Calculus III

Calculus Early Transcendental 6th edition

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Calculus III

4C's Rule:

- Communication
- Collaboration
- Creativity
- Critical Thinking

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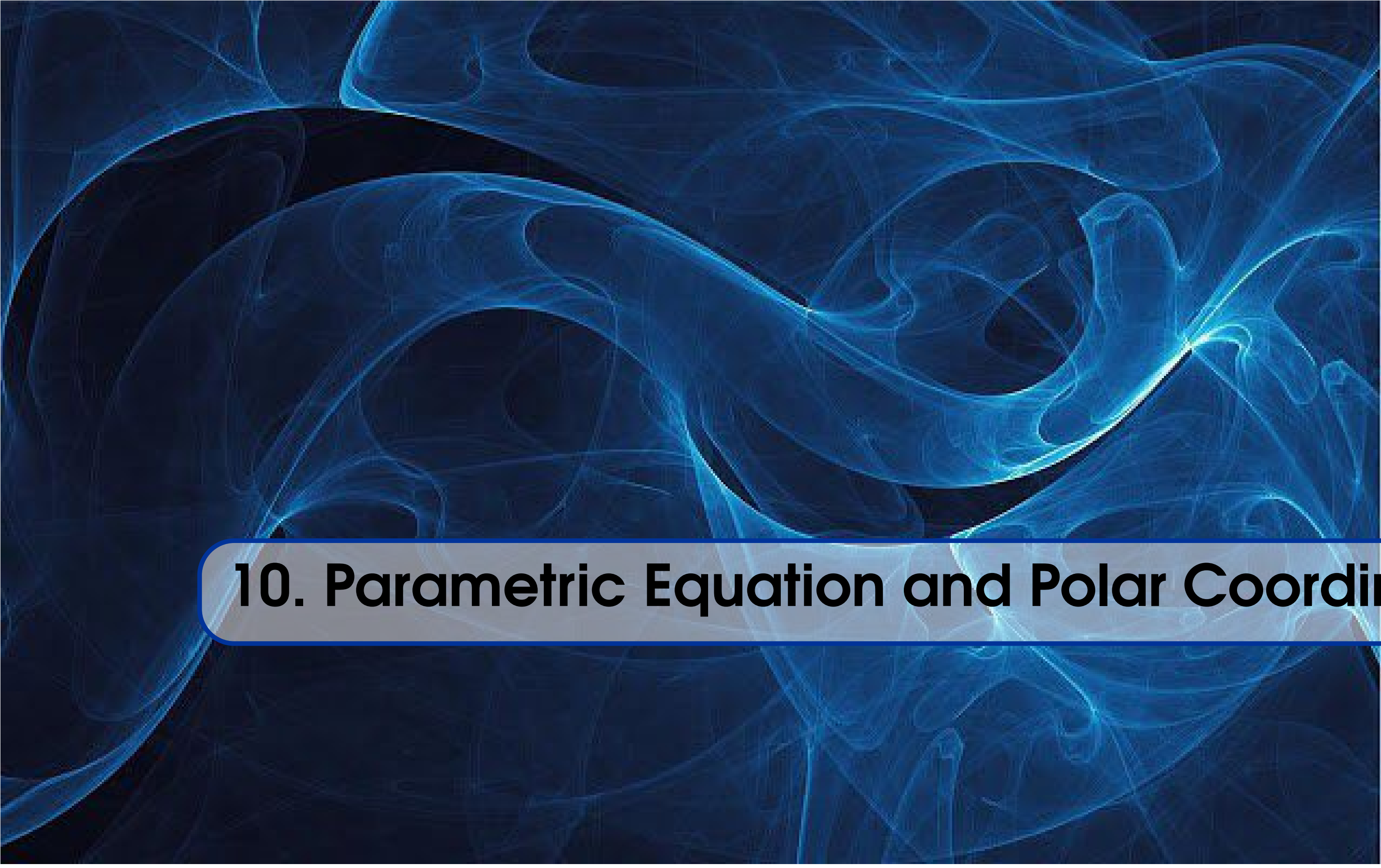
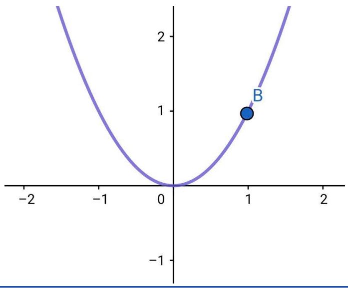


10. Parametric Equation and Polar Coordi

10.1 Parametric Equation

$$x = 1 \Rightarrow y = 1$$
$$x = ? \Rightarrow y = ?$$
$$y = f(x)$$



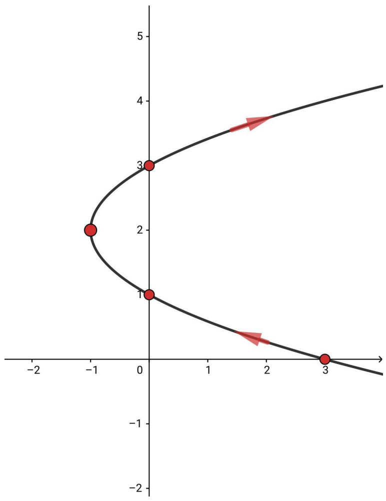
$$y = t + 1$$
$$x = 1 - t^2$$
$$t = 1 \Rightarrow$$
$$x = 0\ , y = 2, (0, 2)$$

■ **Example 10.1** Sketch the curve define by the **Parametric equation** $x = t^2 - 2t\ ,\ y = t + 1.$

Solution

t	x	y
2	0	3
1	-1	2
0	0	1
-1	3	0
-2	8	-1
⋮	⋮	⋮

$$t = y - 1$$
$$\Downarrow$$
$$x = (y-1)^2 - 2(y-1)$$
$$= y^2 - 2y + 1 - 2y + 2$$
$$x = y^2 - 4y + 2$$



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■ **Example 10.2** Sketch $x = t^2 - 2t$, $y = t + 1$, $-1 \leq x \leq 2$.
Solution

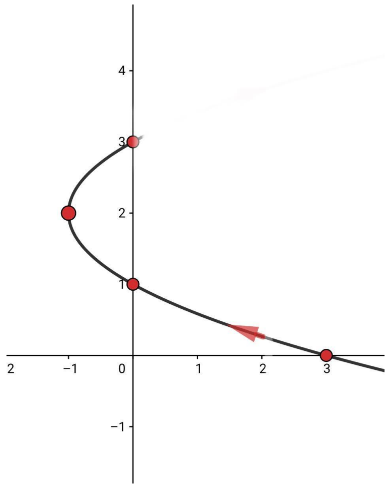
Parametric equation :

$$x = f(t)\ ,\ y = g(t)$$

$$a \leq t \leq b$$

$$Initial\ point\ (f(a), g(a))$$

$$Terminal\ point\ (f(b), g(b))$$



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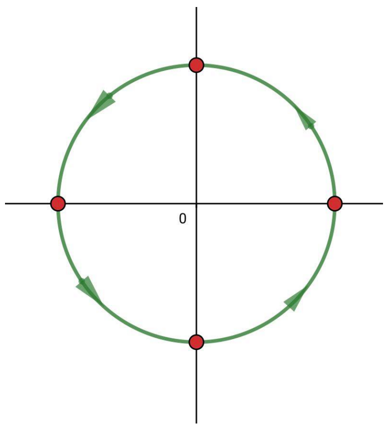
■ **Example 10.3** Sketch $x = \cos t$, $y = \sin t$
Solution

$$\cos^2 t + \sin^2 t = 1$$

$$x^2 + y^2 = 1$$

$$t = 0 \Rightarrow (1, 0)$$

$$t = \frac{\pi}{2} \Rightarrow (0, 1)$$



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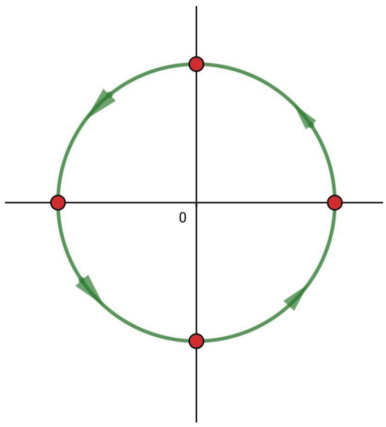
■ **Example 10.4** Sketch $x = \cos 2t$, $y = \sin 2t$
Solution

$$\cos^2 2t + \sin^2 2t = 1$$

$$x^2 + y^2 = 1$$

$$t = 0 \Rightarrow (0, 0)$$

$$t = \frac{\pi}{2} \Rightarrow (-1, 0)$$



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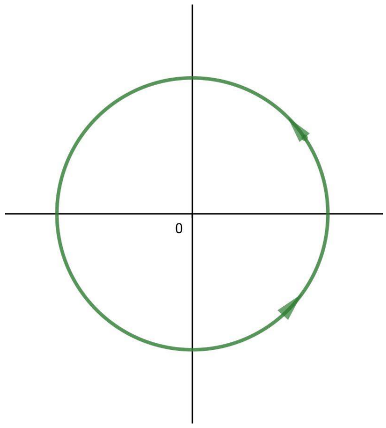
Exercise 10.1 Sketch $x = \cos\frac{1}{2}t$, $y = \sin\frac{1}{2}t$ ■

■ **Example 10.5** Sketch $x = 2\cos t$, $y = 2\sin t$

Solution

$$x^2 + y^2 = 4\cos^2 t + 4\sin^2 t$$
$$x^2 + y^2 = 4$$



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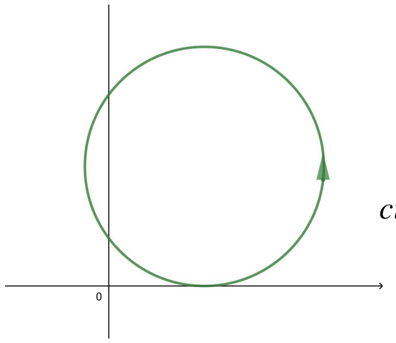
■ **Example 10.6** Sketch $x = 2\cos t + 1$, $y = 2\sin t + 2$

Solution

$$x = 2\cos t + 1 \Rightarrow x - 1 = 2\cos t$$
$$y = 2\sin t + 2 \Rightarrow y - 2 = s\sin t$$
$$(x-1)^2 + (y-2)^2 = 4(\cos^2 t + \sin^2 t)$$
$$(x-1)^2 + (y-2)^2 = 4$$



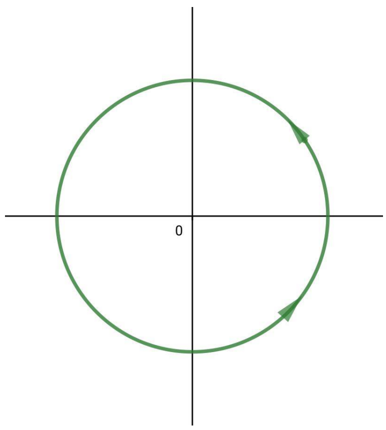
$$x = r\cos t + h$$
$$y = r\sin t + k$$
$$\textit{circle center } (h,k)$$
$$\textit{radius } r$$

■

■ **Example 10.7** Sketch $x = 2\cos t$, $y = 3\sin t$

Solution

$$x = 2\cos t \Rightarrow \cos t = \frac{x}{2}$$
$$y = 3\sin t \Rightarrow \sin t = \frac{y}{3}$$
$$\Rightarrow (\frac{x}{2})^2 + (\frac{y}{3})^2 = 1$$



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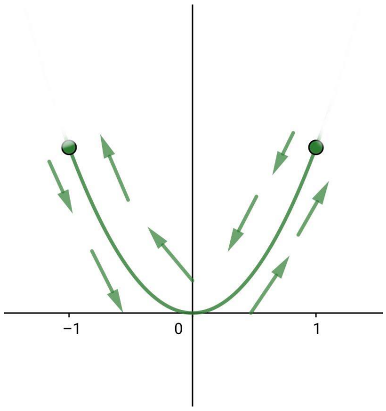
■ **Example 10.8** Sketch $x = \sin t$, $y = \sin^2 t$

Solution

$$y = x^2$$
$$t = 0 \Rightarrow (0,0)$$
$$t = \frac{\pi}{2} \Rightarrow (1,1)$$
$$t = \pi \Rightarrow (0,0)$$
$$t = \frac{3}{2}\pi \Rightarrow (-1,1)$$
$$t = 2\pi \Rightarrow (0,0)$$



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■ **Example 10.9** $\boxed{\frac{37}{628}}$

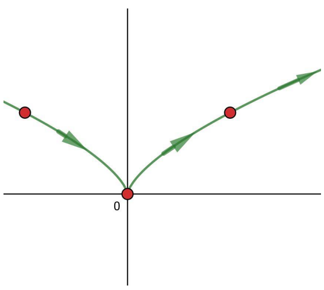
(a) $x = t^3$, $y = t^2$, $\Rightarrow t = x^{\frac{1}{3}} \Rightarrow y = x^{\frac{2}{3}}$
(b) $x = t^6$, $y = t^4$, $t = x^{\frac{1}{6}} \Rightarrow y = x^{\frac{4}{6}} \Rightarrow y = x^{\frac{2}{3}}$
(c) $x = e^{-3t}$, $y = e^{-2t}$, $\Rightarrow x = (e^{-t})^3 \Rightarrow e^{-t} = x^{\frac{1}{3}} \Rightarrow y = (e^{-t})^2 = (x^{\frac{1}{3}})^2 \Rightarrow y = x^{\frac{2}{3}}$

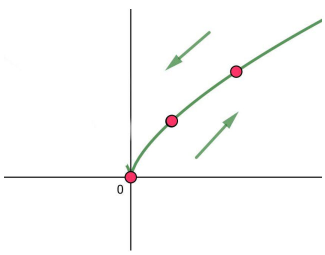
Solution

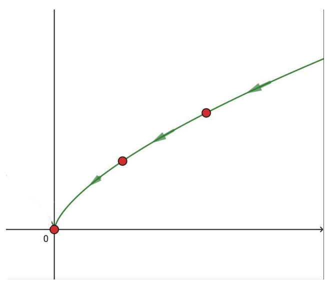
$$(a)\, x = t^6$$
$$y = t^4$$
$$y = x^{\frac{2}{3}}$$

$$(b) x = t^6$$
$$y = t^4$$
$$y = x^{\frac{2}{3}}$$

$$(c)\, x = e^{-3t} = \frac{1}{e^{3t}}$$
$$y = e^{-2t} = \frac{1}{e^{2t}}$$
$$y = x^{\frac{2}{3}}$$







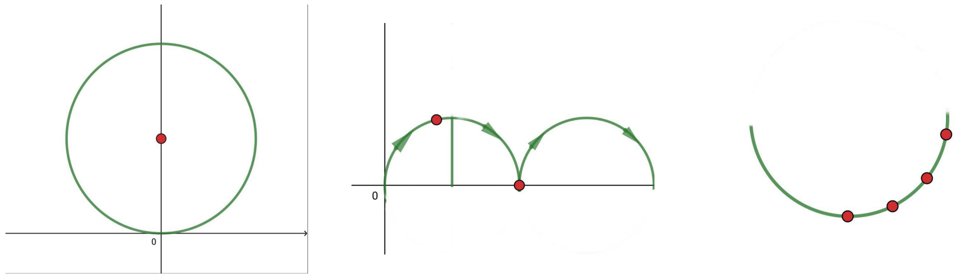
$$t = -1 \Rightarrow (-1,1)$$
$$t = 0 \Rightarrow (0,0)$$
$$t = 1 \Rightarrow (1,1)$$

$$t = -1 \Rightarrow (1,1)$$
$$t = 0 \Rightarrow (0,0)$$
$$t = 1 \Rightarrow (1,1)$$

$$t = -1 \Rightarrow (e^3,, e^2)$$
$$t = 0 \Rightarrow (1,1)$$
$$t = 1 \Rightarrow (\frac{1}{e^3}, \frac{1}{e^2})$$

■

Cycloid



$$\underbrace{x = r(\theta - sin\theta) \;\; y = r(1 - cos\theta)}_{Cycloid} \;\; 0 \leq \theta \leq 2 * \pi$$

Problem 10.1 1, 3, 4, 7, 8, 10, 12, 15, 16, 25, 26, 37.

10.2 Calculus with Parametric Equations.

let $x = f(t), y = g(t)$ Then

$$\frac{dy}{dx} = \frac{\frac{d(y)}{d(t)}}{\frac{d(x)}{d(t)}} = \frac{g'(t)}{f'(t)} \quad , f'(t) \neq 0$$

■ **Example 10.10** if $x = cost, y = sint$

Solution

$$\frac{dy}{dx} = -\frac{cost}{sint} = -cott$$

$$\begin{aligned} \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} \\ = \frac{csc^2 t}{-sint} \\ = -csc^3 t \end{aligned} \tag{10.1}$$

■

■ **Example 10.11** if $x = t^3, y = 3t$, Find $\frac{d^2y}{dx^2}$

Solution

$$\frac{dy}{dx} = \frac{3}{3t^2} = t^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

$$\frac{-2t^{-3}}{3t^2} = \frac{-2}{3}t^{-5}$$

$$\frac{d^2y}{dx^2} = \frac{-2}{3}x^{\frac{-5}{3}}$$

$$y = 3x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = x^{\frac{-2}{3}}$$

$$\frac{d^2y}{dx^2} = \frac{-2}{3}x^{\frac{-5}{3}}$$

■

Exercise 10.2 if $y = cost + t$,
$x = 1 - t + t^2$ Find $\frac{d^2y}{dx^2}$ ■

■ **Example 10.12** Consider the parametric equations : $x = t^2, y = t^3 - 3t$

(a) Show that hte curve has two tangents at (3,0) ,Find the tangent line
(b) Find where the tangent is horizontal ? Vertical ?
(c) Find where the curve is concave up? down ?
(d) Sketch the curve .

