Modern-Analysis 2

Lecture-15

Theorem (Riemann's Condtion):

Let f be a bounded function on $[a,b]$ and let α be an increasing function on $[a,b]$. Then $f \in \mathscr{R}_\alpha[a,b]$ if and only if for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that:

$$U(f,p) - L(f,p) < \epsilon$$

Proof: Let $f \in \mathscr{R}_\alpha[a,b]$, and $\epsilon > 0$, $\exists P, S$ such that:

$$\int_a^b f d\alpha - \frac{\epsilon}{2} < L(f,P), \text{and } U(f,S) < \int_a^b f d\alpha + \frac{\epsilon}{2}$$

$$\boxed{\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha}$$

Let $T = P \cup S$, $U(f,T) \leq U(f,S)$

and $L(f,T) \geq L(f,P) \Rightarrow -L(f,T) \leq -L(f,P)$..."**The Previous Lemma**"

So $U(f,T) - L(f,T) \leq U(f,S) - L(f,P) \leq \int_a^b f d\alpha + \frac{\epsilon}{2} + \frac{\epsilon}{2} - \int_a^b f d\alpha = \epsilon$

"$\Longrightarrow$" Let for any $\epsilon > 0, \exists P$ such that $U(f,P) - L(f,P) < \epsilon$

$$\boxed{U(f,P) \geq \overline{\int_a^b} f d\alpha, , L(f,P) \leq \underline{\int_a^b} f d\alpha, , -\underline{\int_a^b} f \alpha \leq -L(f,P)}$$

So $0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha \leq U(f,P) - L(f,P) < \epsilon$

So $0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \epsilon, \forall \epsilon > 0$ $\boxed{0 \leq a \leq \epsilon, \forall \epsilon > 0 \Rightarrow a = 0}$

$\Rightarrow \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha = 0 \Rightarrow \overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, So $f \in \mathscr{R}[a,b]$

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f, α, P , $U(f,P) = \sum_{i=1}^n M_i \Delta\alpha_i$, $L(f,P) = \sum_{i=1}^n m_i \Delta\alpha_i$

$U = \{$Upper sums for all partitions$\}$, $L = \{$Lower sum for all partition$\}$

$\overline{\int_a^b} f d\alpha = infU$, $\underline{\int_a^b} f d\alpha = supL$

$f \in \mathscr{R} \iff \forall \epsilon > 0, \exists P$ such that: $U(f,p) - L(f,P) < \epsilon$

" $\Longleftarrow$ " $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$

$\exists S$ such that $\int_a^b f d\alpha + \frac{\epsilon}{2} > U(f,S) \geq U(f,T)$

$\exists P$ such that $-\int_a^b f d\alpha + \frac{\epsilon}{2} > L(f,P) \leq L(f,T) \Rightarrow U(f,p) - L(f,T) < \epsilon$

Example: $I = [a,b], c \in (a,b), K_1, K_2$

define $\alpha(x) = \begin{cases} K_1, & a \leq x < c \\ \text{any point in } [K_1, K_2], & x = c \\ K_2, & c < x \leq b \end{cases}$.

Let $f : [a,b] \mapsto \mathbb{R}$ be bounded, continuous , show that $f \in \mathscr{R}[a,b]$ and find $\int_a^b f d\alpha$.

Solution: since f is continuous at c, $\forall \epsilon > 0, \exists x_1, x_2 \in [a,b]$ such that:

$a < x_1 < c < x_2 < b$ and if $x \in [x_1, x_2]$ then $|f(x) - f(c)| < \frac{\epsilon}{3(k_1 - k_2)}, \forall x \in [x_1, x_2]$

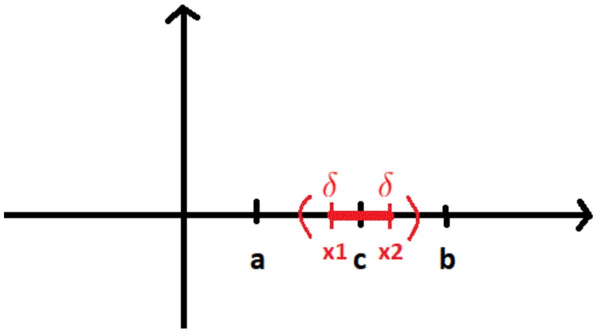

FIGURE 1. $\delta > 0$ such that $|x-c| < \delta \Rightarrow |f(x)-f(c)| < \epsilon$

Now let $P = \{a, x_1, x_2, b\}$, $U(f,P) = \sum_{k=1}^{3} M_k \Delta\alpha_k$

$= M_1(\alpha(x_1)-\alpha(a)) + M_2(\alpha(x_2)-\alpha(x_1)) + M_3(\alpha(b)-\alpha(x_2)) = M_2(k_2-k_1)$

$\leq \left(f(c) + \frac{\epsilon}{3(k_2-k_1)}\right)(k_2-k_1) = f(c)(k_2-k_1) + \frac{\epsilon}{3} \cdots (1)$

$L(f,P) = \sum_{k=1}^{3} m_k \Delta\alpha_k = m_2(\alpha(x_2)-\alpha(x_1)) = m_2(k_2-k_1)$

$\geq \left(f(c) - \frac{\epsilon}{3(k_2-k_1)}\right)(k_2-k_1) = f(c)(k_2-k_1) - \frac{\epsilon}{3}$

So $-L(f,P) \leq -f(c)(k_2-k_1) + \frac{\epsilon}{3} \cdots (2)$

$U(f,P) - L(f,P) \leq f(c)(k_2-k_1) + \frac{\epsilon}{3} - f(c)(k_2-k_1) + \frac{\epsilon}{3} = \frac{2}{3}\epsilon < \epsilon$

$\Rightarrow f \in \mathscr{R}_\alpha[a,b]$ ***(Riemann's condition)***

Now, $\int_a^b f d\alpha = \overline{\int}_a^b f d\alpha \leq U(f,P) \leq f(c)(k_2-k_1) + \frac{\epsilon}{3}$

$\Rightarrow \int_a^b f d\alpha \leq f(c)(k_2-k_1) + \frac{\epsilon}{3}, \forall \epsilon > 0$

$\int_a^b f d\alpha \leq f(c)(k_2-k_1) \cdots (1)$

Also, $\int_a^b f d\alpha = \underline{\int}_a^b f d\alpha \geq L(f,P) = f(c)(k_2-k_1) - \frac{\epsilon}{3}, \forall \epsilon > 0$

$\int_a^b f d\alpha \geq f(c)(k_2-k_1) \cdots (2)$

$$\Rightarrow \int_a^b f d\alpha = f(c)(k_2-k_1)$$

Example: Let $\alpha(x) = \begin{cases} 1 & 1 \leq x < 2 \\ \frac{3}{2} & x = 2 \\ 2 & 2 < x \leq 3 \end{cases}$

Find $\int_1^3 (\sin x)^{x^2+1} d\alpha = (\sin 2)^5 (2-1)$

Anyone with this link can view this project:

https://www.overleaf.com/read/vrmsnmtjsgqg