Modern Analysis II

Solutions

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Chapter 1

VII.METRIC SPACE

1.1 35.The distance Function

Exercise 35.3:

Solution: By *Triangle Ineq* we have:

$d(x,z) \leq d(x,y) + d(y,z)$

$\Leftrightarrow d(x,z) - d(y,z) \leq d(x,y)$. . .(1)

Also by *Triangle Ineq* we have:

$d(y,z) \leq d(y,x) + d(x,z)$

$\Leftrightarrow d(y,z) - d(x,z) \leq d(y,x)$

$\Leftrightarrow -d(y,x) \leq d(x,z) - d(y,z)$. . .(2)

Hence $d(x,y) = d(y,x)$ from (1),(2) we have $|d(x,z) - d(y,z)| \leq d(x,y)$

Exercise 35.5:

Solution:

- Want to show that $d(x,y) = 0 \Leftrightarrow x = y$

  $\sum_{k=1}^{\infty} |x_k - y_k| = 0$

  $\Leftrightarrow x_k - y_k = 0$

  $\Leftrightarrow x = y$, $\forall x, y \in L^1$ and $x_k, y_k \in \mathbb{R}$

- Want to show that $d(x,y) = d(y,x)$

  $|x_k - y_k| = |y_k - x_k|$

  $\Leftrightarrow \sum_{k=1}^{n} |x_k - y_k| = \sum_{k=1}^{n} |y_k - x_k|$, $\forall n = 1, 2, 3, ...$

  But $|x_k - y_k|$ is increasing and bounded $\Rightarrow |x_k - y_k|$ is convergent.

  $\Rightarrow \sum_{k=1}^{\infty} |x_k - y_k| = \sum_{k=1}^{\infty} |y_k - x_k|$

  $\Rightarrow d(x,y) = d(y,x)$

Exercise 35.6:

Solution:

- l^1 denote the set of all seq $\{a_n\} \Rightarrow$ we have $\sum |a_n|$ is convergent

  $\therefore |a_n| \mapsto 0$

  $\Rightarrow a_n \mapsto 0$

  $\therefore \{a_n\} \subset c_0$

  $\therefore l^1 \subset c_0$

- Let $\{a_n\} \in c_0$

  $\Leftrightarrow a_n$ is convergent to 0

  $\therefore \{a_n\}$ is bounded

  $\therefore \{a_n\} \in l^\infty$

  $c_0 \subset l^\infty$

$$\therefore l^1 \subset c_0 \subset l^\infty$$

Exercise 35.8:

$d[(x_1,x_2),(y_1,y_2)] = d_1(x_1,y_2) + d_2(x_2,y_2)$

(1)-$d(x,y) = 0 \iff x = y$ Trivial

(2)-$d(x,y) = d(y,x)$ Trivial

(3)-Triangle inequality $d[(x_1,x_2),(z_1,z_2)] = d_1(x_1,x_2) + d_2(z_1,z_2) \leq$
$d_1(x_1,y_1) + d_1(y_1,z_1) + d_2(x_2,y_2) + d_2(y_2,z_2) =$
$[d_1(x_1,y_1) + d_2(x_2,y_2)] + [d_1(y_1,z_1) + d_2(y_2,z_2)] =$
$d[(x_1,x_2),(y_1,y_2)] + d(y_1,y_2),(z_1,z_2)]$

1.2 36.$\mathbb{R}^n, l^2$

$(\mathbb{R}^n, d)$ is a metric space, $d = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ ::

Solution: To proof triangle inequality we need *Cauchy-Schwarz Inequality:*

$$\left|\sum_{k=1}^n a_k b_k\right| \leq \sqrt{\sum_{k=1}^n a_k^2 \sum_{k=1}^n b_k^2}$$

$d(x,z) \leq d(x,y) + d(y,z)$
$\Leftrightarrow \sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$
$\Leftrightarrow \sum_{i=1}^n (x_i - z_i)^2 \leq \sum_{i=1}^n (x_i - y_i)^2 + \sum_{i=1}^n (y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^n (x_i - y_i)^2 \sum_{i=1}^n (y_i - z_i)^2}$
Let $a_i = y_i - x_i$, $b_i = z_i - y_i$, $\forall i = 1, 2, 3...n$
$\Leftrightarrow z_i - x_i = z_i - y_i + y_i - x_i = b_i + a_i$
$\Leftrightarrow \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2\sum_{i=1}^n a_i b_i \leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2\sqrt{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2} \Leftrightarrow \sum_{i=1}^n b_i a_i \leq \sqrt{\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2}$ *(Cauchy-Schwarz Inequality)*

$(\mathbb{L}^n, d)$ is a metric space, $d = \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2}$ ::

Solution: To proof triangle inequality we need *Cauchy-Schwarz Inequality:*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leq \sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}$$

$d(x,z) \leq d(x,y) + d(y,z)$

$\Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$

$\Leftrightarrow \sum_{i=1}^{n}(x_i - z_i)^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2 \sum_{i=1}^{n}(y_i - z_i)^2}$

Let $a_i = y_i - x_i$, $b_i = z_i - y_i$, $\forall i = 1, 2, 3 \ldots n$

$\Leftrightarrow z_i - x_i = z_i - y_i + y_i - x_i = b_i + a_i$

$\Leftrightarrow \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2} \Leftrightarrow \sum_{i=1}^{n} b_i a_i \leq \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}$ *(Cauchy-Schwarz Inequality)*

$\Rightarrow \sqrt{\sum_{i=1}^{n}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}$

For every positive integer n take $n \mapsto \infty$, then we have:

$d(x,z) = \sqrt{\sum_{i=1}^{\infty}(x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^{\infty}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{\infty}(y_i - z_i)^2}$

$= d(x,y) + d(y,z)$

$l^1 \subset l^2 \subset l^\infty$:

Solution:

1. let $\{ a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$

   $|a_n| \mapsto 0$

   $\forall \epsilon > 0 \exists k \in \mathbb{N}$ such that:

   $a_k < \epsilon, \forall n \geq k$ *(Take $\epsilon = 1$)*

   $\Rightarrow a_k < 1$

   $\Rightarrow a_k^2 < |a_k|$

   $\sum_{k=1}^{\infty} a_k^2 < \sum_{k=1}^{\infty} a_k$

2. let $\{a_n\} \in l^2 \Rightarrow \sum_{k=1}^{\infty} a_n^2 < \infty$

   $\Rightarrow a_n$ is absolutely convergent

   $\Rightarrow a_n$ is bounded

   $\therefore a_n \in l^\infty$

   $\therefore l^2 \subset l^\infty$

Exercise 36.3:.

Solution:

- let $\{ a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$

  $|a_n| \mapsto 0$

  $\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that:

  $a_k < \epsilon$, $\forall n \geq k$ *(Take $\epsilon = 1$)*

  $\Rightarrow a_k < 1$

  $\Rightarrow a_k^2 < |a_k|$

  $\sum_{k=1}^{\infty} a_k^2 < \sum_{k=1}^{\infty} a_k$

- let $\{a_n\} \in l^2 \Rightarrow \sum_{n=0}^{\infty} a_n^2 < \infty$

  $\Leftrightarrow a_n^2 \longmapsto 0$

  $\Rightarrow \{a_n\} \in c_0 \Rightarrow l^2 \subset c_0$

- let $a_n = \frac{1}{n} \Rightarrow a_n \in l^2$, $a_n \notin l^1$

  let $b_n = \frac{1}{\sqrt{n}} \Rightarrow b_n \in c_0$, $b_n \notin l^2$

$$\boxed{l^1 \subset l^2 \subset c_0.}$$

Exercise 36.8:

Solution: Let $\{a_n\} \in l^1 \Rightarrow \sum_{k=1}^{\infty} |a_n| < \infty$

since $\{b_n\} \in l^{\infty} \Leftrightarrow |b_n| < M$

$\sum_{k=1}^{\infty} |a_n b_n| \leq \sum_{k=1}^{\infty} |a_n| M$

$= M \sum_{k=1}^{\infty} |a_n| < M.\infty = \infty$

$\Rightarrow \sum_{k=1}^{\infty} |a_n b_n|$ is convergent.

$$\boxed{\{a_n b_n\} \in l^1}$$

Exercise 36.9:

Solution: Let $\{a_n\} \in c_0 \Leftrightarrow a_n \longmapsto 0$

$\forall \epsilon > 0$, $\exists k \in \mathbb{N}$ such that $|a_n| < \epsilon_0 \ \forall n \geq k$

let $\{b_n\} \in l^\infty \Leftrightarrow |b_n| \leq M$

let $\epsilon_0 = \frac{\epsilon}{M}$

$\Rightarrow |a_n b_n| \leq M|a_n| < M\frac{\epsilon}{M} = \epsilon$

$$\boxed{\{a_n b_n\} \in c_0}$$

Give an example:

Let $a_n = \frac{1}{\sqrt{n}} \in C_0$, and let $b_n = (-1)^n \in l^\infty \Rightarrow$

$a_n b_n = \frac{(-1)^n}{\sqrt{n}} \Rightarrow \sum (a_n b_n)^2 = \sum \frac{1}{n} \notin l^2 \Rightarrow \{a_n b_n\} \notin l^2$

Exercise 36.10:

Solution: Let $\{a_n\} \in l^\infty \Leftrightarrow |a_n| \leq M$

Let $\{b_n\} \in l^\infty \Leftrightarrow |b_n| < N$, $\forall N, M \in \mathbb{R}$

$\Rightarrow |a_n b_n| \leq M.N \Rightarrow$

$$\boxed{\{a_n b_n\} \in l^\infty}$$

Give an example:

Let $\{a_n\} = (-1)^n$

Let $\{b_n\} = (-1)^{1-n} \Rightarrow$

$a_n b_n = (-1)^n (-1)^{1-n} = (-1)^{n+1-n} = -1$

$a_n b_n = -1 \Rightarrow a_n b_n \longmapsto -1$

$\{a_n b_n\} \notin c_0$

1.3 37. Sequences in Metric Spaces

Exercise 37.7:

Solution: Let $\{a_n^{(k)}\}$ be a sequence in l^1.

