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Combinatorial species and Lagrange Inversion

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Contents

Abstract

The goal of this report is to prove the combinatorial version of the Lagrange inversion formula introduced by Ira M. Gessel and Gilbert Labelle [6]. Prior to this, however, the report covers preliminary results. In section 1, combinatorial species are discussed as well as generating functions, combinatorial equivalence and operations. Section 3 covers the Lagrange inversion formula for power series. Section 4 covers multisort species, operations and the combinatorial Lagrange inversion formula.

1 Introduction

This report explores combinatorial species, a concept first introduced by Joyal in his 1981 paper [7]. My main sources for this report include Joyal's work [7], the book by Bergeron, Labelle, and Leroux [2], and their praised 'Red Book' [1]. The main goal of this report is to understand and showcase the combinatorial version of the Lagrange inversion formula, due to Gessel and Labelle [6]. This formula is a combinatorial analogue of the Lagrange inversion formula for power series.

I chose to tackle a topic on species because it lies at the intersection of algebra and combinatorics, an area I've yet to explore fully in my undergrad studies. Combinatorics has always excited me with its broad applications in probability, algebra, and computer science, and equipped with the benefits of algebra, I could see no fault.

The elegance of the theory of combinatorial species is large in fact due to its generality and simplicity. With others having worked to formalise the theory of structures by putting emphasis on properties, Joyal instead puts emphasis on the transport of structures, a general definition of relabelling labelled structures. In addition, the formal theory of combinatorial species provides us with a categorical framework for understanding combinatorial structures through algebraic expressions and equations. Through this, Joyal can explain simply the effectiveness of the use of generating functions to solve enumerative problems, as in the theory of species, algebraic identities on power series correspond to combinatorial identities on species.

We begin this report by introducing the notion of species in the context of combinatorics. Examples are given throughout to provide context and avoid pitfalls. Following this, we introduce the different series associated to a species; generating functions, type generating functions, and cycle index series. As we will see, generating functions, commonly referred to as "[clotheslines] on which we hang up a sequence of numbers" [9], offer us an algebraic tool to enumerate labelled structures. In a similar fashion, type generating functions can be used to enumerate unlabelled structures, while cycle index series are a generalisation of both of these series with similar properties. Generating functions and their properties have been studied and reported on extensively [9], [3], [4]. There exists many different types of generating functions, each with their own use cases, however, here we will only be concerned with exponential generating functions.

Closely following this, we introduce combinatorial equivalence allowing us to identify structurally similar species despite their differences in representation. We conclude this section by introducing the various well-defined operations such as addition, multiplication, substitution, and differentiation, which allow us to construct further species from simpler ones. By the end of this section, the reader should have a sufficient understanding of the algebra of combinatorial species.

In Section 3, we introduce power series [4] and their properties and conclude this section with the Lagrange inversion formula for power series [5]. This formula relates the coefficients of one power series to another and is the main motivation for our goal theorem.

In Section 4, we extend our discussion further to multisort species [1] and using as motivation the Lagrange inversion formula for power series, we conclude this report by stating and proving the Lagrange inversion formula for species [6].

2 Combinatorial Species

In this section, we will give the definition of a combinatorial species first introduced by Joyal [7]. We will also present many examples in order to clarify any confusion that has arisen from the definition. Following this, we will cover the different types of series associated to a species and demonstrate their properties. We then cover combinatorial equivalence and conclude this section by introducing various operations used to build further species from simpler ones.

2.1 Definition of species and examples

Definition 2.1. A *Species* F is a rule that produces:

1. For every finite set U, a finite set $F[U]$ (called the *set of structures of F on U*, or simply the *set of F-structures on U*).
2. For every bijection $\sigma: U \to V$, a function $F[\sigma]: F[U] \to F[V]$, called the *transport of F-structures along σ*.

In addition, $F[\sigma]$ must satisfy the following properties:

1. For all bijections $\sigma: U \to V$ and $\tau: V \to W$, we have:
$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$
2. For the identity map: $\mathrm{Id}_U: U \to U$:
$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$$

Remark 2.2. Sometimes, $F[\sigma]$ will be abbreviated to σ when there is no ambiguity to its meaning. $\diamondsuit$

Remark 2.3. In the language of category theory, a species $F: \mathcal{B} \to \mathcal{E}$ is a functor from the category of sets with *bijections* to the category of sets with *functions.* $\diamondsuit$

Remark 2.4. One can see by the properties that $F[\sigma]$ has an inverse, namely: $\mathrm{F}[\sigma^{-1}]$, and so $F[\sigma]$ is necessarily a bijection. $\diamondsuit$

Definition 2.5. Let n be a positive integer, then the set $\{1, 2, 3, ..., n\}$ containing the positive integers from 1 to n will be denoted by $[n]$. We will also denote the empty set, $\emptyset$, by $[0]$, and given a species F, the F-structures on $[n]$, $F[[n]]$, will simply be denoted $F[n]$.

Definition 2.6. A *partition*, π on a finite set U is a finite collection of distinct and pairwise-disjoint nonempty subsets of U whose union is U. The elements of π are called classes.

Example 2.7. Some examples of species.

- the species $\wp$ of *subsets*: $\wp[U]$ is the set of all subsets of U.
- the species $\mathcal{P}ar$ of *partitions*:
$$\mathcal{P}ar[U] = \{\pi | \pi \text{ is a partition on } U\}$$
where we describe the transport along a bijection $\sigma: U \to V$ by:
$$\mathcal{P}ar[\sigma](\pi) = \{\sigma(C) \in \wp(V) \mid C \in \pi\}$$
- the species $\mathcal{L}$ of *total (or linear) orders*
$$\mathcal{L}[U] = \{\leq \mid \leq \text{ is a total linear order on } U\}$$
A *total order*, or sometimes a *chain*, on U is a relation $\leq$ on U that is

1. *reflexive* ($x \leq x$)
2. *antisymmetric* ($x \leq y$ and $y \leq x \implies x = y$),
3. *transitive* ($x \leq y, y \leq z \implies x \leq z$),
4. *strongly-connected* ($\forall x, y \in U, x \leq y$ or $y \leq x$).

We describe the transport along a bijection $\sigma\colon U \to V$ by:

$$\mathcal{L}[\sigma](x_1 \leq x_2 \leq ... \leq x_n) = \sigma(x_1) \leq' \sigma(x_2) \leq' ... \leq' \sigma(x_n)$$

where $x_1, x_2, ..., x_n$ are the distinct elements of U.

- the species a of trees, i.e. connected simple graphs without cycles.
- the species $\mathcal{A}$ of *rooted trees*, i.e. trees with a 'root' node. We define $\mathcal{A}[U] = \mathrm{a}[U] \times U$.
- the species End of *endofunctions*:

$$\mathrm{End}[U] = \{\psi \mid \psi\colon U \to U\}$$

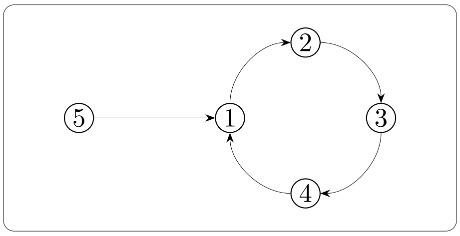
where we describe the transport along a bijection $\sigma\colon U \to V$ by:

$$\mathrm{End}[\sigma](\psi) = \sigma \circ \psi \circ \sigma^{-1}$$

for each $\psi \in \mathrm{End}[U]$.

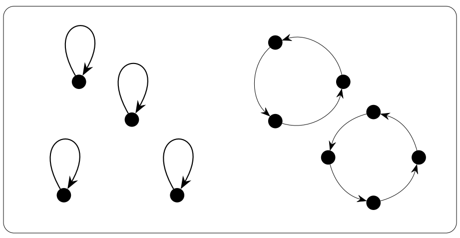
The definition of the transport function may seem a bit peculiar so I would like to motivate it by describing the species End using a *functional digraph*:

Let $U = \{1, 2, 3, 4, 5\}$ and let us represent the function $\psi\colon U \to U$, given by; $\psi(1) = 2$, $\psi(2) = 3$, $\psi(3) = 4$, $\psi(4) = 1$, $\psi(5) = 1$, using a directed graph:



Then, for a bijection $\sigma\colon U \to V$, we would like for the new endofunction $\psi' = \mathrm{End}[\sigma](\psi)$ to be described by the above directed graph relabelled, in the natural sense, by σ. If we take an element $s \in V$, to find where out where it maps to under ϕ', we first find $\sigma^{-1}(s)$, it then maps to $\psi(\sigma^{-1}(s))$, we then map back to our new labels $\sigma(\psi(\sigma^{-1}(s)))$ to find the image of s under ψ'. Thus $\psi' = \sigma \circ \psi \circ \sigma^{-1}$.

- the subspecies $\mathcal{S} \subset \mathrm{End}$ of *Permutations*, i.e. bijective endofunctions.



- the subspecies $\mathrm{Der} \subset \mathcal{S}$ of *Derangements*, i.e. those with no fixed points.

- the subspecies $\mathcal{C} \subset \mathcal{S}$ of *cycles*, i.e. cyclic permutations.
- The species E of *sets*: $\mathrm{E}[U] = \{U\}$.
- The species ϵ of *elements*: $\epsilon[U] = U$.
- The species X, characteristic of *singletons*, defined by:

$$X[U] = \begin{cases} \{U\}, & \text{if } |U| = 1, \\ \emptyset, & \text{otherwise.} \end{cases}$$

- The species 1, characteristic of the *empty set*, defined by:

$$1[U] = \begin{cases} \{U\}, & \text{if } U = \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

- The *empty species* 0, defined by: $0[U] = \emptyset$

$\Diamond$

2.2 Associated series

2.2.1 The generating series

Definition 2.8. Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence of real numbers. Then we say the *generating series generated by the sequence* $\{a_n\}_{n\in\mathbf{N}}$, denoted by $A(x)$, is the power series:

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

Definition 2.9. Let F be a species, and let us define the *generating series $F(x)$ of the species F* by:

$$F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$$

where: $f_n = |F[n]|$.

Example 2.10. Let us look at some examples:

a) $\mathcal{L}(x) = \frac{1}{1-x}$ b) $\mathcal{S}(x) = \frac{1}{1-x}$ c) $\mathrm{E}(x) = e^x$

d) $\epsilon(x) = xe^x$ e) $X(x) = x$ f) $1(x) = 1$

g) $\mathcal{C}(x) = -\log(1-x)$ h) $0(x) = 0$ i) $\mathrm{End}(x) = \sum_{n\geq 0} n^n \frac{x^n}{n!}$

$\Diamond$

a) For the species $\mathcal{L}$ of total orders, we construct a bijective map from the set $\mathcal{L}[n]$ to the set of n-tuples on $[n]$ containing no two identical components. We map the chain $x_1 \leq x_2 \leq ... \leq x_n$ to the n-tuple: $(x_1, x_2, ..., x_n)$. There are $n!$ elements in the codomain (n choices for the first component, $n-1$ for the second, etc.). Thus we have:

$$\mathcal{L}_n = n!, \ \forall n \geq 0$$

and so:

$$\mathcal{L}(x) = \sum_{n\geq 0} n! \frac{x^n}{n!} = \frac{1}{1-x}$$

b) For the species $\mathcal{S}$ of permutations, we can similarly construct a bijective map from the $\mathcal{S}$-structures on $[n]$ to the set of n-tuples on $[n]$ containing no two identical components. We map the permutation $x_1x_2x_3...x_n$ given in one-line notation, to the n-tuple $(x_1, x_2, x_3, ..., x_n)$. As before, the codomain has $n!$ elements and so:

$$\mathcal{S}_n = n!,\ \forall n \geq 0$$

and so:

$$\mathcal{S}(x) = \sum_{n\geq 0} n!\frac{x^n}{n!} = \frac{1}{1-x}$$

c) The species E of sets:

$$\mathrm{E}[n] = \{[n]\}\ \forall n \geq 0$$

Thus:

$$\mathrm{E}_n = |\mathrm{E}[n]| = 1,\ \forall n \geq 0$$

and so:

$$\mathrm{E}(x) = \sum_{n\geq 0} \frac{x^n}{n!}$$

which we recognise as the Taylor series expansion of the function e^x, and so we write $\mathrm{E}(x) = e^x$.

