MA585-HW6

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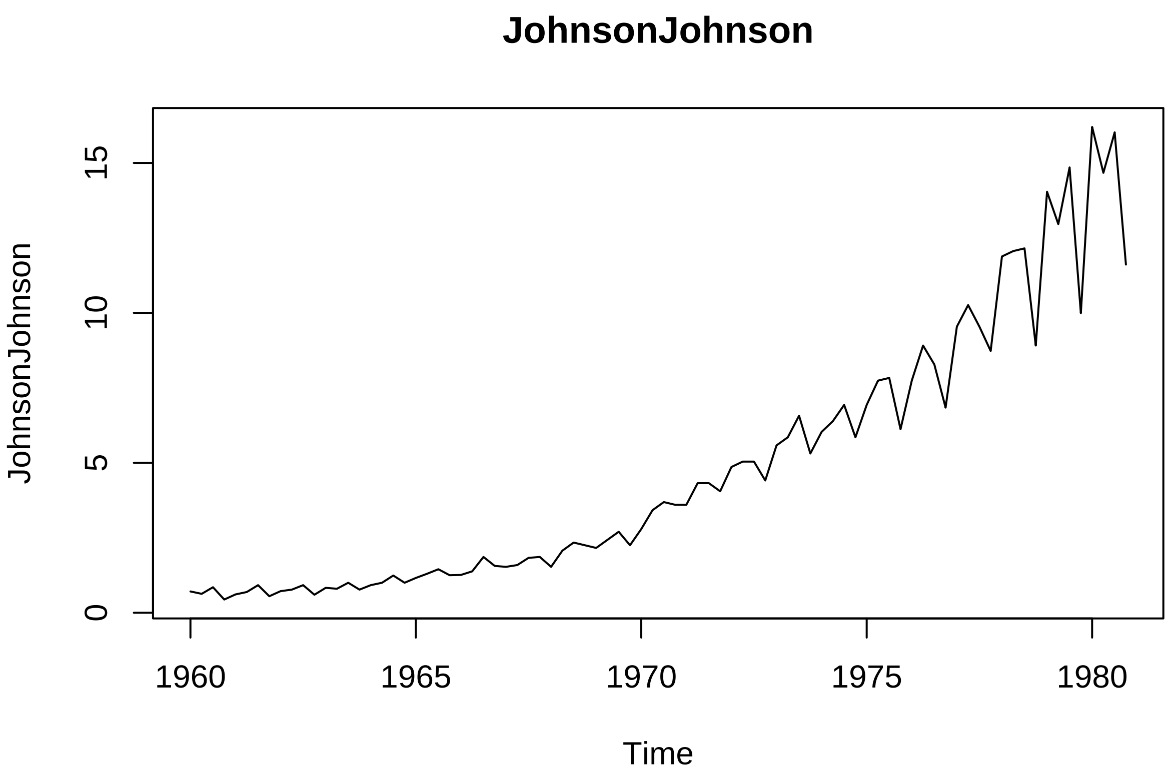
2022/3/23

1

a. Plot the data. Describe the features of the data. Do the data look stationary? Explain your answer.

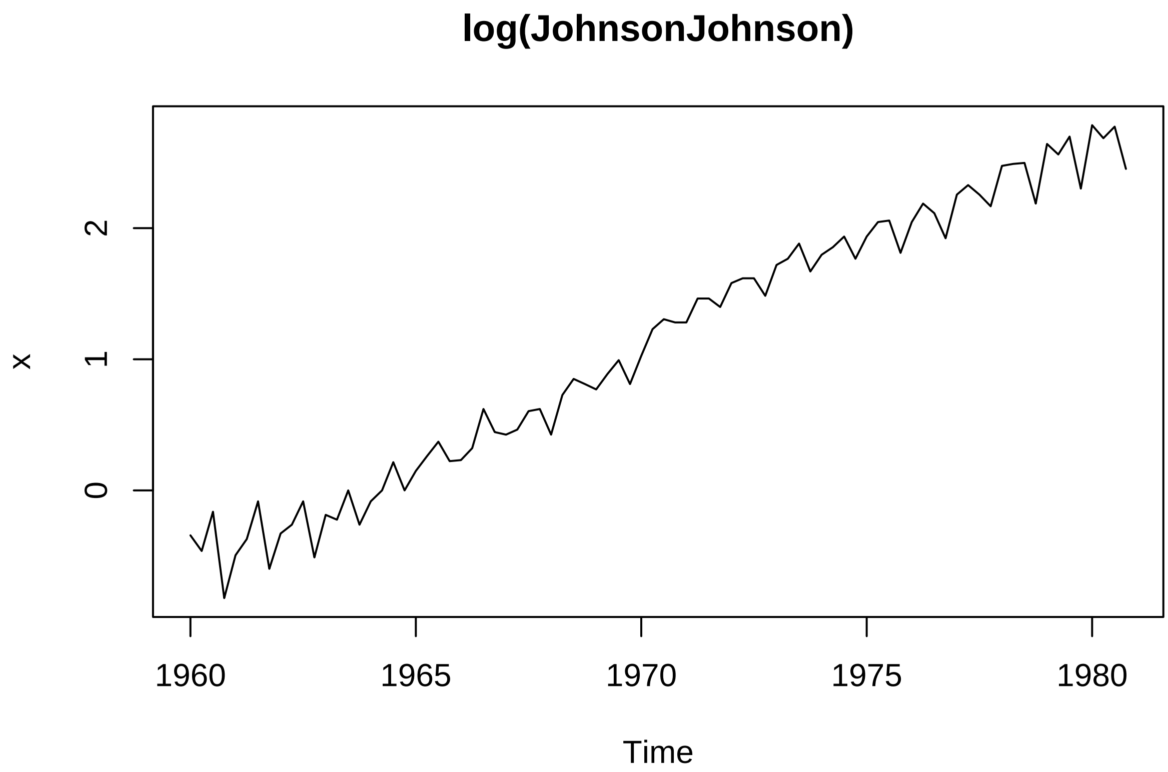
```
plot.ts(JohnsonJohnson, main='JohnsonJohnson')
```