$a \in l^1$, $a = (a_1, a_2, a_3, ...)$

if $\{a^{(k)}\}$ convergent to a then $\lim a_j^{(k)} = a_j$, $\forall j = 1, 2, 3..,$

$|a_j^{(k)}| - |a_j| < |a_j^{(k)} - a_j| < \epsilon$, $\forall j = 1, 2, 3..,$

Let $\epsilon = 1$

$\Rightarrow |a_j^{(k)}| < 1 + |a_j| = M$

$\Rightarrow |a^{(k)}| < M$

$$\boxed{\{a^{(k)}\} \in l^\infty}$$

Exercise 37.9 (a):

Solution: $d : \mathbb{R}^n \times \mathbb{R}^n \longmapsto [0, \infty)$

1. $d(x, y) = 0 \Leftrightarrow x = y$ "Trivial"

2. $d(x, y) = d(y, x)$"Trivial"

3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$
   $\sum_{i-1}^{n} |x_i - z_i| = \sum_{i=1}^{n} |x_i - y_i + y_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + |y_i - z_i| = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = d(x, y) + d(y, z)$

Exercise 37.9 (b):

Solution: Let $\{a^{(k)}\}$be a sequence in $\mathbb{R}^n$

$d(a^{(k)}, a) < \epsilon, \forall \epsilon > 0$

" $\Rightarrow$ " Let $\{a^{(k)}\}$ convergent to a

$d(a^{(k)}, a) < \epsilon$

$d(a^{(k)}, a) = \sqrt{\sum_{j=1}^{n}(a_j^{(k)} - a_j)^2}$

Let $\epsilon_0 = \frac{\epsilon}{n}$

By Theorem: $|a_j^{(k)} - a_j| \leq \sum_{j=1}^{n}(a_j^{(k)} - a_j)^2 = d(a^{(k)}, a) < \epsilon_0 \Rightarrow$

$$\boxed{d'(a^{(k)}, a) = \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \sum_{j=1}^{n} \frac{\epsilon}{n} = \frac{\epsilon}{n} n = \epsilon}$$

" $\Leftarrow$ " Let $\{a^{(k)}\}$ convergent to a

$d'(a^{(k)}, a) = \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \epsilon_0$

$|a_j^{(k)} - a_j| < \sum_{j=1}^{n} |a_j^{(k)} - a_j| < \epsilon_0$

Let $\epsilon_0 = \frac{\epsilon}{\sqrt{n}}$

$$\boxed{d(a^{(k)}, a) = \sqrt{\sum_{j=1}^{n}(a_j^{(k)} - a_j)^2} \leq \sqrt{\sum_{j=1}^{n}(\frac{\epsilon^2}{n})} = \sqrt{\sum_{j=1}^{n} \frac{\epsilon^2}{n}} = \epsilon}$$

1.4 38. Closed Set

Exercise 38.5(a):

Prove that x is closed $\iff$ $x^{\alpha} \subseteq x$

Proof:

" $\Rightarrow$ " let x be a closed set $\Rightarrow \overline{x} = x$

$x^{\alpha} \subseteq \overline{x} \Longrightarrow x^{\alpha} \subseteq x$

" $\Leftarrow$ " Let $x^{\alpha} \subseteq x$

let a be a limit point then $\exists\{x_n\}$ such that $\lim x_n = a$

- $x_n = a$ for some n

  $\Rightarrow a \in x$

- $x_n \neq a$ for some n

  $\Rightarrow a \in x^{\alpha}$ and we suppose that $x^{\alpha} \subseteq x$

  $\Rightarrow a \in x$

$\therefore$ x is closed

Exercise 38.5(b):

Proof:

Let $x \subseteq \mathbb{R}$ and x is an infinite and bounded set then we have:

$$a_1 \in x$$
$$a_1 \neq a_2 \in x$$
$$\vdots$$
$$a_2 \neq a_k \in x$$
$$\{a_k\} \subseteq x \subseteq \mathbb{R}$$

$\exists\{a_{k_l}\}$ that convergent to a

$$\boxed{\therefore a \in x^{\alpha} \Rightarrow x^{\alpha} \neq \phi}$$

Exercise 38.5(c):

Proof: Suppose the contrary,

Let $X \subseteq \mathbb{R}$ be an uncountable and contains non of accumulation points.

$\Rightarrow \forall x \in X$, $\exists \epsilon_x > 0$ such that:

$$\nu_\epsilon(x) \cap X = \{x\}$$

$\Rightarrow \exists n \in \mathbb{N}$ such that $X^\alpha = \{x \in X : \epsilon_x > \frac{1}{n}\}$ is uncountable.

consider the family:

$$\boxed{\{(x - \frac{1}{2n}, x + \frac{1}{2n}) : x \in X^\alpha\}}$$

this is an uncountable family of pairwise disjoint open subsets of $\mathbb{R}$ which contradicts that the countable set $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

Exercise 38.13:

(a)-$\overline{X} = \overline{\overline{X}}$.

It's clear that $\overline{X} \subseteq \overline{\overline{X}}$

Now want to show that $\overline{\overline{X}} \subseteq \overline{X}$, let $a \in \overline{\overline{X}} \Rightarrow \exists \{x_n\} \in \overline{X}$such that $x_n \mapsto a$ so $\{x_n\}$ is a limit point of $X \Rightarrow \exists \{y_k\}_{k=1}^{\infty}$ is a sequence in X such that $y_k^{k_n} \mapsto x_n$.claim that $y_k^{(k_n)} \mapsto a$ as $n \mapsto \infty$.

proof the claim : let $\epsilon_0 = \frac{\epsilon}{2} > 0, d(y_k^{(k_n)}, a) \leq d(y_k^{(k_n)}, x_n) + d(x_n, a) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Rightarrow y_k^{(k_n)} \mapsto a \Rightarrow a \in \overline{X}$

(b)-$\overline{X}$ is closed in M:

Let $a \in \overline{\overline{X}}$ i.e (a is a limit point of $\overline{X}$ but $\overline{X} = \overline{\overline{X}} \Rightarrow a \in \overline{X}$

(c)-if $X \subset Y \subset M \Rightarrow \overline{X} \subset \overline{Y}$.

Let $a \in \overline{X} \Rightarrow \exists \{x_n\} \in X$ such that $x_n \mapsto a$, since $\{x_n\} \subseteq X \subset Y \Rightarrow \{x_n\} \in Y \Rightarrow a \in \overline{Y}$.

(d)-$\overline{X \cup Y} = \overline{X} \cap \overline{Y}$.

Let $a \in \overline{X \cup Y} \Rightarrow a$ is a limit point of $X \cup Y \Rightarrow \exists \{x_n\} \subset X \cup Y$ such that $x_n \mapsto a$ in Y or $x_n \mapsto a$ in $X \Rightarrow a \in \overline{X}$ or $a \in \overline{Y} \Rightarrow a \in \overline{X} \cup \overline{Y} \Rightarrow \overline{X \cup Y} \subseteq \overline{X} \cup \overline{Y}$

Now $X \subseteq X \cup Y$ and $Y \subseteq X \cup Y \Rightarrow \overline{X} \subseteq \overline{X \cup Y}$ and $\overline{Y} \subseteq \overline{X \cup Y} \Rightarrow \overline{X} \cup \overline{Y} \subseteq \overline{X \cup Y}$

(e)-) If Y is a closed subset of M such that $\overline{X} \subset Y$, then $X \subset Y$.

since Y is closed $\Rightarrow Y$ contains all limit points.

and $X \subset \overline{X} \subset Y \Rightarrow X \subset Y$.