d) The species ϵ of elements:

$$\epsilon[n] = [n]$$

Thus:

$$\epsilon_n = |\epsilon[n]| = n,\ \ \forall n \geq 0$$

and so:

$$\epsilon(x) = \sum_{n\geq 0} n\frac{x^n}{n!} = x\sum_{n\geq 0} \frac{x^n}{n!} = xe^x$$

e) The species X of singletons:

$$X[1] = \{1\},\ \text{and}\ X[n] = \emptyset\ \forall n \neq 1$$

Thus:

$$X_1 = 1,\ \text{and}\ X_n = 0\ \forall n \neq 1$$

and so:

$$X(x) = x$$

f) The species 1, characteristic of the empty set:

$$1[0] = \{\emptyset\}\ \text{and}\ 1[n] = \emptyset\ \forall n > 0$$

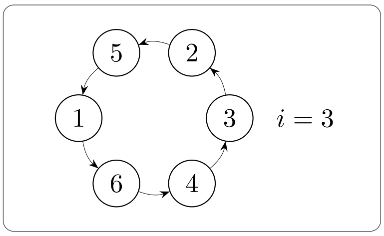
Thus:

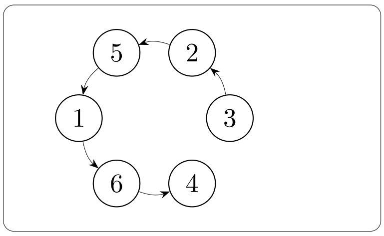
$$1_0 = 1,\ \text{and}\ 1_n = 0\ \forall n > 0$$

and so:

$$1(x) = 1$$

g) For the species $\mathcal{C}$ of cycles, we can construct an n-to-one surjective map from the set $\mathcal{C}[n]\times[n]$ to the set $\mathcal{C}[n]$. We can then construct a bijective map from $\mathcal{C}[n] \times [n]$ to $\mathcal{L}[n]$. The first map sends the ordered cycle-integer pair (c, i) to c. This is clearly n-to-one and onto. The second map sends the ordered cycle-integer pair (c, i) to the chain given by following along c starting at i and neglecting the final edge. An example is given below:





Thus:

$$\mathcal{C}_0 = 0, \text{ and } \mathcal{C}_n = \frac{n!}{n} = (n-1)! \; \forall n > 0$$

and so:

$$\mathcal{C}(x) = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n}$$

which we recognise as the Taylor series expansion of $-\log(1-x)$, and so we write $\mathcal{C}(x) = -\log(1-x)$.

h) The empty species 0:

$$0[n] = \emptyset \; \forall n \geq 0$$

Thus:

$$0_n = |0[n]| = 0, \; \forall n \geq 0$$

and so:

$$0(x) = 0$$

i) For the species End of endofunctions, we can construct a bijective map from the set End$[n]$ to the set of all n-tuples on $[n]$. Where we send the endofunction $x_1x_2...x_n$, written in one-line notation, to the n-tuple $(x_1, x_2, ..., x_n)$. This is clearly bijective. We note the set of n-tuples on $[n]$ contains n^n elements (n choices for the first component, n for the second, etc.). Thus:

$$\mathrm{End}_n = |\mathrm{End}[n]| = n^n, \; \forall n \geq 0$$

and so:

$$\mathrm{End}(x) = \sum_{n \geq 0} n^n \frac{x^n}{n!}$$

2.2.2 The isomorphism type generating series

Definition 2.11. Let U and V be two finite sets, and let $s \in F[U]$, and $t \in F[V]$. Then if there exists a bijection $\sigma\colon U \to V$, such that: $F[\sigma](s) = t$, we say that s and t are *isomorphic* or of the same *isomorphism type*, and we call σ an *isomorphism* from s to t. An isomorphism from s to itself is called an *automorphism* of s.

Remark 2.12. It is not hard to show that the relation: $\sim$ on a finite set U defined by:

$$s \sim t \iff \text{there exists an isomorphism from } s \text{ to } t$$

is an equivalence relation. ♢

Definition 2.13. Given the finite set $U = [n]$, the equivalence classes of the equivalence relation $\sim$ on U defined above are called the *isomorphism classes*. We denote by $T(F_n)$ the set of these isomorphism classes.

Remark 2.14. The elements of $T(F_n)$ are sometimes called the *unlabelled structures* of F on the set $[n]$. ♢

Definition 2.15. Let F be a species and let us define the *type generating series* of F, $\widetilde{F}(x)$, by:

$$\widetilde{F}(x) = \sum_{n \geq 0} \tilde{f}_n x^n$$

where: $\tilde{f}_n = |T(F_n)|$.

Example 2.16. Some simple examples:

a) $\widetilde{\mathcal{L}}(x) = \frac{1}{1-x}$

b) $\widetilde{\mathcal{S}}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$

c) $\widetilde{\mathcal{C}}(x) = \frac{x}{1-x}$

d) $\widetilde{\epsilon}(x) = \frac{x}{1-x}$

e) $\widetilde{X}(x) = x$

f) $\widetilde{\mathrm{E}}(x) = \frac{1}{1-x}$

g) $\widetilde{\wp}(x) = \frac{1}{(1-x)^2}$

h) $\widetilde{0}(x) = 0$

i) $\widetilde{1}(x) = 1$

$\diamondsuit$

a) The species $\mathcal{L}$ of total orders: We note that all elements of $\mathcal{L}[n]$ have the same isomorphism type. As given elements $s = x_1 \leq x_2 \leq ... \leq x_n$ and $t = x'_1 \leq' x'_2 \leq' ... \leq' x'_n$ in $\mathcal{L}[n]$, the bijection $\sigma\colon [n] \to [n]$ sending x_i to x'_i for all $1 \leq i \leq n$ ensures that $\mathcal{L}[\sigma](s) = t$. Thus:

$$\widetilde{\mathcal{L}}_n = 1, \ \forall n \geq 0$$

and so:

$$\widetilde{\mathcal{L}}(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

b) This will be discussed at the end of this subsection.

c) The species $\mathcal{C}$ of cycles: Given two cycles $u, v \in \mathcal{C}[n]$, we can construct a bijection $\sigma\colon [n] \to [n]$ such that $\mathcal{C}[\sigma](u) = v$ by identifying nodes in the functional digraph. Thus:

$$\widetilde{\mathcal{C}}_n = 1, \ \forall n \geq 1, \text{ and } \widetilde{\mathcal{C}}_0 = 0$$

and so:

$$\widetilde{\mathcal{C}}(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x}$$

d) The species ϵ of elements: Given $s, t \in \epsilon[n]$, we take any bijection $\sigma\colon [n] \to [n]$ such that $\sigma(s) = t$, then $\epsilon[\sigma](s) = \sigma(s) = t$. Thus:

$$\widetilde{\epsilon}_n = 1, \ \forall n \geq 1, \text{ and } \tilde{\epsilon}_0 = 0$$

and so:

$$\widetilde{\epsilon}(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x}$$

e) The species X of singletons: For $n \neq 1$, $X[n] = \emptyset$ and so $\widetilde{X}_n = 0$. When $n = 1$, we have $X[1] = \{[1]\}$, and so we may take $\sigma\colon [1] \to [1]$ to be defined by $\sigma(1) = 1$, which ensures $X[\sigma]([n]) = [n]$. Thus:

$$\widetilde{X}(x) = x$$

f) The species E of sets: $\mathrm{Id}_{[n]}\colon [n] \to [n]$ ensures that $\mathrm{E}[Id_{[n]}]([n]) = [n]$. Thus:

$$\widetilde{\mathrm{E}}_n = 1, \ \forall n \geq 0$$

and so:

$$\widetilde{\mathrm{E}}(x) = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

g) The species $\wp$ of subsets: We note that $s, t \in \wp[n]$ are isomorphic if and only if they have the same cardinality. Thus:

$$\widetilde{\wp}_n = n + 1, \ \forall n \geq 0$$

and so:

$$\widetilde{\wp}(x) = \sum_{n \geq 0} (n+1)x^n = \frac{1}{(1-x)^2}$$

h) The empty species 0: $0[n] = \emptyset$ for all $n \geq 0$. Thus:

$$\widetilde{0}_n = 0, \ \forall n \geq 0$$

and so:

$$\widetilde{0}(x) = 0$$

i) The species 1, characteristic of the empty set: $1[n] = \emptyset$ for all $n > 0$, and when $n = 1$; $1[Id_{[0]}]([0]) = [0]$. Thus:

$$\widetilde{1}(x) = 1$$

Let us take a look at a very interesting example. but first let us introduce some terminology.

Definition 2.17. Let n be a positive integer, and let $1 \leq a_1 \leq a_2 \leq ... \leq a_k$ be a sequence of integers so that $a_1 + a_2 + ... + a_k = n$. Then we say the sequence $(a_1, a_2, ..., a_k)$ is an *integer partition* of n into k parts. We denote the number of integer partitions of n into k parts by $p_k(n)$, and those into any number of parts by $p(n)$.

Remark 2.18. For convenience, we write $p_0(0) = 1$ and $p_k(0) = 0$ for all $k \geq 1$. $\Diamond$

Lemma 2.19. *Let $\sigma: U \to U$ be a permutation of U with cardinality n for some non-negative integer, and let $x \in U$. Then there exists $1 \leq i \leq n$ such that $\sigma^i(x) = x$.*

Proof. Consider the entries $\sigma(x), \sigma^2(x), ..., \sigma^n(x)$. Suppose none of these are x, then by the pigeonhole principle, two of these are equal; $\sigma^j(x) = \sigma^k(x)$ where without loss of generality we may assume: $1 \leq j < k \leq n$. Then applying σ^{-1} j-times, we get $x = \sigma^{k-j}(x)$. We note that $1 \leq k - j \leq n - 1$ so we arrive at a contradiction. The result follows. $\square$

Definition 2.20. Let $\sigma: U \to U$ be a permutation on a finite set U and let $x \in U$. Let i be the smallest positive integer such that $\sigma^i(x) = x$. We then say the sequence $x, \sigma(x), \sigma^2(x), ..., \sigma^{i-1}(x)$ forms an i-cycle in σ or, equivalently, is a cycle of length i.

Remark 2.21. Given a finite set U and a permutation $\sigma: U \to U$, we note that every element $x \in U$ belongs to a unique cycle in σ. The set of cycles in σ is called the decomposition of σ into disjoint cycles. $\Diamond$

Definition 2.22. Let U be a finite set, and σ a permutation of U. Then we define the *cycle type* of σ to be the sequence:

$$(\sigma_1, \sigma_2, ...)$$

where for $k \geq 1$, $\sigma_k =$ number of cycles of length k in the decomposition of σ into disjoint cycles.

Proposition 2.23.

$$\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$$

Proof. We recognise:

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = \prod_{i=1}^{\infty} (1 + x^k + x^{2k} + ...)$$

To form x^n after multiplying out the right-hand side, we must choose from each i-th parenthesis a term, x^{ij_i} such that: $n = \sum_{i=1}^{\infty} ij_i$. Each choice corresponds to an integer partition of n, namely if $(j_1, j_2, ..., j_n, 0, 0, ...)$ is our choice, then the corresponding integer partition is described by j_1-many 1s, j_2-many 2s, ... and so on. The result follows. $\square$

Example 2.24. By the previous proposition, to show

$$\widetilde{\mathcal{S}}(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

we need to show that $\tilde{\mathcal{S}}_n = p(n)$. Representing $\mathcal{S}$ by a functional digraph, it is clear that we can only have an isomorphism between $\mathcal{S}$ structures if they have the same cycle type and because it is easy to construct an isomorphism between two $\mathcal{S}$ structures of the same cycle type, we have that $\tilde{\mathcal{S}}_n$ = number of possible cycle types for the permutations on a set of n elements. There is a clear bijection between the set of cycle types of the permutations on $[n]$ and the integer partitions of n. Namely, for the cycle-type $(\sigma_1, \sigma_2, ...)$, the corresponding integer partition is that described by σ_1-many 1s, σ_2-many 2s, ... and so on. Thus $\tilde{\mathcal{S}}_n = p(n)$ as desired.

