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Cooperative games with restricted formation of coalitions

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ABSTRACT

In the study of cooperative games, restricted cooperation between players is typically modeled by a set system of feasible coalitions of the players. In this paper, we go one step further and allow for a distinction among players within a feasible coalition, between those who are able to form the coalition and those who are not. This defines a contracting map, a choice function. We introduce the notion of quasi-building system and require that such a choice function gives rise to a quasi-building system. Many known set systems and structures studied in the literature are covered by quasi-building systems. For transferable utility games having a quasi-building system as cooperation structure we take as a solution the average of the marginal vectors that correspond to the set of rooted trees that are compatible with the quasi-building system. Properties of this solution, called the AMV-value, are studied. Relations with other solutions in the literature are also studied. To establish that the AMV-value is an element of the core, we introduce appropriate convexity-type conditions for the game with respect to the underlying quasi-building system. In case of universal cooperation, the AMV-value coincides with the Shapley value.

1. Introduction

The classical model of cooperative games with transferable utility (TU-games) fits the case of universal cooperation. Any subset of players is assumed to be able to form a coalition and obtain some worth which can be freely distributed as payoff amongst its members. A solution determines how much payoff each player receives.

In many situations, universality of cooperation fails owing to the existence of some restrictions on forming feasible coalitions. One of the most well-known examples with non-universal cooperation is due to Myerson [18], where, for a given graph of communication links between players, the possibility of forming coalitions is modeled by connectivity in the graph. Later on, other restrictions on cooperation among players have been modeled by certain combinatorial structures, such as precedence constraints or distributive lattices in [13,14], permission structures in [9], convex geometries in [4,5], antimatroids in [1], augmenting systems in [3], building sets in [17], directed communication structure in [15], and dominance structure in [16]. In each aforementioned model, marginal vectors are defined, although using different methods, and Shapley-type values are studied as solutions, that is, the average of these marginal vectors is taken as a solution concept to determine how to distribute the worth of the grand coalition of all players.

In this paper, we go one step further and assume not only that cooperation between players is restricted to a set of feasible coalitions, but also that instead of all players only a subset of the players in a feasible coalition is able to form the coalition

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and receive a marginal contribution if they do so. Applications which describe such a situation are the water distribution problem along a river in [2] and the problem of payoff distribution in economical hierarchies and networks in [8].

To model cooperative games with restricted formation of coalitions we introduce quasi-building systems as combinatorial structure. A quasi-building system on a set of players consists of a set system of feasible coalitions and a choice function that assigns to every feasible coalition a set of players that are able to form the coalition, satisfying the conditions (Q1) and (Q2) in Section 2. A quasi-building system gives rise to a nonempty collection of rooted trees on the players set. Each such a rooted tree represents a hierarchical structure compatible with the quasi-building system. Namely, the coalition consisting of any player and his successors in the tree is a feasible coalition and that player belongs to the choice set of this coalition.

Given a TU-game on a quasi-building system, for each compatible rooted tree we define a marginal vector, which assigns to every player as payoff how much this player contributes in worth when forming a coalition with his successors in the tree. We call the average of all such marginal vectors the Average Marginal Vector value (the AMV-value). We study its properties and relations to other solutions in the literature. We define union stable, intersection-closed, and chain quasi-building systems and for each of these subclasses we give conditions under which the AMV-value is an element of the core.

The paper is organized as follows. In Section 2 quasi-building systems are introduced and studied. In Section 3 the AMV-value is defined and several properties are shown. In Section 4 subclasses of union stable, intersection-closed, and chain quasi-building systems are introduced and convexity-type conditions for each of these classes are given under which the AMV-value is in the core. In Section 5 we give examples of these subclasses and their relations with other known combinatorial structures.

2. Cooperative games on quasi-building systems

A cooperative game with transferable utility, or TU-game, is a pair $([n], v)$, where $[n] = \{1, \ldots, n\}$ is a set of players and $v : 2^{[n]} \to \mathbb{R}$ is a characteristic function with $v(\emptyset) = 0$. The function v assigns a real number, the worth $v(S)$, to every coalition $S \in 2^{[n]}$ of players, which can be freely distributed as payoff among its members. We consider a vector $x \in \mathbb{R}^n$ as a payoff vector which assigns a payoff x_i to player $i \in [n]$. The problem is how much payoff every player should receive. The most well-known single-valued solution concept is the Shapley value [20] assigning to each TU-game the average of all marginal vectors of the game. Each linear ordering on the players set induces as marginal vector a payoff vector at which each player receives as payoff his contribution to the worth when he joins the coalition consisting of his predecessors in the linear ordering to form a larger coalition.

