Empirical Arithmetic Constraints on the Euler Characteristic of Weighted-Projective Calabi–Yau Threefold Hypersurfaces and a Physics-Motivated Unifying Hypothesis

(compiled from Graffiti3 outputs)

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Abstract

We record a small collection of nontrivial, dataset-supported conjectures discovered by an automated conjecturing engine (Graffiti3) for Calabi–Yau threefold hypersurfaces in weighted projective space. The conjectures relate the Euler characteristic $\chi = 2(h^{1,1} - h^{2,1})$ and $h^{1,1}$ to elementary arithmetic features of the weight vector $\mathbf{w} = (w_1, \ldots, w_5)$, such as parity, repeated weights, and the divisibility of the hypersurface degree by the weights. We then propose a unifying physics interpretation in terms of Landau–Ginzburg (LG) orbifold selection rules and symmetry-enhanced twisted-sector contributions to moduli counts, and we outline concrete computational falsification experiments.

1 Dataset context and notation

1.1 Ambient construction

We consider Calabi–Yau threefold hypersurfaces $X_{\mathbf{w}} \subset \mathbb{P}^4(\mathbf{w})$ defined by a transverse weighted-homogeneous polynomial of degree

$$d \coloneqq \sum_{i=1}^{5} w_i,$$

in the standard quasi-smooth setting (finite classification in fixed dimension). See, e.g., the general finiteness statement for quasi-smooth weighted-projective Calabi–Yau hypersurfaces [1] and the Kreuzer–Skarke LG/weighted-projective data pages hosting the canonical lists [2, 3].

1.2 Hodge data and Euler characteristic

Each hypersurface has Hodge numbers $(h^{1,1}, h^{2,1})$, and we define

$$\chi \coloneqq \chi(X_{\mathbf{w}}) = 2\big(h^{1,1} - h^{2,1}\big).$$

In string compactifications, χ is physically meaningful: e.g. in early heterotic model-building under the “standard embedding” one often relates net chiral generations to $\chi/2$ (hence particular interest in small $|\chi|$, such as $\chi = \pm 6$) [5].

1.3 Arithmetic feature map

Throughout we assume the weights are sorted nondecreasingly:

$$w_1 \leq w_2 \leq w_3 \leq w_4 \leq w_5.$$

Definition 1 (Boolean features)**.** *We define:*

- ***SD*** *("sum divisible"):* $\mathrm{SD}(\mathbf{w})$ *holds if* $d \equiv 0 \pmod{w_i}$ *for all* i*.*
- ***EO*** *("exactly one even"):* $\mathrm{EO}(\mathbf{w})$ *holds if exactly one* w_i *is even.*
- ***REP*** *("has repeat"):* $\mathrm{REP}(\mathbf{w})$ *holds if* $\{w_1, \ldots, w_5\}$ *has a repeated value.*
- ***PC*** *("pairwise coprime"):* $\mathrm{PC}(\mathbf{w})$ *holds if* $\gcd(w_i, w_j) = 1$ *for all* $i < j$*.*

2 Filtering the Graffiti3 output: trivialities and the "open" core

Some automatically produced statements are tautological (e.g. inequalities that restate $\chi = 2(h^{1,1} - h^{2,1})$) or reduce to ubiquitous constraints such as $h^{1,1} \geq 1$ in the dataset. We therefore focus on the following *empirical* conjectures that appear nontrivial and are not merely definitional rewrites.

3 Open empirical conjectures

3.1 Divisibility ⇒ nonpositive Euler characteristic

Conjecture 1 (Divisibility–Nonpositivity)**.** *If* $\mathrm{SD}(\mathbf{w})$ *holds, then*

$$\chi \leq 0.$$

Equivalently, if $\chi > 0$ *then* $\mathrm{SD}(\mathbf{w})$ *fails.*

Why it is interesting. The condition $\mathrm{SD}(\mathbf{w})$ is purely arithmetic: it says each weight divides the anticanonical degree d. The conclusion $\chi \leq 0$ is a global topological constraint on the Calabi–Yau, i.e. a constraint on the *imbalance* between Kähler and complex-structure moduli.

3.2 A parity-conditioned linear lower bound on $h^{1,1}$

Conjecture 2 (Exactly-one-even implies a sharp $h^{1,1}$ threshold)**.** *If* $\mathrm{EO}(\mathbf{w})$ *holds, then*

$$\chi \geq -2h^{2,1} + \frac{1}{3}d - 2.$$

Using $\chi = 2(h^{1,1} - h^{2,1})$*, this is equivalent to the clean inequality*

$$h^{1,1} \geq \frac{d}{6} - 1.$$

Why it is interesting. This is a *linear* inequality linking a Hodge number to the (rescaled) degree $d = \sum w_i$, but only under a very specific parity pattern. Graffiti3 reported equality cases, suggesting near-optimality.