$\diamondsuit$

2.2.3 The Cycle Index Series

Definition 2.25. Let σ be a permutation of a finite set U, then:

$$\text{Fix } \sigma = \{u \in U \mid \sigma(u) = u\}$$

$$\text{fix } \sigma = |\text{Fix } \sigma|$$

where Fix σ denotes the set of *fixed points* of σ, and fix σ denotes the *number* of fixed points.

Definition 2.26. The *cycle index series* of a species F is the formal power series:

$$Z_F(x_1, x_2, ...) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathcal{S}[n]} \text{fix } F[\sigma] \; x_1^{\sigma_1} x_2^{\sigma_2} ... \right)$$

Theorem 2.27. *For any species of structures F, we have:*

$$F(x) = Z_F(x, 0, 0, 0, ...) \tag{1}$$

$$\tilde{F}(x) = Z_F(x, x^2, x^3, ...) \tag{2}$$

Proof. To establish 1:

$$Z_F(x, 0, 0, ...) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S[n]} \text{fix } F[\sigma] \; x^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} ... \right)$$

The term: $x^{\sigma_1} 0^{\sigma_2} 0^{\sigma_3} ...$ is nonzero only when $\sigma_2 = \sigma_3 = ... = 0$ which means σ in this case fixes all points and so is the identity, and therefore $F[\sigma] = \text{Id}_U$;

$$Z_F(x, 0, 0, ...) = \sum_{n \geq 0} \frac{1}{n!} (f_n \; x^n) = F(x)$$

To establish 2:

$$Z_F(x, x^2, x^3, ...) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S[n]} \text{fix } F[\sigma] \; x^{\sigma_1} x^{2\sigma_2} x^{2\sigma_2} ... \right)$$

Notice that for $k > n$, $\sigma_k = 0$, and $\sigma_1 + 2\sigma_2 + 3\sigma_3 + ... + n\sigma_n = n$:

$$Z_F(x, x^2, x^3, ...) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in S[n]} \text{fix } F[\sigma] \; x^n \right)$$

By Burnside's lemma:

$$\frac{1}{n!} \sum_{\sigma \in S[n]} \text{fix } F[\sigma] = |F[n]/\sim|$$

The result follows. $\square$

Lemma 2.28. *Let F be a species and let $\sigma\colon[n]\to[n]$ be a permutation. Then the cycle type of $F[\sigma]\colon F[n]\to F[n]$ only depends on the cycle type of σ.*

Proof. Let $\sigma\colon[n]\to[n]$ and $\tau\colon[n]\to[n]$ be two permutations on $[n]$. Then there exists a permutation $\mu\colon[n]\to[n]$ such that:

$$\sigma=\mu\circ\tau\circ\mu^{-1}$$

This then means that:

$$F[\sigma]=F[\mu]\circ F[\tau]\circ F[\mu]^{-1}$$

Thus $F[\sigma]$ and $F[\tau]$ have the same cycle type. □

Remark 2.29. Let F be a species, then by the previous lemma, we can then rewrite:

$$Z_F(x_1,x_2,...)=\sum_{n\geq 0}\frac{1}{n!}\left(\sum_{\sigma\in\mathcal{S}[n]}\text{fix } F[\sigma]\ x_1^{\sigma_1}x_2^{\sigma_2}...\right)$$

as

$$Z_F(x_1,x_2,...)=\sum_{n_1+2n_2+...<\infty}\text{fix } F[(n_1,n_2,...)]\frac{x_1^{n_1}x_2^{n_2}...}{n_1!1^{n_1}n_2!2^{n_2}...}$$

where $(n_1,n_2,...)$ is a permutation of $[n]$ with that cycle type. Here, we have used the fact that the number of permutations on $[n]$ with cycle type $(n_1,n_2,...)$ is

$$\frac{n!}{n_1!1^{n_1}n_2!2^{n_2}...}$$

. ◊

Example 2.30. Some simple examples:

a) $Z_{\mathcal{L}}(x_1,x_2,...)=\frac{1}{1-x_1}$

b) $Z_{\text{E}}(x_1,x_2,...)=exp(x_1+\frac{x_2}{2}+\frac{x_3}{3}+...)$

c) $Z_{\mathcal{S}}(x_1,x_2,...)=\frac{1}{(1-x_1)(1-x_2)(1-x_3)...}$

d) $Z_{\epsilon}(x_1,x_2,...)=x_1exp(x_1+\frac{x_2}{2}+\frac{x_3}{3}+...)$

◊

a) The species $\mathcal{L}$ of total orders: Let $\sigma\colon[n]\to[n]$ be a bijection from a finite set U to a finite set V. We note that Fix $F[\sigma]=\emptyset$ if $\sigma\neq\text{Id}_{[n]}$, and in the case when $\sigma=\text{Id}_{[n]}$ all elements of $\mathcal{L}[n]$ are fixed. Thus:

$$Z_{\mathcal{L}}(x_1,x_2,...)=\sum_{n\geq 0}\frac{1}{n!}(n!x_1^n)=\frac{1}{1-x_1}$$

b) The species E of sets: We note that Fix $\text{E}[n]=\{[n]\}$ for all $n\geq 0$. Thus:

$$Z_E(x_1,x_2,...)=\sum_{n_1+2n_2+...<\infty}\frac{(x_1/1)^{n_1}(x_2/2)^{n_2}...}{n_1!n_2!...}$$

One may recognise this as

$$exp\Big(x_1+\frac{x_2}{2}+\frac{x_3}{3}+...\Big)$$

c) The species $\mathcal{S}$ of permutations: An element $\mu\in\mathcal{S}[n]$ is fixed by a permutation $\sigma\colon[n]\to[n]$ if and only if $\mu=\sigma\circ\mu\circ\sigma^{-1}$. We note that if $(u_1,u_2,...,u_k)$ is a cycle in σ, then $(\mu(u_1),\mu(u_2),...,\mu(u_k))$ is a cycle in σ. In other words μ sends cycles to cycles of equal length. We note that if μ does not do this, it cannot satisfy $\sigma\circ\mu=\mu\circ\sigma$. Thus any fixed element $\mu\in\mathcal{S}[n]$ under σ can be chosen by first permuting the cycles of equal length in

σ and then choosing an element in the image of each cycle for which the smallest element of the input cycle gets mapped to under μ. All other elements in the new cycles are then uniquely determined. By reviewing the steps, this can be done in:

$$\sigma_1!1^{\sigma_1}\sigma_2!2^{\sigma_2}...\sigma_n!n^{\sigma_n}$$

ways, where $(\sigma_1, ..., \sigma_n)$ is the cycle type of σ. Thus:

$$\begin{aligned}
Z_{\mathcal{S}}(x_1, x_2, ...) &= \sum_{n_1+2n_2+...<\infty} n_1!1^{n_1}n_2!2^{n_2}...\frac{x_1^{n_1}x_2^{n_2}...}{n_1!1^{n_1}n_2!2^{n_2}...} \\
&= \sum_{n_1+2n_2+...<\infty} x_1^{n_1}x_2^{n_2}... \\
&= \frac{1}{(1-x_1)(1-x_2)(1-x_3)...}
\end{aligned}$$

d) The species ϵ of elements: An element $s \in \epsilon[n]$ is fixed by $\epsilon[\sigma]\colon \epsilon[n] \to \epsilon[n]$ if and only if $\sigma(s) = s$. Thus:

$$\begin{aligned}
Z_{\epsilon}(x_1, x_2, ...) &= \sum_{n_1+2n_2+...<\infty} n_1\frac{x_1^{n_1}x_2^{n_2}...}{n_1!1^{n_1}n_2!2^{n_2}...} \\
&= x_1 \sum_{n_1+2n_2+...<\infty : n_1>0} \frac{x_1^{n_1-1}x_2^{n_2}...}{(n_1-1)!1^{n_1}n_2!2^{n_2}...} \\
&= x_1 \sum_{n_1+2n_2+...<\infty} \frac{x_1^{n_1}x_2^{n_2}...}{n_1!1^{n_1}n_2!2^{n_2}...} \\
&= x_1 exp\left(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + ...\right)
\end{aligned}$$

2.2.4 Isomorphism of Species

Definition 2.31. Let F and G be two species. An *equipotence* α of F to G is a family of bijections $\alpha_U\colon F[U] \to G[U]$. In this case, we say F and G are *equipotent* and write $F \equiv G$

Remark 2.32. Equipotence is a powerful tool in combinatorics and is used throughout for the enumeration of labelled structures. In Example 2.10, we used equipotences to count different structures, although we did not call them that. However, for further analysis, we will need something stronger. A notion of equivalence that identifies unlabelled structures as well. $\Diamond$

Definition 2.33. Let F and G be two species of structures. We say F and G are identical if $F[U] = G[U]$ for all finite sets U, and for all bijections $\sigma\colon U \to V$ between two finite sets U and V, we have $F[\sigma] = G[\sigma]$.

Remark 2.34. While this definition does ensure an identification between unlabelled structures of identical species. It is too strong a condition. We would like a weaker definition of equivalence. Thus we introduce the following. $\Diamond$

Definition 2.35. Let F and G be two species. An *isomorphism* of F to G is a family of bijections $\alpha_U\colon F[U] \to G[U]$ which satisfies the *naturality condition*: For every bijection $\sigma\colon U \to V$ between two finite sets, the following diagram commutes:

$$\begin{tikzcd}
F[U] \arrow[r, "\alpha_U"] \arrow[d, "F[\sigma]"'] & G[U] \arrow[d, "G[\sigma]"] \\
F[V] \arrow[r, "\alpha_V"] & G[V]
\end{tikzcd}$$

In other words, for any F-structure $s \in F[U]$, one must have $(G[\sigma] \circ \alpha_U)(s) = (\alpha_V \circ F[\sigma])(s)$ In this case, one writes: $F \simeq G$.