(f)-$\overline{X} = \cap\{Y|Y$ is closed and $X \subseteq Y\}$.

*$X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y} = Y \Rightarrow \overline{X} \subseteq Y \subseteq \overline{X} \subseteq \cap Y$.

*$X \subseteq \overline{X}$ and $\overline{X}$ is closed $\Rightarrow \cap Y \subseteq \overline{X}$.

Exercise 38.14:

Let z be a limit point of $\{x_n : n \in \mathbb{N}\}$. So there is a sequence $\{z_k\}$ such that $z_k \in \{x_n : n \in \mathbb{N}\}$ for all k and $\lim_{k \mapsto \infty} z_k = z$.

Suppose for a contradiction that $z \notin \{x_n : n \in \mathbb{N}\}$. By induction on m, we define a sequence $\{a_m\}$ which is a subsequence of both $\{x_n\}$ and $\{z_k\}$. For the base case, set $a_1 = z_1 = x_n$ for some integer n. For the inductive step, suppose we have defined $a_1, ..., a_m$ and $a_m = z_k = x_n$. Note the set $\{z_{k+1}, z_{k+2}, ...\}$ is infinite for otherwise some x_j appears in this set an infinite number of times, contradicting the fact that $\lim_{k \mapsto \infty} z_k = z \neq x_j$. Since $x_1, x_2, ...$ is an enumeration of $\{x_n : n \in P\}$, and since the set $\{z_{k+1}, z_{k+2}, ...\}$ is infinite but $\{x_1, ..., x_n\}$ is finite, there exists some $n' > n$ such that $x_{n'} = z_{k'}$ for some $k' > k$. Set $a_{m+1} = z_{k'} = x_{n'}$. Note that $\{a_m\}$ is a subsequence of both $\{z_k\}$ and $\{x_n\}$. Since $\{z_k\}$ converges, so does $\{a_m\}$, contradicting the assumption that $\{x_n\}$ has no convergent subsequence.

Prove that $B_\epsilon(x)$is open set:

Proof: Let $y \in B_\epsilon(x)$, want to find $\delta > 0$ such that:

$B_\delta(y) \subseteq B_\epsilon(x)$

consider $\delta = \epsilon - d(x, y) > 0$

$\Rightarrow d(x, y) < \epsilon \Rightarrow \epsilon - d(x, y) > 0$

Let $z \in B_\delta(y) \Rightarrow d(z, y) < \delta$

$\Rightarrow d(z, y) < \epsilon - d(x, y)$

$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \epsilon$

$= d(x, y) + \epsilon - d(x, y) = \epsilon$

$\therefore d(x, z) < \epsilon \Rightarrow z \in B_\epsilon(x) \Rightarrow B_\delta(y) \subseteq B_\epsilon(x)$

so $B_\epsilon(x)$ is an open set of M

1.5 39. Open Set

Exercise 39.9::

Proof:We want to show that X is open subset of $M \iff X = \bigcup B_\epsilon(x)$, $\forall x \in X$.

"$\Rightarrow$"

suppose that X is open, then by definition $\forall x \in X, \exists \epsilon > 0$ such that $B_\epsilon(x) \subset X$

since $x \in X \Rightarrow X = \bigcup_{x\in X}\{x\} \subset \bigcup B_\epsilon(x) \subset X \Longrightarrow X = \bigcup B_\epsilon(x)$

"$\Leftarrow$"

Let $X = \bigcup B_\epsilon(x), \forall x \in X$, but each ball is open and by theorem 39.6(ii) $\Rightarrow X =$ union of open sets $\Rightarrow X$ is open.

Exercise 39.10:

Proof: "$\Rightarrow$"

Suppose X is closed $X = \overline{X}$, so let $a \in M$ such that $B_{\frac{1}{k}}(a) \cap X \neq \phi$ pick $X_k \in B_{\frac{1}{k}} \cap X$

we have $\{x_k\}_{k=1}^{\infty}$ is a sequence in X and $x_k \in B_{\frac{1}{k}}(a), d(x_k, a) < \frac{1}{k}$

$x_k \mapsto a$ as $k \mapsto \infty$, so a is a limit point of x

$a \in \overline{X} \Rightarrow a \in X$.

"$\Leftarrow$"

Let $a \in M$ such that if $B_\epsilon(\alpha) \cap X \neq \phi, \forall \epsilon > 0 \Rightarrow \alpha \in X$

Let α be a limit point of $X \Rightarrow \exists\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \mapsto \alpha$, so $\forall \epsilon > 0 \exists k \in \mathbb{N}$ such that $d(x_n, \alpha) < \epsilon \Rightarrow x_n \in B_\epsilon(\alpha) \cap X, \forall n \geq k$

$\Rightarrow B_\epsilon(\alpha) \cap X \neq \phi$ and $\alpha \in X \Rightarrow X$ is closed.

Exercise 39.11:

(a)-$X^0 \subset X$ for $X \subset M$.

Let $x \in X^0, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq X \Rightarrow x \in X$.

(b)-X is open $\iff$ $X^0 = X$.

"$\Rightarrow$" Let X be an open subset of $M \iff X = \bigcup B_\epsilon(x) \Rightarrow X^0 = X$

"$\Leftarrow$" Let $X^0 = X \Rightarrow \forall x \in X, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq X \Rightarrow X$ open.

(c)-$(X^0)^0 = X^0$.

"$\Rightarrow$"Let $x \in (X^0)^0 \Rightarrow B_\epsilon(x) \subseteq X^0 \Rightarrow (X^0)^0 \subseteq X^0$

"$\Leftarrow$"Let $x \in X^0 \Rightarrow B_\epsilon(x) \subset X \Rightarrow x$ is interior point of $X^0 \Rightarrow x \in B_{\frac{\epsilon}{2}}(x)$ $\Rightarrow x \in (X^0)^0$ so $X^0 \subseteq (X^0)^0$

$X^0 = (X^0)^0$.

(d)-X^0 is open for all $X \in M$.

Let $x \in X^0 \Rightarrow B_\epsilon(x) \subseteq X$, by definition the union of open set is open$\Rightarrow$ $\bigcup B_\epsilon(x) = X^0$ is open.

(e)- if $X \subset Y \subset M$ then $X^0 \subset Y^0$, Proof:

Let $x \in X^0 \Rightarrow B_\epsilon(x) \subseteq X \subset Y$, since $X \subset Y \Rightarrow \exists x \in X$ then $x \in Y$ and $B_\epsilon(x) \subset Y \Rightarrow x \in Y^0, \Rightarrow X^0 \subset Y^0$

(f)-$X^0 \cap Y^0 = (X \cap Y)^0$.

"$\Rightarrow$" Let $a \in (X \cap Y)^0 \Rightarrow B_\epsilon(a) \subseteq X \cap Y \Rightarrow B_\epsilon(a) \subseteq X$ and $B_\epsilon(a) \subseteq Y \Rightarrow a \in X^0$ and $a \in Y^0 \Rightarrow a \in X^0 \cap Y^0 \cdots (1)$

"$\Leftarrow$" Let $a \in X^0 \cap Y^0 \Rightarrow a \in X^0$ and $a \in Y^0 \Rightarrow B_\epsilon(a) \subseteq X$ and $B_\epsilon(a) \subseteq Y \Rightarrow B_\epsilon(a) \subset X \cap Y \Rightarrow a \in (X \cap Y)^0 \cdots (2)$

from (1) and (2) we have $X^0 \cap Y^0 = (X \cap Y)^0$

(g)-If Y is an open subset of M such that $Y \subset X \subset M$, then $Y \subset X^0$.

Let $Y \subset X$ and Y be an open $\Rightarrow .\forall y \in Y, \exists \epsilon > 0$ such that $B_\epsilon(y) \subseteq Y$, since $y \in Y \subset X \Rightarrow y \in X$ and $B_\epsilon(y) \subset X \Rightarrow y \in X^0 \Longrightarrow Y \subset X^0$

(h)- If $X \subset M$, then $X^0 = \cup\{Y|Y \subset X$ and Y is open$\}$.

since X^0 is open then $X^0 \subseteq X$ and we know that $X^0 \subseteq \cup\{Y|Y \subset X$ and Y is open $\}$. Now let $y \in Y \Rightarrow y \in \cup Y$, since Y is open $\Rightarrow \forall y \in Y, \exists \epsilon > 0$ such that $B_\epsilon(y) \subseteq Y \subseteq \cup Y$ and $\cup Y \subset X \Rightarrow B_\epsilon(y) \subseteq X \Rightarrow y \in X^0$.

(i)-$\overline{X^c} = (X^0)^c$ for all $X \subset M$.