Solution

(a) $(3,0) \Rightarrow x = 3, y = 0$

$t^2 = 3$ $\qquad$ $t^3 - 3t = 0$

$\downarrow$ $\qquad$ $\downarrow$

$t = \sqrt{3}, -\sqrt{3}$ $\qquad$ $t = 0, \sqrt{3}, -\sqrt{3}$ $\qquad$ Refuse t=0

The curve passes the point (3,0) two times at $t = \sqrt{3}, -\sqrt{3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^{-}3}{2t}$$

$$\text{at } t = \sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{3\sqrt{3}^2 - 3}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$\text{at } t = -\sqrt{3} \Rightarrow \frac{dy}{dx} = \frac{3*3-3}{-2\sqrt{3}} = -\sqrt{3}$$

So equation (1)

$y - 0 = \sqrt{3}(x-3)$

And equation (2)

$y - 0 = -\sqrt{3}(x-3)$

■

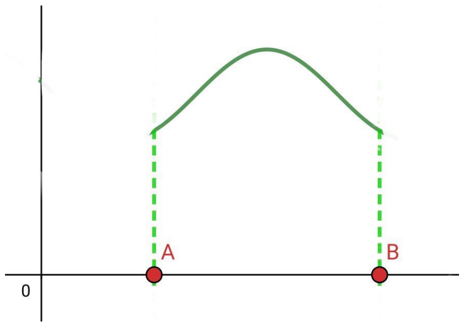
Area: If $y = g(t)$, $x = f(t)$

- Then the area between the curve C and $x - axis$ is

$$A = \int_a^b g(t)f'(t)dt.$$

- The area with the $y - axis$ is

$$A = \int_a^b f(t)g'(t)dt.$$



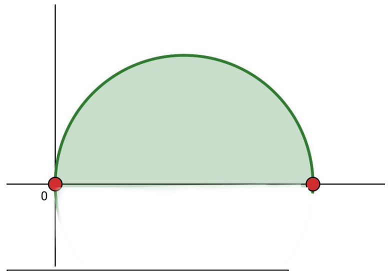
■ **Example 10.13** Find the area under one are of cycloid
$x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$
Solution

$$A = \int_0^{2\pi} g(\theta)f'(\theta)d\theta = \int_0^{2\pi} r^2(1-\cos\theta)^2 d\theta$$

$$= r^2 \int_0^{2\theta} 1 - 2\cos\theta + \cos^2\theta d\theta$$

$$= r^2 \int_0^{2\theta} 1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}cos2\theta d\theta$$

$$r^2[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{1}{4}\sin 2\theta]|_0^{2\pi} = \boxed{3\pi r^2}$$



$\boxed{f'(\theta) = r(1-\cos\theta)}$

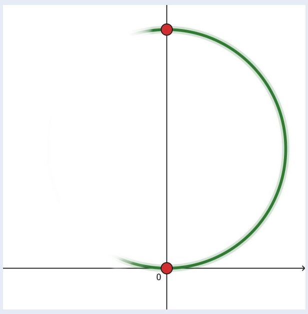
■

Exercise 10.3 Q32 page 637 Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the $y - axis$
Solution

$$A = \int_0^2 f(t)g'(t)dt$$

$$= \int_0^2 (t^2 - 2t)\frac{1}{2}t^{-\frac{1}{2}}dt$$

$$\vdots \; \vdots$$



$Let\ x = 0 \Rightarrow$

$t^2 - 2t = 0$

$t = 0$, $t = 2$

■

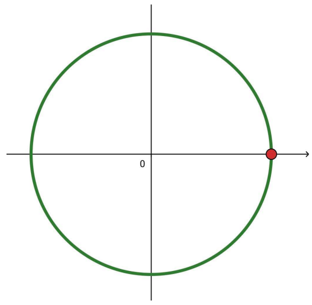
Arc Length: Let $x = f(t)$, $y = g(t)$

$$L = \int_a^b \sqrt{f'^2(t) + g'^2(t)}dt$$

R the curve should be traversed once in $a \le t \le b$

■ **Example 10.14** Find the length of $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$
Solution

$$L = \int_0^{2\pi} \sqrt{\cos^2\theta + \sin^2\theta d\theta}$$
$$\int_0^{2\pi} 1 = 2\pi$$



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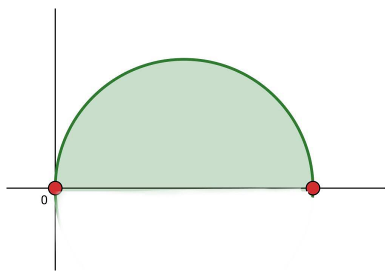
■ **Example 10.15** Find the length of one are of the cycloid
$x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.
Solution

$$L = \int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 d\theta}$$
$$= \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2(\sin\theta)^2 d\theta}$$
$$= r\int_0^{2\pi} \sqrt{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta d\theta}$$
$$= r\int_0^{2\pi} \sqrt{2 - 2cos\theta d\theta}$$
$$= r\int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2} d\theta}$$
$$= 2r\int_0^{2\pi} |\sin\frac{\theta}{2}| d\theta$$
$$= 2r\int_0^{2\pi} \sin\frac{\theta}{2}$$

$$2 - 2\cos\theta$$
$$= 2(1-\cos\theta)$$
$$= 4(\frac{1}{2} - \frac{1}{2}\cos\theta)$$
$$= 4(\sin^2\frac{1}{2}\theta)$$

$$2r(2)\cos\frac{\theta}{2}|_{2\pi}^{0}$$
$$4r(1+1) = 8r$$



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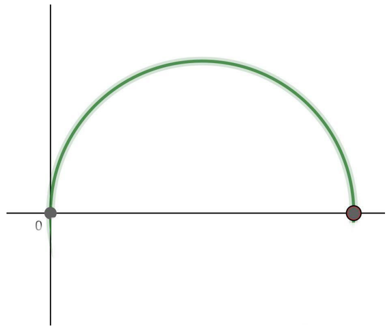
Surfaces Area:

$$S = 2\pi\int_a^b g(t)\sqrt{f'^2(t) + g'^2(t)} dt \ about \ the \ x - axis$$
$$about \ the \ x - axis \ = 2\pi\int_a^b ydL$$
$$S = 2\pi\int_a^b f(t)\sqrt{f'^2(t) + g'^2(t)} dt \ about \ the \ y - axis$$

■ **Example 10.16** Find the area of the surface obtained by revolving one arc of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ about $x - axis$

Solution:

$$S = 2\pi \int_0^{2\pi} r(1 - \cos\theta \sqrt{2(1 - \cos\theta)} d\theta$$
$$\vdots$$



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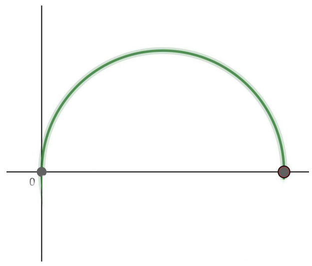
■ **Example 10.17** Find the area of a sphere of radius r $\boxed{S = 4\pi r^2}$

Solution

$$S = 2\pi \int_0^{\pi} r\sin\theta \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$
$$= 2\pi r \int_0^{\pi} \sin\theta \sqrt{r^2} d\theta$$
$$= 2\pi r^2 \int_0^{\pi} \sin\theta d\theta$$
$$= 2\pi r^2 \cos\theta |_{\pi}^{0}$$
$$= 2\pi r^2 (1 + 1) = 4\pi r^2$$

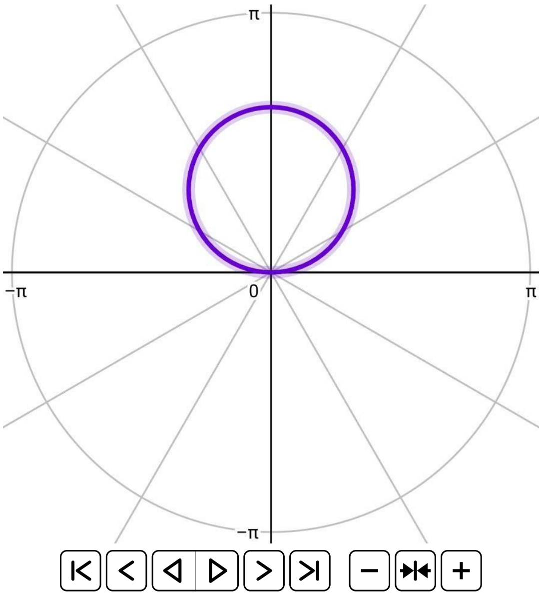


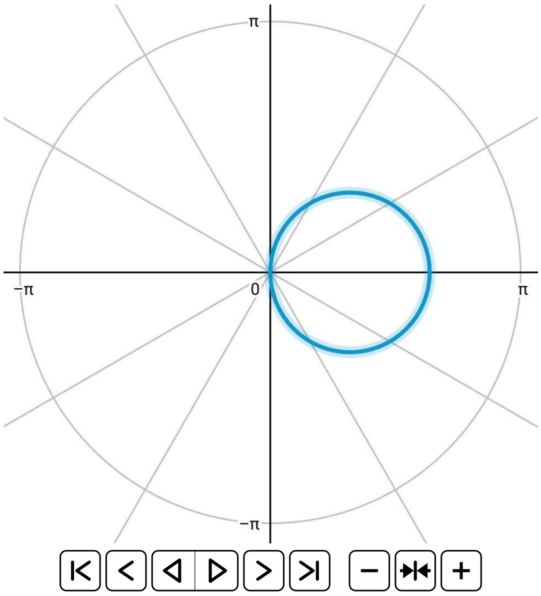
$x = r\cos\theta$
$y = r\sin\theta$
$0 \leq \theta \leq \pi$

■

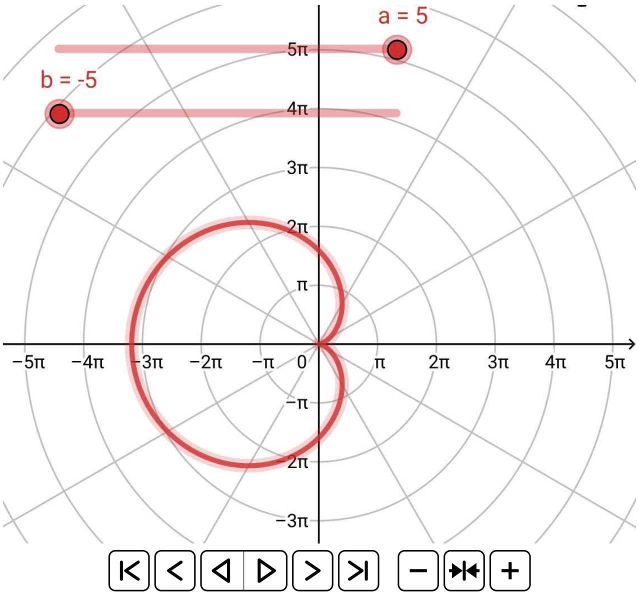
Problem 10.2 1, 3, 5, 7, 11, 12, 13, 15, 17, 18, 19, 28, 29, 30, 33, 34, 37, 39, 40, 41, 43, 57, 59, 60, 65, 69.

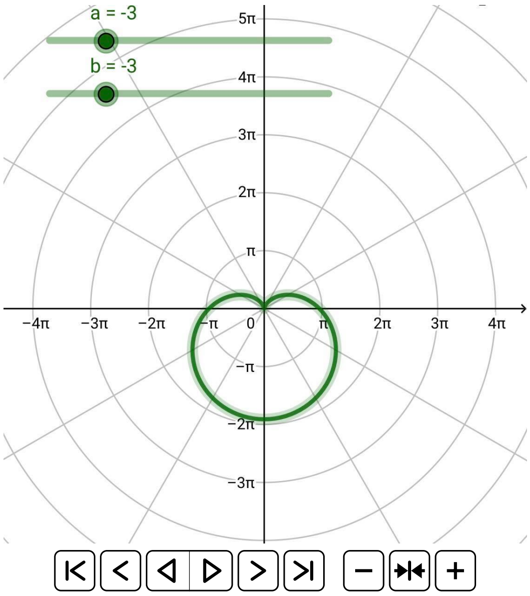
10.3 Polar Coordinates.





. $r = \sin(a\theta)$ $r = \cos(a\theta)$

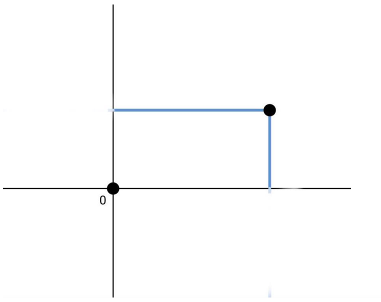


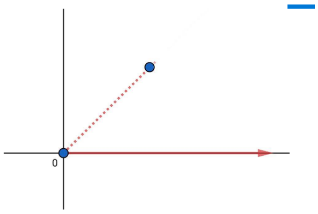


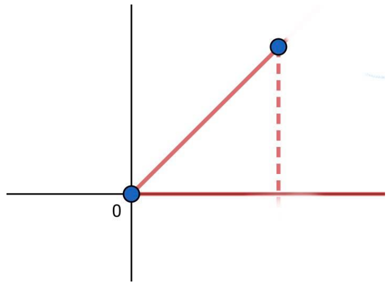
. $r = a + b\cos\theta$ $r = a + b\sin\theta$

■ **Example 10.18** Locate the following polar points .

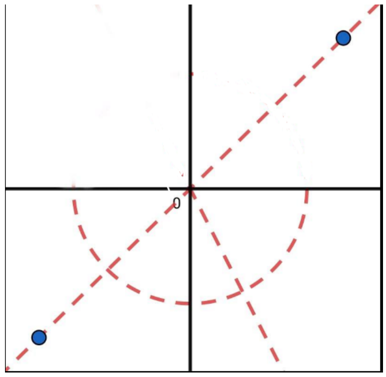
1. $(3, \dfrac{\pi}{4})$
2. $(-3, \dfrac{\pi}{4})$
3. $(1, \pi)$
4. $(-1, 0)$
5. $(1, -\pi)$







Solution



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$x = rcos\theta$ $\quad$ $r^2 = x^2 + y^2$

$y = rsin\theta$ $\quad$ $\theta = tan^{-}\frac{y}{x}$

■ **Example 10.19** Find the cartesion coordinate for the following polar points. 1) $(1, \frac{2\pi}{3})$

$x = 1 * cos\frac{2\pi}{3} = -\frac{1}{2}$

$y = 1 * sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

■

■ **Example 10.20** Converte from cartesion to polar

Solution $(1, -\sqrt{3})$ $r^2 = 1 + 3 = 4 \Rightarrow r = 2,$

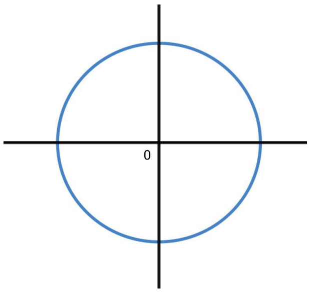
$\theta = tan^{-1}\sqrt{3}/1 \Rightarrow \theta = 300$

■

Polar Curves

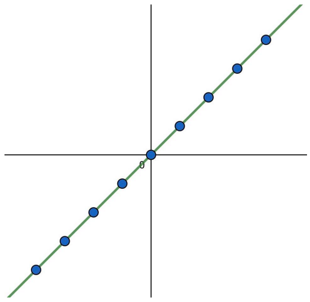
■ **Example 10.21** Sketch the following polar curves .

- $r = 2 \; r^2 = 4 \Rightarrow x^2 + y^2 = 4$

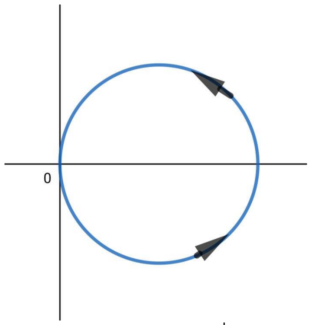


- $\theta = \dfrac{\pi}{4}$

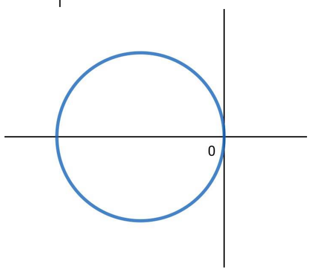
$$tan\theta = \frac{y}{x}$$
$$1 = \frac{y}{x} \Rightarrow y = x$$



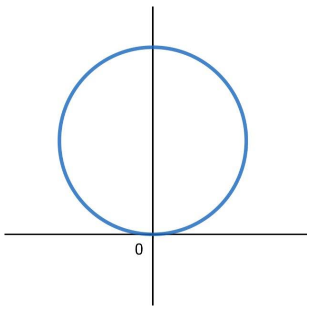
- $r = 2 * cos\theta$



$$r^2 = 2 * r * cos\theta \Rightarrow x^2 + y^2 = 2x$$
$$x^2 - 2x + ...\mathbf{+1} + y^2 = 0...\mathbf{+1}$$
$$(x = 1)^2 + y^2 = 1$$



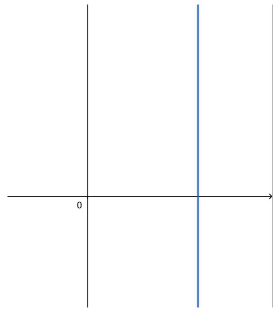
- $r = -4 * cos\theta$



- $r = 2 * sin\theta$

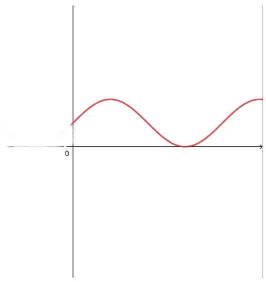
- $r = -4 * sin\theta$ (H.w)
- $r = 2 * sin\theta - 4 * cos\theta$(H.W)

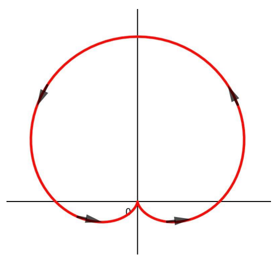
- $r = 3 * sec\theta$
  $r = \dfrac{3}{cos\theta} \Rightarrow r * cos\theta = 3$
  $\Rightarrow x = 3$



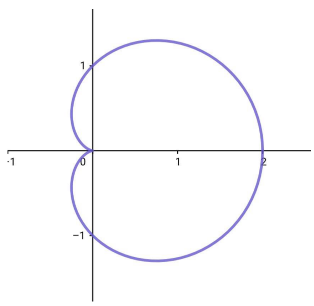
- $r = 2 * sin\theta$
- $r = -3 * \csc\theta$ (H.W)
- $r = 1 + sin\theta$
  First sketch the equation in the cartesion system