Remark 2.36. In the language of category theory, an isomorphism of species would be equivalent to a natural transformation of functors. $\Diamond$

Remark 2.37. It is easy to show that this is an equivalence relation and thus: $F \simeq G \iff G \simeq F$ Informally, two species F and G are isomorphic if they have the same structures on a finite set U up to relabelling. From now on we will also be writing $F = G$ in place of $F \simeq G$. $\Diamond$

Theorem 2.38.

$$F = G \implies \begin{cases} F(x) = G(x) \\ \widetilde{F}(x) = \widetilde{G}(x) \\ Z_F(x_1, x_2, \ldots) = Z_G(x_1, x_2, \ldots) \end{cases}$$

Proof. Let F and G be two isomorphic species. This means there exists a family of bijections $\alpha_U \colon F[U] \to G[U]$ such that for any bijection $\sigma \colon U \to V$, the following diagram commutes:

$$\begin{tikzcd} F[U] \arrow[r, "\alpha_U"] \arrow[d, "F[\sigma]"'] & G[U] \arrow[d, "G[\sigma]"] \\ F[V] \arrow[r, "\alpha_V"] & G[V] \end{tikzcd}$$

In particular, this means that for $U = V$, we have: $G[\sigma] \circ \alpha_U = \alpha_U \circ F[\sigma]$. Let $s \in F[U]$ be fixed by $F[\sigma]$, then: $(G[\sigma] \circ \alpha_U)(s) = \alpha_U(s) \implies \alpha_U(s)$ is fixed by $G[\sigma]$. Similarly, suppose $t \in G[U]$ is fixed by $G[\sigma]$, then choose the unique $s \in F[U]$ such that $\alpha_U(s) = t$. Then $\alpha_U(s) = t = G[\sigma](t) = (G[\sigma] \circ \alpha_U)(s) = (\alpha_U \circ F[\sigma])(s) \implies s$ is fixed by $F[\sigma]$. Therefore: fix $F[U] =$ fix $G[U]$ and so $Z_F(x_1, x_2, \ldots) = Z_G(x_1, x_2, \ldots)$. $F(x) = G(x)$ and $\widetilde{F}(x) = \widetilde{G}(x)$ follow from the former of this proof together with Theorem 2.27. $\square$

2.3 Operations on species

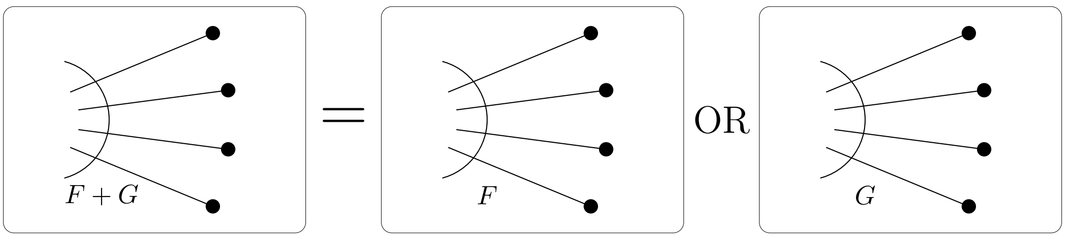
2.3.1 Addition

Definition 2.39. We define the species $F + G$, called the *addition* of F and G, for a finite set U by:

$$(F + G)[U] = F[U] \bigsqcup G[U]$$

The transport of a structure $s \in (F + G)[U]$ along a bijection $\sigma \colon U \to V$ is defined by

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s) \text{ if } s \in F[U] \\ G[\sigma](s) \text{ if } s \in G[U] \end{cases}$$



Lemma 2.40. *Addition is well-defined, i.e.*

$$F_1 = F_2,\ G_1 = G_2 \implies F_1 + G_1 = F_2 + G_2$$

Proof. It is clear that addition is commutative, that is $F + G = G + F$. So let F_1, F_2 and G by species such that $F_1 = F_2$. Then there exists a family of bijections $\alpha_U \colon F_1[U] \to F_2[U]$ such that the following diagram commutes:

$$\begin{tikzcd} F_1[U] \arrow[r, "\alpha_U"] \arrow[d, "F_1[\sigma]"'] & F_2[U] \arrow[d, "F_2[\sigma]"] \\ F_1[V] \arrow[r, "\alpha_V"] & F_2[V] \end{tikzcd}$$

Let us now describe a new family of bijections $\beta_U : (F_1 + G)[U] \to (F_2 + G)[U]$ by

$$\beta(s) = \begin{cases} \alpha(s) & \text{if } s \ \in F[U] \\ s & \text{if } s \ \in G[U] \end{cases}$$

This new family of bijections form the isomorphism from $(F_1 + G)$ to $(F_2 + G)$. $\square$

Theorem 2.41. *Let F and G be two species. Then:*

$$Z_{F+G}(x_1, x_2, ...) = Z_F(x_1, x_2, ...) + Z_F(x_1, x_2, ...)$$

Proof. Note that for any given finite set U and for any permutation $\sigma : U \to U$, fix $(F + G)[\sigma] =$ fix $F[\sigma]$ + fix $G[\sigma]$. The result follows. $\square$

Corollary 2.42.

$$(F + G)(x) = F(x) + G(x)$$

Proof. This is a special case of the above theorem. $\square$

2.3.2 Multiplication

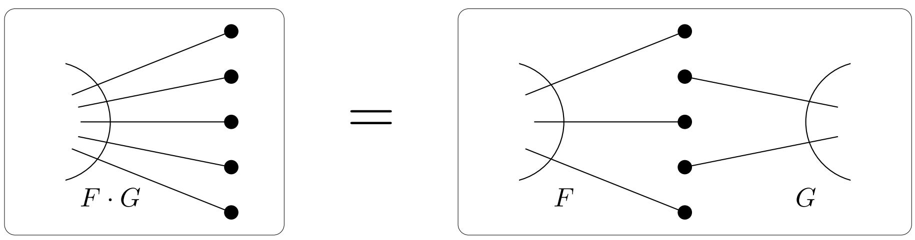
Definition 2.43. We define the species $F \cdot G$, called the *multiplication* of F by G by the following: For a finite set U:

$$(F \cdot G)[U] = \bigsqcup_{\substack{U_1, U_2 \subseteq U \\ U_1 \cap U_2 = \emptyset \\ U_1 \cup U_2 = U}} F[U_1] \times G[U_2]$$

The transport of a structure $s = ((U_1, f), (U_2, g)) \in (F \cdot G)[U]$ by a bijection $\sigma : U \to V$ is defined by:

$$(F \cdot G)[\sigma](s) = ((V_1, F[\sigma_1](f)), (V_2, G[\sigma_2](g)))$$

where: $\sigma_1 = \sigma|_{U_1}$, $\sigma_2 = \sigma|_{U_2}$, and $V_1 = \sigma_1(U_1)$ and $V_2 = \sigma_2(U_2)$.



Remark 2.44. It is important to note here that while in general $F + G$ and $G + F$ are not identical, they are isomorphic and so we will write $F + G = G + F$. Similarly $F \cdot G$ and $G \cdot F$ are not identical in general, but they are isomorphic and so we will write $F \cdot G = G \cdot F$. $\lozenge$

Lemma 2.45. *Let F, G and H be species. Then:*

i) $(F \cdot G) \cdot H = F \cdot (G \cdot H)$

ii) $F \cdot G = G \cdot F$

iii) $F \cdot 1 = 1 \cdot F = F$

iv) $F \cdot 0 = 0 \cdot F = 0$

v) $F \cdot (G + H) = F \cdot G + F \cdot H$

Proof. Let U be a finite set.

i) Let $((f, g), h) \in ((F \cdot G) \cdot H)[U]$. We describe the bijection α_U by:

$$\begin{aligned} \alpha_U \colon ((F \cdot G) \cdot H)[U] &\to (F \cdot (G \cdot H))[U] \\ ((f, g), h) &\mapsto (f, (g, h)) \end{aligned}$$

ii) Let $(f, g) \in (F \cdot G)[U]$. We describe the bijection α_U by:

$$\begin{aligned} \alpha_U \colon (F \cdot G)[U] &\to (G \cdot F)[U] \\ (f, g) &\mapsto (g, f) \end{aligned}$$

iii) $F \cdot 1 = 1 \cdot F$ follows from part ii) of this proof. Let $(f, \emptyset) \in (F \cdot 1)[U]$. We describe the bijection α_U by:

$$\begin{aligned} \alpha_U \colon (F \cdot 1)[U] &\to F[U] \\ (f, \emptyset) &\mapsto f \end{aligned}$$

iv) $F \cdot 0 = 0 \cdot F$ follows from part ii) of this proof. We note that $(F \cdot 0)[U] = \emptyset$ for all finite sets U. Thus $F \cdot 0$ and 0 are identical and in particular, they are isomorphic.

v) Let U be a finite set. We describe the bijection $\alpha_U \colon (F \cdot (G + H))[U] \to (F \cdot G + F \cdot H)[U]$ by:

$$\alpha_U((U_1, f), (U_2, (G, s))) = (FG, ((U_1, f), (U_2, s)))$$

and

$$\alpha_U((U_1, f), (U_2, (H, s))) = (FH, ((U_1, f), (U_2, s)))$$

□

Lemma 2.46. *Multiplication is well-defined, i.e.*

$$F_1 = F_2,\ G_1 = G_2 \implies F_1 \cdot G_1 = F_2 \cdot G_2$$

Proof. Let U be a finite set. Let F_1, F_2, G_1, and G_2 be species such that $F_1 = F_2$ and $G_1 = G_2$. From Lemma 2.45, we have that $F_1 \cdot G_1 = G_1 \cdot F_1$ So we only need to prove that if $F_1 = F_2$, i.e. there exists an isomorphism β_U from F_1 to F_2, we have $F_1 \cdot G_1 = F_2 \cdot G_1$. We describe the bijection α_U by:

$$\begin{aligned} \alpha_U \colon (F_1 \cdot G_1)[U] &\to (F_2 \cdot G_1)[U] \\ ((U_1, f_1), (U_2, g_1)) &\mapsto ((U_1, \beta_{U_1}(f_1)), (U_2, g_1)) \end{aligned}$$

Then:

$$\begin{aligned} \sigma \cdot \alpha_U(f, g) &= \sigma \cdot (\beta_{U_1}(f), g) \\ &= (\sigma|_{U_1} \cdot \beta_{U_1}(f), \sigma|_{U_2}(g)) \\ &= (\beta_{V_1} \cdot \sigma|_{V_1}(f), \sigma|_{V_2}(g)) \\ &= \alpha_V \cdot \sigma(f, g) \end{aligned}$$

□

Lemma 2.47. *Let*

$$A(x) = \sum_{m=0}^{\infty} a_m \frac{x^m}{m!}$$

and

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

be two exponential generating series and let

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Then the exponential generating series defined by the c_n's obeys:

$$C(x) = A(x)B(x)$$

Remark 2.48. Before proving this, it is common to introduce the following notation. For a power series $f(x) = \sum_{i=0}^{\infty} f_i x^i$, we define: $[x^n]f = f_n$. $\lozenge$

Proof. (of Lemma 2.47)

$$A(x) = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x_3}{3!} + ...$$

$$B(x) = b_0 + b_1 \frac{x}{1!} + b_2 \frac{x^2}{2!} + b_3 \frac{x_3}{3!} + ...$$

If we take $a_k \frac{x^k}{k!}$, we must then choose $b_{n-k} \frac{x^{n-k}}{(n-k)!}$ to arrive at an x^n term in $A(x)B(x)$. Thus the term here that contributes to the sum is: $a_k b_{n-k} \frac{x^n}{k!(n-k)!}$ Thus summing over all choices of k, we get:

$$[x^n]C(x) = \sum_{k=0}^{n} \frac{1}{(n-k)!k!} a_k b_{n-k}$$

$$\implies n![x^n]C(x) = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} = c_n$$

$\square$

Theorem 2.49. *Let F and G be two species. Then:*

$$(F \cdot G)(x) = F(x) \cdot G(x)$$

Proof. Let U be a finite set with $|U| = n$. Let us count the elements of $(F \cdot G)[U]$ by considering different cases, namely by the number of elements of the first subset, S. Let us consider those of size k. There are $\binom{n}{k}$ possible subsets S containing k elements, there are f_k F-structures on each of these sets and g_{n-k} G-structures on each of the sets. Thus: $|(F \cdot G)[U]| = \sum_{k=0}^{n} \binom{n}{k} f_k g_{n-k}$
By Lemma 2.47, $n![x^n](F(x) \cdot G(x)) = \sum_{k=0}^{n} \binom{n}{k} f_n g_{n-k}$.
The result follows. $\square$

Theorem 2.50. *Let F and G be two species. Then:*

$$Z_{FG}(x_1, x_2, ...) = Z_F(x_1, x_2, ...) \cdot Z_F(x_1, x_2, ...) \tag{3}$$

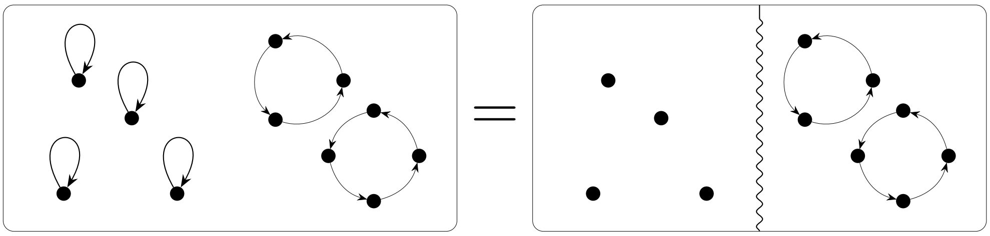
Proof. Let n be a non-negative integer, and let $U = [n]$. Let (U_1, U_2) be some decomposition of U, and let $\sigma \in \mathcal{S}[n]$ with cycle type $(\sigma_1, \sigma_2, ...)$. We note that $((U_1, s), (U_2, t)) \in \text{Fix } (F \cdot G)[\sigma] \iff F[\sigma|_{U_1}](s) = s$ and $G[\sigma|_{U_2}](t) = t$ where $\sigma|_{U_1}: U_1 \to U_1$, and $\sigma|_{U_2}: U_2 \to U_2$ are necessarily permutations. Thus fix $(F \cdot G)[\sigma] = \sum_{\sigma', \sigma''} \text{fix } F[\sigma'] \times \text{fix } G[\sigma'']$, where the sum is taken over all pairs (σ', σ'') with cycle types $(\sigma_1', \sigma_2', ...)$ and $(\sigma_1'', \sigma_2'', ...)$ such that: $\sigma_i' + \sigma_i'' = \sigma_i \; \forall i > 0$.