Let $x \in \overline{X^c} \Rightarrow \exists\{x_n\} \subset X^c$ such that $x_n \mapsto x, \forall \epsilon > 0, \exists x_k \subset X^c$ such that $d(x_k, x) < \epsilon$ that mean $\forall B_\epsilon(x)$ you will find $x_k \nsubseteq X \Rightarrow a \notin X^0 \Rightarrow a \in (X^0)^c \Rightarrow \overline{X^c} \subseteq (X^0)^c$.

now let $x \in (x^0)^c \Rightarrow a \notin X^0 \Rightarrow$ for any ball around $x, \epsilon = \frac{1}{n}, \forall n = 1, 2, 3.., \exists x_n \notin X (x_n \in X^c)$ and $x_n \mapsto x \Rightarrow x \in X^c \Rightarrow (X^0)^c \subseteq \overline{X^c}$

Exercise 39.12:

Proof: $\delta X = \overline{X} \cap \overline{X^c}$

(a)-δX is closed

since δX is equal of union of closed set then δX closed.

(b)-$X \cup \delta X = \overline{X}$

- $X \subset \overline{X}$ and $\delta X \subset \overline{X} \Rightarrow X \cup \delta X \subseteq \overline{X}$.
- Now let $a \in \overline{X} \Rightarrow$ if $a \in X$ we are done, otherwise $a \in X^c$ and $X^c \subseteq \overline{X^c} \Rightarrow a \in \overline{X^c} \Rightarrow a \in \delta X \Rightarrow \overline{X} \subseteq X \cup \delta X$.

(c)-X except $\delta X = X^0$

- Let $a \in X$ except $\delta X \Rightarrow a \in X$ and $a \notin \delta X$, since $X \subseteq \overline{X} \Rightarrow a \in \overline{X}$, by theorem: $\boxed{X^0 \cap \delta X = \phi \Rightarrow X^0 \cup \delta X = \overline{X}}$ and $a \in X, a \notin \delta X \Rightarrow X^0 \cap \delta X = \phi$,so $\overline{X} = X^0 \cup \delta X$ and $a \notin \delta X \Rightarrow a \in X^0 \Rightarrow X$ except $\delta X \subseteq X^0$
- Now if $a \in X^0$ and $X^0 \subseteq X \Rightarrow a \in X$ and since $X^0 \cap \delta X = \phi$, since $a \in X^0 \Rightarrow a \notin \delta X$ therefore $a \in X$ and $a \notin \delta X \Rightarrow a \in X$ except $\delta X \Rightarrow X^0 \subseteq X$ except δX.

(d)-If X is a proper nonempty subset of $\mathbb{R}$, then $\delta X \neq \phi$.

suppose the contrary: $X \neq \phi, X \notin \mathbb{R}^n$ and $\delta = \phi$ since $\overline{X} = X^0 \cup \delta X \Rightarrow \overline{X} = X^0$ since $\delta X = \phi$ but X^0 is open and $\overline{X}$ is closed $\Rightarrow$ contradiction so $\delta X \neq \phi$.

1.6 40. Continuous Functions on Metric Spaces

Exercise 40.6:

Proof: Let $f(x) = c$, f is continuous $\iff \forall \epsilon > 0, \exists \delta > 0$ such that:
if $d_1(x, y) < \delta \Rightarrow d_2(f(x), f(y)) < \epsilon, \forall x, y \in M$
$d_2(f(x), f(y)) = d_2(c, c) = 0 < \epsilon$ so f is continuous.

Exercise 40.7:

Proof:

- $(a) \Rightarrow (b)$

  suppose that f is continuous at a, let U be subset of M_2 containing $f(a)$ be given.since $f(a)$ is continuous$\Rightarrow \forall \epsilon > 0, \exists \delta > 0$ such that $d(x, a) < \delta \Rightarrow d(f(x), f(a)) < \epsilon$, and $B_\epsilon(f(a))$ containing U Take $v := B_\delta(a)$so by theorem:

  > Theorem 39.4: Let M be a metric space. Let $x \in M$ and let $\epsilon > 0$. Then the open ball $B_\epsilon(x)$ is an open subset of M.

  $a \in B_\delta(a)$ and $f(B_\delta(a) \subset B_\epsilon(f(a)) \subset U \Rightarrow B_\delta(a) \subset f^{-1}(U)$

- $(b) \Rightarrow (a)$

  suppose that U is an open subset of M_2 which contains $f(a)$, there exists an open subset V of M_1 which contains a such that contained $f^{-1}(U)$
  Given an arbitrary $\epsilon > 0$, let $U := B_\epsilon(f(a))$. By Theorem 39.4 U is open, so there exists an open subset V containing a contained in $f^{-1}(B_\epsilon(f(a)))$. Since V is open,there exists $\delta > 0$ such that $B_\delta(a) \subset V$. Then:

  $$B_\delta(a) \subset V \subset f^{-1}(B_\epsilon(f(a)))$$

  so for all $x \in M_1$ with $d_1(x, a) < \delta$ we have that $d_2(f(x), f(a)) < \epsilon$. Thus, f is continuous at a.

Exercise 40.8:

Proof: The generalized statement is that if $f_1, ..., f_n$ are continuous functions from $\mathbb{R}^m$ into $\mathbb{R}$.

$h(x) = (f_1, f_2 ... f_i) : \mathbb{R}^m \longmapsto \mathbb{R}^n$, so We prove this generalized statement, which in particular proves the case $m = 1$ and $n = 2$.

let $a \in \mathbb{R}^m$, since f is continuous function for all $i = 1, 2, ...n$.

Definition 40.1: Definition 40.1: Let (M_1, d_1) and (M_2, d_2) be metric spaces, let , and let f be a function from M_1 into M_2. We say that f is continuous at a if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d_1(x, a) < \delta$, then $d_2(f(x), f(a)) < \epsilon$. We say that f is continuous on M_1 if f is continuous at every point of M_1.

$\Longrightarrow \exists \delta_i$ such that if $d(x, a) < \delta_i \Rightarrow d(f_i(x), f_i(a)) < \sqrt{\frac{\epsilon^2}{n}}$ for all $i \Longrightarrow$

$$d(h(x), h(a)) = \sqrt{\sum_{i=1}^{n} |f_i(x) - f_i(a)|^2} < \sqrt{\sum_{i=1}^{n} \frac{\epsilon^2}{n}} = \epsilon$$

Hence h is a continuous function from $\mathbb{R}^m$ into $\mathbb{R}^n$.

Exercise 40.10:

proof Let $\epsilon > 0$ be given

$\Rightarrow \forall \epsilon > 0$, $\exists \delta > 0$ such that:

$d_1(b_n, c_n) < \delta$ whenever $d_2(f(b_n), f(c_n)) < \epsilon$

Let $\{b_n\} \in l^1$ since $\{a_n\} \in l^\infty \Rightarrow |a_n| \leq M$

Let $\{c_n\} \in l^1 \Rightarrow d(\{b_n\}, \{c_n\}) < \delta$

$\sum |b_n - c_n| < \delta$, Let $\delta = \frac{\epsilon}{M}$

$|f(c_n) - f(b_n)| = |\sum a_n c_n - \sum a_n b_n|$

$\leq \sum |a_n||c_n - b_n| < M \frac{\epsilon}{M} = \epsilon$

Exercise 40.11:

proof: Let $\{a_n\} \in l^2 \iff \sqrt{\sum_{n=1}^{\infty} a_n^2} < \epsilon$

want to show that f is continuous at $c = \{c_n\}$ and $b = \{b_n\}$

$\forall \epsilon > 0$, $\exists \delta > 0$ such that:

$|c_n - b_n| < \delta$ whenever $|f(c_n) - f(b_n)| < \epsilon$

$|f(c_n) - f(b_n)| = |\sum_{n=1}^{\infty} c_n a_n - \sum_{n=1}^{\infty} b_n a_n| = |\sum_{n=1}^{\infty}(a_n)(c_n - b_n)|$

$$\leq \sqrt{\sum_{n=1}^{\infty} a_n^2}\sqrt{\sum_{n=1}^{\infty}(c_n - b_n)^2}$$

$$Let : \delta = \frac{\epsilon}{\sqrt{\sum_{n=1}^{\infty} a_n^2}}$$

$$= d(c_n, b_n)\sqrt{\sum_{n=1}^{\infty} a_n^2} < \frac{\epsilon}{\sqrt{\sum_{n=1}^{\infty} a_n^2}}\sqrt{\sum_{n=1}^{\infty} a_n^2} = \epsilon$$

Exercise 40.15:

Proof: suppose that f is continuous. Note that $(-\infty, c)$ and (c, ∞) are open subsets of $\mathbb{R}$. Hence $\{x : f(x) < c\} = f^{-1}((-\infty, c))$ and $\{x : f(x) > c\} = f^{-1}((c, \infty))$ are open in M by Theorem

> Theorem 40.5: Let f be a function from a metric space M_1 into a metric space M_2.The following are equivalent:
>
> 1. f is continuous on M_1.
> 2. $f^{-1}(C)$ is closed whenever C is a closed subset of M_2.
> 3. $f^{-1}(U)$ is open whenever U is an open subset of M_2. f is continuous.

