Homework.2

Raneem Madani

January 23, 2021

36: $\mathbb{R}^n, l^2$

Exercise 36.3:.

Solution:

- let $\{ a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$
  $|a_n| \mapsto 0$
  $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that:
  $a_k < \epsilon$, $\forall n \geq k$ *(Take $\epsilon = 1$)*
  $\Rightarrow a_k < 1$
  $\Rightarrow a_k^2 < |a_k|$
  $\sum_{k=1}^{\infty} a_k^2 < \sum_{k=1}^{\infty} a_k$
- let $\{a_n\} \in l^2 \Rightarrow \sum_{n=0}^{\infty} a_n^2 < \infty$
  $\Leftrightarrow a_n^2 \longmapsto 0$
  $\Rightarrow \{a_n\} \in c_0 \Rightarrow l^2 \subset c_0$
- let $a_n = \frac{1}{n} \Rightarrow a_n \in l^2$, $a_n \notin l^1$
  let $b_n = \frac{1}{\sqrt{n}} \Rightarrow b_n \in c_0$, $b_n \notin l^2$

$$\boxed{l^1 \subset l^2 \subset c_0.}$$

Exercise 36.8:

Solution: Let $\{a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$

since $\{b_n\} \in l^\infty \Leftrightarrow |b_n| < M$

$\sum_{k=1}^{\infty} |a_n b_n| \leq \sum_{k=1}^{\infty} |a_n| M$

$= M \sum_{k=1}^{\infty} |a_n| < M.\infty = \infty$

$\Rightarrow \sum_{k=1}^{\infty} |a_n b_n|$ is convergent.

$$\boxed{\{a_n b_n\} \in l^1}$$

Exercise 36.9:

Solution: Let $\{a_n\} \in c_0 \Leftrightarrow a_n \longmapsto 0$

$\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $|a_n| < \epsilon_0 \ \forall n \geq k$

let $\{b_n\} \in l^\infty \Leftrightarrow |b_n| \leq M$

let $\epsilon_0 = \frac{\epsilon}{M}$

$\Rightarrow |a_n b_n| \leq M|a_n| < M\frac{\epsilon}{M} = \epsilon$

$$\boxed{\{a_n b_n\} \in c_0}$$

Give an example:

Let $a_n = \frac{1}{\sqrt{n}} \in C_0$, and let $b_n = (-1)^n \in l^\infty \Rightarrow$

$a_n b_n = \frac{(-1)^n}{\sqrt{n}} \Rightarrow \sum (a_n b_n)^2 = \sum \frac{1}{n} \notin l^2 \Rightarrow \{a_n b_n\} \notin l^2$

Exercise 36.10:

Solution: Let $\{a_n\} \in l^\infty \Leftrightarrow |a_n| \le M$

Let $\{b_n\} \in l^\infty \Leftrightarrow |b_n| < N$, $\forall N, M \in \mathbb{R}$

$\Rightarrow |a_n b_n| \le M.N \Rightarrow$

$$\boxed{\{a_n b_n\} \in l^\infty}$$

Give an example:

Let $\{a_n\} = (-1)^n$

Let $\{b_n\} = (-1)^{1-n} \Rightarrow$

$a_n b_n = (-1)^n (-1)^{1-n} = (-1)^{n+1-n} = -1$

$a_n b_n = -1 \Rightarrow a_n b_n \longmapsto -1$

$\{a_n b_n\} \notin c_0$

37: Sequences in Metric Spaces

Exercise 37.7:

Solution: Let $\{a_n^{(k)}\}$ be a sequence in l^1.

$a \in l^1$, $a = (a_1, a_2, a_3, ...)$

if $\{a^{(k)}\}$ convergent to a then $\lim a_j^{(k)} = a_j$, $\forall j = 1, 2, 3..,$

$|a_j^{(k)}| - |a_j| < |a_j^{(k)} - a_j| < \epsilon$, $\forall j = 1, 2, 3..,$

Let $\epsilon = 1$

$\Rightarrow |a_j^{(k)}| < 1 + |a_j| = M$

$\Rightarrow |a^{(k)}| < M$

$$\boxed{\{a^{(k)}\} \in l^\infty}$$

Exercise 37.9 (a):

Solution: $d : \mathbb{R}^n \times \mathbb{R}^n \longmapsto [0, \infty)$

1. $d(x, y) = 0 \Leftrightarrow x = y$ "Trivial"

2. $d(x, y) = d(y, x)$"Trivial"

3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
   $\sum_{i-1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i| =$
   $\sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d(x, y) + d(y, z)$

Exercise 37.9 (b):

Solution: Let $\{a^{(k)}\}$be a sequence in $\mathbb{R}^n$

$d(a^{(k)}, a) < \epsilon$, $\forall \epsilon > 0$

" $\Rightarrow$ " Let $\{a^{(k)}\}$ convergent to a

$d(a^{(k)}, a) < \epsilon$

$d(a^{(k)}, a) = \sqrt{\sum_{j=1}^{n} (a_j^{(k)} - a_j)^2}$

Let $\epsilon_0 = \frac{\epsilon}{n}$

By Theorem: $|a_j^{(k)} - a_j| \leq \sum_{j=1}^{n} (a_j^{(k)} - a_j)^2 = d(a^{(k)}, a) < \epsilon_0 \Rightarrow$

$$\boxed{d'(a^{(k)}, a) = \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \sum_{j=1}^{n} \frac{\epsilon}{n} = \frac{\epsilon}{n} n = \epsilon}$$