We then recognise that the coefficient in front of the $x_1^{\sigma_1} x_2^{\sigma_2} ... x_n^{\sigma_n}$ term on the right-hand side of 3 is exactly this sum. Thus:

$$Z_{FG}(x_1, x_2, ...) = Z_F(x_1, x_2, ...) \cdot Z_F(x_1, x_2, ...)$$

$\square$

Example 2.51. Let us now derive our first combinatorial equation.



$\mathcal{S}$ $\qquad$ $\mathrm{E} \cdot \mathrm{Der}$

Let U and V be two finite sets. We can identify each $\mathrm{E} \cdot \mathrm{Der}$-structure, (a, b), on U with an $\mathcal{S}$-structure, s, on U in a one-to-one way. Namely, s is the permutation whose fixed points are those elements of a and the derangement component of s is then described by b. This is a clear bijection, which we will call $\alpha_U : \mathcal{S}[U] \to (\mathrm{E} \cdot \mathrm{Der})[U]$. It is also clear that this bijection satisfies the naturality condition: $(\mathrm{E} \cdot \mathrm{Der})[\sigma] \circ \alpha_U = \alpha_V \circ \mathcal{S}[\sigma]$ for any bijection $\sigma : U \to V$. Thus we have:

$$\mathcal{S} = \mathrm{E} \cdot \mathrm{Der}$$

On applying Theorem 2.38 to the above equation and through division, we get the following generating series for Der:

a) $\mathrm{Der}(x) = \frac{e^{-x}}{1-x}$

b) $\widetilde{\mathrm{Der}}(x) = \prod_{k \geq 2} \frac{1}{1-x^k}$

c) $Z_{\mathrm{Der}}(x_1, x_2, ...) = e^{-(x_1 + \frac{x_2}{2} + \frac{x_3}{3} + ...)} \prod_{k \geq 1} \frac{1}{1-x_k}$

We can then recognise that the formula

$$d_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - ... + \frac{(-1)^n}{n!}\right)$$

for the number of derangements on a set with n elements is directly obtained from a) on carrying out the multiplication and comparing coefficients. $\lozenge$

2.3.3 Composition/Substitution

Definition 2.52. Let F and G be two species. The species $F \circ G$, also denoted $F(G)$, called the *partitional composite* of G in F, is defined as follows: An $F \circ G$-structure on U is a triplet $s = (\pi, \psi, \gamma)$, where:

1. π is a partition of U,

2. ψ is an F-structure on the set of classes of π,

3. $\gamma = (\gamma_p)_{p \in \pi}$, where for each class p of π, γ_p is a G-structure on p.

In other words, for any finite set U,

$$(F \circ G)[U] = \bigsqcup_{\pi \in \mathcal{P}ar[U]} F[\pi] \times \prod_{B \in \pi} G[B]$$

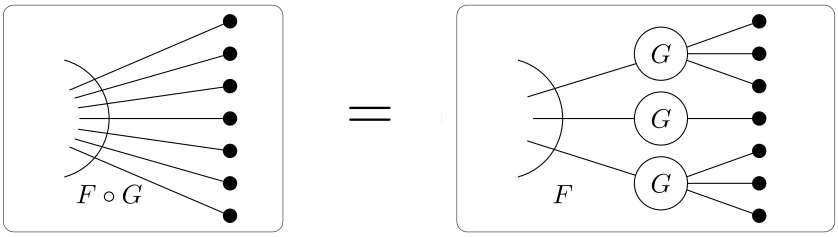
where $\mathcal{P}ar$ is the species of partitions.

The transport along a bijection $\sigma: U \to V$ is carried out by setting, for any $F \circ G$-structure $s = (\pi, \psi, (\gamma_p)_{p\in\pi})$,

$$(F \circ G)[\sigma](s) = (\bar{\pi}, \bar{\psi}, (\bar{\gamma}_{\bar{p}})_{\bar{p}\in\bar{\pi}})$$

where:

1. $\mathcal{P}ar[\sigma](\pi) = \bar{\pi}$
2. $F[\bar{\sigma}](\psi) = \bar{\psi}$ where $\bar{\sigma}$ is the bijection induced on π by σ
3. For each $\bar{p} = \sigma(p) \in \bar{\pi}$, $G[\sigma|_p](\gamma_p) = \bar{\gamma}_{\bar{p}}$



Lemma 2.53. *Composition is well-defined, i.e.*

$$F_1 = F_2,\ G_1 = G_2 \implies F_1(G_1) = F_2(G_2)$$

Proof. Let U be a finite set. F_1 and F_2 are isomorphic, thus there exists a family of bijections $\alpha_U: F_1[U] \to F_2[U]$ satisfying the naturality condition. Similarly there exists a family of bijections $\beta_U: G_1[U] \to G_2[U]$ satisfying the naturality condition. We then define our family of bijections $\chi_U: (F_1(G_1))[U] \to (F_2(G_2))[U]$ by:

$$\chi_U((\pi, \psi, (\gamma_p)_{p\in\pi})) = (\pi, \alpha_\pi(\psi), (\beta_p(\gamma_p))_{p\in\pi})$$

Then let $\sigma: U \to V$ be a bijection between two finite sets U and V:

$$\begin{aligned}
\sigma \cdot \chi_U((\pi, \psi, (\gamma_p)_{p\in\pi})) &= \sigma \cdot (\pi, \alpha_\pi(\psi), (\beta_p(\gamma_p))_{p\in\pi}) \\
&= (\sigma \cdot \pi, \sigma \cdot \alpha_\pi(\psi), (\sigma \cdot \beta_p(\gamma_p))_{p\in\pi}) \\
&= (\bar{\pi}, \alpha_{\bar{\pi}} \cdot \sigma(\psi), (\beta_{\bar{p}} \cdot \sigma(\gamma_p))_{\bar{p}\in\bar{\pi}}) \\
&= \chi_V(\sigma \cdot \pi, \sigma(\psi), (\sigma(\gamma_{\bar{p}}))_{\bar{p}\in\bar{\pi}}) \\
&= \chi_V \cdot \sigma((\pi, \psi, (\gamma_p)_{p\in\pi}))
\end{aligned}$$

□

Lemma 2.54 (Associativity)**.** *Let F, G, and H be species. Then:*

$$F \circ (G \circ H) = (F \circ G) \circ H$$

Proof. Let U be a finite set. We then define our family of bijections $\chi_U: (F \circ (G \circ H))[U] \to ((F \circ G) \circ H)[U]$ by:

$$\chi_U((\pi, f, ((\pi'_p, g_p, (h_{n,p})_{n\in\pi'_p}))_{p\in\pi})) = (\pi'', (\pi, f, (g_p)_{p\in\pi}), (h_k)_{k\in\pi''})$$

where $\pi'' = \cup_{p\in\pi}\pi'_p$. Note: π can be identified in a canonical way with a partition of π''. It then follows that for any bijection $\sigma: U \to V$ between two finite sets U and V:

$$\sigma \cdot \chi_U((\pi, f, ((\pi'_p, g_p, (h_{n,p})_{n\in\pi'_p}))_{p\in\pi})) = \chi_V \cdot \sigma((\pi, f, ((\pi'_p, g_p, (h_{n,p})_{n\in\pi'_p}))_{p\in\pi}))$$

□

Theorem 2.55. *Let F and G be two species, and suppose: $G[\emptyset] = \emptyset$. Then:*

$$(F \circ G)(x) = F(G(x))$$

Proof. Let us note that because $g_0 = |G[\emptyset]| = 0$, we have the following well-defined composition:

$$F(G(x)) = f_0 + f_1 \frac{G(x)}{1!} + f_2 \frac{G(x)^2}{2!} + \dots$$

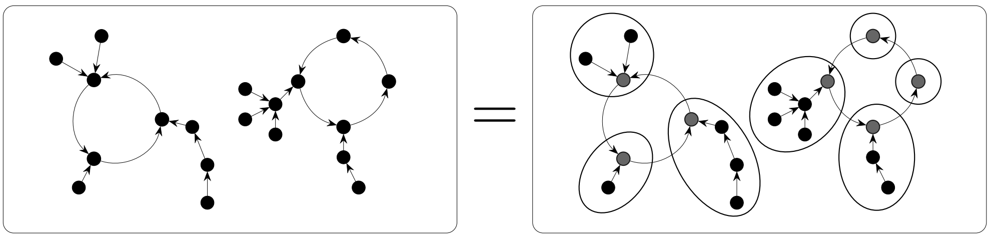
Let U be a finite set with $|U| = n$. Let us count the elements of $(F \circ G)[U]$ by considering the different cases, namely those for different sizes of the partitions (size here means the number of classes of the partition). Let us consider those partitions of size k. There are f_k F-structures on each of these partitions and $\frac{|G^k[n]|}{k!}$ ways to choose the k G-structures on the sets of each partition. Thus: $|(F \circ G)[n]| = \sum_{k=1}^{n} f_k \frac{|G^k[n]|}{k!}$ Note now that:

$$(F \circ G)(x) = f_0 + \sum_{k=1}^{1} f_k \frac{1![x^k]G^k(x)}{k!} \frac{x}{1!} + \sum_{k=1}^{2} f_k \frac{2![x^k]G^k(x)}{k!} \frac{x^2}{2!} + \dots$$

$$= f_0 + f_1 \frac{G(x)}{1!} + f_2 \frac{G^2(x)}{2!} + \dots$$

□

Example 2.56. End $= \mathcal{S} \circ A$



End $\qquad\qquad$ $\mathcal{S} \circ \mathcal{A}$

Let U be a finite set. We can recognise that every element $\psi \in \text{End}[U]$ consists of two types of points:

i) Recurrent points, those $x \in U$ for which there exists $k > 0$ such that $\psi^k(x) = x$. Elements located along cycles.

ii) Non-recurrent points, those $x \in U$ for which there does not exist such a $k > 0$.