Conversely, suppose the sets $\{x : f(x) < c\}$ and $\{x : f(x) > c\}$ are open in M for every $c \in \mathbb{R}$. any open subset U of $\mathbb{R}$ can be written as the union of open balls $U = \cup_\alpha \in A(a_\alpha, b_\alpha)$, where A is an arbitrary indexing set. Note $(a_\alpha, b_\alpha) = (-\infty, b_\alpha) \cup (a_\alpha, \infty)$ and $f^{-1}((a_\alpha, b_\alpha)) = f^{-1}((-\infty, b_\alpha)) \cup f^{-1}((a_\alpha, \infty)) = \{x : f(x) < b_\alpha\} \cap x : f(x) > a_\alpha$. Since the intersection of any two open sets is open, each set $f^{-1}((a_\alpha, b_\alpha))$ is open. Since the arbitrary union of open sets is open, the set $f^{-1}(U) = \cap_{\alpha \in A} f^{-1}((a_\alpha, b_\alpha))$ is open. Hence by Theorem 40.5(iii), f is continuous.

1.7 42.Compact Metric Space

Exercise 42.1:

- $\mathbb{R}^n$: let $U_k = \{B_{(k)}\}_{k=1}^{\infty}$ since U_k is the open ball of radius k, centred at 0.
  so $\mathbb{R}^n \subseteq \bigcup_{k=1}^{\infty}\{U_k\}$
  but there is no subcover U_k^* such that $\bigcup_{k=1}^{\infty} U_k^* = \mathbb{R}^n$

- we know that $l^1 \subset l^2 \subset c_0 \subset l^\infty$, so it To show that the set is not compact if M is l^2, c_0, or l^∞: take

$$\delta^{(1)} = \{1, 0, 0, 0...\}$$

$$\delta^{(2)} = \{0, 1, 0, 0...\}$$

$$\vdots$$

$$\delta^{(k)} = \{0, 0, 0, 0.., 1, ..\}$$

  so we have: $\delta_n^{(k)} = \begin{cases} 1, n = k \\ 0, n \neq k \end{cases}$
  note that $\{\delta^{(k)}\}_{k=1}^{\infty}$ is a sequence of points in l^2, c_0, or l^∞ that has no convergent subsequence. Therefore l^2, c_0, and l^∞ are not compact. By Theorem 43.5.

  > Let M be a metric space. Then M is compact if and only if every sequence in M has a convergent subsequence.

Exercise 42.2:

Proof: To show that X is closed, it suffices to show the complement X^c of X is open.

> Theorem: Let M be a metric space ,$X \subseteq M$, then X is closed $\iff$ X^c is open.

Let $x \in X$ and $y \in X^c$, since $x \neq y \Rightarrow d(x, y) = r$
consider the family:

$$x \in U_x = \{B_{\frac{r}{2}}(x)\}$$

$$y \in V_y = \{B_{\frac{r}{2}}(y)\}$$

and $U_x \cap V_y = \phi$, since $x \in X \Rightarrow X = \bigcup_{i=1}^{n}\{x_i\} \subset \bigcup_{i=1}^{n} U_{x_i}$

> Definition: Let M be a metric space, we say that $U_x \subset M$ is open in M if $\forall x \in U_x, \exists \epsilon = \frac{r}{2} > 0$, such that $B_{\frac{r}{2}} \subset U_x$

so U_x is open.
since X is compact, we have finite subcover, $\exists x_1, x_2 ... x_n \in X \subset \bigcup_{i=1}^{n} U_{x_i}$
since $U_x \cap V_y = \phi \Longrightarrow$

$$\left(\bigcup_{i=1}^{n} U_{x_i}\right) \cap \left(\bigcap_{i=1}^{n} V_{y_i}\right) = \phi$$

> Theorem: Let M be a metric space, if $V_{y_1}, V_{y_2} ... V_{y_n}$ are open set $\Rightarrow \bigcap_{i=1}^{n} V_{y_i}$ is open.

so $V = \bigcap_{i=1}^{n} V_{y_i}$is open.
so for every $y \in X^c, \exists$ an open set V such that $y \in V \subset X^c$, Hence X^c is open$\Rightarrow X$ is closed.

Exercise 42.3:

Proof:

- since $U_k = \{x_k\}_{k=1}^n$ be a finite collection of compact subset of a metric space M, then for all $x_1, x_2, ..., x_n$ there is a finite subcover U^* of $\{x_k\}_{k=1}^n$, so $\bigcup_{k=1}^n U_k$ there exists subcover $\bigcup_{k=1}^n U_k^*$ so $x_1 \cup x_2 \cup ... \cup x_n$ is compact.
- Let $U = \{(n, n + \frac{3}{2}) : \forall n \in \mathbb{N}\}$ there is no finite subcover so U is not compact.

Exercise 42.6:

Proof: $f : M \longmapsto \mathbb{R}$, By corollary:

> Corollary 42.7 If f is a continuous real-valued function on a compact metric space M, there exist $c, d \in M$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in M$.That is, f attains a maximum and a minimum on M.

then f has an infimum value, let $x_0 \in M$ such that $f(x) \geq f(x_0) > 0$, so let $T = \frac{f(x_0)}{2}$ and $f(x) > T > 0$ for all $x, x_0 \in M$.

Exercise 42.12:

Proof: By definition:

A contraction mapping, on a metric space (M, d) is a function f from M to itself, with the property that there is some non negative real number $0 \leq k < 1$,such that for all x and y in M, $d(f(x), f(y)) \leq k\, d(x, y)$.

- consider the function $g(x) = d(f(x), x)$ want to show that $g(x)$ is continuous:(By triangle inequality) we have:
  $d(f(x), x) - d(f(y), y) \leq (d(x, y) + d(y, f(x))) - (d(y, f(x)) + d(f(x), f(y))) = d(x, y) - d(f(x), f(y)) < 2d(x, y)$
  as similar we have $d(f(y), y) - d(f(x), x) < 2d(x, y)$
  $\Rightarrow |d(f(x), x) - d(f(y), y)| < 2d(x, y)$, $\forall \epsilon > 0, \exists \delta > 0$ such that:
  $d(x, y) < \delta$, whenever $d(f(x), f(y)) < \epsilon$ so let $\delta = \frac{\epsilon}{2} \Rightarrow |d(f(x), x) - d(f(y), y)| < 2d(x, y) < 2\delta = 2\frac{\epsilon}{2} = \epsilon$
  so $g(x)$ continuous function.

- since $g(x)$ continuous and compact function $\Rightarrow g(x)$ has a minimum value.
  let c be a minimum value, so $d(f(x_0), x_0) = c$
  suppose the contrary, $(f(x_0) \neq x_0) \Rightarrow c > 0$
  $\Rightarrow d(f(f(x_0)), f(x_0)) < d(f(x_0), x_0) = c$ "contradiction"
  so $f(x_0) = x_0$

- To show that $f(x) = x$ is unique:
  suppose the contrary, let $x \neq y$, $\forall x, y \in M$ such that: $f(x) = x, f(y) = y$, then $d(f(x), f(y)) < f(x, y)$
  but $f(x) = x$ and $f(y) = y$
  so $d(f(x), f(y) = d(x, y)$ "contradiction"

1.8 43.The Bolzano-Weierstrass Characterization

Exercise 43.1:

Proof:

- Want to show that the set $\{x \in M : d(x,0) = 1\}$ is closed: by theorem 40.3, let $f(x) = d(x,0) =$ is cont on M and $f^{-1}(\{1\}) = \{x \in M : d(x,0) = 1\}$ is continuous preimage of a closed set, so $f(x)$ is closed by theorem:

  Theorem 40.5: Let f be a function from a metric space M_1 into a metric space M_2. The following are equivalent:

  (i) f is continuous on M_1.

  (ii) $f^{-1}(C)$ is closed whenever C is a closed subset of M_2.

- Want to show that the set $\{x \in M : d(x,0) = 1\}$ is bounded:
  let $y, z \in M$ so $d(y,z) \leq d(y,0) + d(0,z) = 2$, so $d(y,z) \leq 2$, $\forall y, z \in M$ so by definition43.6.

- To show that the set is not compact if M is l^2, c_0, or l^∞: take

$$\delta^{(1)} = \{1, 0, 0, 0...\}$$

$$\delta^{(2)} = \{0, 1, 0, 0...\}$$

$$\vdots$$

$$\delta^{(k)} = \{0, 0, 0, 0.., 1, ..\}$$

so we have: $\delta_n^{(k)} = \begin{cases} 1, n = k \\ 0, n \neq k \end{cases}$

note that $\{\delta^{(k)}\}_{k=1}^{\infty}$ is a sequence of points in l^2, c_0, or l^∞ that has no convergent subsequence. Therefore l^2, c_0, and l^∞ are not compact. By Theorem 43.5.

