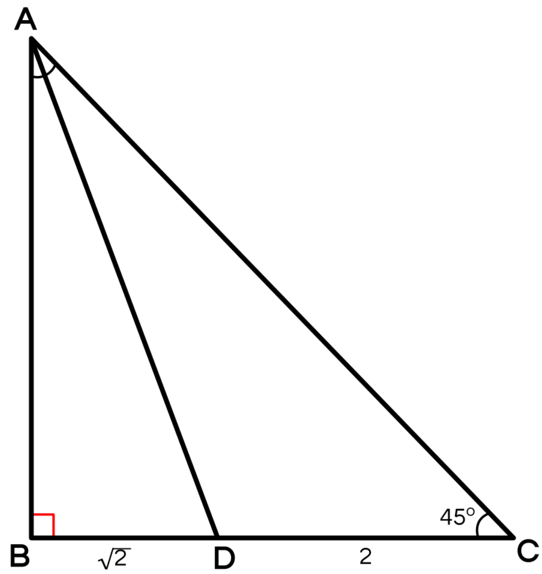
Define a sequence recursively by $x_1 = 2021$ and, for all $n \geq 2$, x_{n+1} is the hypotenuse of a right triangle with legs x_1 and x_n. If $x_{26} - x_{10} = 4042$, what is x_{17}?

Problem 4

5 people are holding a water gun. Each person randomly picks someone else and shoots them. Bob and Joe, two of the 5 people, each get scared with probability $\frac{1}{2}$ and decide not to shoot someone else. What is the expected number of people that do not get shot? If your answer can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers, find $m+n$.

Problem 5

In isosceles right triangle ABC, $\angle ABC = 90°$. D is on BC such that $BD = \sqrt{2}$ and $DC = 2$. Find $10 \cdot \angle ADC$ in degrees.



SOLUTIONS

Solution to Problem 1

Answer 361: Notice that $2040 = 2021 + 19$ and $2002 = 2021 - 19$. Thus, we can use difference of squares to get

$$2040 \cdot 2002 = (2021 + 19)(2021 - 19) = 2021^2 - 19^2.$$

So, the expression we want to compute is

$$2021^2 - (2021^2 - 19^2) = 19^2 = \boxed{361}.$$

Solution to Problem 2

Answer 18: The sum of the measures of the exterior angles of a polygon must add up to 360°, so the sum of the measures of the k exterior angles of our polygon must be less than or equal to 360. The exterior angles are each $180 - (180 - n) = n$ degrees. Thus, we have

$$kn \leq 360.$$

We must have that $k \leq n$, since a polygon with n sides has at most n angles. Therefore, $k^2 \leq nk$, so we have $k^2 \leq 360$. Since $19^2 = 361$, we know that k is at most $\boxed{18}$. An example that works is an 19-gon with 18 exterior angles of 19° and one exterior angle of 18°.

Solution to Problem 3

Answer 8084: The given condition implies

$$x_{n+1}^2 = x_n^2 + 2021^2 \text{ for all } n \geq 2.$$

So, we have

$$\begin{aligned} x_{26}^2 &= x_{25}^2 + 2021^2 \\ &= x_{24}^2 + 2 \cdot 2021^2 \\ &\vdots \\ &= x_{10}^2 + 16 \cdot 2021^2. \end{aligned}$$

Hence,

$$16 \cdot 2021^2 = x_{26}^2 - x_{10}^2 = (x_{26} - x_{10})(x_{26} + x_{10}).$$

Since $x_{26} - x_{10} = 2 \cdot 2021$, we have $x_{26} + x_{10} = 8 \cdot 2021$. Solving this system of equations yields

$$x_{26} = 5 \cdot 2021 \text{ and } x_{10} = 3 \cdot 2021.$$

We have

$$x_{17}^2 = x_{10}^2 + 7 \cdot 2021^2 = 9 \cdot 2021^2 + 7 \cdot 2021^2 = 16 \cdot 2021^2$$

, so $x_{17} = 4 \cdot 2021 = \boxed{8084}$.