We see that those connected elements off a cycle form a tree. Thus there is a clear identification between the elements of End$[U]$ and $(\mathcal{S} \circ \mathcal{A})[U]$. It is also clear that it satisfies the naturality condition. Thus:

$$\text{End} = \mathcal{S} \circ \mathcal{A}$$

On applying Theorem 2.38 and Theorem 2.55 to the above equation, we get the following:

$$\text{End}(x) = (\mathcal{S} \circ \mathcal{A})(x) = \mathcal{S}(\mathcal{A}(x)) = \frac{1}{1 - \mathcal{A}(x)}$$

$$\implies \text{End}(x) = \mathcal{A}(x) \cdot \text{End}(x) + 1$$

If we then multiply out and compare coefficients we get:

$$1 = a_0 + 1 \implies a_0 = 0$$

and for $n > 0$:

$$\frac{n^n}{n!} = \sum_{i=0}^{n} \frac{a_i}{i!} \frac{(n-i)^{n-i}}{(n-i)!}$$
$$\iff n^n = \sum_{i=0}^{n} \binom{n}{i} a_i (n-i)^{n-i}$$

It is clear then that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is uniquely determined by the above. We then propose the following ansatz:

$$a_n = n^{n-1}\ \forall n > 0 \text{ and } a_0 = 0$$

The problem then reduces to showing the following formula holds for all positive integers n:

$$n^n = \sum_{i=0}^{n} \binom{n}{i} i^{i-1} (n-i)^{n-i}$$

The proof is provided below.

Lemma 2.57. *Let k be a positive integer and let j be a non-negative integer strictly less than k. Then*

$$T(k,j) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^j = 0$$

Proof. Before we begin, let us note that we may assume j is positive as $T(k,0)$ is simply the binomial expansion of $(1-1)^k$ which is zero for all positive k. We will prove the statement using induction on k. For $k = 1$, the statement can easily be checked. Assume the statement holds true for k, and let j be a positive integer strictly less than k+1, now let us consider:

$$\begin{aligned} T(k+1,j) &= \sum_{i=0}^{k+1} \binom{k+1}{i} (-1)^{k+1-i} i^j \\ &= \sum_{i=0}^{k+1} \binom{k+1}{i} i (-1)^{k+1-i} i^{j-1} \end{aligned}$$

We may then use the identity $\binom{k+1}{i} i = (k+1)\binom{k}{i-1}$.

$$\begin{aligned} T(k+1,j) &= (k+1) \sum_{i=1}^{k+1} \binom{k}{i-1} (-1)^{k+1-i} i^{j-1} \\ &= (k+1) \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (i+1)^{j-1} \\ &= (k+1) \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \sum_{m=0}^{j-1} \binom{j-1}{m} i^m \\ &= (k+1) \sum_{m=0}^{j-1} \binom{j-1}{m} \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^m \end{aligned}$$

By the induction hypothesis, the term

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^m$$

vanishes, and so we get $T(k+1,j) = 0$. □

Lemma 2.58. *For all positive integers n, we have:*

$$n^n = \sum_{i=1}^{n} \binom{n}{i} i^{i-1}(n-i)^{n-i}$$

Proof. Let n be a positive integer.

$$\begin{aligned}
\sum_{i=1}^{n} \binom{n}{i} i^{i-1}(n-i)^{n-i} &= \sum_{i=1}^{n} \binom{n}{i} i^{i-1} \sum_{j=0}^{n-i} \binom{n-i}{j} n^j (-i)^{n-i-j} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} n^j (-1)^{n-i-j} i^{n-j-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=1}^{n-j} \binom{n}{j} \binom{n-j}{i} n^j (-1)^{n-i-j} i^{n-j-1} \\
&= \sum_{j=0}^{n-1} \binom{n}{j} n^j \sum_{i=1}^{n-j} \binom{n-j}{i} (-1)^{n-i-j} i^{n-j-1}
\end{aligned} \tag{4}$$

An application of Lemma 2.57 to the term:

$$\begin{aligned}
T(n-j, n-j-1) &= \sum_{i=0}^{n-j} \binom{n-j}{i} (-1)^{n-i-j} i^{n-j-1} \\
&= \sum_{i=1}^{n-j} \binom{n-j}{i} (-1)^{n-i-j} i^{n-j-1}
\end{aligned}$$

when $j < n-1$ reduces Equation 4 to

$$\binom{n}{n-1} n^{n-1} \binom{1}{1} (-1)^0 1^0 = n^n$$

□

What this means for our example is that the number of rooted trees on a set with n elements, a_n, is: 0 when $n = 0$ and n^{n-1} otherwise. From this, we easily get Cayley's formula, that for any positive integer n, the number of trees on $[n]$ is n^{n-2}. Our theory has thus given us an alternative proof of Cayley's formula, as compared to the standard explicit bijection between doubly-rooted labelled trees and endofunctions. ◊

2.3.4 Differentiation

Definition 2.59. Let F be a species. Then the species F' (also denoted $\frac{d}{dX}F(X)$, called the *derivative* of the species F is defined as follows: An F'-structure on U is an F-structure on $U^+ = U \cup \{*\}$, where $* = *_U$ is an element chosen outside of U, i.e. For any finite set U, one sets

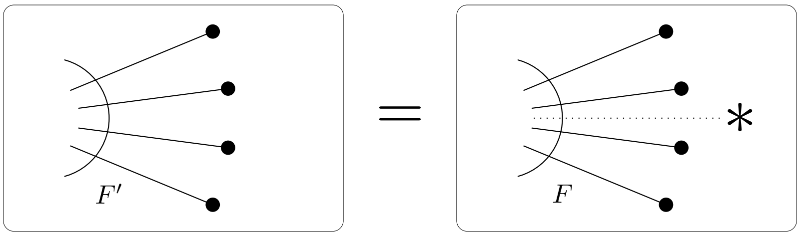
$$F'[U] = F[U^+], \text{ where } U^+ = U + \{*\}$$

The transport along a bijection $\sigma: U \to V$ is carried out by setting, for any F'-structure s on U:

$$F'[\sigma](s) = F[\sigma^+](s)$$

where $\sigma^+: U^+ \to V^+$ is the canonical extension of σ given by:

$$\sigma^+(u) = \sigma(u) \text{ if } u \in U, \text{ and } \sigma^+(*) = *$$



Lemma 2.60. *Differentiation is well-defined. i.e.*

$$F_1 = F_2 \implies F_1' = F_2'$$

Proof. Let U and V be finite sets, and let F_1 and F_2 be isomorphic species. Then there exists bijections $\alpha_{U^+}: F_1[U^+] \to F_2[U^+]$ and $\alpha_{V^+}: F_1[V^+] \to F_2[V^+]$ such that the following diagram commutes:

$$\begin{tikzcd} F_1[U^+] \arrow[r, "\alpha_{U^+}"] \arrow[d, "F_1[\sigma^+]"'] & F_2[U^+] \arrow[d, "F_2[\sigma^+]"] \\ F_1[V^+] \arrow[r, "\alpha_{V^+}"] & F_2[V^+] \end{tikzcd}$$

for any bijection $\sigma: U^+ \to V^+$. We can restrict ourselves to those bijections that fix $*$. Then the following commutative diagram follows from the previous.

$$\begin{tikzcd} F_1'[U] \arrow[r, "\alpha_{U^+}"] \arrow[d, "F_1'[\sigma]"'] & F_2'[U] \arrow[d, "F_2'[\sigma]"] \\ F_1'[V] \arrow[r, "\alpha_{V^+}"] & F_2'[V] \end{tikzcd}$$

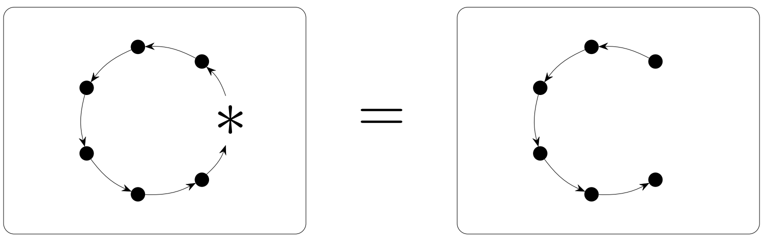
and so our family of isomorphisms $\beta_U: F_1'[U] \to F_2'[U]$ are those α_{U^+}. □

Theorem 2.61. *Let F be a species and $F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$ be its generating series. Then:*

$$F'(x) = \frac{d}{dx}F(x)$$

Proof. $\frac{d}{dx}F(x) = \sum_{n\geq 0} n f_n \frac{x^n}{n!} = \sum_{n\geq 1} f_n \frac{x^n}{(n-1)!} = \sum_{n\geq 0} f_{n+1} \frac{x^n}{n!}$, and $|F'[n]| = |F[n+1]| = f_{n+1}$. The result follows. □

Example 2.62. $\mathcal{C}' = \mathcal{L}$



Let U and V be finite sets. We can identify each C'-structure, c, on U with a $\mathcal{L}$-structure, l on U in a one-to-one way. l is the $\mathcal{L}$-structure formed by following along c starting at $*$ and ending just before $*$. One can easily see that the naturality condition is satisfied. Thus:

$$\mathcal{C}' = \mathcal{L}$$

On applying Theorem 2.38 and Theorem 2.61 to the above combinatorial equation, we get the following:

$$\mathcal{C}'(x) = \mathcal{L}(x)$$

which we can verify is true by Example 2.10. ◊

3 Lagrange Inversion Formula

In this section, we will cover the proof of the Lagrange inversion formula for power series due to Ira M. Gessel [5] which is used as motivation for our main theorem. Formal Laurent series and properties have been taken from [8].

3.1 Preliminaries

Let us recall the definition of a formal Laurent series

Definition 3.1. A *formal Laurent series* in x is any series of the form:

$$L(x) = \sum_{n=n_0}^{\infty} c_n x^n$$

Remark 3.2. Addition is defined termwise, and multiplication is defined by the distributivity property. It is clear that the set $Q((x))$ of Laurent series forms a commutative ring under these operations. Similarly as before, we will also define $[x^m]L(x) = c_m$ for any integer m. $\Diamond$

Lemma 3.3. $Q((x))$ *is not only a ring but a field.*

Proof. Let $f(x) = \sum_{n=n_0}^{\infty} f_n x^n$ be a nonzero Laurent series. Let $g(x) = \sum_{k=0}^{\infty} g_k x^k$ be a Laurent series with $g_0 \neq 0$. Without loss of generality, we can assume that $n_0 = 0$ and $f_0 \neq 0$. Then:

$$[x^m](fg)(x) = \sum_{i=0}^{m} f_i g_{m-i}$$

For g to be f's inverse, one must have: $f_0 g_0 = 1$ and $[x^m](fg)(x) = \sum_{i=0}^{m} f_i g_{m-i} = 0$ for $m > 0$ which we can solve recursively for the $g_i's$ since $f_0 \neq 0$.