θ	r
$0 \rightarrow 2$	$1 \rightarrow 2$
$\frac{\pi}{2} \rightarrow \pi$	$2 \rightarrow 1$
$\pi \rightarrow \frac{2\pi}{3}$	$1 \rightarrow 0$
$\frac{2\pi}{3} \rightarrow \pi$	$0 \rightarrow 1$



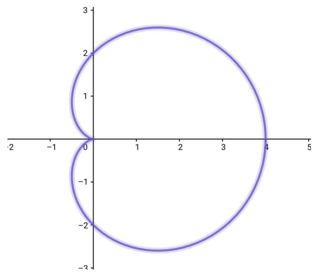


- $r = 1 - sin\theta$ (H.W)
- $r = 1 + cos\theta$ (H.W)
- $r = 1 - cos\theta$ (H.W)



- $r = 2 + 2cos\theta$ (H.W)
- $r = 2 + sin\theta$ (H.W)

θ	r
$0 \to 2$	$2 \to 3$
$\frac{\pi}{2} \to \pi$	$3 \to 2$
$\pi \to \frac{2\pi}{3}$	$2 \to 1$
$\frac{2\pi}{3} \to \pi$	$1 \to 2$

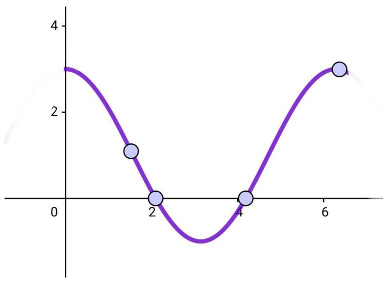


- $r = 3 + 2cos\theta$ (H.W)
- $r = 3 - 2cos\theta$ (H.W)

■

■ **Example 10.22** Sketch $r = 1 + 2\cos\theta$

Solution



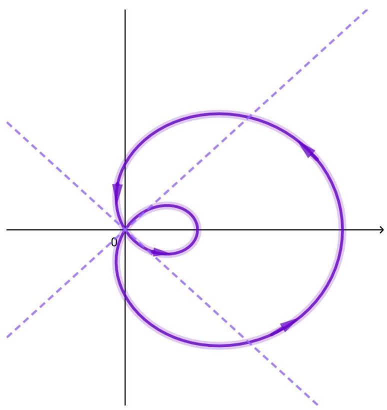
$$r = 0 \Rightarrow$$

$$1 + 2\cos\theta = 0$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2}{3}\pi \,,\; \frac{4}{3}\pi$$

θ	r
$0 \to \frac{\pi}{2}$	$3 \to 1$
$\frac{\pi}{2} \to \frac{\pi}{3}$	$1 \to 0$
$\frac{2}{3}\pi \to \pi$	$0 \to -1$
$\pi \to \frac{4}{3}\pi$	$0 \to -1$
$\frac{4}{3}\pi \to \frac{3}{2}\pi$	$0 \to 1$
$\frac{3}{2}\pi \to 2\pi$	$1 \to 3$

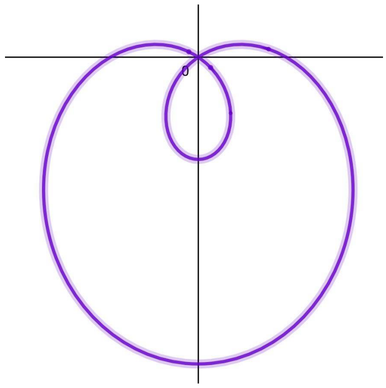


■

$$\boxed{r = a + b\sin\theta}$$

■ **Example 10.23** Sketch $r = -3 - 6\sin\theta$

Solution

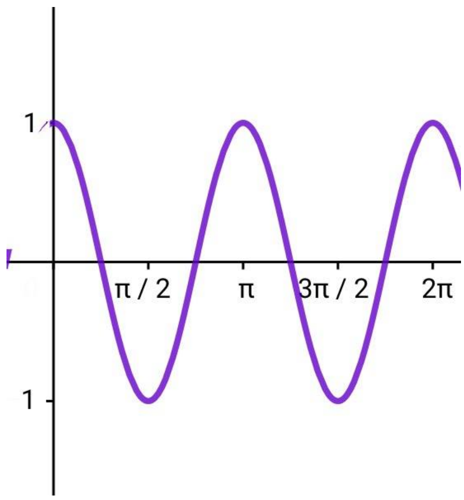


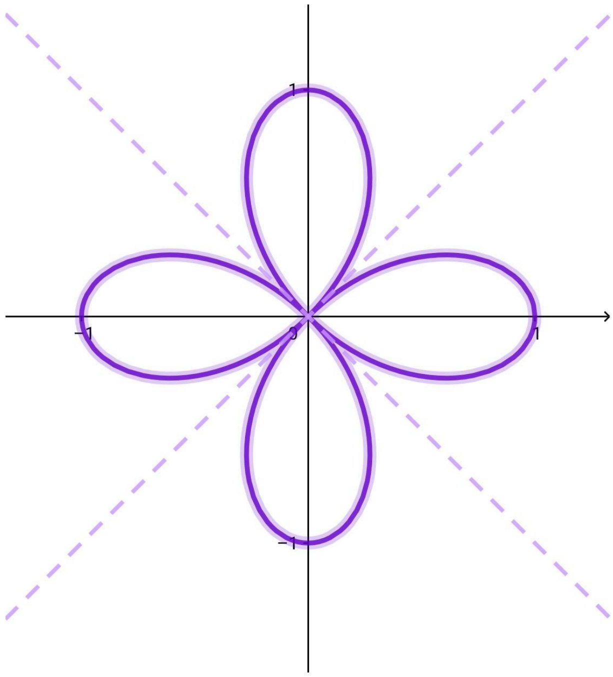
■

■ **Example 10.24** Sketch $r = \cos 2\theta$

Solution

θ	r
$0 \to \frac{\pi}{4}$	$1 \to 0$
$\frac{\pi}{4} \to \frac{\pi}{2}$	$0 \to -1$
$\frac{\pi}{2} \to \frac{3}{4}\pi$	$-1 \to 0$
$\frac{3}{4}\pi \to \pi$	$0 \to 1$
$\pi \to \frac{5}{4}\pi$	$1 \to 0$
$\frac{5}{4}\pi \to \frac{3}{2}\pi$	$0 \to -1$
$\frac{3}{2}\pi \to \frac{7}{4}\pi$	$-1 \to 0$
$\frac{7}{4}\pi \to 2\pi$	$0 \to 1$





■

Exercise 10.4 Sketch:
1. $r = \sin 2\theta$
2. $r = \cos 3\theta$
3. $r = \cos 4\theta$

■

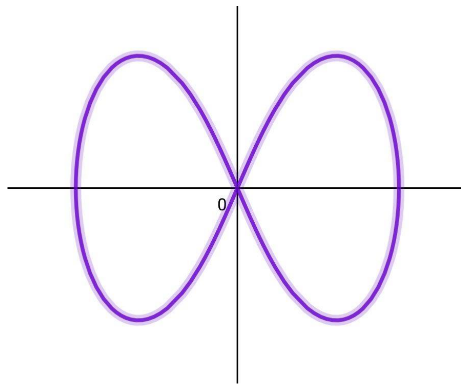
■ **Example 10.25** Sketch $r^2 = \cos 2\theta$

Solution

$$r = \sqrt{\cos 2\theta}$$

$$r = -\sqrt{\cos 2\theta}$$

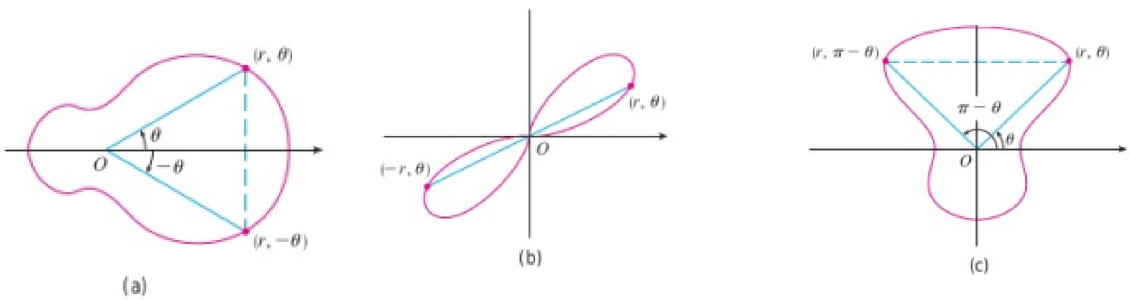
θ	$r=\cos 2\theta$	$r^2=\cos 2\theta$
$0 \to \frac{\pi}{4}$	$1 \to 0$	$1 \to 0$
$\frac{\pi}{4} \to \frac{\pi}{2}$	$0 \to -1$	$\times$
$\frac{\pi}{2} \to \frac{3}{4}\pi$	$-1 \to 0$	$\times$
$\frac{3}{4}\pi \to \pi$	$0 \to 1$	$0 \to 1$
$\pi \to \frac{5}{4}\pi$	$1 \to 0$	$1 \to 0$
$\frac{5}{4}\pi \to \frac{3}{2}\pi$	$0 \to -1$	$\times$
$\frac{3}{2}\pi \to \frac{7}{4}\pi$	$-1 \to 0$	$\times$
$\frac{7}{4}\pi \to 2\pi$	$0 \to 1$	$0 \to 1$



Symmetry:

When we sketch polar curves, it is sometimes helpful to take advantage of symmetry ,The following three rules are explained below :

1. If a polar equation is unchanged when θ is replaced by - θ , the curve is symmetric about the polar axis .
2. If the equation is unchanged when r is replaced by -r , or when θ is replaced by $\theta + \pi$ the curve is symmetric about the pole.(This means that the curve remains unchanged if we rotate it through 180^0 about the origion)
3. If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \dfrac{\pi}{2}$



■ **Example 10.26** $r = cos\theta$ is symmetric

1. about the polar axis .
2. about the origin .
3. about $\theta = \frac{\pi}{2}$

■

Tangents in polar system $\frac{dy}{dx}$

$x = rsoc\theta \qquad y = rsin\theta$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta} * sin\theta + rcos\theta}{\frac{dr}{d\theta} * cos\theta - r * sin\theta}$$

■ **Example 10.27** Let $r = 1 + sin\theta$

1. Find the slope at $\theta = \frac{\pi}{3}$
2. Find where the tangent is horizontal? Vertical ?

Solution: $x = rcos\theta \qquad ,y = rsin\theta$

1. $$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} * sin\theta + rcos\theta}{\frac{dr}{d\theta} * cos\theta - r * sin\theta}$$

$$\frac{dy}{d\theta} = cos\theta$$

$$cos\frac{\pi}{3} = \frac{1}{2}$$

$$sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{cos\frac{\pi}{3} * sin\frac{\pi}{3} + cos\frac{\pi}{3}(1 + sin\frac{\pi}{3})}{cos\frac{\pi}{3} * cos\frac{\pi}{3} + -(1 + sin\frac{\pi}{3}) * sin\frac{\pi}{3}}$$

$$= \frac{\frac{\sqrt{3}}{4} + \frac{1}{2} + \frac{\sqrt{3}}{4}}{\frac{1}{4} - \frac{\sqrt{3}}{2} - \frac{3}{4}} = -1$$

Equation of the tangent:

Slope=-1

$x_0 = (1 + sin\frac{\pi}{3} * cos\frac{\pi}{3} = (\frac{1}{2} + \frac{\sqrt{3}}{4}))$

$y_0 = (1 + sin\frac{\pi}{3} * sin\frac{\pi}{3}) = (1 + \frac{\sqrt{3}}{4}) * \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} + \frac{3}{4}$

Equation : $y - \frac{\sqrt{3}}{2} + \frac{3}{4} = -1(x - (\frac{1}{2} + \frac{\sqrt{3}}{4}))$

2. $\frac{dy}{dx} = \frac{cos\theta * sin\theta + (1 + sin\theta) * cos\theta}{cos\theta * cos\theta - (1 + sin\theta) * sin\theta}$

$\frac{2 * cos\theta * sin\theta + cos\theta}{cos^2\theta - sin^2\theta - sin\theta} = \frac{sin2\theta + cos\theta}{cos2\theta - sin\theta}$

$\frac{dy}{dx} = 0 \Rightarrow 2 * cos\theta * sin\theta + cos\theta = 0 \Rightarrow cos\theta(2sin\theta + 1) = 0$

$cos\theta = 0$ or $sin\theta = -\frac{1}{2}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\theta = \frac{7pi}{6}, \frac{11\pi}{6}$

$\Rightarrow \frac{dx}{d\theta} = 0 \Rightarrow cos^2\theta - sin^2\theta - sin\theta = 0$

$1 - sin^2\theta - sin\theta - sin\theta = 0$

$(2sin\theta - 1)(sin\theta + 1) = 0$

$sin\theta = \frac{1}{2}$ $, sin\theta = -1$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

H.T $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{\pi}{2}$

V.T $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

at $\theta = \frac{3\pi}{2}$

$\lim_{\theta \to \frac{3\pi}{2}^+} \frac{dy}{dx} = \lim_{\theta \to \frac{3\pi}{2}} \frac{sin2\theta + cps\theta}{cos2\theta - sin\theta}$ $(\frac{0}{0})$

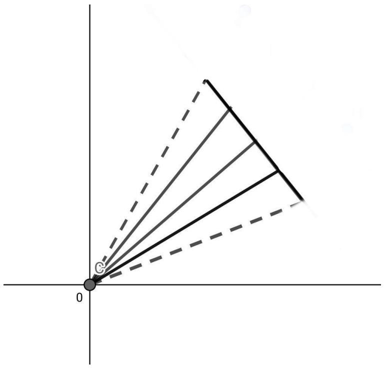
L'Hopital $= \lim_{\theta \to \frac{3\pi}{2}^+} = \frac{2\cos 2\theta - \sin\theta}{2\sin 2\theta - \cos\theta} = -\infty$

■

Problem 10.3 1, 3, 7, 8, 9, 11, 13, 15, 16, 17, 19, 21, 24, 25, 29, 30, 31, 34, 37, 39, 40, 42, 43, 47, 57, 58, 61, 63, 65, 67, 70.

10.4 Area and length

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$



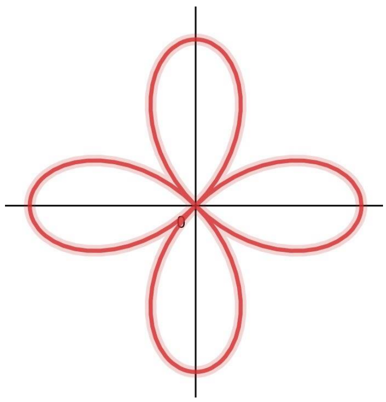
■ **Example 10.28** Find the area of one leaf of the rose $r = \cos 2\theta$

$$A = 2 * \frac{1}{2} * \int_0^{\frac{\pi}{4}} (\cos 2\theta)^2 d\theta$$

Solution:
$$A = \int_0^{\frac{\pi}{2}} (\frac{1}{2} + \frac{1}{2} * \cos 4\theta) d\theta$$

$$A = \frac{\theta}{2} + \frac{1}{8}\sin 4\theta|_0^{\frac{\pi}{2}}$$

$$A = \frac{\pi}{4}$$

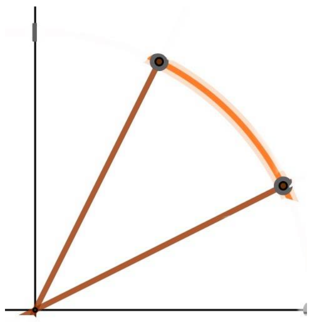


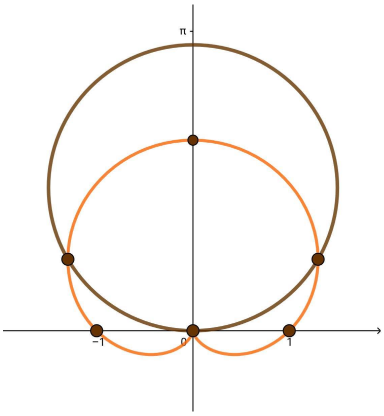
■

■ **Example 10.29** Find the area of the region that lies inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.

Solution:
$$r = f(\theta)$$

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$





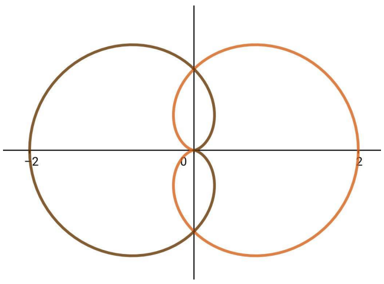
$$A=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}(3\sin\theta)^2-(1+\sin\theta)^2d\theta$$

$$=2.\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(3\sin\theta)^2-(1+\sin\theta)^2d\theta$$

■

■ **Example 10.30** Find the area inside both $r=1+\cos\theta$, $r=1-\cos\theta$

Solution:



$$A=4.\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-\cos\theta)^2d\theta$$

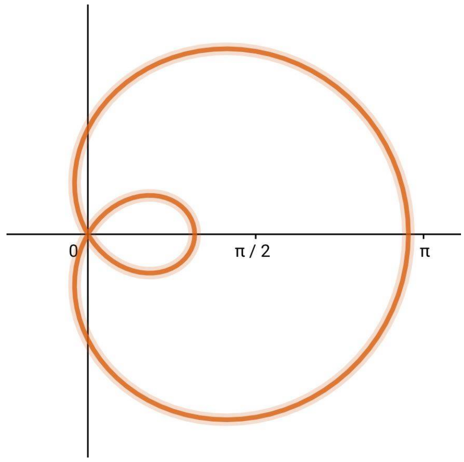
$$or\,A=4.\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(1-\cos\theta)^2d\theta$$

■

■ **Example 10.31** Find the area inside the inner loop of $r=1+2\cos\theta$

Solution:

$$r = 0$$
$$\Rightarrow 1 + 2\cos\theta = 0$$
$$\Rightarrow \cos\theta = -\frac{1}{2}$$
$$\Rightarrow \theta = \frac{2}{3}\pi\ ,\frac{4}{3}\pi$$
$$A = 2.\frac{1}{2}\int_{\frac{2}{3}\pi}^{\pi}(1+2\cos\theta)^2 d\theta$$
$$= 2.\frac{1}{2}\int_{\pi}^{\frac{4}{3}\pi}(1+2\cos\theta)^2 d\theta$$
$$= \frac{1}{2}\int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi}(1+2\cos\theta)^2 d\theta$$

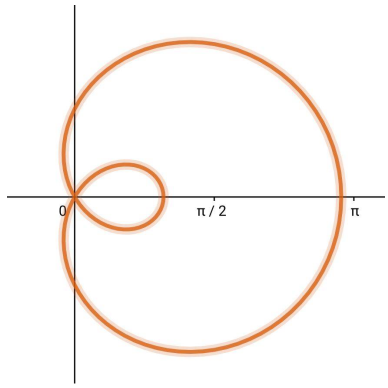


■

■ **Example 10.32** Find the area the lies between the inner and the outer loop of $r = 1 + 2\cos\theta$.

