Modern Analysis

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Contents

Chapter 3
Sequences and Series

3.1: Sequences and their Limits

(1) **The sequence (x_n) is defined by the following formulas for the n the term. Write the first five terms in each case:**

(a) $x_n := 1 + (-1)^n$

$0, 2, 0, 2, ...$

(b) $x_n := \frac{(-1)^n}{n}$

$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, ...$

(c) $x_n := \frac{1}{n(n+1)}$

$x_1 = \frac{1}{2}, x_2 = \frac{1}{6}, x_3 = \frac{1}{12} ...$

(d) $x_n := \frac{1}{n^2+2}$

$\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, \frac{1}{18}, \cdots$

$\boxed{\frac{5}{62}}$ **Prove that:** $\lim \left(\frac{n}{n^2+1}\right) = 0$

solution:

Let $\epsilon > 0$ be given. Then by the Archemedian property there is k $\in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$.

Now, if $n \geq k$, then we have:

$\left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} \leq \frac{n}{n^2} = \frac{1}{n} < \frac{1}{k} < \epsilon.$

$\therefore \lim \frac{n}{n^2+1} = 0 \quad \square$

3.3: Monotone Sequence

$\boxed{\frac{3}{77}}$ **Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that(x_n) is decreasing and bounded below by 2. Find the limit.**

solution:

- Claim 1: Let $x_n \geq 2$

  Proof the claim 1 *(By Induction(PMI))*

  $x_1 \geq 2$, if $x_k \geq 2$ *(for some k)*

  $\Rightarrow x_{k+1} = 1 + \sqrt{x_k - 1} \geq 1 + \sqrt{2-1} = 2$ *(for some k)*

  $\therefore x_n$ is bounded.

- Claim 2: x_n is decreasing

  Proof the claim 2:

  We know that $x_1 \geq 2$

  If $x_{k+1} < x_k$

  *Want to show that $x_{k+2} < x_{k+1}$ *(for some k)*

  $\Rightarrow x_{k+2} = 1 + \sqrt{x_{k+1} - 1} < 1 + \sqrt{x_k - 1} = x_{k+1}$ *(for some k)*

  $\therefore x_{k+2} < x_{k+1}$ *(for some k)*

  $\Rightarrow x_n$ is decreasing.

- x_n is bounded and decreasing.

  *By the **(MCT)*** x_n is convergent. □

- So, since x_n is convergent we have: $\boxed{\lim(x_{n+1}) = \lim(x_n) = x}$

  $\Rightarrow \lim(x_{n+1}) = 1 + \lim\sqrt{x_n - 1}$

  $\Rightarrow x = 1 + \sqrt{x-1}$

  $\Rightarrow (x-1)^2 + 1 - x = 0$

  $\Rightarrow x^2 - 3x + 2 = 0$

  $\Rightarrow (x_2)(x_1) = 0$

  $\Rightarrow x = 1$ or $x = 2$

  But $x_n \geq 2 \Rightarrow \boxed{\lim x_n = 2}$
