Homework-5

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55.Riemann-Stieltjes Integration with Respect to Functions of Bounded Variation

Exercise-55.3:.

(a) $\int_0^3 \sqrt{x}dx^3$

since $\alpha(x) = x^3$ is continuous and differentiable on $[0,3] \Rightarrow$

$$= \int_0^3 \sqrt{x}dx^3 = \int_0^3 \sqrt{x}3x^2dx = 3\int_0^3 \sqrt{x}x^2dx$$

$$3\left(\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1}\right) = 6\frac{\sqrt{3^7}}{7}$$

(b) $\int_1^4 \sqrt{x^2+1}d(x^2+3)$

since $\alpha(x) = (x^2+3)$ is continuous and differentiable on $[1,4] \Rightarrow$

$$\int_1^4 \sqrt{x^2+1}d(x^2+3) = \int_1^4 \sqrt{x^2+1}2xdx = 2\int_1^4 \sqrt{x^4+x^2}dx = 44.8$$

(c) $\int_1^4 x-[x]dx^2$

since $\alpha(x) = x^2$ is continuous and differentiable on $[1,4] \Rightarrow$

$$\int_1^4 x-[x]dx^2 = \int_1^2 x-[x]dx^2 + \int_2^3 x-[x]dx^2 + \int_3^4 x-[x]dx^2$$

$$\int_1^4 x-[x]dx^2 = \int_1^2 x-1dx^2 + \int_2^3 x-2dx^2 + \int_3^4 x-3dx^2$$

$$= \int_1^2 2x^2-2xdx + \int_2^3 2x^2-4xdx + \int_3^4 2x^2-6xdx$$

$$\frac{2x^3}{3} - x^2|_1^2 + \frac{2x^3}{3} - 2x^2|_2^3 + \frac{2x^3}{3} - 3x^2|_3^4 = 8$$

Exercise-55.6:.

Solution: Since $\alpha \in BV[a,b]$ and f continuous, then by theorem $f \in \mathscr{R}[a,b]$

Now it's clearly that:

$$L(f,P,T) \leq S(f,p,T) \leq U(f,P)$$

$\Rightarrow \int_a^b f d\alpha - \epsilon < L$

and $\int_a^b f d\alpha + \epsilon < U \Rightarrow \int_a^b f d\alpha - \epsilon < L(f,p) \leq S(f,p,T) \leq U(f,p) < \int_a^b f d\alpha + \epsilon$

$\Rightarrow \int_a^b f d\alpha - \epsilon < S(f,p,T) < int_a^b f d\alpha + \epsilon$

$|S(f,p,T) - \int_a^b f d\alpha| < \epsilon \Rightarrow$

$$\lim_{norm\ p \to 0} S(f,p,T) = \int_a^b f d\alpha$$

Exercise-55.9:.

Solution:

$$Max\{f,g\} = \frac{f+g+|f-g|}{2}$$

By theorem if $f,g \in \mathscr{R}[a,b], c \in \mathbb{R} \Rightarrow$

- $f+g \in \mathscr{R}[a,b]$
- $|f| \in \mathscr{R}[a,b]$
- $cf \in \mathscr{R}[a,b]$

So $Max\{f,g\} = \frac{f+g+|f-g|}{2}$

60.Pointwise Convergence and Uniform Convergence

Exercise-60.2:.

Solution:

$$f_n(x) = \frac{1}{1+n^2x^2} \Longrightarrow$$

$\{f_n\}$ converges pointwise to f on $[0,1]$, where:

$$f(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1 & x = 0 \end{cases}$$

Since f is not continuous at point $0 \Rightarrow \ \{f_n\}$ is not uniformly convergent.

$$g_n(x) = xn(1-x)^n \Longrightarrow$$

$\{g_n\}$ converges pointwise to g on $[0,1]$, where:

$$g(x) = \begin{cases} 0 & 0 < x \leq 1 \\ c & x = 0 \text{ , } where\ c \in [0,1] \end{cases}$$

Since g is not continuous at point $0 \Rightarrow \ \{g_n\}$ is not uniformly convergent.

Exercise-60.5:.

Solution: Let $\{f_n\}$ be a sequence of bounded functions on a set X.and $\{f_n\}$ converges uniformly to f on X

So since $f_n \rightrightarrows f \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|f_n - f| < \epsilon$.

$$|f| = |f - f_n + f_n| \leq |f_n - f| + |f_n| < \epsilon + M$$

So f is bounded.

Let $f_n = \frac{x}{n} + \frac{1}{x}\{0 < x \leq 1\}$, so $f_n \to f$ such that $f = \frac{1}{x}\{0 < x \leq 1\}$ and f is unbounded.

Exercise-60.10:.

Solution: let $C[a,b]$ denote the set of continuous real-valued functions on $[a,b]$. We define a metric d on $C[a,b]$ by the formula:

$$d(f,g) = sup\{|f(x) - g(x)|, x \in [a,b]\}$$

Let f_n be a Cauchy sequence in $C[a,b]$, then $\forall \epsilon > 0$, there is N such that $||f_n - f_m|| < \epsilon$ for $n, m \geq N \implies |f_n - f_m|| = sup|f_n - f_m| < \epsilon$.
$|f_n - f_m| \leq sup|f_n(x) - f_m(x)| < \epsilon, \forall n \geq N$.
So $f_n(x)$ converges uniformly to $f(x)$.
And each f_n is continuous on $[a,b]$, and $f_n \to f$ uniformly on $[a,b]$.
Thus, $f \in C[a,b]$. So $C[a,b]$ is complete.

61. Integration and Differentiation of Uniformly Convergent Sequences

Exercise-61.1:.

Solution: Let $f_n := \frac{x+n[x]}{n}|0 \leq x \leq 1$, f_n is convergent pointwise to f, such that:$f(x) = [x]$, and
$\lim_{n\to\infty} \int_0^2 \frac{x+n[x]}{n} = \lim_{n\to\infty} \int_0^1 \left(\frac{x}{n} + \int_1^2 \frac{x+n}{n}\right) =$
$\lim_{n\to\infty} \left(\frac{x^2}{2n}|_0^1 + (\frac{x^2}{2n} + x|_1^2)\right) = \lim_{n\to\infty} \left(\frac{1}{2n} + \frac{4}{2n} + 2 - \frac{1}{2n} - 1\right) = 1$
and $\int_0^2 [x] = \int_0^1 0 + \int_1^2 1 = x|_1^2) = 1 \Rightarrow$

$$\lim_{n\to\infty} \int_0^2 f_n = \int_0^2 f$$