Solution:

$$A = 2.\frac{1}{2}\left(\int_0^{\frac{2}{3}\pi}(1+2\cos\theta)^2 d\theta - \int_{\frac{2}{3}\pi}^{\pi}(1+\cos\theta)^2 d\theta\right)$$



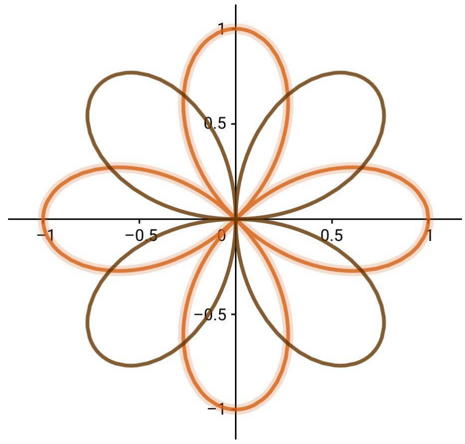
■

■ **Example 10.33** Find the area... inside both $r = \cos 2\theta\ ,\ r = \sin 2\theta$.

Solution:

$$A = 8.2.\frac{1}{2}\int_0^{\frac{\pi}{8}} (\sin 2\theta)^2 d\theta$$

$$or \ = 16.\frac{1}{2}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} (\cos 2\theta)^2 d\theta$$

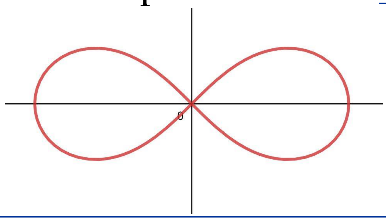


■

Problem 10.4 2, 3, 5, 6, 7, 8, 9, 11, 13, 17, 21, 23, 24, 25, 26, 28, 29, 31, 33, 35, 37, 41, 45, 47.

■ **Example 10.34** Find the area of the loop $r^2 = 9\cos 2\theta$ Solution:

$A = 2 * \frac{1}{2}\int_0^{\frac{\pi}{4}} 9\cos 2\theta d\theta$
$A = \frac{9}{2}\sin 2\theta|_0^{\frac{\pi}{4}}$
$A = \frac{9}{2}$



Parametric Equation

- $x = f(\theta) * cos(\theta)$ $\qquad r = f(\theta)$
- $y = f(\theta) * sin(\theta)$

■

Arc Length :

$$L = \int_a^b \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$

$$= \int_a^b \sqrt{(r' * cos(\theta) - r * sin(\theta))^2 + (r' * sin(\theta) + r * cos(\theta))^2} d\theta$$

$= \int_a^b \sqrt{(r')^2 * cos(\theta)^2 - 2 * r' * r * cos(\theta) * sin(\theta) + (r')^2 * sin(\theta)^2 + 2 * r' * r * cos(\theta) * si$

$$L = \int_a^b \sqrt{(r')^2 + r^2} d\theta$$

■ **Example 10.35** Find the length of the cardiod $r = 1 + \sin\theta$ Solution:

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(r')^2 + r^2 d\theta} \\
&= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2(\theta) d\theta} \\
&= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2(\theta) d\theta} \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta d\theta} * \sqrt{\frac{2-2\sin\theta}{2-2\sin\theta}} \\
&= \int_0^{2\pi} \sqrt{\frac{4(1-\sin^2\theta)}{2(1-\sin\theta)}} \\
&= \int_0^{2\pi} \frac{2\sqrt{\cos^2\theta}}{\sqrt{2(1-\sin\theta)}} \\
&= \int_0^{2\pi} \frac{2\left|\cos\theta\right|}{\sqrt{2(1-\sin\theta)}}
\end{aligned}
$$

■

■ **Example 10.36** 1. $\delta = \int_a^b 2\pi r * \sin\theta * \sqrt{(r')^2 + r^2} d\theta$ about the polar axis.

2. Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis .

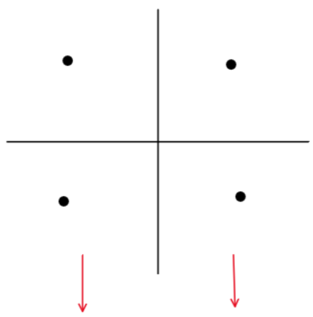
Solution:

$$
\begin{aligned}
\delta &= 2\int_0^{\frac{\pi}{4}} 2\pi r * \sin\theta * \sqrt{(r')^2 + r^2} d\theta \\
&= 4\pi \int_0^{\frac{\pi}{4}} r * \sin\theta * \sqrt{(\cos 2\theta) + \frac{sin^2(2\theta)}{cos(2\theta)}} d\theta \\
&= 4\pi \int_0^{\frac{\pi}{4}} r * \sin\theta * \sqrt{\frac{1}{cos(2\theta)}} d\theta \\
&= 4\pi \int_0^{\frac{\pi}{4}} r * \sin\theta * \frac{1}{r} d\theta \\
&= 4\pi * \cos\theta * (0 - \frac{\pi}{4}) \\
&= 4\pi * (1 - \frac{1}{\sqrt{2}})
\end{aligned}
$$

■

12. Vectors and Geometry of Space

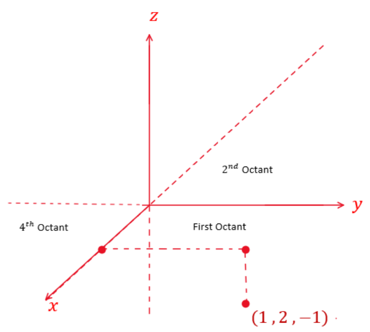
12.1 3 Dimensional coordinate system



$\cancel{y = f(x)}$

$z = f(x, y)$

- $XZ - plane$
- $XY - plane$
- $YZ - plane$

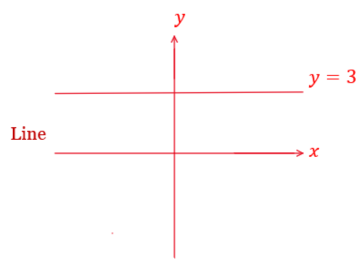


■ **Example 12.1** Describe the following equations :

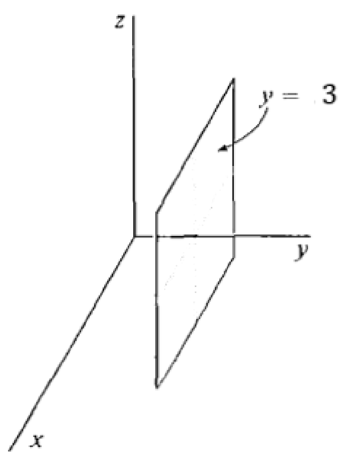
1. (a) $y = 3$ in $2D(R^2)$
   (b) $y = 3$ in $3D(R^3)$ (Plane parallel $xz - plane$)
2. $z^2 = 1$ in R^3
   - z=1 plane above parallel xy-plane
   - z=-1 plane below parallel xy-plane
3. $x^2 + y^2 = 1$ &$z = 2$
4. $y = x$ in R^3
   - plane $\dfrac{\pi}{4}$ with XZ-plane , YZ-plane

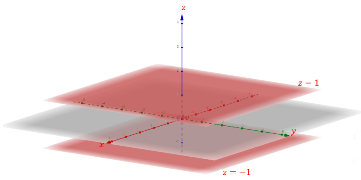
Solution:

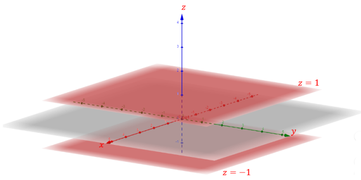
1. (a)

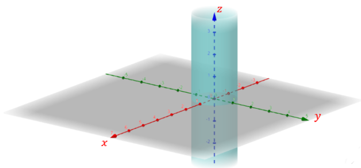


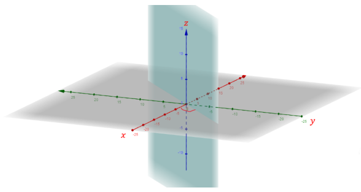
(b)



(c) 

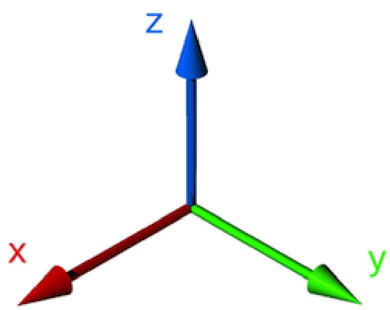
2. 

3. 

4. 

■

3D Coordinate System



R^3 , V_3

- (x_1, y_1, z_1) , $(x_2, y_2\ ,\ z_2)$, Distance:
  $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- (h, k, l) , r

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

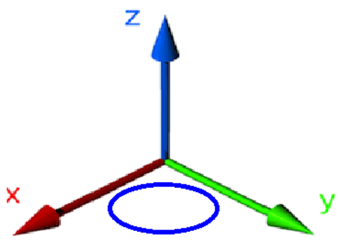
$$Sphere\ center\ (h, k, l)$$

radius r

■ **Example 12.2** Describe the region represented by: $1 \leq x^2 + y^2 + z^2 \leq 4$
Solution
The equation represented the region between:
The sphere of center (0,0,0), radios 2
and the sphere of center (0,0,0), radios 1 ■

■ **Example 12.3** Find an equation of the largest sphere of center (5,4,9) in the first octant.
Solution

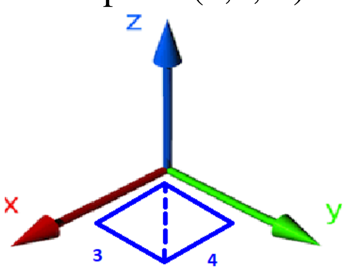
$$r = 4$$
$$(x-5)^2 + (y-4)^2 + (z-9)^2 = 16$$



■

■ **Example 12.4** Find the distance between the point (2,3,-4) and :

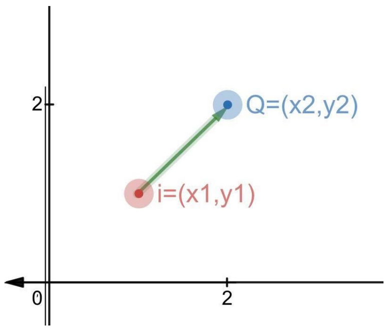
1. the $x - axis$
$$d = \sqrt{3^2 + 4^2} = 5$$
2. the $xz - plane$
$$d = 3$$



■

Problem 12.1 2,3,7,9,10,11,13,14,15,17,19,20,21,23,25,29,32,33,35,38

12.2 Vectors



$$\overrightarrow{v} = \overrightarrow{PQ} = < x_2 - x_1, y_2 - y_1 >$$

in $2D$ $\vec{a} =< x,y >$ $\qquad |\vec{a}| = \sqrt{x^2+y^2}$

in $3D$ $\vec{a} =< x,y,z >$ $\qquad |\vec{a}| = \sqrt{x^2+y^2+z^2}$

if $\vec{a} =< a_1,a_2,a_3, \ \vec{b} =< b_1,b_2,b_3 > \Rightarrow$

$$\vec{a}+\vec{b} =< a_1+b_1,a_2+b_2,a_3+b_3 >$$

$$c\vec{a} =< ca_1,ca_2,ca_3 >$$

$$\vec{a} =< a_1,a_2,a_3 >= a_1\underbrace{< 1,0,0 >}_{i}+a_2\underbrace{< 0,1,0 >}_{j}+a_3\underbrace{< 0,0,1 >}_{k}$$

$$= a_1 i + a_2 j + a_3 k$$

Definition 12.2.1 — Unit vector:. a vector $\vec{u}$ is called a unit if $|\vec{u}| = 1$

Example 12.5 Find a unit vector in the opposite direction of $\vec{v} =< 2,-2,1 >$.

Solution: $|\vec{u}| = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{-1}{\sqrt{4+4+1}} < 2,-2,1 >=< \dfrac{-2}{3},\dfrac{2}{3},\dfrac{-1}{3} >$ ■

Problem 12.2 7,11,13,15,17,19,21,23,24,25,37,41,42

12.3 The Dot Product

Definition 12.3.1 if $\vec{a} =< a_1,a_2,a_3 >$, $\vec{b} =< b_1,b_2,b_3 > \Rightarrow$

$$\vec{a}.\vec{b} = a_1b_1+a_2b_2+a_3b_3$$

Example 12.6 if $\vec{a} =< 2,1,-2 >$, $\vec{b} =< 1,1,3 >$

Solution

$\vec{a}.\vec{b} = 2+1-6 = -3$

$\vec{a}.\vec{a} = a_1^2+a_2^2+a_3^2 = (\sqrt{a_1^2+a_2^2+a_3^2})^2 = |\vec{a}|^2$ ■

Properties:

1. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
2. $\vec{a}.\vec{a} = |\vec{a}|^2$
3. $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b}+\vec{a}.\vec{c}$
4. $(c\vec{a}).\vec{b} = \vec{a}.(c\vec{b}) = c(\vec{a}.\vec{b})$
5. $\vec{0}.\vec{a} = 0$

Theorem 12.3.1 if θ is the angle between $\vec{a}$ & $\vec{b}$ then $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$\theta \in [0,\pi]$

Corollary 1: $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $|\vec{a}| \neq 0$, $|\vec{b}| \neq 0$

■ **Example 12.7** Find the angle between $\vec{a} =< 2,2,-1 >$, $\vec{b} =< 5,-3,-2 >$

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{10-6-2}{3\sqrt{25+9+4}} = \frac{2}{3\sqrt{38}}$$

$$g = cos^{-1}(\frac{2}{3\sqrt{38}} \approx 1.46(84)^\circ$$ ■

Corollary 2: Two non-zero vectors are orthogonal iff $\vec{a}.\vec{b} = 0$

■ **Example 12.8** if $\vec{a} =< 2,1,-2 >$, $\vec{b} =< c,2,1 >$
Find c such that $\vec{a} \perp \vec{b} = 0$
$\vec{a}.\vec{b} = 0 \Leftrightarrow 2c+2-2 = 0 \Leftrightarrow c = 0$ ■

$\vec{a} =< a_1, a_2, a_3 >\Rightarrow$
$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
$\vec{b} =< b_1, b_2, b_2 >$ $\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|\ \ cos\theta$ $\vec{a}.\vec{b} =$
$0 \leftrightarrow \vec{a} \perp \vec{b}$

- $-|a||b| \leq \vec{a}.\vec{b} \leq |a||b|$
  $|\vec{a}.\vec{b}| \leq |a||b|$

■ **Example 12.9** if $|\vec{a}| = 2$, $|\vec{b}| = 3$, $\theta = \frac{2}{3}\pi$. Find $|\vec{a} - 2\vec{b}|$. $|\vec{u}|^2 = \vec{u}.\vec{u}$

$$|\vec{a} - 2\vec{b}|^2 = (\vec{a} - 2\vec{b}).(\vec{a} - 2\vec{b})$$
$$= \vec{a}.\vec{a} - 2\vec{a}\vec{b} - 2\vec{b}.\vec{a} + 4\vec{b}.\vec{b}$$
$$= |\vec{a}|^2 - 4|\vec{a}||\vec{b}|cos\theta + 4|\vec{b}|^2$$
$$= 4 - 4.2.3\left(\frac{-1}{2}\right) + 4.9$$
$$= 16 + 36 = 52\ |\vec{a} - 2\vec{b}| = \sqrt{52}$$

$$\boxed{|\vec{a} + \vec{b}| = \sqrt{|a|^2 + |b|^2 - 2\ |a|\ |b|\ cos\theta}}$$ ■

■ **Example 12.10** Prove that $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

Pf:

$|\vec{a} + \vec{b}|^2 = (\vec{a} + \vec{b}).(\vec{a}.\vec{b})$

$$= |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}.\vec{b} \leq |\vec{a}|^2 + |\vec{b}|^2 + 2\,|a|\,|b|$$
$$= \underbrace{(|a|+|b|)^2}$$

$\Rightarrow\ |\vec{a}+\vec{b}|\ \leq\ |\vec{a}|\ +\ |\vec{b}|$

■

Direction Angles.
is the angle b/w $\vec{v}$ & the $x-axis$ β
γ

Direction Cosines. $cos\alpha = \dfrac{a_1}{|\vec{v}|}$
$cos\beta = \dfrac{a_2}{|\vec{v}|}$ $cos\gamma = \dfrac{a_3}{|\vec{v}|}$

$a_1 = cos\alpha|\vec{v}|$, $a_2 = cos\beta|\vec{v}|$, $a_3 = cos\gamma|\vec{v}|$
$\vec{v} = |\vec{v}| < cos\alpha, cos\beta, < \gamma >$
$\dfrac{\vec{v}}{|\vec{v}|} =< cos\alpha, cos\beta, cos\gamma >$
$\Rightarrow cos^2\alpha + cos^2\beta + cos^2\gamma = `1$

■ **Example 12.11** can $\alpha = \dfrac{\pi}{4}$, $\beta = \dfrac{\pi}{4}$, $\gamma = \dfrac{\pi}{4}$ be direction cosines. Answer. No, because
$cos^2\dfrac{\pi}{4} + cos^2\dfrac{\pi}{4} + cos^2\dfrac{\pi}{4} = \dfrac{3}{2} \neq 1$ ■

Exercise 12.1 if $\vec{a} =< 1,2,3 >$ Find direction angles. ■

Projections

- Scalar Projection:

$$comp_{\vec{a}}\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2} \qquad \frac{|\vec{a}|\,|\vec{b}|\,cos\theta}{|\vec{a}|} = \mathscr{L} \qquad \vec{v} = \mathscr{L}\frac{\vec{a}}{|\vec{a}|}$$
$$\frac{\vec{a}.\vec{b}}{\vec{a}} = \qquad\qquad = \frac{\vec{a}.\vec{a}}{|\vec{a}|^2}\,\vec{a}$$