This time series plot seems not stationary, since it has an obvious trend pattern and the variance seems to change over the time.

b. Apply an appropriate variance stabilizing transformation, if necessary.

```r
x <- log(JohnsonJohnson)
plot.ts(x, main = 'log(JohnsonJohnson)')
```

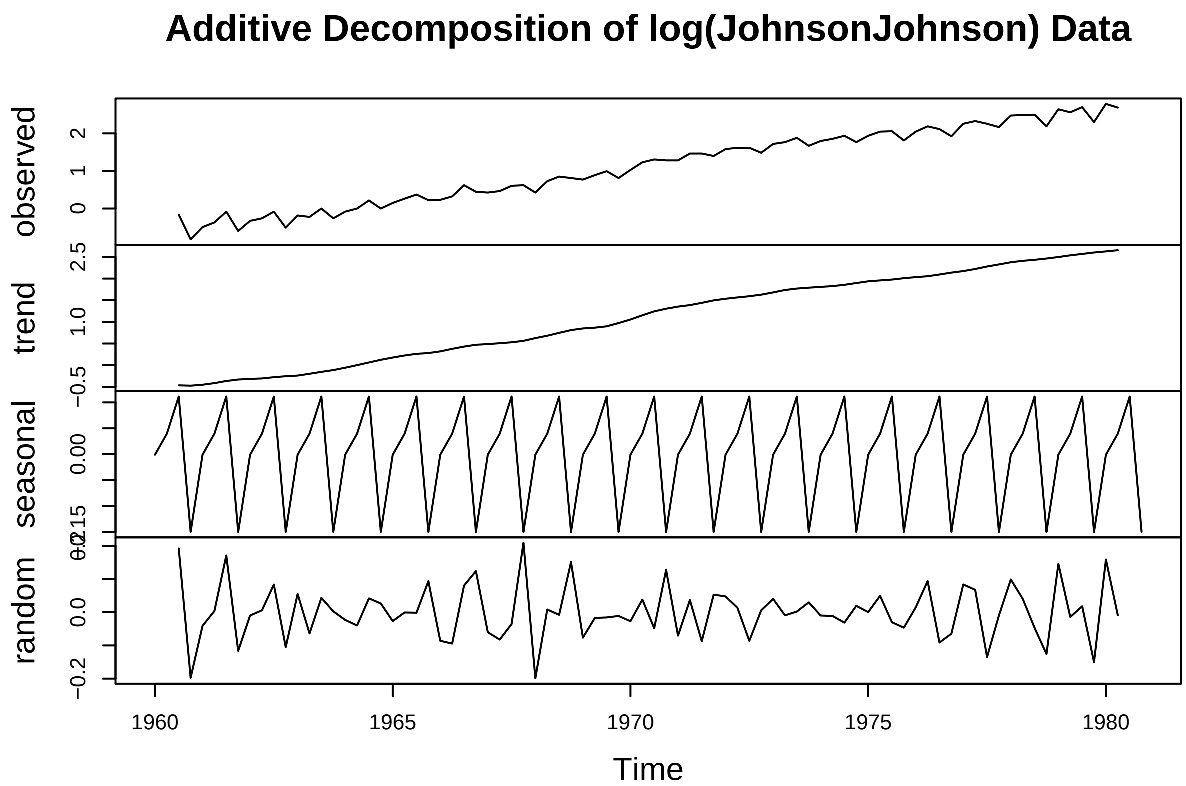


After I apply a log transformation, the variance seems to be stabilized.

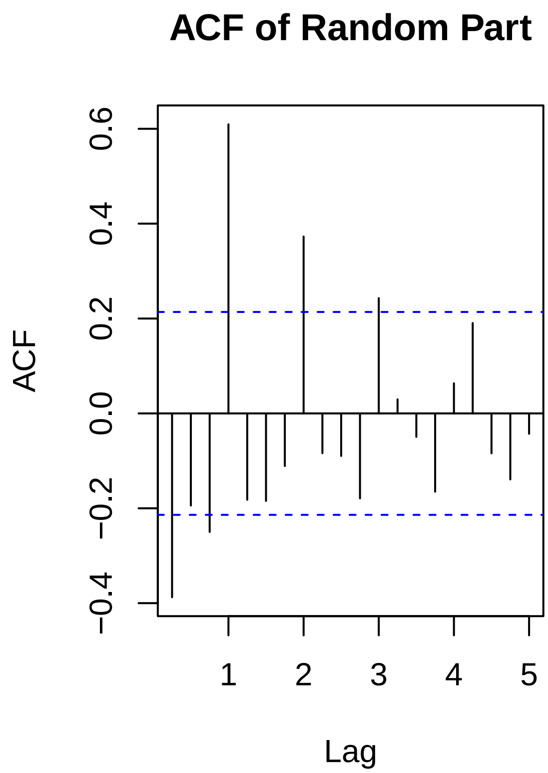
c. Carry out classical decomposition of the data, plot the transformed series along with the ACF and PACF.