Let M be a metric space. Then M is compact if and only if every sequence in M has a convergent subsequence.

Exercise 43.4:

Proof: consider continuous function:

$$d : M \times M \longmapsto \mathbb{R} : (a_1, a_2) \longmapsto d(a_1, a_2)$$

Corollary 42.7: If f is a continuous real-valued function on a compact metric space M, there exist $c, d \in M$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in M$. That is, f attains a maximum and a minimum on M.

so, since d defined on compact $M \times M$ then d has a maximum value.
Let $D = diam(M) = lup\{d(x, y) : \forall x, y \in M$
By definition of supremum $\exists \{x_n\}, \{y_n\} \subset M$ such that:
$\lim_{n \mapsto \infty} d(x_n, y_n) = lup\{d(x_n, y_n)\}$.
since (M, d) is compact then we have a subsequence $\{(x_{n_k}, y_{n_k}) : \forall k \in \mathbb{N}\}$ is convergent to some $(a_1, a_2) \in M \times M \Longrightarrow$

$$diam(M) = D = \lim_{n \mapsto \infty} d(x_n, y_n) = \lim_{n \mapsto \infty} d(x_{n_k}, y_{n_k})$$

$$= d\left(\lim_{n \mapsto \infty} x_{n_k}, \lim_{n \mapsto \infty} y_{n_k}\right) = d(a_1, a_2)$$

Chapter 2

IX.The Riemann-Stieltjes Integral

2.1 51.Riemann-Stieltjes Integration with Respect to an Increasing Integrator

Exercise 51.2:

Solution: Let $p = \{x_0, x_1, \cdots, x_n\}$, suppose S has one more point than p such that $S = \{p \cup \{x^*\} | x^* \notin p\}$, $x^* \in [x_{i-1}, x_i]$.

Let $m^* = inf\{f(x)\ , x_{i-1} \leq x \leq x^*\}$, $m^{**} = inf\{f(x)\ , x^* \leq x \leq x_i\}$.

So $m_i \leq m^*$ and $m_i \leq m^{**}$.

Now $L(f, p) = \sum_{i=1}^{n} m_i \Delta\alpha_i$

$= m_1\Delta\alpha_1 + \cdots + m_i(\alpha(x_i) - \alpha(x_{i-1})) + \cdots + m_n\Delta\alpha_n$

$= m_1\Delta\alpha_1 + \cdots + m_i(\alpha(x_i) - \alpha(x^*) + \alpha(x^*) - \alpha(x_{i-1})) + \cdots + m_n\Delta\alpha_n$

$= m_1\Delta\alpha_1 + \cdots + m_i(\alpha(x_i)\alpha(x^*)) + m_i(\alpha(x^*) - \alpha(x_{i-1})) + \cdots + m_n\Delta\alpha_n$

$\leq = m_1\Delta\alpha_1 + \cdots + m^*(\alpha(x_i)\alpha(x^*)) + m^{**}(\alpha(x^*) - \alpha(x_{i-1})) + \cdots + m_n\Delta\alpha_n = L(f, S)$. So $L(f, p) \leq L(f, S)$.

Exercise 51.3:

Solution:

" $\Longrightarrow$ "Let $f \in \mathscr{R}_\alpha[a,b] \iff \forall \epsilon > 0,\ \exists P$ partition of $[a,b]$ such that:

$$U(f,P) - L(f,P) < \epsilon$$

Now we know that $L(f,P) \leq U(f,P) \Rightarrow 0 \leq U(f,P) - L(f,P)$, so

$$U(f,P) - L(f,P) < \epsilon_1$$

$$U(f,S) - L(f,S) < \epsilon_2$$

Since $0 \leq U(f,P) - L(f,P) \Rightarrow$
$U(f,P) - L(f,S) < [U(f,P) - L(f,S)] + [U(f,S) - L(f,P)] < \epsilon_1 + \epsilon_2 = \epsilon$

$$U(f,P) - L(f,S) < \epsilon.$$

" $\Longleftarrow$ " Let that there exist partitions P and S of $[a,b]$ such that:

$$U(f,P) - L(f,S) < \epsilon$$

Let $T = S \cup P$ so:

$$U(f,T) \leq U(f,P) \cdots (1)$$

$$(L(f,T) \geq L(f,S)) - 1$$

$$-L(f,T) \leq -L(f,S) \cdots (2)$$

From 1 and 2 we have $U(f,T) - L(f,T) \leq U(f,P) - L(f,S) < \epsilon \Rightarrow$

$$U(f,T) - L(f,T) < \epsilon \Rightarrow f \in \mathscr{R}_\alpha[a,b].$$

Exercise 51.4:

Solution: Let P be a partition of $[a, b]$, let $P^* = P \cup \{c\}$,

and $P_1 = P^* \cap [a, c]$ $P_2 = P^* \cap [c, b]$

$$U(f, P) = \sum_{k=1}^{n} M_k \Delta\alpha_k =$$

$$M_1 \Delta\alpha(x_1) + \cdots + M_k[\alpha(x_k) - \alpha(x_{k-1})] + \cdots + M_n \alpha(x_n) =$$

$$M_1 \Delta\alpha(x_1) + \cdots + M_k[\alpha(x_k) - \alpha(c) + \alpha(c) - \alpha(x_{k-1})] + \cdots + M_n \alpha(x_n) =$$

$$M_1 \Delta\alpha(x_1) + \cdots + M_k[\alpha(x_k) - \alpha(c)] + M_k[\alpha(c) - \alpha(x_{k-1})] + \cdots + M_n \alpha(x_n) =$$

$$\sum_{i=1}^{k} M_i \Delta\alpha(x_i) + \sum_{i=k}^{n} M_i \Delta\alpha(x_i) =$$

$$U(f, P_1) + U(f, P_2) \Longrightarrow$$

$$\overline{\int_a^b} f d\alpha = \overline{\int_a^c} f d\alpha + \overline{\int_c^b} f d\alpha.$$

Let P be a partition of $[a, b]$, let $P^* = P \cup \{c\}$,

and $P_1 = P^* \cap [a, c]$ $P_2 = P^* \cap [c, b]$

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta\alpha_k =$$

$$m_1 \Delta\alpha(x_1) + \cdots + m_k[\alpha(x_k) - \alpha(x_{k-1})] + \cdots + m_n \alpha(x_n) =$$

$$m_1 \Delta\alpha(x_1) + \cdots + m_k[\alpha(x_k) - \alpha(c) + \alpha(c) - \alpha(x_{k-1})] + \cdots + m_n \alpha(x_n) =$$

$$m_1 \Delta\alpha(x_1) + \cdots + m_k[\alpha(x_k) - \alpha(c)] + m_k[\alpha(c) - \alpha(x_{k-1})] + \cdots + m_n \alpha(x_n) =$$

$$\sum_{i=1}^{k} m_i \Delta\alpha(x_i) + \sum_{i=k}^{n} m_i \Delta\alpha(x_i) =$$

$$L(f, P_1) + L(f, P_2) \Longrightarrow$$

$$\underline{\int_a^b} f d\alpha = \underline{\int_a^c} f d\alpha + \underline{\int_c^b} f d\alpha.$$

Exercise 51.5:

Solution: f is bounded function on $[a,b]$ and α increasing on $[a,b]$. We want to show that:

$$\overline{\int}_a^b f d\alpha = -\underline{\int}_a^b (-f) d\alpha$$

Let $p = \{x_0, x_1 ... x_n\}$ be any partion of $[a,b]$.

We know that $\overline{\int}_b^a f d\alpha = infU(f,p)$, so $\overline{\int}_b^a f d\alpha \leq U(f,p)$

And $\underline{\int}_a^b (-f) d\alpha = supL(-f,p)$, so $\underline{\int}_a^b (-f) d\alpha \geq L(-f,p)$.

Claim-1: $\overline{\int}_a^b f d\alpha \geq -\underline{\int}_a^b (-f) d\alpha$

Proof the claim: we know that $\left(\overline{\int}_a^b f d\alpha \geq \underline{\int}_a^b f d\alpha\right) \times -1 \Rightarrow$
$\left(-\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b f d\alpha = \underline{\int}_a^b - f d\alpha\right) \times -1 \Rightarrow$
$\overline{\int}_a^b f d\alpha \geq -\underline{\int}_a^b - f d\alpha$.

Claim-2: $\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b (-f) d\alpha$

Proof the claim: we know that $\left(\overline{\int}_a^b - f d\alpha \geq \underline{\int}_a^b - f d\alpha\right) \times -1 \Rightarrow$
$\overline{\int}_a^b f d\alpha = \overline{\int}_a^b - (-f) d\alpha = -\overline{\int}_a^b - f d\alpha \leq -\underline{\int}_a^b - f d\alpha \Rightarrow$
$\overline{\int}_a^b f d\alpha \leq -\underline{\int}_a^b (-f) d\alpha$.