$\square$

Definition 3.4. The residue of a formal Laurent series:

$$L(x) = \sum_{n=n_0}^{\infty} c_n x^n$$

is $\mathrm{Res}(L_{z_0}) = c_{-1}$

Definition 3.5. The derivative of a Laurent series $f(x) = \sum_{n=n_0}^{\infty} f_n x^n$ is the Laurent series $g(x) = \sum_{n=n_0-1}^{\infty} n f_n x^{n-1}$

Remark 3.6. The residue of a derivative is zero. This is easily observed from the definition. $\Diamond$

3.2 The Lagrange Inversion Formula

Theorem 3.7. *Let f be a Laurent series in x and let $g(t) = \sum_{n=1}^{\infty} g_n t^n$ be a Laurent series in t with $g_1 \neq 0$. Then:*

$$Res\ f(x) = Res\ f(g(x))g'(x)$$

Proof. By linearity, it is sufficient to prove the statement for $f(x) = x^k$ for any integer k. If $k \neq -1$, then:

$$\begin{gathered} 0 = \mathrm{Res}\ f(x) \\ \mathrm{Res}\ g(t)^k g'(t) = \mathrm{Res}\ \left(g(t)^{k+1}/(k+1)\right)' = 0 \end{gathered} \tag{5}$$

If $k = -1$, then:

$$\begin{aligned} 1 &= \text{Res } f(x) \\ \text{Res } g(t)^{-1}g'(t) &= \text{Res } \frac{g_1 + 2g_2t + 3g_3t^2 + \ldots}{g_1t + g_2t^2 + g_3t^3 + \ldots} \\ &= \text{Res } \frac{1}{t}\frac{g_1 + 2g_2t + \ldots}{g_1 + g_2t + \ldots} = 1 \end{aligned} \tag{6}$$

□

Theorem 3.8 (Lagrange Inversion Formula)**.** *Let $f(x) = \sum_{n\geq 1} f_n x^n$ be a Laurent series, and let $g(t) = \sum_{n\geq 1} g_n t^n$ be its composition inverse (meaning that $g(f) = x$ and $f(g) = t$). Let $\phi(t)$ be any Laurent series in t, then:*

$$[x^n]\phi(f(x)) = \mathit{Res}\ \frac{\phi(t)g'(t)}{g^{n+1}(t)} \tag{7}$$

Proof. (Lagrange Inversion Formula) Note that $g_1 \neq 0$ as it has a compositional inverse. Then applying Theorem 3.7 to the Laurent series $\phi(f(x))/x^{n+1}$, we get:

$$[x^n]\phi(f(x)) = \text{Res } \frac{\phi(f(x))}{x^{n+1}} = \text{Res } \frac{\phi(f(g))g'}{g^{n+1}} = \text{Res } \frac{\phi(t)g'(t)}{g(t)^{n+1}}$$

□

Corollary 3.9 (Equivalent Formulas)**.** *Let $R(t)$ be a power series in t. Then there is a unique power series $f(x)$ such that: $f(x) = xR(f(x))$, and for any Laurent series $\phi(t)$ and $\psi(t)$ and for any integer n, we have:*

$$[x^n]\phi(f) = \frac{1}{n}[t^{n-1}]\phi'(t)R(t)^n, \text{ where } n \neq 0 \tag{8}$$

$$[x^n]\phi(f) = [t^n](1 - tR'(t)/R(t))\phi(t)R(t)^n \tag{9}$$

$$\phi(f) = \sum_n x^n[t^n](1 - xR'(t))\phi(t)R(t)^n \tag{10}$$

$$[x^n]\frac{\psi(f)}{1 - xR'(f)} = [t^n]\psi(t)R(t)^n \tag{11}$$

$$[x^n]\frac{\psi(f)}{1 - fR'(f)/R(f)} = [t^n]\psi(t)R(t)^n \tag{12}$$

Remark 3.10. Notice that in proving these, we may assume R has nonzero constant term. As in the case that it has zero constant term, $f \equiv 0$ and then the equations are trivial. ◊

Proof. (11) $\iff$ (12): This is shown on noticing that $x = \frac{f}{R(f)}$.

(9) $\iff$ (12): We can take $\psi(t) = \left(1 - t\frac{R'(t)}{R(t)}\right)\phi(t)$, we find that that these two equations are equivalent.

(10) $\iff$ (11): We can rewrite (11) as:

$$\frac{\psi(f)}{1 - xR'(f)} = \sum_n x^n[t^n]\psi(t)R(t)^n$$

and make the substitution $\psi(t) = \left(1 - x\frac{R'(t)}{R(t)}\right)\phi(t)$

(8) $\iff$ (9): Making the substitution $g(t) = \frac{t}{R(t)}$, we see that $g(f) = x$ is equivalent to the equation $f = xR(f)$, and so:

$$\begin{aligned}
[x^n]\phi(f) &= [t^n](1 - tR'(t)/R(t))\phi(t)R(t)^n \\
&= [t^n](1 - t\left(\frac{g(t) - tg'(t)}{g(t)^2}/(t/g(t))\right))\phi(t)t^n/g(t)^n \\
&= [t^n]\frac{g(t) - g(t) + tg'(t)}{g(t)}\frac{\phi(t)t^n}{g(t)^n} \\
&= [t^n]\frac{\phi(t)g'(t)t^{n+1}}{g(t)^{n+1}} \\
&= \text{Res}\ \frac{\phi(t)g'(t)}{g(t)^{n+1}}
\end{aligned} \tag{13}$$

and:

$$\begin{aligned}
[x^n]\phi(f) &= \frac{1}{n}[t^{n-1}]\phi'(t)R(t)^n \\
&= \frac{1}{n}[t^{n-1}]\phi'(t)\frac{t^n}{g(t)^n} \\
&= \frac{1}{n}\text{Res}\ \frac{\phi'(t)}{g(t)^n}
\end{aligned} \tag{14}$$

We can then observe that (14) and (13) are equivalent by:

$$\begin{aligned}
\frac{d}{dt}\frac{\phi(t)}{g(t)^n} &= \frac{\phi'(t)}{g(t)^n} - n\frac{\phi(t)g'(t)}{g(t)^{n+1}} \\
\implies\ 0 = \text{Res}\ \frac{d}{dt}\frac{\phi(t)}{g(t)^n} &= \text{Res}\ \frac{\phi'(t)}{g(t)^n} - n\text{Res}\ \frac{\phi(t)g'(t)}{g(t)^{n+1}} \\
\implies\ \frac{1}{n}\text{Res}\ \frac{\phi'(t)}{g(t)^n} &= \text{Res}\ \frac{\phi(t)g'(t)}{g(t)^{n+1}}
\end{aligned}$$

□

Definition 3.11. We define the *diagonal operator* ∇ from the formal power series in x and t to the formal power series in x by:

$$\nabla\left(\sum_{m,n=0}^{\infty} a_{m,n}\frac{x^m}{m!}\frac{t^n}{n!}\right) = \sum_{n=0}^{\infty} a_{n,n}\frac{x^n}{n!}$$

Remark 3.12. Given the above definition, we may rewrite 11 as:

$$\frac{\phi(f)}{1 - xR'(f)} = \nabla\left(\phi(t)e^{xR(t)}\right)$$

This will become useful in the next section. ◊

4 Combinatorial Lagrange Inversion Formula

In this section, we introduce multisort species and their operations. We conclude this section with the Lagrange inversion formula for species due to Ira M. Gessel and Gilbert Labelle [6].

4.1 Multisort species

Before generalising the theory of species to k-sort species, we must introduce some terminology:

Definition 4.1. Let $k \geq 1$ be an integer. A *multiset* U (with k sorts of elements) is a k-tuple of sets:

$$U = (U_1, U_2, ..., U_k).$$

U may also be called a k-set for brevity. An element of U_i is called an element of U of *sort* i. The *multicardinality* of U is the k-tuple of cardinalities

$$|U| = (|U_1|, |U_2|, ..., |U_k|).$$

The *total cardinality* of U is the sum

$$\|U\| = |U_1| + |U_2| + ... + |U_k|$$

.

Definition 4.2. A *multifunction* f from $(U_1, U_2, ..., U_k)$ to $(V_1, V_2, ..., V_k)$, denoted by

$$f \colon (U_1, U_2, ..., U_k) \to (V_1, V_2, ..., V_k)$$

is a k-tuple of functions $f = (f_1, f_2, ..., f_k)$ so that f_i maps from U_i to V_i for all $i = 1, 2, ..., k$. The *composition* of two multifunctions is defined component-wise. The multifunction f is said to be *bijective* if each function f_i is bijective.

Definition 4.3. Let $k \geq 1$ be an integer. A *species* of k sorts (or a k-sort species) is a rule F which produces:

1. for each finite multiset $U = (U_1, U_2, ..., U_k)$, a finite set

$$F[U_1, U_2, ..., U_k]$$

2. for each bijective multifunction

$$\sigma = (\sigma_1, \sigma_2, ..., \sigma_k) \colon (U_1, ..., U_k) \to (V_1, ..., V_k)$$

a function

$$F[\sigma] = F[\sigma_1, \sigma_2, ..., \sigma_k] \colon F[U_1, U_2, ..., U_k] \to F[V_1, V_2, ..., V_k]$$

In addition, $F[\sigma]$ must satisfy the following:

1. For all bijective multifunctions $\sigma \colon U \to V$ and $\tau \colon V \to W$, we have:

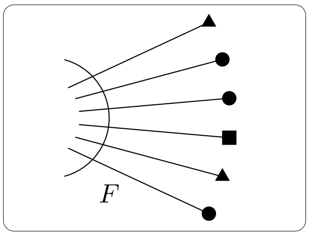
$$F[\tau \circ \sigma] = F[\tau] \circ F[\sigma]$$

2. For the identity multifunction: $\mathrm{Id}_U \colon U \to U$:

$$F[\mathrm{Id}_U] = \mathrm{Id}_{F[U]}$$

Remark 4.4. A species, as before, is a one-sorted species. $\Diamond$

The usual graphical convention to represent F-structures is similar as before but now we assign to each label an identifier such as a number, colour or shape.



Definition 4.5. For each i, $1 \leq i \leq k$, the k-sort species X_i of singletons of sort i is defined by:

$$X_i[U] = \begin{cases} \{U\}, & \text{if } |U_i| = 1 \text{ and } U_j = \emptyset, \text{ for } j \neq i \\ \emptyset, & \text{otherwise.} \end{cases}$$

Remark 4.6. Other variables can be used to designate the species of singletons. Commonly, X, Y, Z, and T are used in place of X_1, X_2, X_3 and X_4 respectively. ◊

4.2 Operations on multisort species

Again, we need to introduce some terminology.

Definition 4.7. A *dissection* of a k-set $U = (U_1, U_2, ..., U_k)$ is an ordered pair of k-sets $(V,\ W)$ so that for $i = 1, 2, ..., k$, $V_i \cup W_i = U_i$ and $V_i \cap W_i = \emptyset$. The set of dissections of U is denoted $\Delta[U]$.

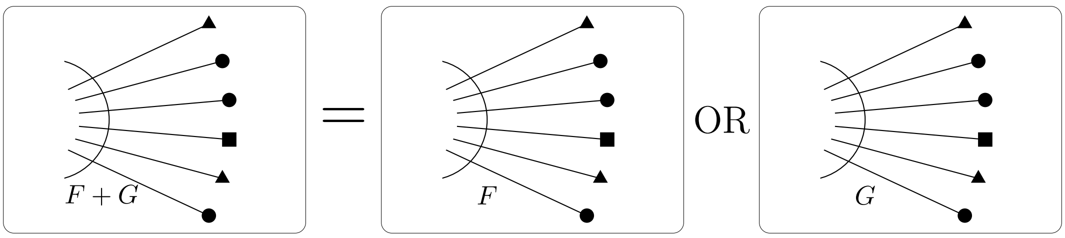
Definition 4.8. A *partition* of a k-set $U = (U_1, U_2, ..., U_k)$ is a partition of the total set $U_1 + U_2 + ... + U_k$. $\mathrm{Par}[U]$ denotes the set of partitions of U. Every class C of an element of $\mathrm{Par}[U]$ can be regarded as a k-set, $(C_1, C_2, ..., C_k)$, where $C_i = C \cup U_i$.