" $\Leftarrow$ " Let $\{a^{(k)}\}$ convergent to a

$d'(a^{(k)}, a) = \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \epsilon_0$

$|a_j^{(k)} - a_j| < \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \epsilon_0$

Let $\epsilon_0 = \frac{\epsilon}{\sqrt{n}}$

$$\boxed{d(a^{(k)}, a) = \sqrt{\sum_{j=1}^{n} (a_j^{(k)} - a_j)^2} \leq \sqrt{\sum_{j=1}^{n} (\frac{\epsilon^2}{n})} = \sqrt{\sum_{j=1}^{n} \frac{\epsilon^2}{n}} = \epsilon}$$

38: Closed Sets

Exercise 38.5(a):

Prove that x is closed $\iff$ $x^\alpha \subseteq x$

Proof:

" $\Rightarrow$ " let x be a closed set $\Rightarrow \overline{x} = x$

$x^\alpha \subseteq \overline{x} \Longrightarrow x^\alpha \subseteq x$

" $\Leftarrow$ " Let $x^\alpha \subseteq x$

let a be a limit point then $\exists \{x_n\}$ such that $\lim x_n = a$

- $x_n = a$ for some n

  $\Rightarrow a \in x$

- $x_n \neq a$ for some n

  $\Rightarrow a \in x^\alpha$ and we suppose that $x^\alpha \subseteq x$

  $\Rightarrow a \in x$

$\therefore x$ is closed

Exercise 38.5(b):

Proof:

Let $x \subseteq \mathbb{R}$ and x is an infinite and bounded set then we have:

$$a_1 \in x$$

$$a_1 \neq a_2 \in x$$

$$\vdots$$

$$a_2 \neq a_k \in x$$

$$\{a_k\} \subseteq x \subseteq \mathbb{R}$$

$\exists \{a_{k_l}\}$ that convergent to a

$$\boxed{\therefore a \in x^\alpha \Rightarrow x^\alpha \neq \phi}$$

Exercise 38.5(c):

Proof: Suppose the contrary,

Let $X \subseteq \mathbb{R}$ be an uncountable and contains non of accumulation points.

$\Rightarrow \forall x \in X$, $\exists \epsilon_x > 0$ such that:

$$\nu_\epsilon(x) \cap X = \{x\}$$

$\Rightarrow \exists n \in \mathbb{N}$ such that $X^\alpha = \{x \in X : \epsilon_x > \frac{1}{n}\}$ is uncountable.

consider the family:

$$\boxed{\{(x - \frac{1}{2n}, x + \frac{1}{2n}) : x \in X^\alpha\}}$$

this is an uncountable family of pairwise disjoint open subsets of $\mathbb{R}$ which contradicts that the countable set $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

Prove that $B_\epsilon(x)$is open set:

Proof: Let $y \in B_\epsilon(x)$, want to find $\delta > 0$ such that:

$B_\delta(y) \subseteq B_\epsilon(x)$

consider $\delta = \epsilon - d(x, y) > 0$

$\Rightarrow d(x, y) < \epsilon \Rightarrow \epsilon - d(x, y) > 0$

Let $z \in B_\delta(y) \Rightarrow d(z, y) < \delta$

$\Rightarrow d(z, y) < \epsilon - d(x, y)$

$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon$

$= d(x, y) + \epsilon - d(x, y) = \epsilon$

$\therefore d(x, z) < \epsilon \Rightarrow z \in B_\epsilon(x) \Rightarrow B_\delta(y) \subseteq B_\epsilon(x)$

so $B_\epsilon(x)$ is an open set of M

40: Continuous Functions on Metric Spaces

Exercise 40.10:

Solution: Let $\epsilon > 0$ be given

$\Rightarrow \forall \epsilon > 0$, $\exists \delta > 0$ such that:

$d_1(b_n, c_n) < \delta$ whenever $d_2(f(b_n), f(c_n)) < \epsilon$

Let $\{b_n\} \in l^1$ since $\{a_n\} \in l^\infty \Rightarrow |a_n| \leq M$

Let $\{c_n\} \in l^1 \Rightarrow d(\{b_n\}, \{c_n\}) < \delta$

$\sum |b_n - c_n| < \delta$, Let $\delta = \frac{\epsilon}{M}$

$|f(c_n) - f(b_n)| = |\sum a_n c_n - \sum a_n b_n|$

$\leq \sum |a_n||c_n - b_n| < M\frac{\epsilon}{M} = \epsilon$

Exercise 40.11:

Solution: Let $\{a_n\} \in l^2 \iff \sqrt{\sum_{n=1}^{\infty} a_n^2} < \epsilon$

want to show that f is continuous at $c = \{c_n\}$ and $b = \{b_n\}$

$\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$|c_n - b_n| < \delta$ whenever $|f(c_n) - f(b_n)| < \epsilon$

$|f(c_n) - f(b_n)| = |\sum_{n=1}^{\infty} c_n a_n - \sum_{n=1}^{\infty} b_n a_n| = |\sum_{n=1}^{\infty} (a_n)(c_n - b_n)|$

$$\leq \sqrt{\sum_{n=1}^{\infty} a_n^2} \sqrt{\sum_{n=1}^{\infty} (c_n - b_n)^2}$$

$$Let : \delta = \frac{\epsilon}{\sqrt{\sum_{n=1}^{\infty} a_n^2}}$$

$$= d(c_n, b_n)\sqrt{\sum_{n=1}^{\infty} a_n^2} < \frac{\epsilon}{\sqrt{\sum_{n=1}^{\infty} a_n^2}}\sqrt{\sum_{n=1}^{\infty} a_n^2} = \epsilon$$