To model cooperation with restricted formation of coalitions, we introduce the following combinatorial structure.

Definition 2.1. A pair $\mathcal{Q} = (\mathcal{H}, U)$ is a *quasi-building system* on $[n]$ if it satisfies the following conditions:

(Q1) $\mathcal{H} \subseteq 2^{[n]}$ is a set system on $[n]$ containing both $\emptyset$ and $[n]$, and $U : \mathcal{H} \to 2^{[n]}$ is a choice function, that is, for every $H \in \mathcal{H}$ it holds that $U(H) \subseteq H$ and if $H \neq \emptyset$ then $U(H) \neq \emptyset$.

(Q2) For each $H \in \mathcal{H}$ and $h \in U(H)$ there exists a unique maximal partition of $H \setminus \{h\}$ into elements of $\mathcal{H}$. Every $S \subseteq H \setminus \{h\}$ satisfying $S \in \mathcal{H}$ is a subset of one of the members of this partition.

In Section 5 we show that quasi-building systems can express restricted cooperative situations that are induced by graphs, precedence constraints, or dominance structures, or are described by set systems like building sets, augmenting systems, distributive lattices, convex geometries, and antimatroids.

For $\mathcal{H}$ being a set of feasible coalitions on players set $[n]$, condition (Q1) says that both the empty set and the grand coalition of all players are feasible[1] and that every nonempty feasible coalition contains a nonempty choice set. Condition (Q2) states that when a player in the choice set of a feasible coalition is removed, the remaining players can avoid conflicts in forming subcoalitions. This is because there is a unique maximal partition of the remaining players into feasible subcoalitions and, moreover, subsets of two or more of these subcoalitions are not able to form a feasible coalition. Note that, for a given set system, there may exist different choice functions which satisfy condition (Q2). By making use of different choice functions, more distinctions between different cooperative situations can be expressed, as will be discussed in Section 5 for augmenting systems and convex geometries.

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on $[n]$ and $v : \mathcal{H} \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. We consider $\mathcal{H}$ as a coalition structure on a set of n players and $v(H)$, $H \in \mathcal{H}$, the worth of feasible coalition H. We call the pair $(v, \mathcal{Q})$ a quasi-building system game on the players set $[n]$. The collection of all quasi-building system games on $[n]$ is denoted by $\mathcal{V}$. A value on $\mathcal{V}$ is a mapping $f : \mathcal{V} \to \mathbb{R}^n$, assigning the payoff vector $f(v, \mathcal{Q})$ to any quasi-building system game $(v, \mathcal{Q}) \in \mathcal{V}$.

In the next section we propose a value on $\mathcal{V}$. For this we need the notion of a rooted tree compatible with a quasi-building system. Recall that a tree is a graph on a set of nodes such that any two different nodes are joined by a unique path in the graph. A rooted tree T is a tree with one of the nodes designated as the root, denoted by $r(T)$. For a rooted tree T, node j is a successor of node i in T if i belongs to the path in T joining j and $r(T)$. Node j is an immediate successor of node i in T if j is a

[1] In case the grand coalition of all players is not feasible, we assume that it has a unique maximal partition into feasible coalitions satisfying that every feasible coalition is a subset of one of the partition members. The analysis can then be applied separately to every partition member.

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successor of i in T and $\{i, j\}$ is an edge of T. The set of immediate successors of node i in tree T is denoted by $S^T(i)$, the set of successors of node i in T by $F^T(i)$, and $\bar{F}^T(i) = F^T(i) \cup \{i\}$ denotes the set of i's successors including himself.

Given a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$, for any $H \in \mathcal{H}$ and $h \in U(H)$, let $P(H \setminus \{h\})$ denote the unique maximal partition of the set $H \setminus \{h\}$ into feasible subcoalitions.

Definition 2.2. For a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$, a rooted tree T on $[n]$ is *compatible with* $\mathcal{Q}$ if for each $i \in [n]$ it holds that $\bar{F}^T(i) \in \mathcal{H}$, $i \in U(\bar{F}^T(i))$, and $P(F^T(i)) = \{\bar{F}^T(j) \mid j \in S^T(i)\}$.

In words, a rooted tree T on $[n]$ is compatible with a quasi-building system on $[n]$ if for every node $i \in [n]$ it holds that the coalition consisting of i and all his successors in T is a feasible coalition and i is in the choice set of this coalition. Moreover, each member of the unique maximal partition of the set of successors of i consists of an immediate successor of i and all his successors in T. In economics, a rooted tree represents a hierarchy on the set of players (see, for example, [8]). Thus, a rooted tree compatible with a quasi-building system represents a hierarchy on the set of players that is compatible with the restriction that only players in choice sets are able to form coalitions as described by the quasi-building system.

