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35:Metric Space

Let $M = \mathbb{R}^2$ and $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ prove triangle inequality in different way.

Solution: Let $x = (x_1, x_2)$, $y = (y_1, y_2)$, $z = (z_1, z_2)$

To show that $d(x, z) \leq d(x, y) + d(y, z)$

$$\Leftrightarrow \sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} \leq \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} + \sqrt{(z_1 - y_1)^2 + (z_2 - y_2)^2}$$

Let $a_i = y_i - x_i$, $b_i = z_i - y_i$, $\forall i = 1, 2, 3...$

$\Leftrightarrow z_1 - x_1 = z_1 - y_1 + y_1 - x_1 = b_1 + a_1$

and $z_2 - x_2 = z_2 - y_2 + y_2 - x_2 = b_2 + a_2$

$\Leftrightarrow \sqrt{(b_1 + a_1)^2 + (b_2 + a_2)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2}$ (squared the two sides)

$\Leftrightarrow (b_1 + a_1)^2 + (b_2 + a_2)^2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$

$\Leftrightarrow b_1^2 + a_1^2 + 2b_1a_1 + b_2^2 + a_2^2 + 2b_2a_2 \leq a_1^2 + a_2^2 + b_1^2 + b_2^2 + 2\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$

$\Leftrightarrow b_1a_1 + b_2a_2 \leq \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$ (squared the two sides)

$\Leftrightarrow b_1^2a_1^2 + b_2^2a_2^2 + 2a_1a_2b_1b_2 \leq a_1^2b_1^2 + a_1^2b_2^2 + a_2^2b_1^2 + a_2^2b_2^2$

$\Leftrightarrow 0 \leq a_1^2b_2^2 + a_2^2b_1^2 - 2a_1a_2b_1b_2$

$\Leftrightarrow 0 \leq (a_1b_2 - a_2b_1)^2$

So $d(x, z) \leq d(x, y) + d(y, z)$ $\square$

35.1-Verify that the function d of Example 35.5 satisfies Definition 35.1 (i) and (ii).

Solution:

- Want to show that $d(x,y)=0 \Leftrightarrow x=y$

  $\sum_{k=1}^{\infty}|x_k-y_k|=0$

  $\Leftrightarrow x_k-y_k=0$

  $\Leftrightarrow x=y,\ \forall x,y\in L^1$ and $x_k,y_k\in\mathbb{R}$ $\square$

- Want to show that $d(x,y)=d(y,x)$

  $|x_k-y_k|=|y_k-x_k|$

  $\Leftrightarrow \sum_{k=1}^{n}|x_k-y_k|=\sum_{k=1}^{n}|y_k-x_k|,\ \forall n=1,2,3,...$

  But $|x_k-y_k|$ is increasing and bounded $\Rightarrow |x_k-y_k|$ is convergent.

  $\Rightarrow \sum_{k=1}^{\infty}|x_k-y_k|=\sum_{k=1}^{\infty}|y_k-x_k|$

  $\Rightarrow d(x,y)=d(y,x)$ $\square$

35.3-Let d be a metric on a set M. Prove that

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

Solution:

By *Triangle Ineq* we have:

$d(x,z) \leq d(x,y) + d(y,z)$

$\Leftrightarrow d(x,z) - d(y,z) \leq d(x,y). \; . \; .(1)$

Also by *Triangle Ineq* we have:

$d(y,z) \leq d(y,x) + d(x,z)$

$\Leftrightarrow d(y,z) - d(x,z) \leq d(y,x)$

$\Leftrightarrow -d(y,x) \leq d(x,z) - d(y,z). \; . \; .(2)$

Hence $d(x,y) = d(y,x)$ from (1),(2) we have $|d(x,z) - d(y,z)| \leq d(x,y)$ $\square$

35.6-Let l^∞ denote the set of all bounded real sequences, and let c_0 denote the set of all real sequences which converge to 0.

Prove that $l^1 \subset c_0 \subset l^\infty$

Solution:

- l^1 denote the set of all seq $\{a_n\}$ $\Rightarrow$ we have $\sum |a_n|$ is convergent

  $\therefore |a_n| \mapsto 0$

  $\Rightarrow a_n \mapsto 0$

  $\therefore \{a_n\} \subset c_0$

  $\therefore l^1 \subset c_0$ $\square$

- Let $\{a_n\} \in c_0$

  $\Leftrightarrow a_n$ is convergent to 0

  $\therefore \{a_n\}$ is bounded

  $\therefore \{a_n\} \in l^\infty$

  $c_0 \subset l^\infty$ $\square$

$$\therefore l^1 \subset c_0 \subset l^\infty$$

36.$\mathbb{R}^n, \mathscr{L}^2$

$$\boxed{(\mathbb{R}^n, d) \text{ is a metric space, } d = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} :}$$

Proof: To proof triangle inequality we need *Cauchy-Schwarz Inequality:*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}$$

$d(x, z) \leq d(x, y) + d(y, z)$

$\Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$

$\Leftrightarrow \sum_{i=1}^{n}(x_i - z_i)^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2}$

Let $a_i = y_i - x_i$, $b_i = z_i - y_i$, $\forall i = 1, 2, 3...n$

$\Leftrightarrow z_i - x_i = z_i - y_i + y_i - x_i = b_i + a_i$

$\Leftrightarrow \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$

$\Leftrightarrow \sum_{i=1}^{n} b_i a_i \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$ *(Cauchy-Schwarz Inequality)* $\square$

$$\boxed{(\mathbb{L}^n, d) \text{ is a metric space, } d = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} :}$$

Proof: To proof triangle inequality we need *Cauchy-Schwarz Inequality:*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}$$

$d(x, z) \leq d(x, y) + d(y, z)$

$\Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$

$\Leftrightarrow \sum_{i=1}^{n}(x_i - z_i)^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2}$

Let $a_i = y_i - x_i$, $b_i = z_i - y_i$, $\forall i = 1, 2, 3...n$

$\Leftrightarrow z_i - x_i = z_i - y_i + y_i - x_i = b_i + a_i$

$\Leftrightarrow \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$

$\Leftrightarrow \sum_{i=1}^{n} b_i a_i \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$ *(Cauchy-Schwarz Inequality)*

$\Rightarrow \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$

For every positive integer n take $n \mapsto \infty$, then we have:

$d(x, z) = \sqrt{\sum_{i=1}^{\infty}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{\infty}(y_i - z_i)^2}$

$= d(x, y) + d(y, z)$

Prove that:

- $l^1 \subset l^2$
- $l^2 \subset l^\infty$

Proof:

1. let $\{ a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$

   $|a_n| \mapsto 0$

   $\forall \epsilon > 0 \exists k \in \mathbb{N}$ such that:

   $a_k < \epsilon, \forall n \geq k$ *(Take $\epsilon = 1$)*

   $\Rightarrow a_k < 1$

   $\Rightarrow a_k^2 < |a_k|$

   $\sum_{k=1}^{\infty} a_k^2 < \sum_{k=1}^{\infty} a_k$

2. let $\{a_n\} \in l^2 \Rightarrow \sum_{k=1}^{\infty} a_n^2 < \infty$

   $\Rightarrow a_n$ is absolutely convergent

   $\Rightarrow a_n$ is bounded

   $\therefore a_n \in l^\infty$

   $\therefore l^2 \subset l^\infty$