Solution to Problem 4

Answer 3103: We proceed by Linearity of Expectation. Since 2 of the people have different probabilities of shooting another person, we can use LoE to compute the sum of the probabilities that each person gets shot or not. To do this, we can take cases on whether Bob/Joe does not get shot and if any of the other 3 people gets shot.

Case 1: The person is Bob/Joe. Consider Bob, since the situation for Joe is equivalent. Since each person other than Bob and Joe chooses one out of 4 people to shoot and 3 of these people are not Bob, the probability that Bob is not shot by a person other than Joe is $\frac{3}{4}$. Since each of these 3 people has to not choose Bob, they account for a probability of $\left(\frac{3}{4}\right)^3$. Now, we also need to take into consideration the probability that Joe shoots Bob. He first chooses to shoot with probability $\frac{1}{2}$ and then chooses Bob with probability $\frac{1}{4}$, so the probability he doesn't shoot Bob is $\frac{7}{8}$. Now since Bob and Joe have equivalent situations, this sums to

$$2 \cdot \left(\frac{3}{4}\right)^3 \cdot \frac{7}{8} = \frac{189}{256}.$$

Case 2: The person is not Bob or Joe. Name this person Ann. Since both Bob and Joe can shoot Ann, the probability that neither of them shoots Ann is $\left(\frac{7}{8}\right)^2$. There are 2 people other than Bob, Joe, and Ann, each of whom doesn't shoot Ann with probability $\frac{3}{4}$, so the probability that Ann is not shot by these two people is $\left(\frac{3}{4}\right)^2$. Since there are 3 people other than Bob and Joe, we multiply the probability that a single person does not get shot by 3 to get the expected value of

$$3 \cdot \left(\frac{7}{8}\right)^2 \cdot \left(\frac{3}{4}\right)^2 = \frac{1323}{1024}.$$

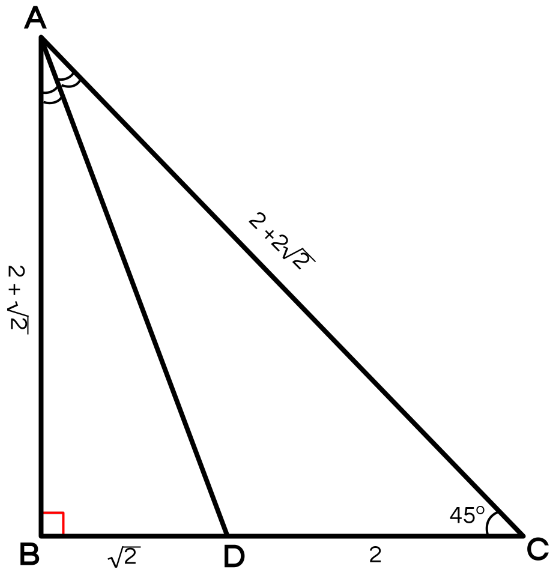
Adding up the expected values, we get

$$\frac{189}{256} + \frac{1323}{1024} = \frac{2079}{1024}$$

for an answer of $2079 + 1024 = \boxed{3103}$. ■

Solution to Problem 5

Answer 1125:



Since $\triangle ABC$ is a 45-45-90 triangle, we have $AB = 2+\sqrt{2}$ and $AC = \sqrt{2}\cdot(2+\sqrt{2}) = 2+2\sqrt{2}$. Notice that

$$\frac{AB}{BD} = \frac{2+\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}\cdot(2+\sqrt{2})}{\sqrt{2}\cdot\sqrt{2}} = \frac{2\sqrt{2}+2}{2} = \frac{AC}{CD}.$$

Thus, by the converse of the Angle Bisector Theorem, AD is an angle bisector of $\angle BAC$ and therefore $\angle DAC = \frac{45}{2} = 22.5°$. Since $\angle C = 45°$, we have $\angle ADC = 180 - 22.5 - 45 = 112.5°$, our answer is $10 \cdot 112.5 = \boxed{1125}$.