Homework fourth Week

Raneem Madani

Student Number: 11820975

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IX.The Riemann-Stieltjes Integral

51.Riemann-Stieltjes Integration with Respect to an Increasing Integrator

Exercise 5:.

Solution: f is bounded function on $[a, b]$ and α increasing on $[a, b]$. We want to show that:

$$\overline{\int_a^b} f d\alpha = -\underline{\int_a^b} (-f) d\alpha$$

Let $p = \{x_0, x_1 ... x_n\}$ be any partion of $[a, b]$.

We know that $\overline{\int}_b^a f d\alpha = inf U(f, p)$, so $\overline{\int}_b^a f d\alpha \leq U(f, p)$

And $\underline{\int}_a^b (-f) d\alpha = sup L(-f, p)$, so $\underline{\int}_a^b (-f) d\alpha \geq L(-f, p)$

Claim-1: $\overline{\int}_a^b f d\alpha \geq -\underline{\int}_a^b (-f) d\alpha$

Proof the claim: we know that $\left(\overline{\int}_a^b f d\alpha \geq \underline{\int}_a^b f d\alpha\right) \times -1 \Rightarrow$

$\left(-\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b f d\alpha = \underline{\int}_a^b - f d\alpha\right) \times -1 \Rightarrow$

$\overline{\int}_a^b f d\alpha \geq -\underline{\int}_a^b - f d\alpha$

Claim-2: $\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b (-f) d\alpha$

Proof the claim: we know that $\left(\overline{\int}_a^b - f d\alpha \geq \underline{\int}_a^b - f d\alpha\right) \times -1 \Rightarrow$

$\overline{\int}_a^b f d\alpha = \overline{\int}_a^b - (-f) d\alpha = -\overline{\int}_a^b - f d\alpha \leq -\underline{\int}_a^b - f d\alpha \Rightarrow$

$\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b (-f) d\alpha$

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so from claim 1 and claim 2 we have $\overline{\int}_a^b f d\alpha = -\underline{\int}_a^b (-f) d\alpha$, so we are done.

Exercise 12:.

Solution: (a)-Let $p = \{x_0, x_1 ... x_n\}$ be any partion of $[0, 2]$.
$U(f, p) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{n} \Delta\alpha_k$
$= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 1 - 0 = 1$
and $L(f, p) = \sum_{k=1}^{n} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{n} \Delta\alpha_k = 0$, since $U(f, p) \neq L(f, P) \Rightarrow f \notin \mathscr{R}_\alpha[0, 2]$

(b)-Let $s = \{x_0, x_1 ... x_n\}$ be any partion of $[0, 2]$.
$U(g, s) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{n} \Delta\alpha_k$
$= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 1 - 0 = 1$
and $L(g, s) = \sum_{k=1}^{n} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{n} \Delta\alpha_k = 0$ since $U(g, s) \neq L(g, s) \Rightarrow g \notin \mathscr{R}_\alpha[0, 2]$

(c)-Let $p = \{x_0, x_1 ... x_n\}$ be any partion of $[0, 2]$, $\alpha(x) = x$.
$U(f, p) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{n} \Delta\alpha_k$
$= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 2 - 0 = 2$
and $L(f, p) = \sum_{k=1}^{n} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{n} \Delta\alpha_k = 0$, since $U(f, p) \neq L(f, p) \Rightarrow f \notin \mathscr{R}_\alpha[0, 2]$

Let $s = \{x_0, x_1 ... x_n\}$ be any partion of $[0, 2]$, $\alpha(x) = x$.
$U(g, s) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{n} \Delta\alpha_k$
$= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 2 - 0 = 2$
and $L(g, s) = \sum_{k=1}^{n} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{n} \Delta\alpha_k = 0$, since $U(g, s) \neq L(g, s) \Rightarrow g \notin \mathscr{R}_\alpha[0, 2]$

Exercise 18:.

Solution:

- Let $\alpha(x) = x$ and $f(x) = \begin{cases} -1, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

  Now $\overline{\int}_a^b f dx = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{n} \Delta\alpha_k$

  $= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 1 - 0 = 1$

  and $\underline{\int}_a^b = \sum_{k=1}^{n} m_k \Delta\alpha_k$, and $m_k = 1$, so $-1 \sum_{k=1}^{n} \Delta\alpha_k =$

  $-1[x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1}] = -1[x_n - x_0] = -1[1-0] = -1$

  so $\overline{\int}_a^b = 1 \leq -1 = \underline{\int}_a^b$ so $f \notin \mathscr{R}_\alpha[0,1]$

- But $|f(x)| = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

  so $|f(x)| := \{1, x \in [0,1]\}$, and $\overline{\int}_a^b = 0 = \underline{\int}_a^b$, so $|f| \in \mathscr{R}_\alpha[0,1]$

$$\boxed{\text{So } |f| \in \mathscr{R}_\alpha[a,b], \text{ but } f \notin \mathscr{R}_\alpha[a,b]}$$

54.Functions of Bounded Variation

Exercise 5:.

Solution: Since $\alpha(x) \in BV[a,b]$ so $\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1}| \leq c, c \in \mathbb{R}$, and we have $|\alpha(x)| > M$, so $\frac{1}{\alpha(x)} \leq \frac{1}{M}$.

Now $\sum_{k=1}^{n} |\frac{1}{\alpha(x_k)} - \frac{1}{\alpha(x_{k-1})}| = \sum_{k=1}^{n} |\frac{\alpha(x_{k-1}) - \alpha(x_k)}{\alpha(x_{k-1})\alpha(x_k)}| \leq \frac{c}{M^2}$.

so $\sum_{k=1}^{n} |\frac{1}{\alpha(x_k)} - \frac{1}{\alpha(x_{k-1})}|$ bounded.