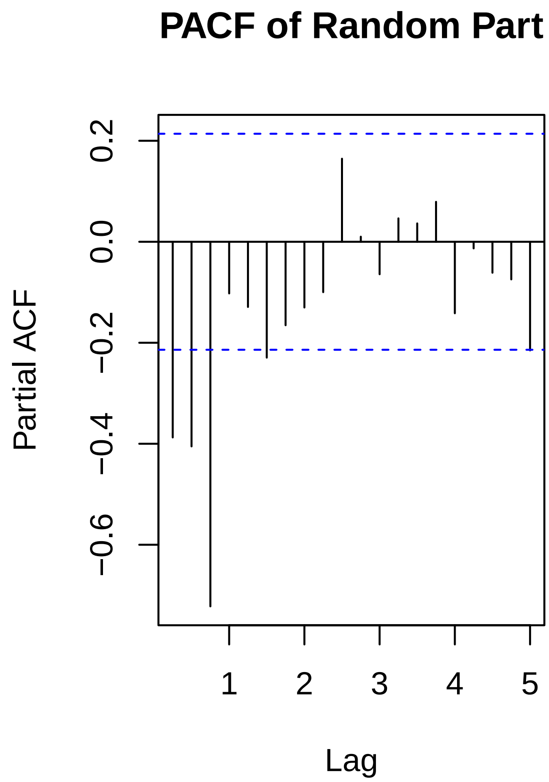
```r
z <- decompose(x, type="additive")

# decomp.plot function from lecture notes
decomp.plot <-  function(x, main = NULL, ...)
{
  if(is.null(main))
    main <-  paste("Decomposition of", x$type, "time series")
  plot(cbind(observed = x$random + if (x$type == "additive")
    x$trend + x$seasonal
    else x$trend * x$seasonal, trend = x$trend, seasonal = x$seasonal,
    random = x$random), main = main, ...)
}
decomp.plot(z,main="Additive Decomposition of log(JohnsonJohnson) Data")
```




```r
# plot ACF and PACF
par(mfrow=c(1,2))
acf(z$random, na.action = na.pass, lag.max=20, main = 'ACF of Random Part')
pacf(z$random, na.action = na.pass, lag.max=20, main = 'PACF of Random Part')
```





d. Identify an ARMA model for the transformed data.

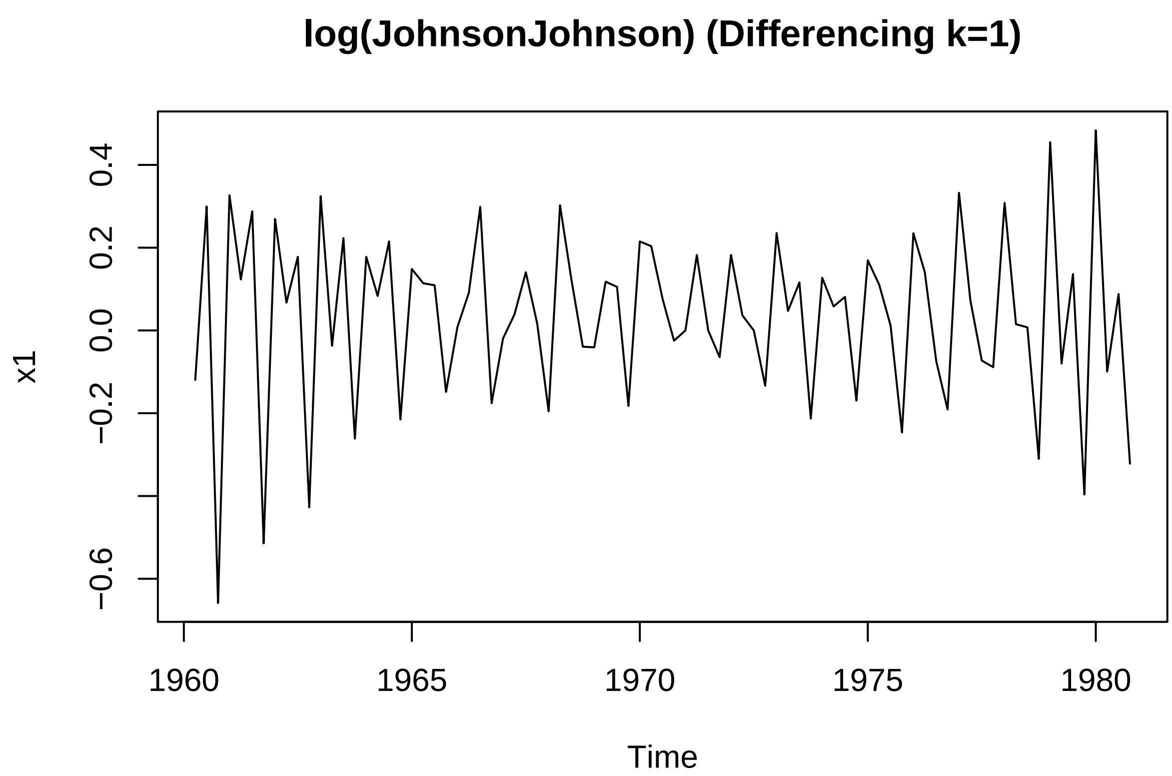
```
# Check with EACF
TSA::eacf(na.omit(z$random))
```

```
## AR/MA
##   0 1 2 3 4 5 6 7 8 9 10 11 12 13
## 0 x o x x o o o x o o o  x  o  o
## 1 x o o x x o o x o o o  o  o  o
## 2 x o x x x o o x o o x  x  o  o
## 3 o o x o o o x o o o o  o  o  o
## 4 x o o o o o x o o o o  o  o  o
## 5 x o o o o o o o o o o  o  o  o
## 6 x o o o o o x o o o o  o  o  o
## 7 x x o o o o o o o o o  o  o  o
```