- Vector Projection:

$$Proj_{\vec{a}}\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{a}|^2}\,\vec{a}$$

■ **Example 12.12** if $\vec{a} =< 2,1,-1 >$
Find
$\vec{b} =< 2,-1,2 >$

1. $comp_{\vec{b}}^{\vec{a}} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{4-1-2}{3} = \dfrac{1}{3}$
2. $Proj_{\vec{b}}^{\vec{a}} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\,\vec{b} = \dfrac{1}{9} < 2,-1,2 >$

■

■ **Example 12.13** if $Proj_{\vec{b}}^{\vec{a}} = < 2,1,-2 >$, $\underbrace{\theta}_{\text{angle b/w } \vec{a} \,\&\, \vec{b}} = \dfrac{8}{15}\pi$

Find:

1. $Proj_{\vec{b}}^{2\vec{a}} = 2\dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\,\vec{b} = 2 < 2,1,-2 > = < 4,2,-4 >$
2. $Proj_{2\vec{b}}^{\vec{a}} = \dfrac{\vec{a}.2\vec{b}}{4|\vec{b}}|\,2\vec{b} = < 2,1,-2 >$
3. $Proj_{-2\vec{b}}^{\vec{a}} = \dfrac{\vec{a}.(-2\vec{b})}{4\,|\vec{b}|}\,(-2\vec{b}) = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|^2}\,\vec{b} = < 2,1,-2 >$
4. $comp_{\vec{b}}^{\vec{a}} = -3$

■

■ **Example 12.14** Find x such that the angle between $< 2,1,-1 >$, $< 1,x,0 >$ is 45°

$\cos 45^\circ = \dfrac{2+x-0}{\sqrt{6}\sqrt{1+x^2}}$ $\qquad |\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

$\Leftrightarrow \dfrac{\sqrt{6}}{\sqrt{2}}\sqrt{1+x^2} = 2+x$

$\Rightarrow 3(1+x^2) = (2+x)^2$

$3+3x^2-4x-1=0$

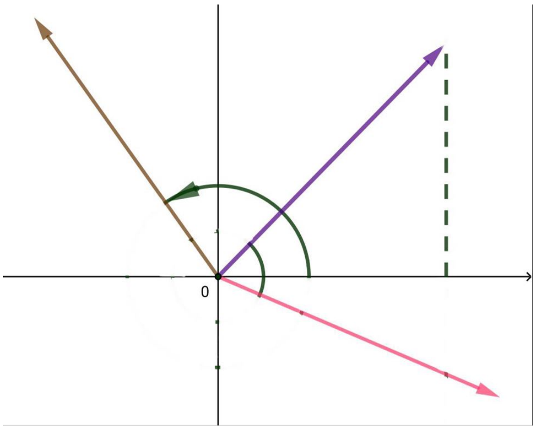
$x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

■

■ **Example 12.15** Find the two unit vectors that make an angle 60° with $\vec{v} =< 3,4 >$.

$\overrightarrow{u} =< a,b >$
$a^2+b^2 = 1 \cdots (1)$

$$\cos 60 = \frac{\overrightarrow{u}.\overrightarrow{v}}{|\overrightarrow{u}|.|\overrightarrow{v}|}$$
$$\frac{1}{2} = \frac{3a+4b}{1.5}$$
$$3a+4b = \frac{5}{2} \cdots (2)$$

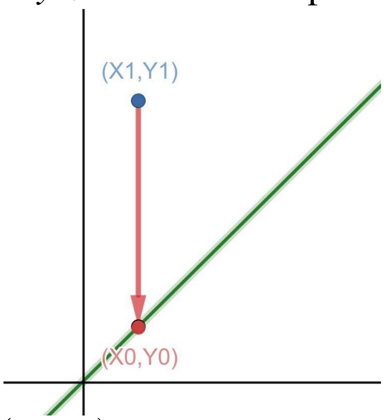


$$\theta^\circ = \tan\frac{4}{3} = x_\circ$$

■

53: the distance between the fine $ax+by+c=0$ and the point (x_1,y_1) is

$$D = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

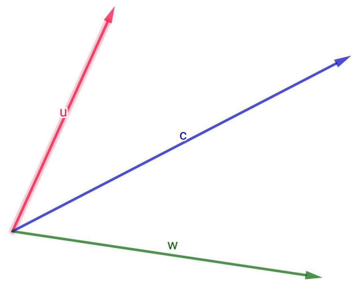


Question: Find the distance between $(-1,3)$& the line $3x-4y+5=0$

$$D = \frac{3(-2)-4(3)+5}{\sqrt{3^2+4^2}} = \frac{13}{5}$$

■ **Example 12.16** if $\overrightarrow{c} = |\overrightarrow{a}|\overrightarrow{b} + |\overrightarrow{b}|\overrightarrow{a}$, $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ not zero vectors
Show that $\overrightarrow{0}$ bisects $\overrightarrow{a}$ & $\overrightarrow{b}$
Solution:

$$\cos\alpha = \frac{\vec{a}\,.\,\vec{c}}{|\vec{a}||\vec{c}|} = \frac{\vec{a}\,.[|\vec{a}|\vec{b}+|\vec{b}\,\vec{a}}{|\vec{a}||\vec{c}|}$$
$$= \frac{\vec{a}\,.\,\vec{b}+|\vec{a}||\vec{b}|}{|\vec{c}|}$$



$$\cos\beta = \frac{\vec{b}\,.\,\vec{c}}{|\vec{b}||\vec{c}|} = \frac{\vec{b}.[|\vec{a}|\vec{b}+|\vec{b}|\vec{a}}{|\vec{b}||\vec{c}|}$$
$$= \frac{\vec{a}\,.\,\vec{b}+|\vec{a}||\vec{b}|}{|\vec{c}|}$$
$$\Rightarrow \alpha = \beta$$ ■

Exercise 12.2 Show that $proj_{\vec{b}}^{\vec{a}}.proj_{\vec{a}}^{\vec{b}} = (\vec{a}\,.\,\vec{b})\cos^2\theta$ ■

Problem 12.3 1,3,7,9,10,11,15,19,20,21,23,25,26,27,31,35,39,43,45,49,54,59

12.4 The Cross Product

Definition 12.4.1 if $\vec{a} =< a_1,a_2,a_3 >$, $\vec{b} =< b_1,b_2,b_3 >$ then the cross product of $\vec{a}$ & $\vec{b}$ is:

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$
$$=< a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1 >$$

■ **Example 12.17** If $\vec{a} =< 1,2,-1 >$, $\vec{b} =< 2,2,-3 >$ Find $\vec{a}\times\vec{b}$

Solution:

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & 2 & -3 \end{vmatrix} = (-4)i + 1(j) + (-2)k =< -4,1,-2 > .$$ ■

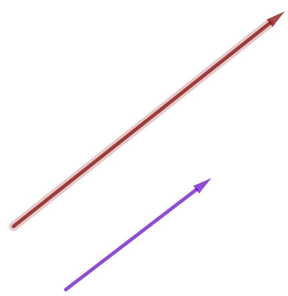
Theorem 12.4.1 $\vec{a} \times \vec{b} \perp \vec{a}$ & $\vec{a} \times \vec{b} \perp \vec{b}$

proof: $(\vec{a} \times \vec{b}).\vec{a} =< a_2b_3 - a_3b_2, -a_1b_3 + a_3b_1, a_1b_2 - a_2b_2 > . < a_1, a_2, a_3 >$
$= a_1a_2b_3 - a_1a_3b_2 - a_2a_3b_1 + a_1a_3b_2 - a_2a_3b_2 = Zero$
$\Rightarrow (\vec{a} \times \vec{b} \perp \vec{a})$.

Theorem 12.4.2 $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta. \quad 0 \le \theta \le 180^\circ$

proof: $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$
$= a_2^2b_3^2 + a_3^2b_2^2 - 2a_2a_3b_2b_3 + a_1^2b_3^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2$
$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$
$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a}\,\vec{b})^2$
$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos\theta^2$

Corollary: if $\vec{a}, \vec{b}$ non-zero vectors, then
$\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0} \iff |\vec{a} \times \vec{b}| = 0 \iff \vec{a} = c\vec{b}$ for some c



- $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- $a \times \vec{b} \perp \vec{a} \& \vec{a} \times \vec{b} \perp \vec{b}$
- $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\,sin\theta \qquad g \in [0, \pi]$
- $|\vec{a} \times \vec{b}$ = area of the parallelogram that determined by $\vec{a} \times \vec{b}$

■ **Example 12.18** find the area of the triangle with vertices
Solution :

- $P(2,1,3)$
- $Q(1,-1,1)$
- $R(3,2,-2)$
- $\overrightarrow{PQ} =< -1,-2,-2 >$
- $\overrightarrow{PR} =< 1,1,-5 >$
- $A =\mid \overrightarrow{PQ} \times \overrightarrow{PR} \mid$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & -2 & -2 \\ 1 & 1 & -5 \end{vmatrix} =< 12,-7,1 >$$

$\Rightarrow A =\mid \overrightarrow{PQ} \times \overrightarrow{PR} \mid$
$= \sqrt{144+49+1}$
$= \sqrt{194}$

■

■ **Example 12.19** Find

1. $i \times i$
2. $i \times j$
3. $k \times j$
4. $< -1,-2,-2 > \times < 1,1,-5 >$

Solution :

1. $\overrightarrow{0} =< 0,0,0 >$
2. k
3. $-i$
4. $= (-i-2j-2k) \times (i+j-5k)$
   $= -k-5j+2k+10i-2j+2i$
   $=< 12,-7,1 >$

■

■ **Example 12.20** True or False :

1. $\overrightarrow{a} \times \overrightarrow{b} = \overrightarrow{b} \times \overrightarrow{a}$
2. $(\overrightarrow{a} \times \overrightarrow{b}) \times \overrightarrow{c} = \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c})$
3. $i \times (i \times j) = i \times k = -j$
   $(i \times i) \times j = \overrightarrow{0} \times i = \overrightarrow{0}$

Solution :

1. False : $\quad i \times j = k \neq j \times i = -k$
2. False

- $i \times (i \times j) = i \times k = -j$
- $(i \times i) \times j = \overrightarrow{0} \times i = \overrightarrow{0}$

■

Properties :

1. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$
2. $(c\overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (c\overrightarrow{b}) = c(\overrightarrow{a} \times \overrightarrow{b})$
3. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$
4. $(\overrightarrow{a} + \overrightarrow{b}) \times \overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{c} + \overrightarrow{b} \times \overrightarrow{c}$
5. $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}$
6. $\overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \cdot \overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a} \cdot \overrightarrow{b})\overrightarrow{c}$

Pf(5) if :

- $\overrightarrow{a} =< a_1, a_2, a_3 >$
- $\overrightarrow{b} =< b_1, b_2, b_3 >$
- $\overrightarrow{c} =< c_1, c_2, c_3 >$

L.H.S

= $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$

$=< a_1, a_2, a_3 > \cdot < b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 >$

$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$

R.H.S

= $(\overrightarrow{a} \times \overrightarrow{b}) \cdot \overrightarrow{c}$

$=< a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_2 > \cdot < c_1, c_2, c_3 >$

$= a_2b_3c_1 - a_3b_2c_1 + a_3b_1c_2 - a_1b_3c2 + a_1b_2c_3 + a_2b_1c_3$

$\Rightarrow$ L.H.S = R.H.S

■ **Example 12.21** If $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c}) = 2$,

Find $2\overrightarrow{b} \cdot (\overrightarrow{a} \times 2\overrightarrow{c})$

Solution :

$= 4\overrightarrow{b} \cdot (\overrightarrow{a} \times \overrightarrow{c})$

$= 4(\overrightarrow{a} \times \overrightarrow{c}) \cdot \overrightarrow{b}$

$= 4\overrightarrow{a} \cdot (\overrightarrow{c} \times \overrightarrow{b})$

$= -4\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$

$= -8$ ■

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$
volume of the parallelepiped :

$$\vec{v} = |\ \vec{a}\cdot(\vec{b}\times\vec{c})\ |$$

■ **Example 12.22** find the volume of the parallelepiped that determine by
$\vec{a} = <1,2,-1>$
$\vec{b} = <2,1,1>$
$\vec{c} = <3,2,-2>$
Solution :

$$\vec{v} = |\ \vec{a}\cdot(\vec{b}\times\vec{c})\ | = \left\|\begin{matrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{matrix}\right\|$$
$$= |\ 1(-4) - 2(-7) + -1(1)\ |$$
$$= 9$$

if $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$ then we say that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are called **coplaner** ■

■ **Example 12.23** show that the following vectors are coplaner
$\vec{a} = <2,1,-1>$
$\vec{b} = <-1,3,2>$
$\vec{c} = <0,7,3>$
Solution :

$$\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 3 & 2 \\ 0 & 7 & 3 \end{vmatrix}$$
$$= 2(-5) - 1(-3) + -1(-7)$$
$$= -10 + 3 + 7 = 0$$

Thus **coplaner** ■

Problem 12.4 1,5,7,9,13,18,19,27,29,31,34,35,38,43

12.5 Equation of linear & planes

in 3D :

To determine a line , we need :

1. point $(x.,y.,z.)$
2. parallel vector $\overrightarrow{v} =< a,b,c >$

find it is equations !

note that $\overrightarrow{v} \parallel \overrightarrow{r}$

$\overrightarrow{v} = t\overrightarrow{a}$,$t \in R$

vector equation of the line .

$< x-x_0, y-y_0, z-z_0 >=< ta,tb,tc >$

parametric equations of the line :

- $x = x. + at$
- $y = y. + bt$
- $z = z. + ct \qquad -\infty < t < \infty$

Example 12.24 1. find the parametric equations of the line that passes through the point $(2,1,3)$& parallel vector $\overrightarrow{v} =< 2,1,-1 >$.

2. Find the point on the line
3. does the point (0,0,5) lie on the line ?

Solution :

1. $x = 2+2t$
   $y = 1+t$
   $z = 3-2t \qquad -\infty < t < \infty$
2. $t = 2 \Rightarrow (6,3,-1)$
   $t = \frac{7}{2} \Rightarrow (9, \frac{9}{2}, -4)$
3. $0 = 2+2t \Rightarrow t = -1$
   $0 = 1+t \Rightarrow t = -1$
   $5 = 3-2t \Rightarrow t = -1 \qquad$ yes.

■

- $x = x_0 + at \Rightarrow t = \dfrac{x-x_0}{a} \qquad a \neq 0$
- $y = y_0 + bt \Rightarrow t = \dfrac{y-y_0}{b} \qquad b \neq 0$
- $z = z_0 + ct \Rightarrow t = \dfrac{z-z_0}{c} \qquad c \neq 0$

So, if

- $a \neq 0$
- $b \neq 0$
- $c \neq 0$

$$\Rightarrow \frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \qquad a \neq 0 \ , b \neq 0 \ , c \neq 0$$

symmetric equation

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

if $a = 0$

$$x = x_0 \ , \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

■ **Example 12.25**

1. find the equation of the line that passes thorough the point $P(1,2,-1) \& Q(3,1,2)$
2. find where the line intersected the xy-plane !

Solution :

1. We need
   (a) point P(1,2,-1)
   (b) $\vec{v} = <2,-1,3>$

   Parametric eq.s

$$x = 1+2t$$

$$y = 2-t$$

$$z = -1+3t$$

   Symmetric eq.

$$\frac{x-1}{2} = \frac{y-2}{-1} = \frac{z+1}{3}$$

2. z= 0
   - $\Rightarrow \frac{x-1}{2} = \frac{1}{3}$
     $\Rightarrow x-1 = \frac{2}{3}$
     $\Rightarrow x = \frac{5}{3}$
   - $\frac{y-2}{-1} = \frac{1}{3} \rightarrow y-2 = \frac{-1}{3} \rightarrow y = \frac{5}{3}$

- $(\frac{5}{3}, \frac{5}{3}, 0)$

■

■ **Example 12.26** Find the parametric equations of the line that passes through the point $(-2,1,1)$ & parallel to the line :
$L_1 = \dfrac{x-2}{1} = \dfrac{2-y}{1} = \dfrac{2z+1}{1}$.
Solution: We need

1. point $(-2,1,1)$
2. $\overrightarrow{v} =< 1,-1,\frac{1}{2} >$

$$x = 2+t$$
$$y = 1-t$$
$$z = 1+\frac{1}{2}t \qquad t \in R$$

■

---

line

1. point $(x.,y.,z.)$
2. parallel vector $\overrightarrow{v} =< a,b,c >$

Parametric equation

- $x = x_0 + at$
- $y = y_0 + bt$
- $z = z_0 + ct \qquad -\infty < t < \infty$

Symmetric equations

- $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

R two lines are parallel iff their vector are parallel

Definition 12.5.1 two lines are called skew if they are not parallel & they do not intersect .

■ **Example 12.27** show that the following lines are skew
$L_1 : x = 1+t, y = -2+3t, z = 4-t / \overrightarrow{v_1} =< 1,3,-1 >$

$L_2 : x = 2s, y = 3+s, z = -3+4s / \overrightarrow{v_2} =< 2,1,4 >$

Solution

$v_1 \neq v_2 \Rightarrow L_1 \neq L_2$ $\quad$ ($L_1 \& L_2$ are not parallel)

$1+t = 2s$ $\quad$ $t-2s = -1 \rightarrow (1)$

$-2+2t = 3+s$ $\quad$ $3t-s = 5 \rightarrow (2)$

$4-t = -3+4s$ $\quad$ $-t-4s = -7 \rightarrow (3)$

Solve $(1) \& (3)$

$$0-6s = -8 \Rightarrow s = \frac{8}{6} = \frac{4}{3}$$

$$t = -1+2s \Rightarrow t = -1+\frac{8}{3} = \frac{5}{3}$$

$$s = \frac{4}{3} \ , t = \frac{5}{3}$$

in Equation 2

$$3\frac{5}{3} - \frac{4}{3} \neq 5 \Rightarrow 5 - \frac{4}{3} \neq 5$$

L_1 & L_2 do not intersect $\Rightarrow L_1 \& L_2$ are skew ■

Planes : to determine a plane we need

1. point $(x_0, y_0 z_0)$
2. normal vector $\overrightarrow{n} =< a,b,c >$ $\quad$ Note that $\overrightarrow{v} \perp \overrightarrow{n}$
   $\Rightarrow \overrightarrow{v} \cdot \overrightarrow{n} = 0$

$\Rightarrow < x-x_0, y-y_0, z-z_0 > \cdot < a,b,c >= 0$

$\Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0) = 0$

$\Rightarrow ax+by+cz+-(ax_0+by_0+cz_0) = 0$

$\Rightarrow ax+by+cz+d+0$

$\Rightarrow d = -(ax_0+by_0+cz_0)$

■ **Example 12.28** 1. find the equation of the plane that passes through the point $(1,-1,3)$& normal vector $\overrightarrow{v} =< 2,1,-1 >$.