For a quasi-building system $\mathcal{Q}$, we denote by $\mathcal{T}(\mathcal{Q})$ the set of rooted trees that are compatible with $\mathcal{Q}$.

Theorem 2.3. *Let $\mathcal{Q}$ be a quasi-building system on $[n]$, then $\mathcal{T}(\mathcal{Q}) \neq \emptyset$.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. We construct a rooted tree T on $[n]$ that is compatible with $\mathcal{Q}$ as follows. From (Q1) it follows that $[n] \in \mathcal{H}$ and $U([n]) \neq \emptyset$. As root $r(T)$ we take any $r \in U([n])$. According to (Q2) there exists a unique maximal partition $S_1, \ldots, S_k$ of $[n] \setminus \{r\}$ for some $k \geq 1$ such that $S_j \in \mathcal{H}$ for $j = 1, \ldots, k$. In each $S_j, j = 1, \ldots, k$, there exists according to (Q1) an element $r_j \in U(S_j)$. We take r_j as immediate successor of r in T, for $j = 1, \ldots, k$. According to (Q2), for every $j = 1, \ldots, k$ there exists a unique maximal partition $S_{j,1}, \ldots, S_{j,k_j}$ of $S_j \setminus \{r_j\}$ for some $k_j \geq 1$ such that $S_{j,h} \in \mathcal{H}$ for $h = 1, \ldots, k_j$. In each $S_{j,h}$, $h = 1, \ldots, k_j, j = 1, \ldots, k$, there exists according to (Q1) an element $r_{j,h} \in U(S_{j,h})$, becoming in T an immediate successor of r_j, and so on. By this construction we obtain a rooted tree T compatible with $\mathcal{Q}$. □

The next example shows that two quasi-building systems with the same set system but different choice functions may lead to different collections of compatible rooted trees.[2]

Example 2.4. Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on the set [3], where $\mathcal{H} = 2^{[3]}$ and $U(H) = H$ for all $H \in \mathcal{H}$. All six rooted line-trees on the set [3] are compatible with this system.

Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on the set [3], where $\mathcal{H} = 2^{[3]}$ and $U([3]) = [3]$, $U(\{1, 2\}) = \{1\}$, $U(\{1, 3\}) = \{3\}$, and $U(\{2, 3\}) = \{2\}$. There are only three rooted trees which are compatible with this system. One tree has node 1 as root and $\{1, 2\}$ and $\{2, 3\}$ as edges, a second tree has node 2 as root and $\{2, 3\}$ and $\{3, 1\}$ as edges, and the third tree has node 3 as root and $\{3, 1\}$ and $\{1, 2\}$ as edges.

3. The average marginal vector value

As a solution concept for a quasi-building system game, we take the average of the marginal vectors corresponding to all rooted trees that are compatible with the quasi-building system. At a marginal vector corresponding to a compatible rooted tree every player receives as payoff what he contributes in worth when joining his successors in the tree. For a quasi-building system game $(v, \mathcal{Q}) \in \mathcal{V}$ and compatible rooted tree $T \in \mathcal{T}(\mathcal{Q})$, the marginal vector $m^T(v, \mathcal{Q}) \in \mathbb{R}^n$ is the payoff vector given by

$$m_i^T(v, \mathcal{Q}) = v(\bar{F}^T(i)) - \sum_{j \in S^T(i)} v(\bar{F}^T(j)), \quad i \in [n].$$

Since $i \in U(\bar{F}^T(i))$ for all $i \in [n]$, a player can only receive a marginal contribution if he belongs to the choice set of the feasible coalition that consists of himself and all his successors in some compatible rooted tree T.

Definition 3.1. On the class of quasi-building system games $\mathcal{V}$ the *Average Marginal Vector value*, or *AMV-value*, assigns to every quasi-building system game $(v, \mathcal{Q}) \in \mathcal{V}$ the payoff vector

$$AMV(v, \mathcal{Q}) = \frac{1}{|\mathcal{T}(\mathcal{Q})|} \sum_{T \in \mathcal{T}(\mathcal{Q})} m^T(v, \mathcal{Q}).$$

The AMV-value is well-defined on the class of quasi-building system games, since according to Theorem 2.3 every quasi-building system has at least one rooted tree compatible with it. The AMV-value takes into account that only players that are in the choice set of a coalition can receive a marginal contribution as an outcome of the formation of coalitions.

2 Since for a feasible singleton coalition in a quasi-building set the choice set always consists of the player himself, we do not write down in examples the choice sets of feasible singleton coalitions.