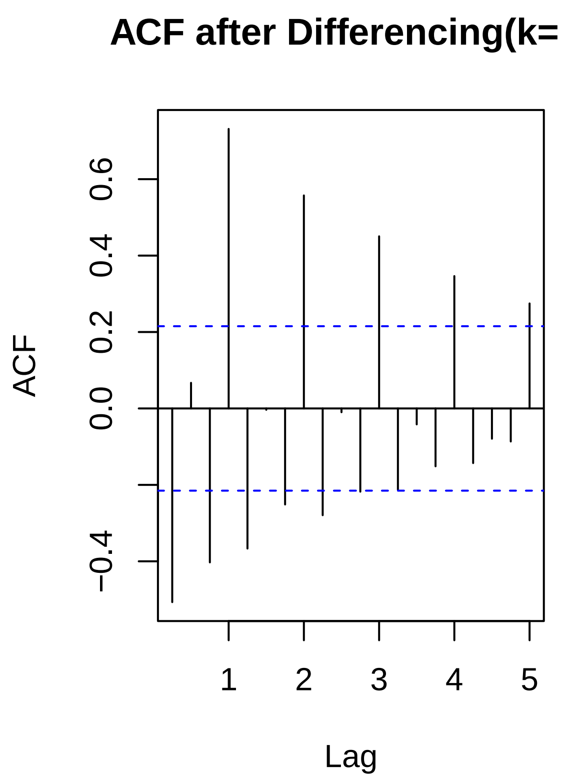
According to the ACF and PACF plots, we can see ACF cuts-off after lag 1, and PACF cuts-off after lag 3. So the model might be MA(1), AR(3), or ARMA(3,1). Based on the result of EACF table, the model is more likely to be MA(1).

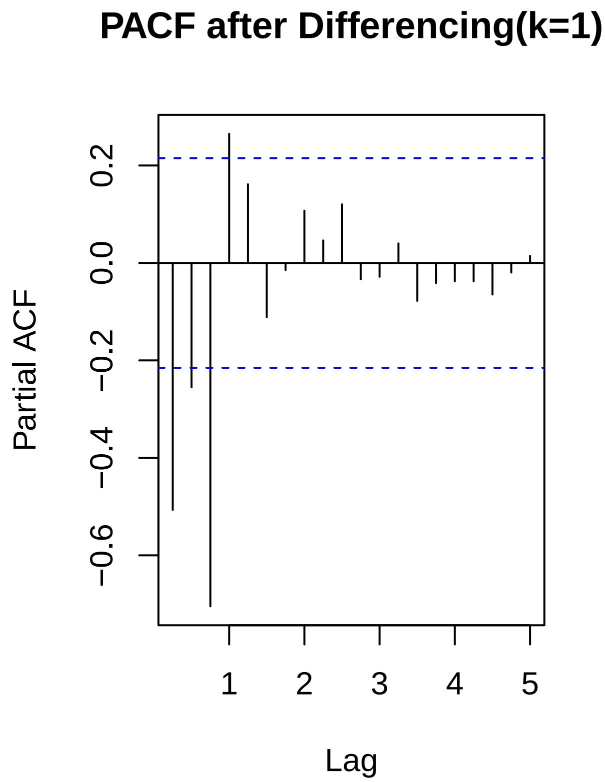
e. Repeat c and d, but instead of classical decomposition, use differencing to make the data stationary.

```
# differencing with k=1
x1 <- diff(x)
plot.ts(x1, main = 'log(JohnsonJohnson) (Differencing k=1)')
```




```
# plot ACF and PACF
par(mfrow=c(1,2))
acf(x1, na.action = na.pass, lag.max=20, main = 'ACF after Differencing(k=1)')
pacf(x1, na.action = na.pass, lag.max=20, main = 'PACF after Differencing(k=1)')
```






```r
# check EACF
TSA::eacf(x1)
```

```
## AR/MA
##   0 1 2 3 4 5 6 7 8 9 10 11 12 13
## 0 x o x x x o x x x o o  x  o  o
## 1 x o o x x o o x x o o  x  o  o
## 2 x x x x x x o x o x o  x  o  o
## 3 x o o o o o x o o o o  o  o  o
## 4 x x o o o o x o o o o  o  o  o
## 5 x x o o o o x o o o o  o  o  o
## 6 x o x x o o x o o o o  o  o  o
## 7 o x x x o o x o o o o  o  o  o
```

According to the ACF and PACF plots, we can see ACF decays to zero, and PACF cuts-off after lag 3 (significantly. So the model might be AR(3). The result of EACF table is a valid proof.

2

Consider the time series of the numbers of users connected to the Internet through a server every minute (R data WWWusage). Carry out a test for unit root. Apply necessary transformation and identify plausible ARMA models.

```r
x <- WWWusage
adf.test(x)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  x
## Dickey-Fuller = -2.6421, Lag order = 4, p-value = 0.3107
## alternative hypothesis: stationary
```

The result of Augmented Dickey-Fuller Test shows that p-value = 0.310, which fails to reject H0, thus the process is not stationary. So I apply a log transformation for WWWusage data.

```r
# log transformation
x1 <- log(WWWusage)
adf.test(x1)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  x1
## Dickey-Fuller = -2.7591, Lag order = 4, p-value = 0.2622
## alternative hypothesis: stationary
```

```r
# still fail to reject H0, try differencing
```

The result of Augmented Dickey-Fuller Test shows that p-value = 0.2622, which still fails to reject H0, thus I apply differencing transformation to stationary the data.