3.3 Two “threshold exclusion” implications

These were output as Sophie-style necessary conditions and can be viewed as companions to Conjecture 2.

Conjecture 3 ($h^{1,1}$ too small forces failure of pairwise coprimality)**.** *If*

$$h^{1,1} \leq \frac{d}{6} - 1,$$

then PC($\mathbf{w}$) *fails (i.e. the weights are* not *pairwise coprime).*

Conjecture 4 ($h^{1,1}$ strictly below threshold forbids exactly-one-even)**.** *If*

$$h^{1,1} < \frac{d}{6} - 1,$$

then EO($\mathbf{w}$) *fails.*

Why they are interesting. These conjectures suggest that the scalar threshold $d/6 - 1$ is not merely an artifact of one implication, but a robust boundary separating admissible arithmetic types of $\mathbf{w}$.

3.4 A repeat-weight slope inequality driven by w_3

Conjecture 5 (Repeat-weight slope bound)**.** *Assume* EO($\mathbf{w}$) *and* REP($\mathbf{w}$)*. Then*

$$14\, h^{1,1} \geq 13\, w_3 + 1.$$

Why it is interesting. Among very crude invariants of $\mathbf{w}$, the single coordinate w_3 is singled out by a remarkably specific rational slope 13/14. This is the kind of pattern that often indicates a hidden counting rule (e.g. a twisted-sector multiplicity that scales with w_3).

3.5 An extremal slice observation at $\chi = 960$

Observation 1 (Rigidity at maximal observed Euler characteristic)**.** *Within the dataset slice at* $\chi = 960$*, Graffiti3 observed that* REP($\mathbf{w}$) *and* EO($\mathbf{w}$) *and* PC($\mathbf{w}$) *all fail (i.e. no repeats, not exactly-one-even, and not pairwise coprime).*

Remark. As stated, this is best treated as a dataset-specific rigidity phenomenon rather than a universal theorem claim.

4 A physics-motivated unifying hypothesis

We now propose a *possible* physics theory (in the model-building sense) that, if true, would make the above conjectures natural consequences.

4.1 Background: the LG orbifold viewpoint

Weighted-projective Calabi–Yau hypersurfaces admit a closely related description via $N = (2, 2)$ Landau–Ginzburg models orbifolded by diagonal symmetries. In this perspective, the weights determine the $U(1)_R$ charges of the LG fields and the structure of the orbifold group; twisted sectors contribute to the spectrum of marginal operators, which correspond to geometric moduli and therefore govern $h^{1,1}$ and $h^{2,1}$. Classic discussion of such constructions appears in early work on superconformal compactifications in weighted projective space [4] and in the Kreuzer–Skarke LG/weighted-projective compilation pages [2].

4.2 Proposed theory: Symmetry-Enhanced Twisted-Sector Dominance (SETSD)

SETSD (informal statement). *For weighted-projective Calabi–Yau hypersurfaces, the sign and coarse magnitude of χ and the lower envelope of $h^{1,1}$ are dominated by the existence (or absence) of symmetry-enhanced twisted sectors in the associated diagonal LG orbifold. Arithmetic conditions on $\mathbf{w}$ (divisibility, parity, repetitions) act as selection rules for which twisted sectors exist and how many marginal operators they contribute.*

More concretely:

- **Divisibility and sign of χ.** If $\mathrm{SD}(\mathbf{w})$ holds, then each ratio d/w_i is an integer. In the LG picture this suggests a particularly large diagonal symmetry group with "integer-period" phase rotations on each field. SETSD posits that this symmetry enhancement biases the marginal spectrum toward complex-structure-type deformations (or, equivalently, suppresses Kähler-type blow-up modes), yielding $h^{2,1} \geq h^{1,1}$ and hence $\chi \leq 0$, explaining Conjecture 1.
- **Exactly-one-even and a universal $h^{1,1}$ floor.** If exactly one weight is even, there is a canonical $\mathbb{Z}_2$ phase symmetry affecting a single field, producing a robust twisted sector whose marginal content scales with the overall degree d. SETSD asserts that this sector contributes at least $\sim d/6$ Kähler moduli, producing Conjecture 2. The companion conjectures (3–4) then reflect that to have $h^{1,1}$ below the predicted floor, one must be in a different arithmetic universality class of orbifold group (nontrivial gcds among weights or different parity patterns).
- **Repeated weights and a w_3-driven slope.** If weights repeat, there is an additional permutation symmetry (at least an S_2 on repeated coordinates) which can combine with diagonal phases to produce extra twisted sectors or multiplicities. SETSD posits that, under $\mathrm{EO} \wedge \mathrm{REP}$, the effective multiplicity of a particular family of marginal operators is controlled by the median weight w_3, yielding a near-linear inequality of the form in Conjecture 5.

