Modern-Analysis 2

Lecture-22

January 23, 2021

55:Riemann-Stieltjes Integration with Respect to Functions of Bounded Variation

Theorem

α continuous and differentiate on $[a,b]$, if $f, \alpha' \in \mathscr{R}[a,b] \Rightarrow f \in \mathscr{R}_\alpha[a,b]$ and

$$\boxed{\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx}$$

Example Find:

1. $\int_0^1 x^2 dx^2 = \int_0^2 x^2(2x)dx = \frac{1}{2}x^4|_0^2 = \frac{16}{2} = 8$

2. $\int_0^2 [x]dx^2 = \int_0^1 [x]dx^2 + \int_1^2 [x]dx^2 = \int_0^1 0dx^2 + \int_1^2 1dx^2 = 0 + \int_1^2 2xdx = 3$

Corollary "Fundamental theorem"

Let f be continuous and differentiable on $[a,b]$. If $f' \in \mathscr{R}[a,b] \Longrightarrow$

$$\boxed{\int_a^b f'(x) = f(b) - f(a)}$$

Proof: $\int_a^b f'(x)dx = \int_a^b 1f'(x)dx = \int_a^b 1df(x) = f(b) - f(a)$

Example: Find $\int_1^4 \sqrt{x^2+1}d(x^2+3) = \int_1^4 \sqrt{x^2+1}(2x)dx = \frac{2}{3}(17^{\frac{3}{2}} - 1)$

Chapter X: Sequences and Series of Functions

sec-60:Pointwise Convergence and Uniform Convergence

$$\{f_n\}_{n=1}^{\infty}, f_n : x \mapsto \mathbb{R}, d(x,y) = |x-y|$$

Definition:

$\{f_n\}$, $f_n : x \longmapsto \mathbb{R}$, $f : x \longmapsto \mathbb{R}$, We say that $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on X if $\lim_{n\mapsto\infty} f_n(x) = f(x), \forall x \in X$.

Example: $f_n(x) = x^2, 0 \leq x \leq 1$

$x = \frac{1}{2}, \frac{1}{4}, \ldots$

$f(\frac{1}{2}), f(\frac{1}{4}), \ldots \mapsto 0$

$f(1) \mapsto 1$

so f_n converges pointwise to f, where $f(x) = \begin{cases} 0 & , 0 \leq x < 1 \\ 1 & , x = 1 \end{cases}$

Definition:

$\{f_n\}$, $f_n : x \longmapsto \mathbb{R}$, $f : x \longmapsto \mathbb{R}$, We say that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on X if $\epsilon > 0, \exists \mathrm{N} \in N$ s.t $|f_n(x) - f(x)| < \epsilon$

$\forall n \geq N, \forall x \in X$.

Theorem

Uniform Converges $\Rightarrow$ pointwise convergence .

$$\boxed{Notation : f_n \to f(point-wise)\ \ , f_n \rightrightarrows f(uniform)}$$

Theorem:

if $\{f_n\}$, $f_n : M \longmapsto \mathbb{R}$, $f : M \longmapsto \mathbb{R}$, We say that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f on M if f_n is cont. at a Then f is cont. at a

PF: Let d be the metric of M

$f_n \rightrightarrows f$

f_n is cont. at $a \Rightarrow f$ is cont. at a

So $\forall \epsilon > 0 \exists$ N s.t $|f_n(x) - f(x)| < \frac{\epsilon}{3}$, $\forall x \in M$

$\forall \epsilon > 0, |f(x) - f(a)| = |f(x) - f_N(X) + f_N(X) - f_N(a) + f_N(a) - f(a)|$

$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)|$

$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \qquad$ if $d(x,a) < \delta$

Thus f is cont. at a

Note :

* Uniform $\Rightarrow$ Pointwise but The converse is false .

Counter example : f_n converges pointwise to f, where $f(x) = \begin{cases} 0 & , 0 \leq x < 1 \\ 1 & , x = 1 \end{cases}$

but if you take the limit it's not cont. so it.s not convergence uniformly