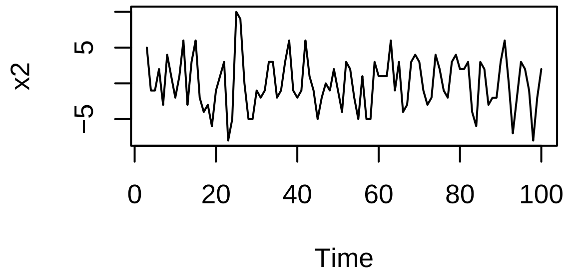
```r
# apply differencing with k=2
x2 <- diff(diff(WWWusage))
adf.test(x2)
```

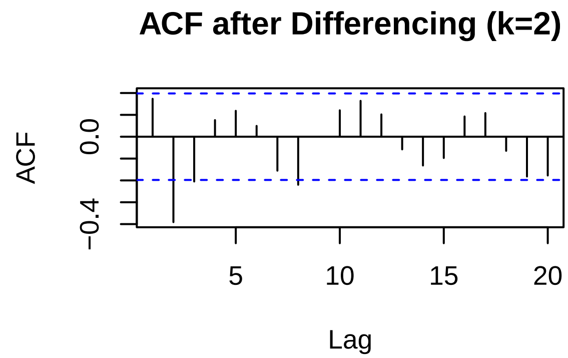
```
## Warning in adf.test(x2): p-value smaller than printed p-value
```

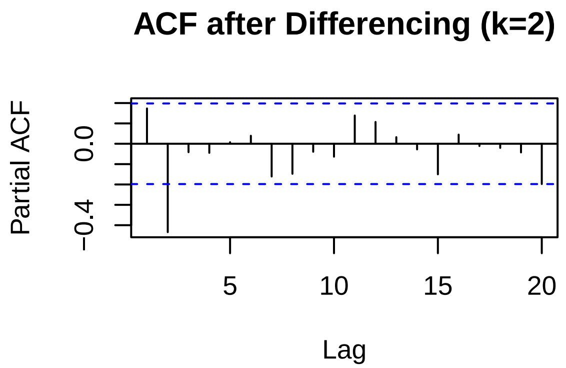
```
##
##  Augmented Dickey-Fuller Test
##
## data:  x2
## Dickey-Fuller = -4.828, Lag order = 4, p-value = 0.01
## alternative hypothesis: stationary
```

After apply second order difference, the result of Augmented Dickey-Fuller Test finally shows a p-value which is small enough to reject H0. Thus the process is stationary now.

```r
par(mfrow=c(2,2))
plot.ts(x2)
acf(x2, lag.max=20, main = 'ACF after Differencing (k=2)')
pacf(x2, lag.max=20, main = 'ACF after Differencing (k=2)')
```





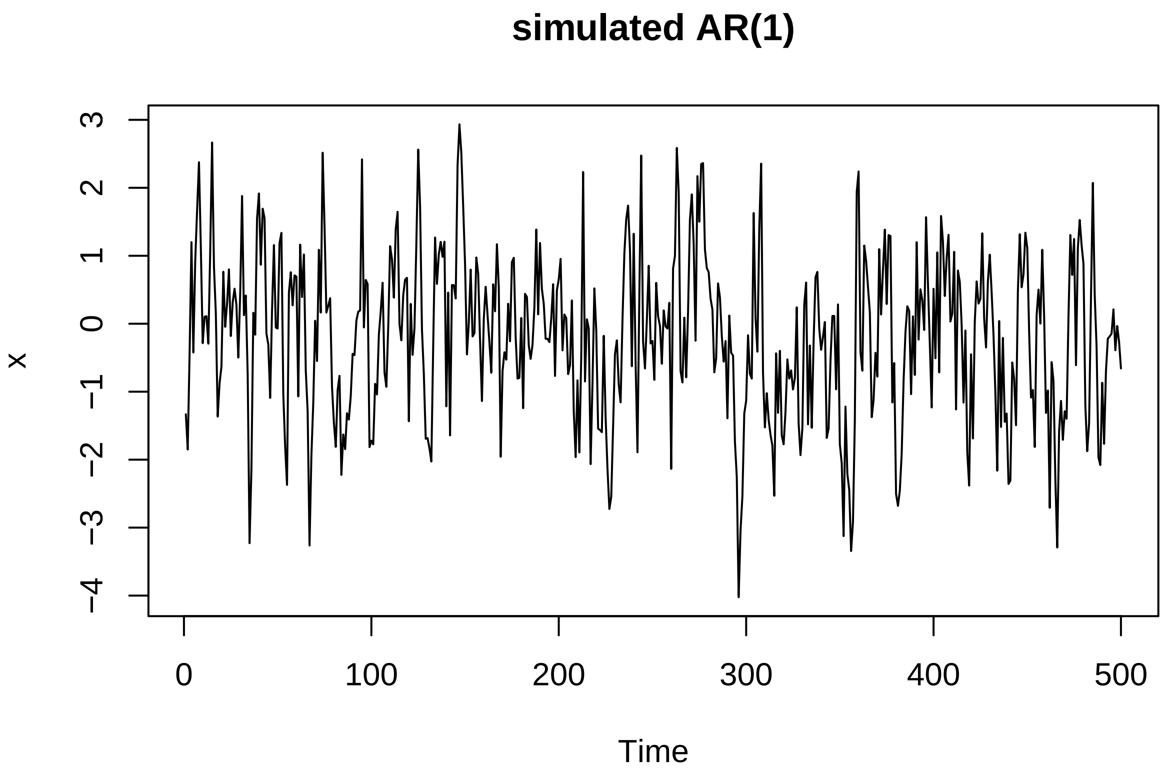


According to the ACF and PACF plots, the WWWusage data after second order difference might be an ARMA(0,0) process, which means it might be a white noise process.