2. find two points on the plane .

Solution

1. point p(1,-1,3)
2. normal vector $\overrightarrow{v} =< 2,1,-1 >$.
   $2(x-1)+1(y+1)+-1(z-3) = 0$
   $2x+y-z+6 = 0$
   2) (0,0,6) (-3,0,0) (0,-6,0)

■

■ **Example 12.29** find the equation of the plane that passes through the points $P(2,1,-2)\quad Q(1,1,-1)\quad R(3,-2,1)$

Solution

1. Point $(2,1,-2)$
2. $\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$
   $=< 3,4,3 >$
   $= 3x+4y+3z+-4$
   $= 0$
   $\overrightarrow{PQ} < -1,0,1 >$
   $\overrightarrow{PR} =< 1,-3,3 >$

$$\overrightarrow{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 1 & -3 & 3 \end{vmatrix}$$
$$=< 3,4,3 > .$$

■

Plane

- $p(x_0,y_0,z_0)$
- $\overrightarrow{n} =< a,b,c >$
  $ax+by+cz+d=0$
  $d = -(ax.+by.+cz.)$

■ **Example 12.30** Find the equation of the plane that passes through the point p(1,2,1), Q(2,3,2), R(-1,-1,3)

Solution

$ax+by+cz+d=0$

- if $d \neq 0$
  $Ax+By+Cz+1=0$
  $A+2B+C+1=0$
  $2A+3B+2C+1=0$
  $-A-B-3C+1=0 \qquad$ (Rejected)
- $d=0$
  $ax+by+cz+1=0$

■

■ **Example 12.31** Find the equation of the plane that passes through the point (1,-1,2) & contains the line
$L_1 : \frac{x-1}{2} = \frac{y+1}{2} = \frac{z-1}{3}.\overrightarrow{v} =< 2,3,1 >$
Solution

1. Point $(1,-1,2)$
2. normal vector

$$\begin{aligned}\overrightarrow{n} &= \overrightarrow{r} \times \overrightarrow{v}\\ &= \overrightarrow{RQ} \times \overrightarrow{v}\\ &=< 0,0,1 > \times < 2,2,3 >\\ &-2x+2y+0z+4=0\\ &\Rightarrow x-y-2=0\end{aligned}$$

■

■ **Example 12.32** find the equations of the line of intersection of the following planes .
$P_1 : 2x-y+z_3 = 0.\overrightarrow{n_1} =< 2,-1,1 >$
$P_2 : x-3y-z-1 = 0.\overrightarrow{n_2} =< 1,-3,-1 >$
Solution

- Point :$P(0,-1,2)$
- Parallel vector

$$\overrightarrow{v} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & -3 & -1 \end{vmatrix}$$

$=< 4,3,-5 >$ let x=0
$-y+z=3$
$+$
$-3y-z=1$
$\Rightarrow -4y=4 \Rightarrow y=-1$
$\Rightarrow z=2$
$x=0+4t$
$y=-1+3t$
$z=2-5t$

Solution 2 : Pick two points on the line :

- $P(0,-1,2)$
- $Q(\frac{4}{3},0,\frac{1}{3})$

Let $y=0$

$$2x+z=3$$
$$+$$
$$x-z=1$$
$$\Rightarrow 3x=4$$
$$\Rightarrow x=\frac{4}{3}$$
$$\Rightarrow z=\frac{4}{3}-1=\frac{1}{3}$$

$\overrightarrow{v}=\overrightarrow{PQ}=<\frac{4}{3},1,\frac{-5}{3}>$

- $x=0+\frac{4}{3}t$
- $y=-1+t$
- $z=2-\frac{5}{3}t$

Solution 3 :

$2x-y+z-3=0$

$x-3y-z-1=0$

let $x=t$

$\Rightarrow 2t-y+z-3=0$

$\Rightarrow t-3y-z-1=0$

$\downarrow$

$-y+z=3-2t$

$+$

$-3y-z=1-t$

$\Rightarrow -4y=4-3t$

$\Rightarrow y=-1+\frac{3}{4}t$

$-3(-1+\frac{3}{4}t)-z=1-t$
$z=3-\frac{9}{4}t-1+t$
$z=2-\frac{5}{4}t$ ■

R

- two plane are parallel if their normal vectors are parallel .
- the angle blw too plane is defined to be the active angle blw $\overrightarrow{n_1}\&\overrightarrow{n_2}$.

■ **Example 12.33** find the angle btw the following plane : $2x-2y=z-1=0$
$x=3y-z=7=0,$. Solution:

- $\overrightarrow{n_1}=<2,-2,1>$
- $\overrightarrow{n_2}=<1,3,-1>$

$\theta: cos\theta = \frac{\overrightarrow{n_1}\cdot\overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|} = \frac{2-6-1}{3\sqrt{11}} = -\frac{5}{3\sqrt{11}} \approx 120.1$
$\Rightarrow \alpha = 59.9$ ■

■ **Example 12.34** find the intersection btw the following line :
$P: 2x-2y+z_1=0$
$L: x=1+t, y=1-t, z=t.$
Solution:
$2(1+t)-2(1-t)+t-1=0$
$2+2t-2+2t+t=1$
$5t=1 \Rightarrow t=\frac{1}{5}$
$x=1+\frac{1}{5}\Rightarrow\frac{6}{5}$
$y=1-\frac{1}{5}\Rightarrow\frac{4}{5}$
$z=\frac{1}{5}$ ■

Distances :

$$\frac{1}{2}*|\overrightarrow{QR}|*D=\frac{1}{2}*|\overrightarrow{QR}*\overrightarrow{PQ}|$$

$$D = \frac{|\overrightarrow{QR}| * |\overrightarrow{PQ}|}{|QR|}$$

OR

$$D = \frac{|\overrightarrow{v} * \overrightarrow{PQ}|}{|\overrightarrow{v}|}$$

■ **Example 12.35** Find the distance below $(1,2-1)$& the line
$x = 1+t$
$y = 1-t$
$z = t$
Solution : Pick two points on the line
$t = 0 \Rightarrow \quad Q(1,1,0)$
$t = 1 \Rightarrow \quad R(2,0,1)$
$D = \dfrac{|\overrightarrow{QR}| * |\overrightarrow{PQ}|}{|QR|}$
$\overrightarrow{PQ} =< 0,-1,1 >$
$\overrightarrow{QR} =< 1,-1,1 >$

$$\overrightarrow{PQ} * \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 0 & -1 & 1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= \frac{\sqrt{0+1+1}}{\sqrt{1+1+1}} = \sqrt{\frac{2}{3}}$$ ■

■ **Example 12.36** Find the distance below the point (x,y,z)& the plane
$ax+by+cz = 0$
Solution :

$$\begin{aligned} D &= \left|\text{Comp}_{\overrightarrow{n}}^{\overrightarrow{r}}\right| \\ &= \frac{|\overrightarrow{r}.\overrightarrow{n}|}{|\overrightarrow{n}|} \qquad \overrightarrow{r} =< x_0-x_1, y_0-y_1, z_0-z_1 > \\ &= \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} \qquad \overrightarrow{n} =< a,b,c > \\ &= \frac{|ax_1+by_1+cz_1+-(ax_0+by_0+cz_0)|}{\sqrt{a^2+b^2+c^2}} \\ D &= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \end{aligned}$$ ■

■ **Example 12.37** Find the distance between the point $(1,2,-1)$ and the plane $2x+2y-z-3=0$

Solution :
$$\begin{aligned} D &= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}} \\ &= \frac{|2(1)+2(2)-1(-1)-3|}{\sqrt{9}} \\ &= \frac{4}{3} \end{aligned}$$
■

■ **Example 12.38** Find the distance between the following planes
$P_1 : x+y-z+1=0$
$P_2 : 2x-2y+3z+7=0$
Solution : $\overrightarrow{n_1} =< 1,1,-1 >$
$\overrightarrow{n_2} =< 2,-2,3 >$
$\Rightarrow \overrightarrow{n_1} \nparallel \overrightarrow{n_2}$
$\Rightarrow P_1 \nparallel P_2$
$\Rightarrow P_1 \; and \; P_2 areintersected$
$\Rightarrow D=0$ ■

■ **Example 12.39** Find the distance between the following plane
$P_1 : x-2y+2z+3=0$
$P_2 : -2x+4y-4z-5=0$
Solution :
$\overrightarrow{n_1} =< 1,-2,2 >$
$\overrightarrow{n_2} =< -2,4,-4 >$
Thus $\overrightarrow{n_1} \parallel \overrightarrow{n_2}$
Pick any point on $P_1 \Rightarrow p(-3,0,0)$
$$D = \frac{|-2(-3)+4(0)-4(0)-5|}{\sqrt{4+16+16}} = \frac{1}{6}$$
■

■ **Example 12.40** Find the distance between the line and plane
$L : x=1+t, \quad y=1-t, \quad z=t$
$P : 2x-y+z+3=0$
Solution (1):
$\overrightarrow{v} =< 1,-1,1 >$

$\overrightarrow{n} =< 2, -1, 1 >$
$P \parallel L \Longleftrightarrow \overrightarrow{v} \perp \overrightarrow{n} \Longleftrightarrow \overrightarrow{n} . \overrightarrow{v} = 0$
Note that

$$\begin{aligned} \overrightarrow{n} . \overrightarrow{v} &= 2+1+1 \\ &= 4 \\ &\neq 0 \end{aligned}$$

$\Rightarrow \overrightarrow{n} \not\perp \overrightarrow{v}$
$\Rightarrow P \nparallel L$
$\Rightarrow D = 0$

Solution (2):
Try to find an intersection point
$2(1+t) - (1-t) + t + 3 = 0$
$2 + 2t - 1 + t + t + 3 = 0$
$4t = -4 \Rightarrow t = -1$
P & L are intersecting
$\Rightarrow D = 0$ ∎

■ **Example 12.41** Find the distance between the following plane & line
$L : x = 1 - t, \quad y = t, \quad z = 2 - t$
$P : 2x + y - z + 3 = 0$
Solution :
$\overrightarrow{v} =< -1, 1, -1 >$
$\overrightarrow{n} =< 2, 1, -1 >$
$\overrightarrow{n} . \overrightarrow{v} = 0 \Rightarrow P \parallel L$
pick a point on the line $\quad t = 0 \Rightarrow (1, 0, 2)$

$$\begin{aligned} D &= \frac{|2+0-2+3|}{\sqrt{4+1+4}} \\ &= \frac{3}{\sqrt{6}} \\ &= \sqrt{\frac{3}{2}} \end{aligned}$$

∎

■ **Example 12.42** Find the distance between the following lines
$L_1 : x = 1 - t, \quad y = 1 + t, \quad z = t$
$L_2 : x = -1 + 2t, \quad y = -2t, \quad z = 1 - 2t$
Solution :

$\overrightarrow{v_1} =< -1, 1, 1 >$
$\overrightarrow{v_2} =< 2, 2, -2 >$
$\overrightarrow{v_1} \parallel \overrightarrow{v_2} \Rightarrow L_1 \parallel L_2$

Exercise 12.3 Pick a point on L_1 $t = 0 \Rightarrow p(1, 1, 0)$ ■

■

■ **Example 12.43** If
$L_1 : x = 1 + t, \quad y = 2 + 3t, \quad z = 4 - t$
$L_2 : x = 2s, \quad y = 3 + s, \quad z = -3 + 4s$
are skew , Find the distance between them .
Solution : Construct two parallel planes that contain $L_1 \& L_2$ respectively

P_1
$point \;\; t = 0 \Rightarrow (1, -2, 4)$
$\overrightarrow{n} = \overrightarrow{v_1} * \overrightarrow{v_2}$
$\overrightarrow{n} = 13i - 6j - 5k$
$13x - 6y - 5z + -5 = 0$

P_2
$point \;\; s = 0 \Rightarrow (0, 3, -3)$
$\overrightarrow{n} =< 13, -6, -5 >$
$13x - 6y - 5z + 3 = 0$

$d = -(13 + 12 - 20)$

$$D = \frac{|13(0) - 6(3) - 5(-3) + -5|}{\sqrt{13^2 + 6^2 + 25}} = \frac{|-8|}{\sqrt{230}} = \frac{8}{\sqrt{230}}$$ ■

■ **Example 12.44** Determine whether each sentence is true or false .

1. Two lines parallel to a third line are parallel .
2. Two lines perpendicular to a third line are parallel .
3. Two planes parallel to a third plane are parallel .
4. Two planes perpendicular to a third plane are parallel .
5. Two lines parallel to a plane are parallel.
6. Two lines perpendicular to a plane are parallel .
7. Two planes parallel to a line are parallel .
8. Two planes perpendicular to a line are parallel .
9. Two planes either intersect or are parallel .
10. Two lines either intersect or are parallel .
11. A plane and a line either intersect or are parallel .

Solution

1. T	5. F	9. T
2. F	6. T	10. F
3. T	7. F	11. T

4. F 8. T

■

■ **Example 12.45** Show that the distance below the following planes
$P_1 : ax+by+cz+d_1 = 0$
$P_2 : ax+by+cz+d_2 = 0, \ \ is$
$D = \dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$
Solution :
Pick a point on $P_1(x_1,y_1,z_1)$
So $D = \dfrac{|ax_1+by_1+cz_1+d_2|}{\sqrt{a^2+b^2+c^2}} = \dfrac{|d_2-d_1|}{\sqrt{a^2+b^2+c^2}}$ ■

■ **Example 12.46** Find equations of the parallel planes to
$P_1 : x-2y-2z+1 = 0$ and units away from it.
Solution :
$P_2 : x-2y-2z+d = 0$
$D = 2$
$\dfrac{|d-1|}{\sqrt{1+4+4}} = 2$
$\Rightarrow |d-1| = 6$
$\Rightarrow d-1 = -6 \quad OR \quad d-1 = 6$
$\Rightarrow d = -5 \quad OR \quad d = 7$ ■

■ **Example 12.47** Find the projection of the point $(1,2,-1)$ on the plane
$2x-2y+z-1 = 0$
Solution:
Let's construct a line that passes through (1,2,-1) & perpendicular to the plane .
point $(1,2,-1)$
parallel vector $\vec{n} =< 2,-2,1 >$
$x = 1+2t$
$y = 2-2t$
$z = -1+t$
we will find the intersection between the line and the plane
$2(1+2t)-2(2-2t)+(-1+t)-1 = 0$

$9t = 4 \Rightarrow t = \frac{4}{9}$

point $(1+\frac{8}{9}, 2-\frac{8}{9}, -1+\frac{4}{9}) = (-\frac{17}{9}, \frac{10}{9}, -\frac{5}{9})$ ■

■ **Example 12.48**

1. $x+y+z=c$
2. $x+y+z=1$
3. $(\cos c)y+(\sin c)z=1$

■

Problem 12.5 1,3,5,9,11,12,13,14,17,19,21,25,26,29,30,31,35,37,38,39,45,46,48,51-57(odd),61-71(odd),74,76

13. Vector Function

$f : \mathbb{R} \mapsto vector$ (real valued function)
$f(x) = sinx$
$f(\pi/2) = 1$

13.1 Vector Functions and space curres

$\overrightarrow{r}(t) := < f(t), g(t), h(t) >\}$ vector function
$= f(t)i + g(t)j + h(t)k$

■ **Example 13.1** $\overrightarrow{r}(t) = < t^2, 1-t, t^2+1 >$
$\overrightarrow{r}(1) = < 1, 0, 2 >$ ■

■ **Example 13.2** Find the domain of $\overrightarrow{r}(t) = < \sqrt{t}, ln(1-t), t^2 >$
$D_{\overrightarrow{r}(t)} = D_f \cap D_g \cap D_h = [0, 1)$ ■

Limit and continuity :
if $\overrightarrow{r}(t) = < f(t), g(t), h(t) >$
then $\lim_{t \to t_\circ} \overrightarrow{r}(t) = < \lim_{t \to t_\circ} f(t), \lim_{t \to t_\circ} g(t), \lim_{t \to t_\circ} h(t)$

■ **Example 13.3** Find $\lim_{t \to t_\circ} < 2t, \frac{\sin t}{\frac{1-e^t}{t}} > = < 0, 1, -1 >$ ■

$\vec{r}(t)$ is cont at $t_\circ \Leftrightarrow \lim_{t \to t_\circ} \vec{r}(t) = \vec{r}(t_\circ)$

Definition 13.1.1 Space Curve: suppose that f, g, h are cont real-value function on I(interval) then
$x = f(t)$, $y = g(t)$, $z = h(t)$
is called space curve it can be represented using $\vec{r}(t) = < f(t), g(t), h(2) >$

■ **Example 13.4** Describe the curve defined by

1. $\vec{r}(t) = < t, 1+t, 2-t >$
   $x = t$
   $y = 1+t, \quad -\infty < t < \infty$
   $z = 2-t$

2. $\vec{r}(t) = < \cos t, \sin t, t >$
   $x = \cos t, \ y = \sin t, \ z = t$

■

Definition 13.1.2 Line Segment :the line segment from $\vec{r}_\circ$ to $\vec{r}_1$, $0 \le t \le 1$

■ **Example 13.5** Find the vector function that represented the line segment $P(1, 2, -1) \& Q(2, 3, 2)$
$\vec{r}(t) = (1-t) < 1, 2, -1 > +t < 2, 3, 2 >, \quad 0 \le t \le 1$
■

■ **Example 13.6** Find a vector function that represented the intersection of $x^2 + y^2 = 1$ and $y + z = 2$ Solution
$\vec{r}(t) = < f(t), g(t), h(t) >$
$\vec{r}(t) = < \cos t, \sin t, 2 - \sin t, \quad 0 \le t \le 1$
■

