Modern Analysis

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5.4: Uniform Continuity

Theorem 1 *:Recall that a function $f: A \mapsto \mathbb{R}$ is said to be Uniformly Continuous on A if $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that if $x, y \in A,$ satisfy $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

$\boxed{\frac{1}{144}}$ Show that the function $f(x) := \frac{1}{x}$ is uniformly continuous on the set $A := [a, \infty)$, where a is a positive constant.

Solution: $f(x) := \frac{1}{x}$, $\forall x \in [a, \infty)$, $a > 0$

Let us consider:

$|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right|$

$\frac{|c-x|}{|cx|} \leq \frac{|x-c|}{a^2}$

$(a < x, c \Rightarrow \frac{1}{a} > \frac{1}{x}, \frac{1}{a} > \frac{1}{c})$

So, $|f(x) - f(c)| \leq \frac{|x-c|}{a^2}$

$\therefore f(x) := \frac{1}{x}$ is *Uniformly Continuous.*

$\boxed{\frac{9}{144}}$ if f is uniformly continuous on $A \subseteq \mathbb{R}$ and $|f(x)| \geq k > 0$ for all $x \in Ashowthat \frac{1}{f}$ is uniformly continuous on A.

Solution: $f-$ Uniformly Continuous on $A \subset \mathbb{R}$

$|f(x)| \geq k > 0$, for all $x \in A$

To show:$\frac{1}{f}$ is uniformly continuous, let us consider:

$\left|\frac{1}{f(x)} - \frac{1}{f(c)}\right| = \frac{|f(x)-f(c)|}{|f(x)f(c)|} \leq \frac{|f(x)-f(c)|}{k^2}$

$\therefore \left|\frac{1}{f(x)} - \frac{1}{f(c)}\right| \leq \left(\frac{1}{k^2}\right) |f(x) - f(c)|$......(1)

So, for $\epsilon > 0$, choose $\delta > 0$ such that $|f(x) - f(c)| < k^2\epsilon$ whenever $|x - c| < \delta$

From(1):$\left|\frac{1}{f(x)} - \frac{1}{f(c)}\right| < \frac{1}{k^2}(k^2\epsilon)$, whenever $|x - c| < \delta$

$\therefore \frac{1}{f}$ is also *Uniformly Continuous.*

6.1: The Derivative

Theorem 2 *:$f: A \mapsto \mathbb{R}$,$c \in I$ we say that $L \in \mathbb{R}$ is* <u>*Derivative*</u> *of f at c,$(f(c)')$ if $\forall \epsilon > 0, \exists \delta > 0$ such that if $x \in \mathbb{R}$,$0 < |x-c| < \delta$ then:* $\left|\frac{f(x)-f(c)}{x-c} - L\right| < \epsilon$.

$\boxed{\frac{9}{171}}$ Prove that if $f: \mathbb{R} \mapsto \mathbb{R}$ is an **even function** [that is,$f(-x) = f(x)$ for all $x \in \mathbb{R}$] and has a derivative at every point, then the derivative f' is an **odd function** [that is,$f'(x) = -f'(x)$ for all $x \in \mathbb{R}$].Also prove that if $g: \mathbb{R} \mapsto \mathbb{R}$ is a differentiable odd function,then g' is an even function.

Solution:

- let f be an even function $\Rightarrow f(-x) = f(x)\ \forall x$

  $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

  $f'(-c) = \lim_{x \to -c} \frac{f(x)-f(-c)}{x-(-c)}$

  $= \lim_{x \to c} \frac{f(-x)-f(c)}{-x+c}$

  $= \lim_{x \to c} \frac{f(x)-f(c)}{-(x-c)}$, but $f(-x) = f(x) \Rightarrow$

  $= -\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

  $= -f'(c)$

- let g be an odd function $\Rightarrow g(-x) = -g(x)\ \forall x$

  $g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$

  $g'(-c) = \lim_{x \to -c} \frac{g(x)-g(-c)}{x-(-c)}$

  $= \lim_{x \to c} \frac{g(-x)-g(c)}{-x+c}$

  $= \lim_{x \to c} \frac{-g(x)-(-g(c))}{-(x-c)}$, but $g(-x) = -g(x) \Rightarrow$

  $= \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$

  $= g'(c)$

6.2: Mean Value

Theorem 3 *:If f is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that:*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$\boxed{\frac{2}{179}}$ Find the point of relative extrema, the intervals on which the following function are increasing, and those on which they are decreasing:

(b) $g(x) := \frac{x}{x^2+1}$ for $x \in \mathbb{R}$.

Solution:

- Point of relative exterma:

  $g'(x) = \frac{x(2x)-(x^2+1)}{(x^2+1)^2}$

  $= \frac{2x^2-x^2-1}{(x^2+1)^2}$

  $= \frac{1-x^2}{(x^2+1)^2}$

  $g'(x) = 0$

  $1 - x^2 = 0$

  $x = \pm 1$

- Find Increasing or Decreasing:Try to check the signal of function:

$g'(x)$	$-$	$+$	$-$
$-\infty$	-1	1	$+\infty$

  on $(-\infty, -1]$, $g(x)$ is Decreasing.

  on $(-1, 1)$, $g(x)$ is Increasing.

  on $[1, +\infty)$, $g(x)$ is Decreasing.

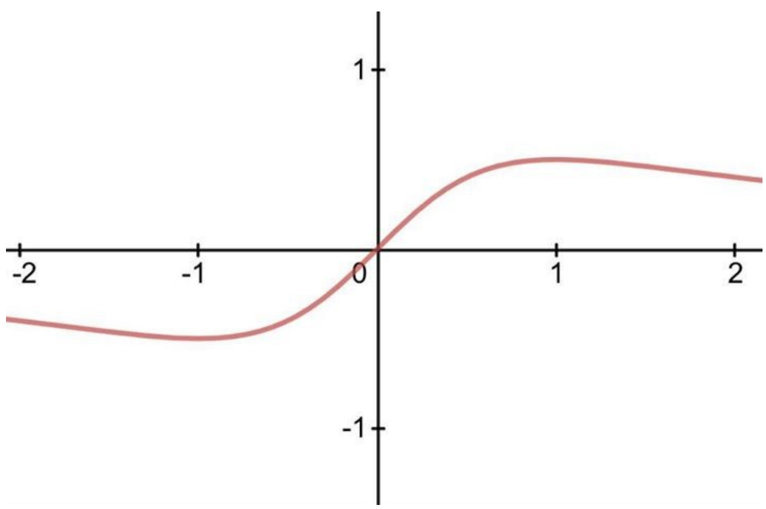

Figure 1: $g(x) = \frac{x}{x^2+1}$