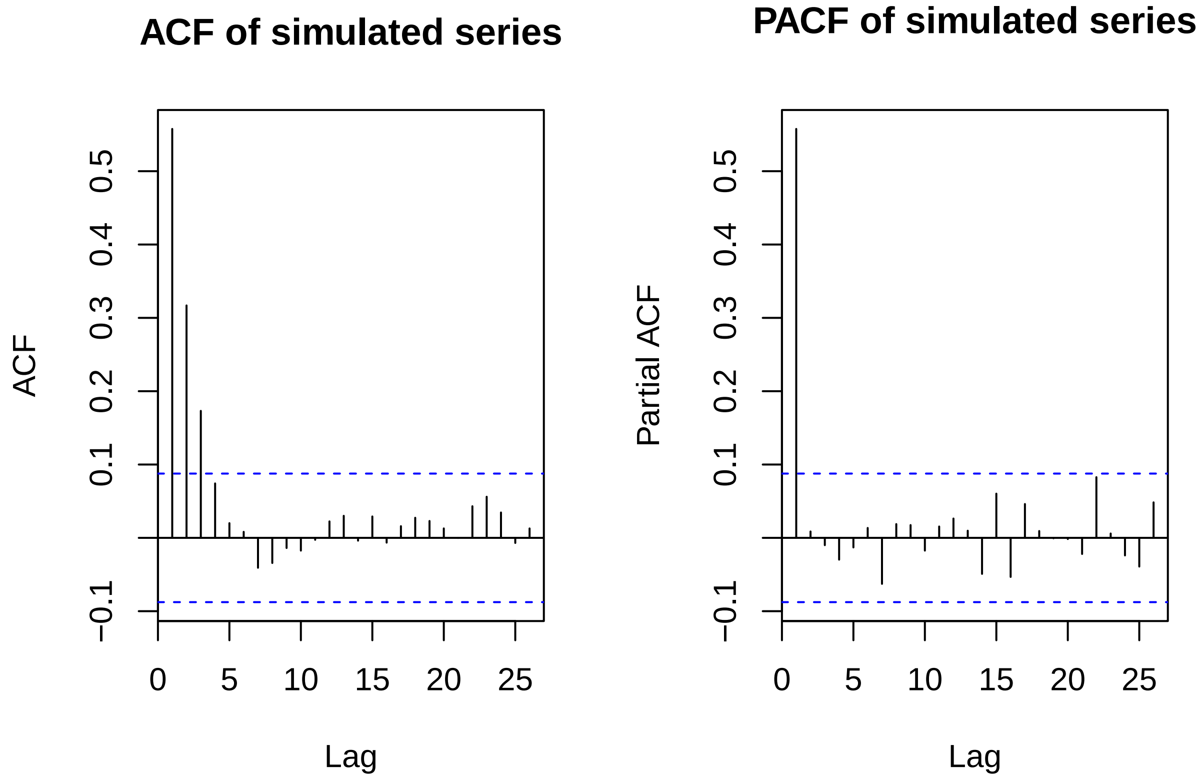
6 (f.)

Write a R code generate a random sample of size 500 from the AR(1) process with phi = 0.6 and sigmaˆ2 = 0.8, Plot the simulated series along with the sample ACF and PACF of the series. Is the sample ACF and PACF consistent with AR(1)?

```
# simulate AR(1) process
AR1.sm <- list(order = c(1,0,0), ar = 0.6, sd = sqrt(0.8))
x <- arima.sim(n=500, AR1.sm)
plot.ts(x, main = 'simulated AR(1)')
```




```
par(mfrow=c(1,2))
acf(x, main = 'ACF of simulated series')
pacf(x, main = 'PACF of simulated series')
```



As we can see, the ACF decay to zero with a damped sine wave, and the PACF cuts-off after lag 1, that exactly consistent with AR(1) process.

3. Identify each of the models below as $ARIMA(p, d, q)$. Specify the order of the models (p, d, q) and the model parameters ϕ and θ.

a. $X_t = 10 + X_{t-1} + e_t + 0.6e_{t-1}$

b. $X_t = 3 + 1.25X_{t-1} - 0.25X_{t-2} + e_t - 0.2e_{t-2}$

c. $X_t - 1.7X_{t-1} + 0.7X_{t-2} = -8 + e_t$

a. $X_t - X_{t-1} = 10 + e_t + 0.6e_{t-1}$

$\underbrace{(1-B)}_{d=1} X_t = 10 + \underbrace{e_t + 0.6e_{t-1}}_{MA.\ q=1}$ $\Rightarrow ARIMA(0,1,1)$

$\theta = 0.6$

b. $X_t - 1.25X_{t-1} + 0.25X_{t-2} = 3 + e_t - 0.2e_{t-2}$

$\underbrace{(1-1.25B + 0.25B^2)}_{(1-B)(1-0.25B)} X_t = 3 + \underbrace{(1-0.2B^2)e_t}_{MA:\ q=2}$

unit root $\Rightarrow$ $p=1.\ d=1$

$\Rightarrow ARIMA(1,1,2)$

$\phi = 0.25,\quad \theta_1 = 0$

$\theta_2 = -0.2$

c. $X_t - 1.7X_{t-1} + 0.7X_{t-2} = -8 + e_t$

$\underbrace{(1-1.7B + 0.7B^2)}_{(1-B)(1-0.7B)} X_t = \underbrace{-8 + e_t}_{q=0}$ $\Rightarrow ARIMA(1,1,0)$

unit root $\rightarrow d=1$, $p=1$

$\phi = 0.7$

4. Consider the two models

$$X_t = 0.9X_{t-1} + 0.09X_{t-2} + e_t$$

and

$$X_t = X_{t-1} + e_t - 0.1e_{t-1}$$

a. Identify both models as $ARIMA(p, d, q)$. Specify (p, d, q) and ARMA parameters ϕ and θ.

b. In what way the two models are different?

c. In what way the two models are similar? What does this tell you about model selection for time series data?