Physical implication (heuristic). Taken together, SETSD implies that *coarse* topological properties of the Calabi–Yau (and thus coarse 4D spectra) can be predicted from the discrete symmetry type of the UV worldsheet SCFT. In heterotic compactifications, where net chirality is tied to an index and (in certain standard settings) closely related to $\chi/2$, Conjecture 1 would mean that the arithmetic condition $\mathrm{SD}(\mathbf{w})$ *forbids* positive net chirality of the "wrong sign" in the simplest embeddings, pushing model searches toward weight systems with $\neg\mathrm{SD}$. See the review discussion of $\chi = \pm 6$ and phenomenological motivation [5].

5 How to falsify (or support) the conjectures

Because the conjectures are empirical, the correct scientific stance is to seek counterexamples systematically.

5.1 Experiment A: direct falsification search in weighted-projective hypersurfaces

1. Enumerate candidate weight vectors $\mathbf{w}$ (including beyond the standard quasi-smooth list, if desired).
2. For each $\mathbf{w}$, compute Hodge numbers $(h^{1,1}, h^{2,1})$ using established computational tools for reflexive/Gorenstein data and weighted hypersurfaces (e.g. the Kreuzer–Skarke ecosystem and associated software such as PALP; see [3]).
3. Check each implication:
   - If $\mathrm{SD}(\mathbf{w})$ and $\chi > 0$ occurs, Conjecture 1 is falsified.
   - If $\mathrm{EO}(\mathbf{w})$ and $h^{1,1} < d/6 - 1$ occurs, Conjecture 2 is falsified.
   - If $\mathrm{EO} \wedge \mathrm{REP}$ and $14h^{1,1} < 13w_3 + 1$ occurs, Conjecture 5 is falsified.

5.2 Experiment B: test universality on broader toric hypersurface datasets

A stronger test is to ask whether analogues of the conjectures survive when moving from weighted projective space to the vastly larger Kreuzer–Skarke toric hypersurface landscape. Public "CY database" efforts built from Kreuzer–Skarke polytopes exist [6], and modern tooling (e.g. CYTools) supports exploring the dataset computationally [7]. If one can define toric analogues of SD/EO/REP/PC (e.g. in terms of lattice-point degrees or phase symmetries of the associated GLSM), then SETSD predicts *the same sign and threshold phenomena* should appear across much broader classes.

5.3 Experiment C: dimension shift

Weighted-projective hypersurface Calabi–Yau fourfolds (in $\mathbb{P}^5(\mathbf{w})$) have been studied with machine learning, including feature sensitivity tied to LG formulas [8]. A sharp test of SETSD is whether:

- SD-like divisibility still biases the sign of χ (or the appropriate topological index),
- parity patterns still enforce linear lower envelopes for moduli-like Hodge numbers,
- repeat-weight sectors still induce near-linear slope constraints tied to a median weight.

6 What would count as a decisive counterexample?

For Conjecture 1. A single weighted-projective Calabi–Yau hypersurface with $\mathrm{SD}(\mathbf{w})$ and $\chi > 0$.

For Conjecture 2. A single example with $\mathrm{EO}(\mathbf{w})$ but $h^{1,1} < d/6 - 1$.

For Conjecture 5. A single example with $\mathrm{EO} \wedge \mathrm{REP}$ but $14h^{1,1} < 13w_3 + 1$.

Interpretation if found. Such counterexamples would refute the strong SETSD version and force a refined theory in which additional invariants (well-formedness, basket data, discrete torsion sectors, non-diagonal symmetries, etc.) enter the selection rules; the Kreuzer–Skarke LG site explicitly distinguishes untwisted, abelian-orbifold, and discrete-torsion cases [2], providing natural axes along which refinements might be organized.

7 Conclusion

We isolated a small "open core" of arithmetic-to-topology conjectures around χ and $h^{1,1}$ for weighted-projective Calabi–Yau threefold hypersurfaces. We proposed SETSD, a physics-motivated hypothesis attributing these patterns to symmetry-enhanced twisted-sector dominance in the associated LG orbifold description. The next step is systematic falsification: extend computations beyond the canonical list (or change the ambient class), and search directly for violating examples.

Acknowledgment. These conjectures were produced by an automated conjecturing engine; the present note is a structured write-up and a physics interpretation proposal.

References

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