---

so from claim 1 and claim 2 we have $\overline{\int}_a^b f d\alpha = -\underline{\int}_a^b (-f) d\alpha$, so we are done.

Exercise 51.6:

- $\int_0^1 x dx = \frac{1}{2}$

  By Theorem 51.14 *(Mean-Value Theorem of the Integral)* since $f(x) = x$ is continuous on $[0, 1]$ and $\alpha(x) = x$ is increasing on $[0, 1]$, there exists $c = \frac{1}{2} \in (0, 1)$ such that:

  $$\int_0^1 x dx = f\left(\frac{1}{2}\right)[\alpha(1) - \alpha(0)] = \frac{1}{2}[1 - 0] = \frac{1}{2}.$$

- $\int_0^1 x^2 dx = \frac{1}{3}$

  By Theorem 51.14 *(Mean-Value Theorem of the Integral)* since $f(x) = x^2$ is continuous on $[0, 1]$ and $\alpha(x) = x$ is increasing on $[0, 1]$, there exists $c = \frac{1}{\sqrt{3}} \in (0, 1)$ such that:

  $$\int_0^1 x^2 dx = f\left(\frac{1}{\sqrt{3}}\right)[\alpha(1) - \alpha(0)] = \frac{1}{3}[1 - 0] = \frac{1}{3}.$$

Exercise 51.7:

Solution: Let $\alpha(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x + 2 & 1 < x \leq 2 \end{cases}$

So since $c = 1 \in [0, 2]$, $f(x) = x$ is continuous on $[0, 2]$ and $\alpha(x)$ increasing on $[0, 2] \Rightarrow \int_0^2 x d\alpha = f(c)[\alpha(b) - \alpha(a)] = 1[4 - 0] = 4.$

Exercise 51.12:

(a)-Let $p = \{x_0, x_n\}$ be a partition of $[0, 2]$.

$U(f, p) = \sum_{k=1}^{2} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{2} \Delta\alpha_k$

$= x_n - x_0 = 1 - 0 = 1$

and $L(f, p) = \sum_{k=1}^{2} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{2} \Delta\alpha_k = 0$, since $U(f, p) \neq L(f, P) \Rightarrow f \notin \mathscr{R}_\alpha[0, 2]$.

(b)-Let $s = \{x_0, x_n\}$ be a partition of $[0, 2]$.

$U(g, s) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{2} \Delta\alpha_k$

$= x_n - x_0 = 1 - 0 = 1$

and $L(g, s) = \sum_{k=1}^{2} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{2} \Delta\alpha_k = 0$ since $U(g, s) \neq L(g, s) \Rightarrow g \notin \mathscr{R}_\alpha[0, 2]$.

(c)-Let $p = \{x_0, x_n\}$ be a partition of $[0, 2]$, $\alpha(x) = x$.

$U(f, p) = \sum_{k=1}^{2} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{2} \Delta\alpha_k$

$= x_n - x_0 = 2 - 0 = 2$

and $L(f, p) = \sum_{k=1}^{2} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{2} \Delta\alpha_k = 0$, since $U(f, p) \neq L(f, p) \Rightarrow f \notin \mathscr{R}_\alpha[0, 2]$.

Let $s = \{x_0, x_n\}$ be a partition of $[0, 2]$, $\alpha(x) = x$.

$U(g, s) = \sum_{k=1}^{n} M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^{2} \Delta\alpha_k$

$= x_n - x_0 = 2 - 0 = 2$

and $L(g, s) = \sum_{k=1}^{2} m_k \Delta\alpha_k$, and $m_k = 0$, so $0 \sum_{k=1}^{2} \Delta\alpha_k = 0$, since $U(g, s) \neq L(g, s) \Rightarrow g \notin \mathscr{R}_\alpha[0, 2]$.

Exercise 51.18:

Solution:

- Let $\alpha(x) = x$ and $f(x) = \begin{cases} -1, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

  Now $\overline{\int}_a^b f dx = \sum_{k=1}^n M_k \Delta\alpha_k$, and $M_k = 1$, so $\sum_{k=1}^n \Delta\alpha_k$
  $= x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1} = x_n - x_0 = 1 - 0 = 1$
  and $\underline{\int}_a^b = \sum_{k=1}^n m_k \Delta\alpha_k$, and $m_k = 1$, so $-1 \sum_{k=1}^n \Delta\alpha_k =$
  $-1[x_1 - x_0 + x_2 - x_1 + ... + x_n - x_{n-1}] = -1[x_n - x_0] = -1[1-0] = -1$
  so $\overline{\int}_a^b = 1 \leq -1 = \underline{\int}_a^b$ so $f \notin \mathscr{R}_\alpha[0,1]$.

- But $|f(x)| = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 1, & x \in \mathbb{Q}^c \cap [0,1] \end{cases}$

  so $|f(x)| := \{1, x \in [0,1]\}$, and $\overline{\int}_a^b = 0 = \underline{\int}_a^b$, so $|f| \in \mathscr{R}_\alpha[0,1]$.

$$\boxed{\text{So } |f| \in \mathscr{R}_\alpha[a,b], \text{ but } f \notin \mathscr{R}_\alpha[a,b].}$$

2.2 54. Functions of Bounded Variation

Exercise 54.1:

Solution:

- $\alpha(x) = x^2[-2, 1]$

  Since $\alpha(x)$ is continuous on $[-2, 1]$

  $\alpha(x)$ is differentiable on $(-2, 1)$

  and $\alpha'(x)$ is bounded on $(-2, 1) \Rightarrow$

$$\alpha \in BV[-2, 1]$$

$$V_{-2}^{1}x^2 = \underbrace{V_{-2}^{0}x^2}_{} + \underbrace{V_{0}^{1}x^2}_{}$$

decreasing increasing

$$\alpha(-2) - \alpha(0) + \alpha(1) - \alpha(0) = 4 + 1 = 5$$

- $\alpha(x) = x^3 + x^2 - x + 1[-2, 2]$

  Since $\alpha(x)$ is continuous on $[-2, 2]$

  $\alpha(x)$ is differentiable on $(-2, 2)$

  and $\alpha'(x)$ is bounded on $(-2, 2) \Rightarrow$

$$\alpha \in BV[-2, 2]$$

$$V_{-2}^{2}(x^3 + x^2 - x + 1) = \underbrace{V_{-2}^{-1}\alpha(x)}_{} + \underbrace{V_{-1}^{.5}\alpha(x)}_{} + \underbrace{V_{.5}^{2}\alpha(x)}_{}$$

increasing decreasing increasing

$$\alpha(-1) - \alpha(-2) + \alpha(-1) - \alpha(.5) + \alpha(2) - \alpha(.5)$$

$$= (2 + 1) + (2 - .875) + (11 - .875) = 14.25$$

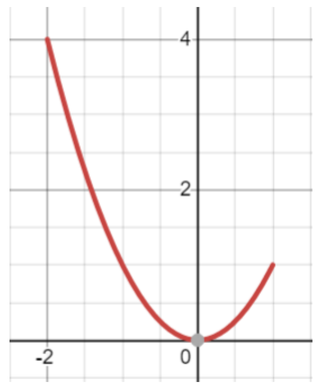

Figure 2.1: $\alpha(x) = x^2[-2, 1]$

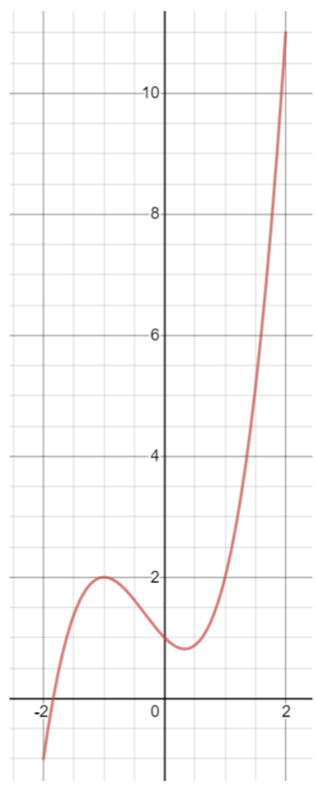

Figure 2.2: $\alpha(x) = x^3 + x^2 - x + 1[-2, 2]$

Exercise 54.4:

Solution: Let $\alpha \in BV[a,b]$

$\alpha(x) \in BV[a,b] \Rightarrow \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \leq M$

So $\Rightarrow \sum_{i=1}^{n} ||\alpha(x_i)| - |\alpha(x_{i-1})|| \leq \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})| \leq M$

$$\Rightarrow \sum_{i=1}^{n} ||\alpha(x_i)| - |\alpha(x_{i-1})|| \leq M$$

$$|\alpha(x)| \in BV[a,b]$$

Exercise 54.5:

Solution: Since $\alpha(x) \in BV[a,b]$ so $\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1}| \leq c, c \in \mathbb{R}$, and we have $|\alpha(x)| > M$, so $\frac{1}{\alpha(x)} \leq \frac{1}{M}$.
Now $\sum_{k=1}^{n} |\frac{1}{\alpha(x_k)} - \frac{1}{\alpha(x_{k-1})}| = \sum_{k=1}^{n} |\frac{\alpha(x_{k-1})-\alpha(x_k)}{\alpha(x_{k-1})\alpha(x_k)}| \leq \frac{c}{M^2}$.
so $\sum_{k=1}^{n} |\frac{1}{\alpha(x_k)} - \frac{1}{\alpha(x_{k-1})}|$ bounded.