(a) $X_t - 0.9X_{t-1} - 0.09X_{t-2} = e_t$

$(1 - 0.9B - 0.09B^2) X_t = e_t$

$B = \dfrac{-0.9 \pm \sqrt{0.9^2 + 4\times0.09}}{2\times0.09}$ $(\sqrt{1.17})$

$B_1 = 11.5$, $B_2 \approx 1.009$

$|B_1| > 1,\ |B_2| > 1 \Rightarrow$ Causal

$\therefore$ the model can be $ARIMA(2,0,0)$ or $AR(2)$ with $\phi_1 = 0.9,\ \phi_2 = 0.09$

$X_t - X_{t-1} = e_t - 0.1e_{t-1}$

$(1-B)X_t = (1-0.1B)e_t$

$\phi(B) = 1 \rightarrow d=1$ $\quad\theta(B) = 10 > 0$

not Causal $\quad\checkmark$ invertible

$\therefore$ the model maybe $ARIMA(0,1,1)$ invertible but not causal with $\theta = -0.1$

(b) The two model is different in the way that ① is causal while ② is not causal.
And due to this property, they belong to different model.

(c) For the ② model, we have

$$\begin{cases} Y_t = X_t - X_{t-1} \\ Y_t = e_t - 0.1 e_{t-1} \leftarrow MA(1) \Rightarrow AR(\infty) \end{cases}$$

$AR(\infty)$

$\underbrace{(1+\pi_1 B + \pi_2 B^2 + \cdots)}_{\pi(B)} Y_t = e_t$ $\quad Y_t = (1+\theta B) e_t$, $\quad e_t = \frac{Y_t}{1+\theta B}$

$\pi(B) Y_t = \frac{Y_t}{1+\theta B}$

$\pi(B)(1+\theta B) = 1$

$(1+\pi_1 B + \pi_2 B^2 + \cdots)(1+\theta B) = 1$

$$\Rightarrow \begin{matrix} \pi_1 + \theta = 0 \\ \pi_2 + \pi_1\theta = 0 \\ \pi_3 + \pi_2\theta = 0 \end{matrix} \Rightarrow \begin{cases} \pi_1 = -\theta \\ \pi_2 = -\pi_1\theta = \theta^2 \\ \pi_3 = -\pi_2\theta = -\theta^3 \\ \vdots \\ \pi_j = (-\theta)^j \end{cases} \Rightarrow \pi_j = (0.1)^j, \begin{cases} \pi_1 = 0.1 \\ \pi_2 = 0.01 \end{cases}$$

general

$\rightarrow \pi_j + \pi_{j-1}\theta = 0$

So the model is $Y_t + 0.1 Y_{t-1} + 0.01 Y_{t-2} = e_t$

$\Rightarrow (X_t - X_{t-1}) + 0.1(X_{t-1} - X_{t-2}) + 0.01(X_{t-2} - X_{t-3}) = e_t$

$X_t - 0.9 X_{t-1} - 0.09 X_{t-2} - 0.01 X_{t-3} = e_t$

which is similar as ① model //

The above process told us that if we have different data, we will have different model identification resule through different process.

However, we prefer the naive model in practice, so in this case, we prefer ② model. //

5. Let X_t be a stationary process with autocovariance function $\gamma(h)$.

a. Show that the process ∇X_t is stationary and find its autocovariance function.

b. Show that the process $\nabla^2 X_t$ is also stationary.

(a) Let $Y_t = \nabla X_t = X_t - X_{t-1}$

$\Rightarrow \quad E(Y_t) = E(\nabla X_t) = E(X_t) - E(X_{t-1}) = 0$

$$\begin{aligned}
cov(Y_t, Y_{t+h}) &= cov(\nabla X_t, \nabla X_{t+h}) \\
&= cov(X_t - X_{t-1}, X_{t+h} - X_{t+h-1}) \\
&= cov(X_t, X_{t+h}) - cov(X_{t-1}, X_{t+h}) - (X_t, X_{t+h-1}) + (X_{t-1}, X_{t+h-1}) \\
&= \gamma(h) - \gamma(h+1) - \gamma(h-1) + \gamma(h) \\
&= 2\gamma(h) - \gamma(h+1) - \gamma(h-1)
\end{aligned}$$

$\because$ X_t is stationary, $\gamma(h)$ doesn't depend on t

$\therefore$ $cov(Y_t, Y_{t+h})$ also doesn't depend on t

thus. $\nabla X_t = Y_t$ process is stationary

(b) let $Z_t = \nabla^2 X_t = X_t - 2X_{t-1} + X_{t-2}$

$\Rightarrow \quad E(Z_t) = E(X_t) - 2E(X_{t-1}) + E(X_{t-2}) = 0$

$$\begin{aligned}
cov(Z_t, Z_{t+h}) &= cov(\nabla^2 X_t, \nabla^2 X_{t+h}) \\
&= cov(X_t - 2X_{t-1} + X_{t-2}, X_{t+h} - 2X_{t+h-1} + X_{t+h-2}) \\
&= cov(X_t, X_{t+h}) - 2\,cov(X_t, X_{t+h-1}) + cov(X_t, X_{t+h-2}) \\
&\quad -2\,cov(X_{t-1}, X_{t+h}) + 4\,cov(X_{t-1}, X_{t+h-1}) - 2\,cov(X_{t-1}, X_{t+h-2}) \\
&\quad + cov(X_{t-2}, X_{t+h}) - 2\,cov(X_{t-2}, X_{t+h-1}) + cov(X_{t-2}, X_{t+h-2}) \\
&= \gamma(h) - 2\gamma(h-1) + \gamma(h-2) \\
&\quad - 2\gamma(h+1) + 4\gamma(h) - 2\gamma(h-1) \\
&\quad + \gamma(h+2) - 2\gamma(h+1) + \gamma(h) \\
&= 6\gamma(h) - 4\gamma(h-1) + \gamma(h-2) - 4\gamma(h+1) + \gamma(h+2)
\end{aligned}$$

