Modern Analysis

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Chapter 3 Sequences and Series

3.1: Sequences and their Limits

(1) The sequence (x_n) is defined by the following formulas for the n the term. Write the first five terms in each case:

(a)
$$x_n := 1 + (-1)^n$$

 $0, 2, 0, 2, \dots$

(b)
$$x_n := \frac{(-1)^n}{n}$$

 $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \dots$

(c)
$$x_n := \frac{1}{n(n+1)}$$

 $x_1 = \frac{1}{2}, x_2 = \frac{1}{6}, x_3 = \frac{1}{12}...$

(d)
$$x_n := \frac{1}{n^2+2}$$

 $\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, \frac{1}{18}, \dots$

 $\left| \frac{5}{62} \right|$ Prove that: $\lim \left(\frac{n}{n^2+1} \right) = 0$

solution:

Let $\epsilon > 0$ be given. Then by the Archemedian property there is $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$.

Now, if $n \geq k$, then we have:

$$\left| \frac{n}{n^2 + 1} - 0 \right| = \frac{n}{n^2 + 1} \le \frac{n}{n^2} = \frac{1}{n} < \frac{1}{k} < \epsilon.$$

$$\therefore \lim \frac{n}{n^2+1} = 0 \quad \Box$$

CHAPTER 3 SEQUENCES AND SERIES

3.3: Monotone Sequence

$$\frac{3}{77}$$

Let $x_1 \ge 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.

solution:

• Claim 1: Let $x_n \ge 2$

Proof the claim 1 (By Induction(PMI))

$$x_1 \ge 2$$
, if $x_k \ge 2$ (for some k)

$$\Rightarrow x_{k+1} = 1 + \sqrt{x_k - 1} \ge 1 + \sqrt{2 - 1} = 2$$
 (for some k)

 $\therefore x_n$ is bounded.

• Claim 2: x_n is decreasing

Proof the claim 2:

We know that $x_1 \geq 2$

If
$$x_{k+1} < x_k$$

*Want to show that $x_{k+2} < x_{k+1}$ (for some k)

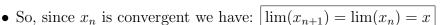
$$\Rightarrow x_{k+2} = 1 + \sqrt{x_{k+1} - 1} < 1 + \sqrt{x_k - 1} = x_{k+1}$$
 (for some k)

$$\therefore x_{k+2} < x_{k+1}$$
 (for some k)

 $\Rightarrow x_n$ is decreasing.

• x_n is bounded and decreasing.

By the (MCT) x_n is convergent.



$$\Rightarrow \lim(x_{n+1}) = 1 + \lim \sqrt{x_n - 1}$$

$$\Rightarrow x = 1 + \sqrt{x-1}$$

$$\Rightarrow (x-1)^2 + 1 - x = 0$$

$$\Rightarrow x^2 - 3x + 2 = 0$$

$$\Rightarrow (x_2)(x_1) = 0$$

$$\Rightarrow x = 1 \text{ or } x = 2$$

But
$$x_n \ge 2 \Rightarrow \overline{\lim x_n = 2}$$