Problem 13.1 1,3,5,15,17,25,27,35,37,42

13.2 13.2

$\overrightarrow{r}'(t) = \lim_{h\to 0} \dfrac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h}$
$=< f'(t), g'(t), h'(t) >$
$f'(t)i + g'(t)j + h'(t)k$

■ **Example 13.7** if $\overrightarrow{r}(t) = (1+t)i + te^{-t}j + \sin 2tk$

a Find $\overrightarrow{r}'$

b unit tangent vector at $t = 0$

Solution

$\overrightarrow{r}'(t) = 2ti + (e^{-t} - te^{-t})j + 2\cos 2tk$
$\overrightarrow{r}(0) =< 0,1,2 >$ tangent vector
$\overrightarrow{T}(t) = \dfrac{\overrightarrow{r}(0)}{|\overrightarrow{r}'(0)|} = \dfrac{1}{\sqrt{0+1+4}} < 0,1,2 >=< 0, \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} >$ ■

■ **Example 13.8** Find parametric equation for tangent line to the helix:
$\overrightarrow{r}(t) =< \cos t, \sin t, t >$ at $t = \pi$

Solution

point :$(-1, 0, \pi)$
parallel vector $\overrightarrow{r}'(t) =< -\sin t, \cos t, 1 >$
$\overrightarrow{r}'(\pi) =< 0, -1, 1 >$
$x = -1 + 0t$
$y = 0 - t$
$z = \pi + t$ ■

Differential Rule:

1. $\dfrac{d}{dt}(\overrightarrow{u}(t) + \overrightarrow{v}(t)) = \dfrac{d}{dt}\overrightarrow{u}(t) + \dfrac{d}{dt}\overrightarrow{v}(t)$
2. $(c\overrightarrow{u}(t))' = c\overrightarrow{u}'(t)$
3. $\dfrac{d}{dt}(f(t)\overrightarrow{u}(t)) = f'(t)\overrightarrow{u}(t) + f(t)\overrightarrow{u}'(t)$
4. $\dfrac{d}{dt}(\overrightarrow{u}(t).\overrightarrow{v}(t)) = \overrightarrow{u}, (t).\overrightarrow{v}(t) + \overrightarrow{u}(t).\overrightarrow{v}(t)$
5. $\dfrac{d}{dt}(\overrightarrow{u}(t) \times \overrightarrow{v}(t)) = \overrightarrow{u}'(t) \times \overrightarrow{v}(t) + \overrightarrow{u}t \times \overrightarrow{v}(t)$
6. $\dfrac{d}{dt}\overrightarrow{u}(f(t)) = f'(t)\overrightarrow{u}'(f(t))$

■ **Example 13.9** if $\overrightarrow{u}'(1) \times \overrightarrow{v}(1) =< 2, -1, 3 >$
$\overrightarrow{v}'(1) \times \overrightarrow{u}(1) =< 2, -1, 3 >$
Find $(\overrightarrow{u} \times \overrightarrow{v})'(1) = \overrightarrow{u}' \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{v}$
$=< 2, -1, 3 > + < -2, 1, -3 >=< 0, 0, 0 >$ ■

■ **Example 13.10** if $|\overrightarrow{r}(t)| = c \quad (constant)$
show that $\overrightarrow{r}'(t) \perp \overrightarrow{r}(t)$
Proof
$|\overrightarrow{r}(t)| = const$
$|\overrightarrow{r}(t)|^2 = const$
$\overrightarrow{r}(t).\overrightarrow{r}(t) = const$ ■

Differentiate:
$\overrightarrow{r}'(t).\overrightarrow{r}(t) + \overrightarrow{r}(t).\overrightarrow{r}'(t) = 0$
$2\overrightarrow{r}'(t).\overrightarrow{r}(t) = 0$
$\overrightarrow{r}'(t).\overrightarrow{r}(t) = 0$
$\overrightarrow{r}'(t) \perp \overrightarrow{r}(t)$

Definition 13.2.1 Integrals: $\overrightarrow{r}(t) =< f(t), g(t), h(t) >$
then $\int \overrightarrow{r}(t)dt =< \int f(t)dt, \int g(t)dt, \int h(t)dt >$

■ **Example 13.11** if $\overrightarrow{r}(t) = 2ti - e^t j + lntk$
Find $\overrightarrow{r}(t)$ where $\overrightarrow{r}(1) =< 0, 0, 1 >$ $\overrightarrow{r}(t) =< t^2 + c_1', -e^t + c_2', tlnt - t + c_3' >$ $< 0, 0, 1 >$ $\overrightarrow{r}(1) =< 1 + c_1, c_2 - e, -1 + c_3 >$
$1 + c_1 = 0 \Rightarrow c_1 = -1$
$c_2 - e = 0 \Rightarrow c_2 = e$
$-1 + c_3 = 1 \Rightarrow c_3 = 2$ ■

Problem 13.2 3,4,5,6,9,11,12,17,19,21,23,25,32,34,37,39,49

13.3 Arc Length

$L = \int_a^b \sqrt{f'^2(t) + g'^2(t) + h'^2(t)}dt$
$L = \int_a^b = |\overrightarrow{r}'(t)|$

■ **Example 13.12** Find the length of the helix $\overrightarrow{r}(t) =< \cos t, \sin t, t > \quad 0 \leq$

$t \leq \pi$

Solution

$L = \int_0^\pi \sqrt{sin^2 t + cos^2 t + 1} = \int_0^\pi \sqrt{2}dt = \pi\sqrt{2}$ ■

■ **Example 13.13** Two particles travel along the curves

$\overrightarrow{r}_1(t) =< t, t^2, t^3 >$

$\overrightarrow{r}_2(t) =< 1+2s, 1+6s, 1+14s >$

1. Do the particles collide?
2. Do their paths intersect?

$t = 1+2s, \;\; t^2 = 1+6s, \;\; t^3 = 1+14s \Rightarrow$

$(1+2s)^2 = 1+6s \Rightarrow 1+4s+4s^2 = 1+6s \Rightarrow$

$s(4s-2) = 0 \Rightarrow s = 0, s = 1/2 \Rightarrow$

$s = 0, t = 1 \Rightarrow 1 \stackrel{?}{=} 1$

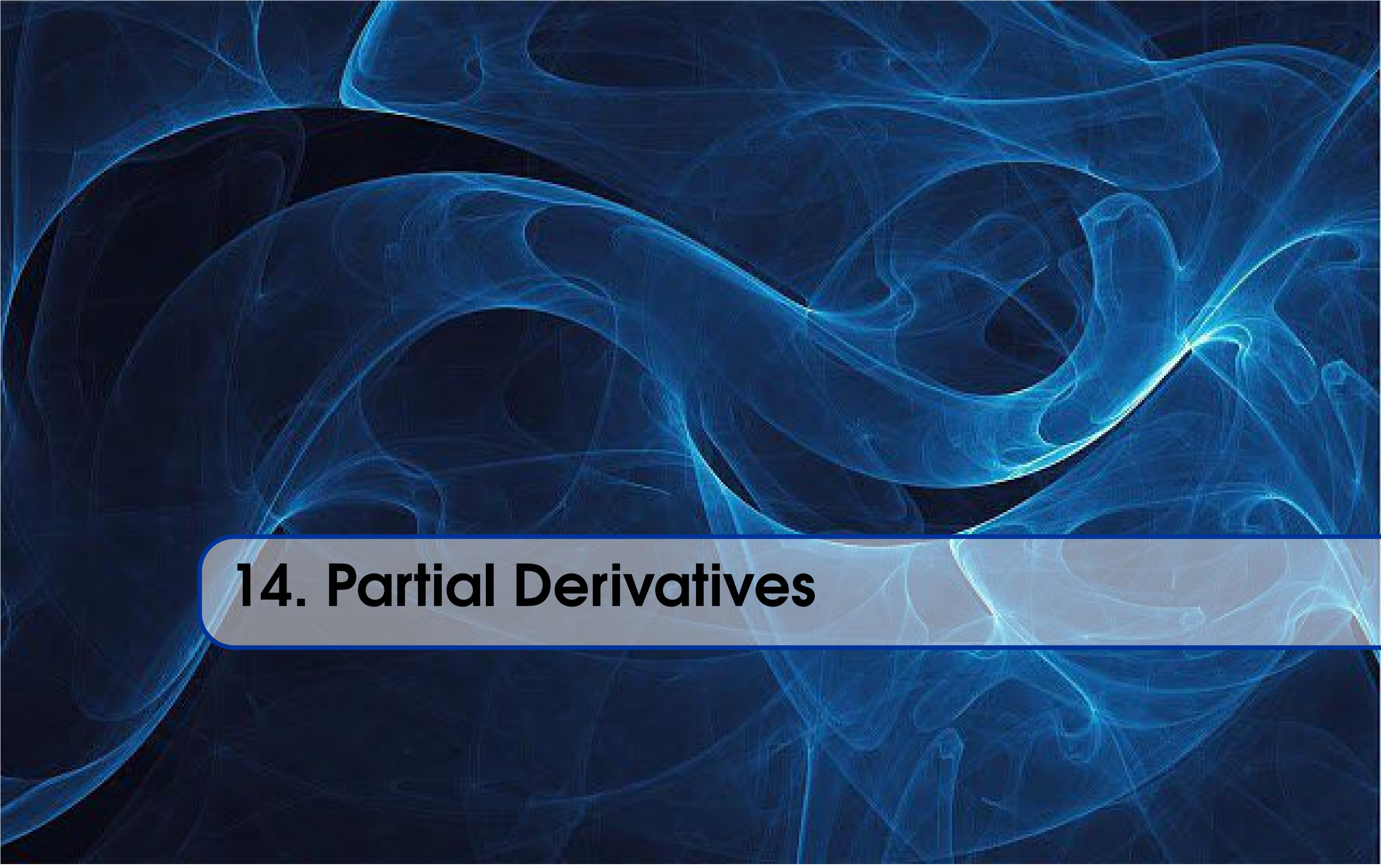
$s = 1/2, t = 2 \Rightarrow 8 \stackrel{?}{=} 8$

the paths intersect two lines, at the point $(1,1,1)$ & $(2,4,8)$

But they do not collide, the paths intersect at different t, s

■

Problem 13.3 1,3,5



14. Partial Derivatives

14.1 Function of several variables

Definition 14.1.1 A function of two variables is a rule that assigns for each (x,y) in the domain one value $z = f(x,y)$ in the range
x, y are called independent variables
z is called an independent variable

■ **Example 14.1** Find the domain of the following function

1. $f(x,y) = \sqrt{y-x+1}$
   $D = \{(x,y) : y-x+1 \geq 0\}$
   Range:$R = [0,\infty)$
2. $f(x,y) = \sqrt{9-x^2-y^2} + \sqrt{x}$
   $D = \{(x,y) : 9-x^2-y^2 \geq 0 \text{ \& } x \geq 0\}$
3. $f(x,y) = ln(x^2+y^2-9)$
   $D = \{(x,y) : x^2+y^2-9 > 0$
   $R = (-\infty,\infty)$
4. $f(x,y) = \dfrac{\sin^{-1}(x-y)}{\sqrt{x-y^2}} \quad | \quad \sin[-\pi/2,\pi/2] \rightarrow [-1,1]$
   $D = \{(x,y) : -1 \leq x-y \leq 1 \text{ \& } > 0\}$
   (a) $-1 \leq x-y$
   (b) $x-y \leq 1$
   (c) $x-y^2 > 0$

■

Example 14.2 Find the range of $f(x,y) = \sqrt{9-x^2-y^2}$
$D = \{(x,y) : x^2+y^2 \leq 9$
$R = [0,3]$
$0 \leq x^2+y^2 \leq 9$
$0 \geq -x^2-y^2 \geq -9$
$9 \geq 9-x^2-y^2 \geq 0$
$3 \geq \sqrt{9-x^2-y^2} \geq 0$
Domain 2D.
Range 1D. ∎

Definition 14.1.2 let $z = f(x,y)$ then the graph of the function is the set:
$G = \{(x,y,z) : (x,y) \in D, z = f(x,y)$

Example 14.3 Sketch the following

1. $f(x,y) = 6-2x-3y$
   $z = 6-2x-3y$
   $2x+3y+z-6 = 0$
   $x-intercept \quad y=0, z=0 \Rightarrow x=3 \; y-intercept \quad x=0, z=0 \Rightarrow y=2$
   $z-intercept \quad x=0, y=0 \Rightarrow z=0$
2. $g(x,y) = \sqrt{9-x^2-y^2}$
   $z = \sqrt{9-x^2-y^2}$
   $z^2 = 9-x^2-y^2$
   $x^2+y^2+z^2 = 9, \; z \geq 0$

∎

Definition 14.1.3 let $z = f(x,y)$ then the level curve of f at k ∈Range is the set

$$L = \{(x,y) : k = f(x,y)\} \supseteq \mathbb{R}^2$$

Example 14.4 Find level curve to $f(x,y) = 6-2x-3y$ at $k = 0, 6, -6...$
Solution

$k = 0 \Rightarrow 0 = 6-2x-3y \Rightarrow y = 2-\frac{2}{3}x$

$k = 6 \Rightarrow 6 = 6-2x-3y \Rightarrow y = \frac{-2}{3}x$ ∎

■ **Example 14.5** Find level curve for $f(x,y) = \sqrt{9-x^2-y^2}$ at:
$k = 1 \Rightarrow 1 = \sqrt{9-x^2-y^2} \Rightarrow x^2+y^2 = 8$
$k = 2 \Rightarrow 2 = \sqrt{9-x^2-y^2} \Rightarrow x^2+y^2 = 5$
$k = 0 \Rightarrow 0 = \sqrt{9-x^2-y^2} \Rightarrow x^2+y^2 = 9$ ■

■ **Example 14.6** Sketch some level curve of the function $f(x,y) = 4x^2+y^2$
$k = 1 \Rightarrow 4x^2+y^2 = 1$
$\frac{x^2}{\frac{1}{4}}+y^2 = 1$
$k = 4 \Rightarrow 4x^2+y^2 = 4$
$x^2+\frac{y^2}{4} = 1$ ■

Definition 14.1.4 Function of three variable
$$\underset{dependent\ variable}{w} = \underset{independent\ variable}{f(x,y,z)}$$
Domain: 3D
Range: 1D
Graph: 4D
Level surface:3D

■ **Example 14.7** Find the domain of $f(x,y,z) = \sqrt{9-x^2-y^2-z^2}$
$D = \{(x,y,z) : x^2+y^2+z^2 \leq 9\}$ ■

■ **Example 14.8** Find the following limits if exists:

1. $\lim_{(x,y)\to(1,2)}(2x+y) = 2(1)+2 = 4$
2. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(0,0)} \frac{(x-y)(x+y)}{x_y} = 0$
3. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ we will take different paths.
   path 1:along $y = 0$
   $\lim_{x\to 0} \frac{x^2}{x^2} = 1$
   path 2:along $x = 0$

$\lim_{y\to 0} \dfrac{0-y^2}{0+y^2} = -1$
$-1 \neq 1 \Rightarrow D.N.E$

4. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$
   path $y=0 \Rightarrow \lim_{x\to 0} \dfrac{0}{x^2} = 0$
   path $x=0 \Rightarrow \lim_{y\to 0} \dfrac{0}{y^2} = 0$
   path $y=x \Rightarrow \lim_{y\to 0} \dfrac{y^2}{y^2+y^2} = 1/2$
   $0 \neq 1/2 \Rightarrow D.N.E$
   path $y=mx \Rightarrow \lim_{x\to 0} \dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$
   depends on m so D.N.E

■

5. $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$
   path $y=mx \to \lim_{x\to 0} \dfrac{xm^2x^2}{x^2+m^4x^4}$
   $= \lim_{x\to 0} \dfrac{x^3}{x^2}\dfrac{m^2}{1+m^4x^2}$
   $= \lim_{x\to 0} \dfrac{x^3}{x^2}\dfrac{m^2}{1+m^4x^2}$
   $\to 0 \to m$
   $=0$
   path $x=my^2$
   $\lim \dfrac{xy^2}{x^2+y^2} = \lim_{y\to 0} \dfrac{my^2y^2}{m^2y^4+y^4} = \dfrac{m}{m^2+1}$ depends on m D.N.E
6. $\lim_{(x,y)\to(0,0)} \dfrac{sin(x^2+y^2)}{x^2+y^2}$
   $r = \sqrt{x^2+y^2}$
   $g = tan^{-1}y(x,y) \to 0$
   $r \to 0^+ g \to ??$
   $= \lim_{r\to 0^+} \dfrac{sinr2}{2} = 1$
7. $\lim_{(x,y)\to(0,0)} \dfrac{1-e^{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \dfrac{1-e^r}{r} = \lim_{r\to 0^+} \dfrac{-e^r}{1} = -1$

8. $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = \lim_{r\to 0^+} \frac{r^3cos^3\Theta - r^3sin3\Theta}{r^2}$
$= \lim_{r\to 0^+} r(cos^3\theta - sin^3\theta) = 0$

9. $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{r\to 0^+} \frac{rcos\theta rsin\theta}{r^2} = cos\theta sin\theta$ depends on θ dose not exist

10. $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{rcos\theta rsin\theta}{r} = 0$

11. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+y^2} = \lim \frac{r^3cos^3\theta + r^3sin^3\theta}{r^2(cos^2\theta - sin^2\theta}$
$= \lim_{r\to 0^+} r(\frac{cos^3\theta + sin^3\theta}{cos^2\theta - sin^2\theta}$ D.N.E / if $\theta = \frac{\pi}{y}$
continuity $z = f(x,y) iscontat(x.,y.)$
if $\lim_{(x,y)\to(x.,y.)} f(x,y) = f(x.,y.)$

Example 14.9 find where the function $f(x,y) : \begin{cases} \frac{x^2-y^2}{x^2+y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$,

is cont

Solution

f is cont for $(x,y) \neq (0,0) at (0,0)$?

- $f(0,0) = 0$
- $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ D.N.E $\rightarrow$ f is not cont at (0,0)

f is cont R^2 $(0,0)$ ■

Problem 14.1 6,7,9,11,13,15,19,20,21,23,25,29,41,61,63

14.2 Partial Derivatives

Definition 14.2.1 if $z = f(x,y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_x = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_y = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Rules:

1. to find f_x treat y as a constant.
2. to find f_y treat x as a constant.