Exercise 54.6:

Solution:

$$Max\{\alpha,\beta\} = \frac{\alpha+\beta+|\alpha-\beta|}{2}$$

$$Min\{\alpha,\beta\} = \frac{\alpha+\beta-|\alpha-\beta|}{2}$$

By theorem if $\alpha, \beta \in BV[a,b], c \in \mathbb{R} \Rightarrow$

- $\alpha + \beta \in BV[a,b]$
- $|\alpha| \in BV[a,b]$
- $c\alpha \in BV[a,b]$

So $Max\{\alpha,\beta\} = \frac{\alpha+\beta+|\alpha-\beta|}{2}$, $Min\{\alpha,\beta\} = \frac{\alpha+\beta-|\alpha-\beta|}{2} \in BV[a,b]$

2.3 55. Riemann-Stieltjes Integration with Respect to Functions of Bounded Variation

Exercise 55.3:

(a) $\int_0^3 \sqrt{x} dx^3$

since $\alpha(x) = x^3$ is continuous and differentiable on $[0, 3] \Rightarrow$

$$= \int_0^3 \sqrt{x} dx^3 = \int_0^3 \sqrt{x} 3x^2 dx = 3 \int_0^3 \sqrt{x} x^2 dx$$

$$3 \left(\frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} \right) = 6 \frac{\sqrt{3^7}}{7}$$

(b) $\int_1^4 \sqrt{x^2+1} d(x^2+3)$

since $\alpha(x) = (x^2+3)$ is continuous and differentiable on $[1, 4] \Rightarrow$

$$\int_1^4 \sqrt{x^2+1} d(x^2+3) = \int_1^4 \sqrt{x^2+1} 2x dx = 2 \int_1^4 \sqrt{x^4+x^2} dx = 44.8$$

(c) $\int_1^4 x - [x] dx^2$

since $\alpha(x) = x^2$ is continuous and differentiable on $[1, 4] \Rightarrow$

$$\int_1^4 x - [x] dx^2 = \int_1^2 x - [x] dx^2 + \int_2^3 x - [x] dx^2 + \int_3^4 x - [x] dx^2$$

$$\int_1^4 x - [x] dx^2 = \int_1^2 x - 1 dx^2 + \int_2^3 x - 2 dx^2 + \int_3^4 x - 3 dx^2$$

$$= \int_1^2 2x^2 - 2x dx + \int_2^3 2x^2 - 4x dx + \int_3^4 2x^2 - 6x dx$$

$$\frac{2x^3}{3} - x^2|_1^2 + \frac{2x^3}{3} - 2x^2|_2^3 + \frac{2x^3}{3} - 3x^2|_3^4 = 8$$

Exercise 55.6:

Solution: Since $\alpha \in BV[a,b]$ and f continuous, then by theorem $f \in \mathscr{R}[a,b]$

Now it's clearly that:

$$L(f,P,T) \leq S(f,p,T) \leq U(f,P)$$

$\Rightarrow \int_a^b f d\alpha - \epsilon < L$
and $\int_a^b f d\alpha + \epsilon < U \Rightarrow \int_a^b f d\alpha - \epsilon < L(f,p) \leq S(f,p,T) \leq U(f,p) < int_a^b f d\alpha + \epsilon$
$\Rightarrow \int_a^b f d\alpha - \epsilon < S(f,p,T) < int_a^b f d\alpha + \epsilon$
$|S(f,p,T) - \int_a^b f d\alpha| < \epsilon \Rightarrow$

$$\lim_{norm\ p \to 0} S(f,p,T) = \int_a^b f d\alpha$$

Exercise 55.9:

Solution:

$$Max\{f,g\} = \frac{f+g+|f-g|}{2}$$

By theorem if $f,g \in \mathscr{R}[a,b], c \in \mathbb{R} \Rightarrow$

- $f+g \in \mathscr{R}[a,b]$
- $|f| \in \mathscr{R}[a,b]$
- $cf \in \mathscr{R}[a,b]$

So $Max\{f,g\} = \frac{f+g+|f-g|}{2}$

Chapter 3

X.Sequences and Series of Functions

3.1 60. Pointwise Convergence and Uniform Convergence

Exercise 60.2:

Solution:

$$f_n(x) = \frac{1}{1+n^2x^2} \Longrightarrow$$

$\{f_n\}$ converges pointwise to f on $[0,1]$, where:

$$f(x) = \begin{cases} 0 & 0 < x \leq 1 \\ 1 & x = 0 \end{cases}$$

Since f is not continuous at point $0 \Rightarrow$ $\{f_n\}$ is not uniformly convergent.

$$g_n(x) = xn(1-x)^n \Longrightarrow$$

$\{g_n\}$ converges pointwise to g on $[0,1]$, where:

$$g(x) = \begin{cases} 0 & 0 < x \leq 1 \\ c & x = 0 \text{ , } where \ c \in [0,1] \end{cases}$$

Since g is not continuous at point $0 \Rightarrow$ $\{g_n\}$ is not uniformly convergent.

Exercise 60.4:

Solution: $f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ 2 - nx, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} \leq x \leq 1 \end{cases}$

Exercise 60.5:

Solution: Let $\{f_n\}$ be a sequence of bounded functions on a set X.and $\{f_n\}$ converges uniformly to f on X

So since $f_n \rightrightarrows f \Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|f_n - f| < \epsilon$.

$$|f| = |f - f_n + f_n| \leq |f_n - f| + |f_n| < \epsilon + M$$

So f is bounded.

Let $f_n = \frac{x}{n} + \frac{1}{x}\{0 < x \leq 1\}$, so $f_n \to f$ such that $f = \frac{1}{x}\{0 < x \leq 1\}$ and f is unbounded.

Exercise 60.9:

Solution:

$$d(f, g) := sup\{|f(x) - g(x)|, x \in [a, b]\}$$

- $sup\{|f(x) - g(x)|, x \in [a, b]\} = 0 \iff |f(x) - g(x)| = 0 \iff f(x) = g(x)$
- $sup\{|f(x) - g(x)|, x \in [a, b]\} = sup\{|g(x) - f(x)|, x \in [a, b]\}$ "Trivial"
- *"Triangle Inequality:* $sup\{|f(x) - h(x)| = sup\{|f(x) - g(x) + g(x) - h(x)| \leq sup\{|f(x) - g(x)| + |g(x) - h(x) = sup\{|f(x) - g(x)| + sup\{|g(x) - h(x)|\}$

Exercise 60.10:

Solution: let $C[a,b]$ denote the set of continuous real-valued functions on $[a,b]$. We define a metric d on $C[a,b]$ by the formula:

$$d(f,g) = sup\{|f(x) - g(x)|, x \in [a,b]\}$$

Let f_n be a Cauchy sequence in $C[a,b]$, then $\forall \epsilon > 0$, there is N such that $||f_n - f_m|| < \epsilon$ for $n, m \geq N \Longrightarrow |f_n - f_m|| = sup|f_n - f_m| < \epsilon$.
$|f_n - f_m| \leq sup|f_n(x) - f_m(x)| < \epsilon, \forall n \geq N$.
So $f_n(x)$ converges uniformly to $f(x)$.
And each f_n is continuous on $[a,b]$, and $f_n \to f$ uniformly on $[a,b]$.
Thus, $f \in C[a,b]$. So $C[a,b]$ is complete.

3.2 61. Integration and Differentiation of Uniformly Convergent Sequences

Exercise 61.1:

Solution: Let $f_n := \frac{x+n[x]}{n}|0 \leq x \leq 1$, f_n is convergent pointwise to f, such that:$f(x) = [x]$, and
$\lim_{n\to\infty} \int_0^2 \frac{x+n[x]}{n} = \lim_{n\to\infty} \int_0^1 \left(\frac{x}{n} + \int_1^2 \frac{x+n}{n}\right) =$
$\lim_{n\to\infty} \left(\frac{x^2}{2n}|_0^1 + (\frac{x^2}{2n} + x|_1^2)\right) = \lim_{n\to\infty} \left(\frac{1}{2n} + \frac{4}{2n} + 2 - \frac{1}{2n} - 1\right) = 1$
and $\int_0^2 [x] = \int_0^1 0 + \int_1^2 1 = x|_1^2) = 1 \Rightarrow$

$$\lim_{n\to\infty} \int_0^2 f_n = \int_0^2 f$$