Similar as (a):

$\because$ X_t is stationary, $\gamma(h)$ doesn't depend on t

$\therefore$ $cov(Z_t, Z_{t+h})$ also doesn't depend on t

thus. $\nabla^2 X_t = Z_t$ process is stationary

6. Let e_t be zero mean Gaussian white noise process with variance σ^2 and let $|\phi| < 1$ be a constant. Consider the process, starting at X_1,

$$X_1 = e_1$$
$$X_t = \phi X_{t-1} + e_t, \qquad t = 2, 3, \ldots$$

a. Express X_t as a linear combination of the white noise process e_t.

b. Use the result in (a) to compute the mean and the variance of the process X_t. Is the process X_t stationary?

c. Show

$$Correlation(X_t, X_{t-h}) = \phi^h \left[\frac{Var(X_{t-h})}{Var(X_t)}\right]^{1/2}$$

for $h \geqslant 0$.

d. Argue that for large t,

$$Var(X_t) \approx \frac{\sigma^2}{1-\phi^2}$$

and

$$Correlation(X_t, X_{t-h}) \approx \phi^h, \quad h \geqslant 0,$$

so in a sense, X_t is "asymptotically stationary."

a. $X_t = \phi X_{t-1} + e_t = \phi(\phi X_{t-2} + e_{t-1}) + e_t$

$= \phi^2 X_{t-2} + \phi e_{t-1} + e_t$

$= \phi^3 X_{t-3} + \phi^2 e_{t-2} + \phi e_{t-1} + e_t = \cdots$

$= \sum_{j=0}^{t-1} \phi^j e_{t-j}$

b. $E(X_t) = E\left(\sum_{j=0}^{t-1} \phi^j e_{t-j}\right) = \sum_{j=0}^{t-1} E(\phi^j e_{t-j}) = \sum_{j=0}^{t-1} \phi^j \cancel{E(e_{t-j})}^{\,0} = 0$

$Var(X_t) = Var\left(\sum_{j=0}^{t-1} \underset{\text{fix}}{\phi^j}\, \underset{\text{random}}{e_{t-j}}\right) = \sum_{j=0}^{t-1} Var(\phi^j e_{t-j}) = \sum_{j=0}^{t-1} \phi^{2j}\, \cancel{Var(e_{t-j})}^{\,\sigma^2}$

uncorrelated

$e \sim WN$

$= \sigma^2 \underbrace{\sum_{j=0}^{t-1} \phi^{2j}}$ $\quad 1+\phi^2+\phi^4+\cdots = \frac{1-\phi^{2t}}{1-\phi^2}$

$= \sigma^2 \cdot \frac{1-\phi^{2t}}{1-\phi^2} \Rightarrow$ non-stationary

c. $cov(X_t, X_{t-h}) = cov\left(\sum_{i=0}^{t-1} \underline{\phi^i e_{t-i}}, \sum_{j=0}^{t-h-1} \underline{\phi^j e_{t-j}}\right)$

$i=0$	e_t	$j=0$	e_{t-h}
$i=1$	e_{t-1}	$j=1$	e_{t-h-1}
$\vdots$	$\vdots$	$\vdots$	
$t-1$	e	$t-h-1$	e_1

$= cov\left(\underline{\sum_{i=0}^{h-1} \phi^i e_{t-i}} + \sum_{i=h}^{t-1} \phi^i e_{t-i}, \underline{\sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}}\right)$ no correlation $\Rightarrow$ cov $=0$

$= cov\left(\underline{\sum_{i=h}^{t-1} \phi^i e_{t-i}}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$ format transformation

$= cov\left(\underline{\sum_{k=0}^{t-h-1} \phi^{k+h} e_{t-h-k}}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$

$= \phi^h cov\left(\sum_{k=0}^{t-h-1} \phi^k e_{t-h-k}, \sum_{j=0}^{t-h-1} \phi^j e_{t-h-j}\right)$

$= \phi^h \underline{Var(X_{t-h})}$ $\quad \left(\frac{Var(X_{t-h})}{Var(X_t)}\right)^{\frac{1}{2}}$

d. when $|\phi| < 1$, $\lim_{t\to\infty} \phi^t = 0$, so $\lim_{t\to\infty} Var(X_t) = \lim_{t\to\infty} \sigma^2 \frac{1-\cancel{\phi^{2t}}^{\,0}}{1-\phi^2} = \frac{\sigma^2}{1-\phi^2}$

$\because \lim_{t\to\infty} Var(X_t) = \lim_{t\to\infty} Var(X_{t-h}) \approx \frac{\sigma^2}{1-\phi^2}$

So $\lim_{t\to\infty} correlation(X_t, X_{t-h}) \approx \phi^h \cdot \cancel{\frac{\frac{\sigma^2}{1-\phi^2}}{\frac{\sigma^2}{1-\phi^2}}}^{\,1} = \phi^h$, $h>0$