Definition 4.9. For two k-sort species F and G, we define the species $F + G$, called the *addition* of F and G by:

$$(F + G)[U] = F[U] + G[U]$$

for all finite k-sets U. The transport of a structure $s \in (F + G)[U]$ along a bijection $\sigma\colon U \to V$ is defined by

$$(F + G)[\sigma](s) = \begin{cases} F[\sigma](s) \text{ if } s \in F[U] \\ G[\sigma](s) \text{ if } s \in G[U] \end{cases}$$

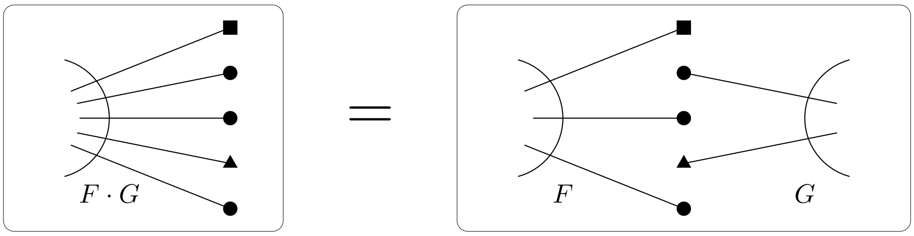


Definition 4.10. For two k-sorts, F and G, we define the species $F \cdot G$, called the *multiplication* of F and G, by

$$(F \cdot G)[U] = \sum_{(V,W) \in \Delta[U]} F[V] \times G[W]$$

for all finite k-sets U. The transport of a structure $s = (f, g) \in (F \cdot G)[U]$ by a bijection $\sigma\colon U \to V$ is defined by:

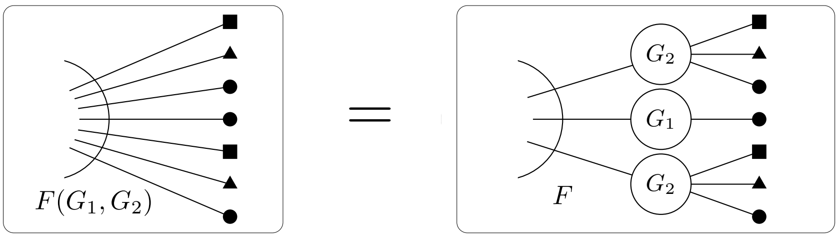
$$(F \cdot G)[\sigma](s) = (F[\sigma|_V](f), G[\sigma|_W](g))$$



Definition 4.11. Let $F = F(Y_1, Y_2, ..., Y_k)$ be an m-sort species, and $(G_j)_{j=1,2,...m}$ a family of k-sort species. Then the *partitional composition* $F(G_1, G_2, ..., G_m)$ is a k-sort species defined by:

$$F(G_1, G_2, ..., G_m)[U] = \sum_{\substack{\pi \in \mathrm{Par}[U] \\ \chi: \pi \to [m]}} F[\chi^{-1}] \times \prod_{\substack{j \in [m] \\ C \in \chi^{-1}(j)}} G_j[C]$$

for all finite k-sets U. χ^{-1} denotes the m-set $(\chi^{-1}(1), \chi^{-1}(2), ..., \chi^{-1}(m))$



Remark 4.12. On careful analysis of the above, we can view χ as a function assigning to each class of π one of the $(G_i)_{i \in [m]}$ species. $\Diamond$

Definition 4.13. We define the *diagonal operator* ∇ from two-sorted species to (one-sorted) species by: Given a two-sorted species $F(X, Y)$, we define ∇F by:

$$(\nabla F)[U] = F[U, U]$$

where the transport function $(\nabla F)[\sigma]: (\nabla F)[U] \to (\nabla F)[V]$ along a bijection $\sigma: U \to V$ is given by:

$$(\nabla F)[\sigma](s) = F[(\sigma, \sigma)](s)$$

for some s in $(\nabla F)[U]$.

Definition 4.14. An R-enriched rooted tree on a finite set U consists of

a) a root $z \in U$.

b) a function $\phi: U - \{z\} \to U$ such that for all $u \in U - \{z\}$ $\phi^k(u) = z$ for some positive integer k.

c) an R-structure on each preimage $\phi^{-1}(u)$ for all $u \in U$.

Remark 4.15. Henceforth, we shall reserve A to be the species of R-enriched rooted trees. $\Diamond$

Remark 4.16. In remark 3.12, we have observed that the Lagrange inversion formula may be rewritten as:

$$\frac{\phi(f)}{1 - xR'(f)} = \nabla \left(\phi(t) e^{xR(t)} \right)$$

With this in mind, we can motivate the combinatorial version of the Lagrange inversion formula by recognising the left hand side as the generating function for

$$\Phi(F)\mathcal{S}(XR'(F))$$

and with further generalisations to generating functions ensures the left hand side is the generating function of

$$\nabla\left(\Phi(Y)\mathrm{E}(XR(Y))\right)$$

with $\mathcal{S}$ the species of permutations, R some given species, F the species which solves $F = XR(F)$ and Φ some arbitrary species. $\Diamond$

4.3 Combinatorial Lagrange Inversion Formula

Theorem 4.17 (Combinatorial Lagrange Inversion Formula)**.** *Let R be any species, then there is a unique species $A = A(X)$ satisfying: $A = XR(A)$, and for any species F,*

$$F(A)S(XR'(A)) = \nabla(F(Y)E(XR(Y))) \tag{15}$$

Proof. Uniqueness: We must show that given two solutions A_1 and A_2 to the combinatorial equation: $A = XR(A)$, and for two finite set U and V of the same cardinality n, they satisfy the naturality condition. We prove this by induction on n. We first note that if A is a solution of $A = XR(A)$. Then: $|A[\emptyset]| = |(X(R(A)))[\emptyset]| = |\emptyset| = 0 \implies A[\emptyset] = \emptyset \implies A_1[\emptyset] = A_2[\emptyset] = \emptyset$ and so the case $n = 0$ is satisfied. Now let $n > 0$. We have the following commutative diagrams:

$$\begin{tikzcd}
A_1[U] \arrow[r, "\alpha_U"] \arrow[d, "A_1[\sigma]"'] & (XR(A_1))[U] \arrow[d, "(XR(A_1))[\sigma]"] \\
A_1[V] \arrow[r, "\alpha_V"] & (XR(A_1))[V]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2[U] \arrow[r, "\beta_U"] \arrow[d, "A_2[\sigma]"'] & (XR(A_2))[U] \arrow[d, "(XR(A_2))[\sigma]"] \\
A_2[V] \arrow[r, "\beta_V"] & (XR(A_2))[V]
\end{tikzcd}$$

and by the induction hypothesis we have:

$$\begin{tikzcd}
A_1[U_1] \arrow[r, "\mu_{U_1}"] \arrow[d, "A_1[\sigma|_{U_1}]"'] & A_2[U_1] \arrow[d, "A_2[\sigma|_{U_1}]"] \\
A_1[V_1] \arrow[r, "\mu_{V_1}"] & A_2[V_2]
\end{tikzcd}$$

for any proper subsets U_1 and V_1 of U and V respectively. From this, we can construct bijections $\gamma_U\colon (XR(A_1))[U] \to (XR(A_2))[U]$ and $\gamma_V\colon (XR(A_1))[V] \to (XR(A_2))[V]$ that commute the following diagram:

$$\begin{tikzcd}
(XR(A_1))[U] \arrow[r, "\gamma_U"] \arrow[d, "(XR(A_1))[\sigma]"] & (XR(A_2))[U] \arrow[d, "(XR(A_2))[\sigma]"] \\
(XR(A_1))[V] \arrow[r, "\gamma_V"] & (XR(A_2))[V]
\end{tikzcd}$$

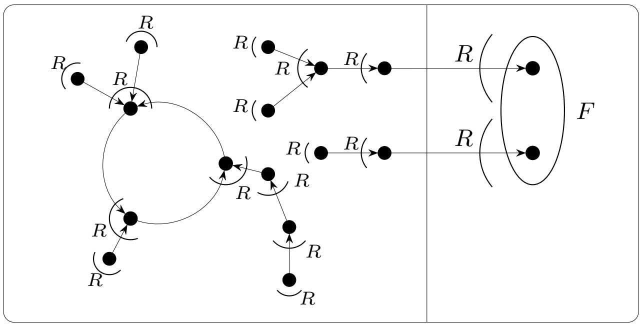
and so we have the following commutative diagram:

$$\begin{tikzcd}
A_1[U] \arrow[r, "\alpha_U"] \arrow[d, "(XR(A_1))[\sigma]"'] & (XR(A_1))[U] \arrow[r, "\gamma_U"] \arrow[d, "(XR(A_1))[\sigma]"] & (XR(A_2))[U] \arrow[r, "\beta_U^{-1}"] \arrow[d, "(XR(A_2))[\sigma]"'] & A_2[U] \arrow[d, "A_2[\sigma]"] \\
A_1[V] \arrow[r, "\alpha_V"] & (XR(A_1))[V] \arrow[r, "\gamma_V"] & (XR(A_2))[V] \arrow[r, "\beta_V^{-1}"] & A_2[V]
\end{tikzcd}$$

Thus A_1 and A_2 are isomorphic.

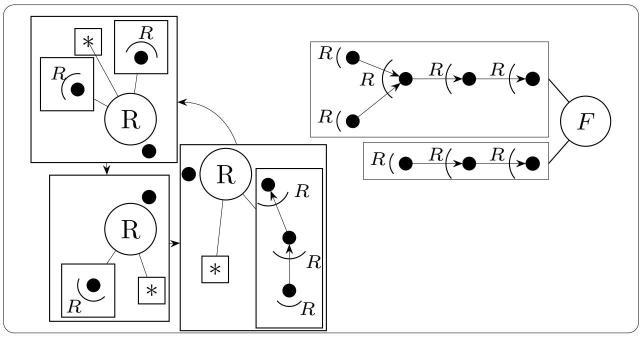
Furthermore, we recognise that the species A of R-enriched rooted trees satisfies the equation $A = XR(A)$ and so this is our unique solution.

To prove Equation 15: We will examine the right hand side: $H = \nabla(F(Y)E(XR(Y))$. If $|U| \neq 1$, then $XR(Y)[U,V] = \emptyset$ so we proceed with $U = \{u\}$. An element of $XR(Y)[U,V]$ will then consist of u together with an R-structure on V which can be thought as the unique function from V to U together with an R structure on the unique preimage. It follows that $E(XR(Y))[U,V]$ can be considered as the set of functions from V to U together with an R-structure on each of the preimages. Finally, an element of $H[U] = \nabla(F(Y)E(XR(Y)))[U,U]$ can then be considered as an F-structure on a subset W of U together with a function from $U - W$ to U with an R-structure on each of the preimages. An example is given below to illustrate such a structure.



$U - W$ $\qquad\qquad\qquad$ W

Suppose we are given an element of $H[U]$ with an F structure on $W \subset U$ and a function $\phi\colon U - W \to U$ with an R-structure on each preimage. Let W^* be the set of elements in U such that $\phi^k(u) \in W$ for some non-negative integer k. We see that $W \subset W^*$ and $\phi|_{W^*-W}$ must acyclic. Thus we can decompose our given structure into an acyclic function from $W^* - W$ to W with an R structure on each preimage and an F structure on W, i.e. an element of $(F \circ A)[W^*]$, and into an R enriched endofunction on $U - W^*$. It is clear that the species of R enriched endofunction is isomorphic to $\mathcal{S}(XR'(A))$ as illustrated in the below diagram.



$U - W$ $\qquad\qquad\qquad$ W

This is a clear bijection which satisfies the naturality condition. The result then follows. $\square$

References

[1] François Bergeron, Gilbert Labelle, and Pierre Leroux. *Combinatorial species and tree-like structures*. Number 67. Cambridge University Press, 1998.

[2] François Bergeron, Gilbert Labelle, and Pierre Leroux. Introduction to the theory of species of structures. *Université du Québeca Montréal, Montreal*, 2008.

[3] Miklós Bóna. *A walk through combinatorics: an introduction to enumeration and graph theory*. World Scientific, 2002.

[4] Ziyue Chen. Introduction to generating functions and their applications. https://math.uchicago.edu/~may/REU2022/REUPapers/Chen,Ziyue.pdf.

[5] Ira M. Gessel. Lagrange inversion. *Journal of Combinatorial Theory, Series A*, 144:212–249, November 2016.

[6] Ira M Gessel and Gilbert Labelle. Lagrange inversion for species. *Journal of Combinatorial Theory, Series A*, 72(1):95–117, 1995.

[7] André Joyal. Une théorie combinatoire des séries formelles. *Advances in mathematics*, 42(1):1–82, 1981.

[8] David G. Wagner. 7. formal power series. https://www.math.uwaterloo.ca/~dgwagner/co430I.pdf.

[9] Herbert S Wilf. *generatingfunctionology*. CRC press, 2005.