■ **Example 14.10** find f_x, f_y:

1. $f(x,y) = x+y+xy$
   $f_x = 1+0+y = 1+y$
   $f_y = 1+x$
2. $f(x,y) = \frac{x}{y}$
   $f_x = \frac{1}{y}$
   $f_y = x(\frac{-1}{y^2}$
   3) $f(x,y)xe^{xy}$
   $f_x = e^{xy} + xye^{xy}$
   $f_y = x^2e^{xy}$
   $f(x,y) = \frac{2}{x} - x\ln y$
   $f_x = \frac{-2}{x^2} - \ln y$
   $f_y = 0 - \frac{x}{y}$

■

Function of three variables:
$w = f(x,y,z)$ to find
f_x treat y,z as constant .
f_y treat x,z as constant .
f_z treat x,y as constant .

■ **Example 14.11** if $f(x,y,z) = xy + \frac{1}{z} - e^{xz}$
find $f_x = y + 0 + -ze^{xz}$
$f_y = x$
$f_z = 0 + \frac{-1}{z^2} - xe^{xz}$ ■

Higher Derivatives:

■ **Example 14.12** if $f(x,y) = x^2y + xy - 2ye^x$
find f_{xx}, f_{xy}, f_{yy}
$f_x = 2xy + y - 2ye^x$
$f_y = x^2 + x - 2e^x$
$f_{xx} = 2y - 2ye^x$
$f_{xy} = 2x + 1 - 2e^x$
$f_{yx} = 2x + 1 - 2e^x$
$f_{yy} = 0$
clairaut's theorem : if f defined on a disc D that contains (a,b) if f_{xy}, f_{yx} cont on D then $f_{xy}(a,b) = f_{yx}(a,b)$ ■

■ **Example 14.13** T/F : there exists a function s.t $f_x = 21x + 3y$
$f_y = x^2 - 2y$
Solution: note that $f_{xy} = 3 f_{yx} = 2x$
$f_{xy} \neq f_{yx}$ false . ■

■ **Example 14.14** if $f(x,y,z) = z^2 cos(x+2y)$ find f_{zxyz}
Solution $f_z = 2zcos(x+2y)$
$f_{zx} = -2zsin(x+2y)$
$f_{zxy} = -4zcos(x+2y)$
$f_{zxyz} = -ycos(x+2y)$ ■

Partial differential Equation:

1. Laplace's Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
   Solution of this equation are called :harmonic equation

   ■ **Example 14.15** show that $u(x,y) = x^2 - y^2$ is harmonic
   $u_x = 2x$, $u_y = -2y$
   $u_{xx} = 2$, $u_{yy} = -2$
   so $u_{xx} + u_{yy} = 0$ ■

   ■ **Example 14.16** show that $u(x,y) = e^x \cos y$ satisfies Laplace's equation
   $u_x = e^x \sin y$, $u_y = e^x \cos y$
   $u_{xx} = e^x \sin y$, $u_{yy} = -e^x \sin y$

$\Rightarrow u_{xx} + u_{yy} = 0$ ■

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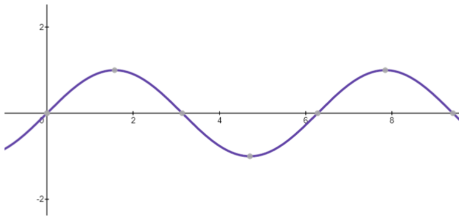
2. Wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$



■ **Example 14.17** show that $u(x,t) = \sin x - at$ satisfies the wave equation
$u_t = -acos(x - at)$, $u_x = cos(x - at)$
L.H.S=$u_{tt} = -a^2 sin(x - at)$, R.H.S=$u_{xx} = -sin(x - at)$
L.H.S=R.H.S ■

Problem 14.2 5,7,8,9,12,13,14,1821,25,29,31,33,37,39,40

14.3 Tangent Plane and Linear Approximation

Let $z = f(x,y)$ then the tangent plane at $(x_\circ, y_\circ)$
$z = z_\circ = f_x(x_\circ, y_\circ)(x - x_\circ) + f_y(x_\circ, y_\circ)(y - y_\circ)$

■ **Example 14.18** Find the tangent plane of $f(x,y) = 2x^2 + y^2$
at $(1,1,3)$
$f_x = 4x \Rightarrow f_x(1,1) = 4$
$f_y = 2y \Rightarrow f_y(1,1) = 2$
$z - 3 = 4(x-1) + 2(y-1)$
$4x + 2y - z - 3 = 0$ ■

Linear Approximation: let $z = f(x,y)$, fx, fy cont
The linear approximation of f at (a,b)
is $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

■ **Example 14.19** let $f(x,y) = xe^{xy}$

1. Find linear approximation at $(1,0)$
2. Approximation $f(1.1, -0.1)$

Solution:
$f(1,0) = 1$

$f_x = xye^{xy}$
$f_x(1,0) = 1$
$f_y = x^2e^{xy}$
$f_y(1,0) = 1$
$\Rightarrow L(x,y) = 1 + 1(x-1) + 1(y-0)$
$L(x,y) = x + y \approx xe^{xy}$ around $(1,0)$
$f(1.1,-0.1) \approx L(1.1,-0.1) = 1.1 - 0.1 = 1$
$f(1.1,-0.1) = 0.9854$ ■

■ **Example 14.20** if the tangent plane to $z = f(x,y)$ at $(2,3)$
is $2x - 3y + z = 1 \Rightarrow L(x,y) = 1 - 2x + 3y$
Approximation $f(2.1,2.9)$
sol $f(2.1,2.9) \approx 1 - 2(2.1) + 3(2.9) = 1 - 4.2 + 8.7 = 5.5$ ■

■ **Example 14.21** Approximation $12\sqrt{8.9} - 12\sqrt[3]{8.1}$
Solution:
$f(x,y) = 12\sqrt{x} - 12\sqrt[3]{y}$ at $(9,8)$
$L(x,y) = 12 + 2(x-9) - 1(y-8)$
$f(8.9,8.1) \approx L(8.9,8.1) = 12 + 2(\frac{-1}{10} + \frac{-1}{10}$
$12 - \frac{3}{10} = 11.7$ ■

Definition 14.3.1 Differentials: if $z = f(x,y)$, then we define the differential
$dz = f_x\partial x + f_y\partial y$
$dz = \frac{\partial f}{\partial x}\partial x + \frac{\partial f}{\partial y}\partial y$
let:
$\partial x = \delta x = x - a$
$\partial y = \Delta y = y - a$
$\partial z = \delta z = z - z_\circ = f(x,y) - f(a,b)$
$dz = \frac{\partial f}{\partial x}|_{(a,b)}\ (x-a) + \frac{\partial f}{\partial y}|_{(a,b)}\ (y-b)$

■ **Example 14.22** let $t = f(x,y) = x^2 + 3xy - y^2$

1. find the differential

$\partial t = (2x+3y)\partial x + (3x-2y)\partial y$

2. if x change from 2 to 2.05
   y change from 3 to 2.96
   com pane $\partial z, \delta z$

$\partial z = (2(2)+3(3))\frac{5}{100} + (3.2.2.3)\frac{-4}{100}$
$= \frac{65}{100} = 0.65$
$(2,3) \to (2.05, 2.96)$
$\Delta z = f((2.05, 2.96) - f(2,3)) = 0.6449$
$dz = \delta z = z - z_\circ$
■

---

functions of three variables
$w = f(x,y,z)$
$\partial w = f_x \partial x + f_y \partial y + f_z \partial z$

Problem 14.3 1-43(odd),44,48,49,50,51,53,59,61,65,71,72(a,d),73,75,77,87*,89*,93,94

14.4 Directional derivatives and Gradient vector

$f_x = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$
$f_y = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$

Definition 14.4.1 the direction derivative of f at $(x_\circ, y_\circ)$ in the direction of the limit vector $\overrightarrow{u} =< a,b > isD_{\overrightarrow{u}} f(x_\circ, y_\circ) = \lim \frac{f(x+ah, y+bh)-f(x,y)}{h}$

Gradient vector:

Definition 14.4.2 if $z = f(x,y), then the gradient of f at (x_\circ, y_\circ) is \nabla f =< f_x(x_\circ, y_\circ), f_y(x_\circ, y_\circ)$

■ **Example 14.23** if $f(x,y) = x^2 - y^2 find \nabla f \mid_(1,2)$
$f_x = 2x \to f_x(1,2) = 2$
$f_y = 2y \to f_y(1,2) = -4$

Solution:
$,\nabla f\mid_(1,2) =< 2,-4 >$ ■

Theorem 14.4.1 $D_{\overrightarrow{u}} f(x_\circ, y_\circ) = \nabla f . \overrightarrow{u}$

■ **Example 14.24** if $f(x,y) = sinx + e^{xy}$ find the directional of f at (0,1) in the direction of $\overrightarrow{u} =< 3,-4 >$
so, $D_{\overrightarrow{u}} f(0,1) = \nabla f . \overrightarrow{u}$
$=< 2,0 > . < \frac{3}{5}, \frac{4}{5} >$
$= \frac{6}{10} + 0\frac{4}{5} = (0,6)$
$\nabla f =< f_x, f_y > . < 2,0 >$
$f_x = cosx + ye^{xy}$
$f_y = xe^{xy}$
$f_x(0,1) = 2$
$f_y(0,1) = 0$
function of three variables $w = f(x,y,z)$
$\nabla f =, f_x, f_y, f_z >$
$D_{\overrightarrow{u}} f(a,b,c) = \nabla f . \overrightarrow{u}$
■

■ **Example 14.25** find the direction derivative of $f(x,y,z) = \frac{x-y}{z} + x^2 + e^y$ at $p(1,0,1) in the directional to the point Q(-1,2,0)$
Solution:
$D_{\overrightarrow{u}} f(-1,2,0) = \nabla f . \overrightarrow{u} =< 3,0,-1 > . < \frac{2}{3}, \frac{2}{3}, \frac{-1}{3} >= -2+0+\frac{1}{3} = -1\frac{2}{3}$
$\overrightarrow{u} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} =< \frac{-2}{3}, \frac{2}{3}, \frac{-1}{0} >$
$f_x = \frac{1}{z} + 2x, f_x(1,0,1) = 1+2 = 3$
$f_y = \frac{-1}{z} + e^y \to f_y(91,0,1) = -1+1 = 0$
$f_y = \frac{y-x}{z^2}, f_y(1,0,1) = \frac{0-1}{1^2} = -1$

$\nabla f =< 3,0,-1 >$

■

■ **Example 14.26** if $D_{\overrightarrow{u}} f = 3\nabla 2f.\overrightarrow{u}$
if $D_{\overrightarrow{u}} 2f = 6$
$D_{\overrightarrow{u}} - 2f = -6$
$D_{\overrightarrow{2u}} f = 3$
$D_{\overrightarrow{-2u}} f = -3$ ■

Question $z = f(x,y)(x_\circ, y_\circ)$
$\overrightarrow{u} =??$
find $\overrightarrow{u}$ that maximize $D_{\overrightarrow{u}} f(x_\circ, y_\circ) = \nabla f.\overrightarrow{u} = |\nabla f||\overrightarrow{u}|cos$

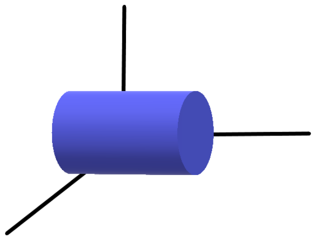
Theorem 14.4.2
1. the max directional derivative of f at $(x_\circ, y_\circ) is |\nabla f|$ and it accrues if $\overrightarrow{u}$ has the same direction of ∇f .
2. the min directional derivative of f is $-|\nabla f|$ and it accrues if $\overrightarrow{u}$ has the opposite direction of ∇f.

■ **Example 14.27** Let $z = f(x.y) = xe^y$ find the max directional derivative at (2,0).
Solution:
$\nabla f =< f_x, f_y >, f_x = e^y f_x(2,0) = 1, f_y = xe^y f_y(2,0) = 2$
$\nabla f =< 1,2 >$
$maxD_{\overrightarrow{u}} f(2,0) = |\nabla f| = \sqrt{5}$
it occurs if $\overrightarrow{u}$ has the same direction of $< 1,2 >$
max directional derivative $\leftrightarrow$ max rate of change
$\leftrightarrow$ increasing most rapidly
min directional derivative $\leftrightarrow$ min rate of change
$\leftrightarrow$ decreasing most rapidly

■

Tangent plane for level surfaces

$k = f(x,y,z)$ level surfaces
to find the plane we need 1.point $(x_\circ, y_\circ, z_\circ)$ 2.$\overrightarrow{n} = \nabla f$



■ **Example 14.28** 1. find the equation of the tangent plane to $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3at(2,1,3)$
2. find the equation of the normal liner

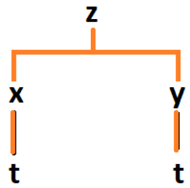
Solution:

1. plane ! point (2,1.3)
   $\overrightarrow{n} = \nabla f =< \frac{2x}{4}, 2y, \frac{2}{9} =< 1, 2, \frac{2}{3}$
   $|(x-2) + 2(y-1) + \frac{2}{3}(z-3)|$
2. point (2,1,3)
   $\overrightarrow{n} =< 1, 2, \frac{2}{3} >, x = 2 + t$
   $, y = 1 + 2t$
   $, z = 3 + \frac{2}{3}t$
   $\nabla f =< f_x, f_y, f_z >$
   $D_{\overrightarrow{u}} f(x_\circ, y_\circ, z_\circ) = \nabla f . \overrightarrow{u}$
   max $D_{\overrightarrow{u}} f = |\nabla f|$ it accrues if $\nabla f, \overrightarrow{u}$ have the same direction
   F(x,y,z) = K the n the normal to the tangent $D_{\overrightarrow{n}} f = |\nabla f|$

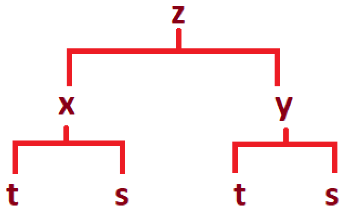
■

Problem 14.4 1,3,5,6,11,13,15,17,19,21,25,30,31,33,35,36,37

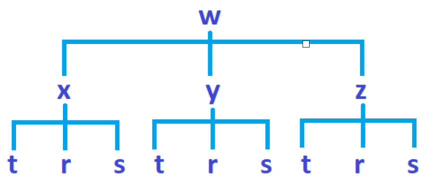
14.5 The Chain Rule



Case I:
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$



Case II:
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial t}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$



Case III:
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

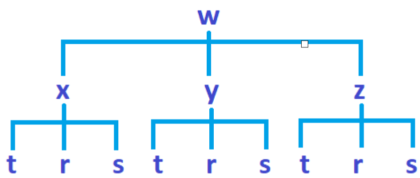
■ **Example 14.29** If $u = x^4y + y^2z^3$

$x = rse^t$

$y = rs^2e^{-t}$

$z = r^2s\sin t$

Find $\frac{\partial u}{\partial s}$ when $r = 2,\ s = 1,\ t = 0$



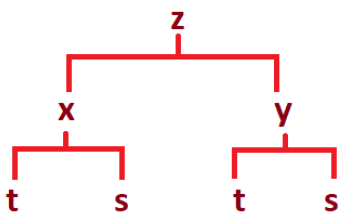
Solution:

$$\frac{\partial u}{\partial s} = u_x\frac{\partial x}{\partial s} + u_y\frac{\partial y}{\partial s} + u_z\frac{\partial z}{\partial s}$$
$$= (4x^3y)re^t + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2t^2)(r^2\sin t)$$
$$\frac{\partial u}{\partial s}\Big|_{(r,s,t)=(2,1,0)} = (64)2 + (16+0)4 + 0 = 128 + 64 = 192$$ ■

■ **Example 14.30** .

if $g(s,t) = f(\overset{x}{s^2 - t^2}, \overset{y}{t^2 - s^2})$

show that g satisfies $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$



Solution:

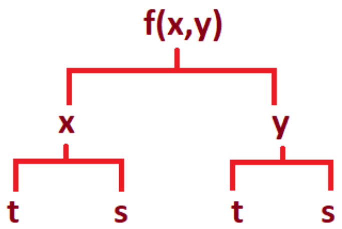
$g(s,t) = f(x,y)$

$$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial g}{\partial s} = f_x(2s) + f_y(-2s) \cdots 1$$
$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t}$$
$$\frac{\partial g}{\partial t} = f_x(-2t) + f_y(2t) \cdots 2$$
so, $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = f_x(2st) + f_y(-2st) + f_x(-2st) + f_y(2st) = 0$

■

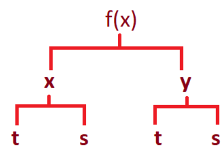
Example 14.31 .
let $z = f(x,y)$, (f has cont second partial derivative)
if $x = r^2 + s^2$, $y = 2rs$
find:

1. $\frac{\partial z}{\partial r}$
2. $\frac{\partial^2 z}{\partial r^2}$

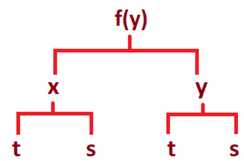


Solution:

$$\frac{\partial z}{\partial r} = f_x\frac{\partial x}{\partial r} + f_y\frac{\partial y}{\partial r} = f_x(2r) + f_y(2s)$$
$$= 2rf_x + 2sf_y$$



$$\frac{\partial^2 z}{\partial r^2} = 2[r(f_{xx}\frac{\partial x}{\partial r} + f_{xy}\frac{\partial y}{\partial r}) + f_x] + 2s[f_{yx}\frac{\partial x}{\partial r} + f_{yy}\frac{\partial y}{\partial r}]$$
$$= (4r^2)f_{xx} + 8rsf_{xy} + 4s^2 f_{yy}$$



■

Definition 14.5.1 Implicit differentiation
if $F(x,y,z) = 0$

then $\dfrac{\partial z}{\partial x} = -\dfrac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$

$\dfrac{\partial z}{\partial y} = -\dfrac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$

Example 14.32 if $x^3 + y^3 + z^3 = 1 - 6xyz$

Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

Solution:

$x^3 + y^3 + z^3 + 6xyz - 1 = 0$

$\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{3x^2 + 6yz}{3z^2 + 6xy}$

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{3y^2 + 6xz}{3z^2 + 6xy}$ ■

Problem 14.5 3,4,5,7,9,11,12,13,14,15,16,17,19,21,23,26,27,28,31,33,35,39,43,46,48,50,51