The AMV-value satisfies the following properties. In case the set system is the collection of all subsets of players, i.e., $\mathcal{H} = 2^{[n]}$, and $U(H) = H$ for all $H \in \mathcal{H}$, the AMV-value of the quasi-building system game $(v, (2^{[n]}, U))$ is the Shapley value [20] of the TU-game v. The next two properties are standard.

Definition 3.2. A value f on $\mathcal{V}$ satisfies *efficiency* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ it holds that

$$\sum_{i\in[n]} f_i(v, \mathcal{Q}) = v([n]).$$

An efficient value assigns to every quasi-building system game a payoff vector which distributes the worth of the grand coalition among the players.

Proposition 3.3. *The AMV-value satisfies efficiency.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$. Since the AMV-value of $(v, \mathcal{Q})$ is the average of all marginal vectors $m^T(v, \mathcal{Q})$, $T \in \mathcal{T}(\mathcal{Q})$, it suffices to show that $\sum_{j=1}^{n} m_j^T(v, \mathcal{Q}) = v([n])$ for all $T \in \mathcal{T}(\mathcal{Q})$. Take any $T \in \mathcal{T}(\mathcal{Q})$. For each $i \in [n]$ it holds that

$$\begin{aligned}\sum_{j\in\bar{F}^T(i)} m_j^T(v, \mathcal{Q}) &= \sum_{j\in\bar{F}^T(i)} v(\bar{F}^T(j)) - \sum_{j\in\bar{F}^T(i)} \sum_{k\in S^T(j)} v(\bar{F}^T(k)) \\ &= \sum_{j\in\bar{F}^T(i)} v(\bar{F}^T(j)) - \sum_{k\in F^T(i)} v(\bar{F}^T(k)) \\ &= v(\bar{F}^T(i)),\end{aligned}$$

since $k \in F^T(i)$ if and only if $k \in S^T(j)$ for some $j \in \bar{F}^T(i)$. Since $[n] = \bar{F}^T(r(T))$, it follows that

$$\sum_{j\in[n]} m_j^T(v, \mathcal{Q}) = \sum_{j\in\bar{F}^T(r(T))} m_j^T(v, \mathcal{Q}) = v(\bar{F}^T(r(T))) = v([n]). \quad \square$$

Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on $[n]$. For any two quasi-building system games $(v, \mathcal{Q})$ and $(w, \mathcal{Q})$ in $\mathcal{V}$ and $a, b \in \mathbb{R}$, the quasi-building system game $(av + bw, \mathcal{Q})$ in $\mathcal{V}$ is defined by $(av + bw)(H) = av(H) + bw(H)$ for all $H \in \mathcal{H}$.

Definition 3.4. A value f on $\mathcal{V}$ satisfies *linearity* if for every $(v, \mathcal{Q}),\ (w, \mathcal{Q}) \in \mathcal{V}$ and $a, b \in \mathbb{R}$ it holds that

$$f(av + bw, \mathcal{Q}) = af(v, \mathcal{Q}) + bf(w, \mathcal{Q}).$$

Proposition 3.5. *The AMV-value satisfies linearity.*

Proof. Let $(v, \mathcal{Q}),\ (w, \mathcal{Q}) \in \mathcal{V}$ and $a, b \in \mathbb{R}$. For each $T \in \mathcal{T}(\mathcal{Q})$ and $i \in [n]$ it holds that

$$\begin{aligned}m_i^T(av + bw, \mathcal{Q}) &= (av + bw)(\bar{F}^T(i)) - \sum_{j\in S^T(i)} (av + bw)(\bar{F}^T(j)) \\ &= a\left(v(\bar{F}^T(i)) - \sum_{j\in S^T(i)} v(\bar{F}^T(j))\right) + b\left(w(\bar{F}^T(i)) - \sum_{j\in S^T(i)} w(\bar{F}^T(j))\right) \\ &= am_i^T(v, \mathcal{Q}) + bm_i^T(w, \mathcal{Q}),\end{aligned}$$

which implies that the AMV-value is linear, since the AMV-value of a quasi-building system game is the average of the marginal vectors for the game induced by all rooted trees that are compatible with the quasi-building system. $\square$

The null player property is a widely known concept. In a standard TU-game, a player is a null player if he never contributes to the worth of any coalition he joins. For a quasi-building system game, however, a player is only able to receive a marginal contribution if he is a member of the choice set of that coalition.

Definition 3.6. For a quasi-building system game $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$, a player $i \in [n]$ is a *restricted null player* if for every $H \in \mathcal{H}$ such that $i \in U(H)$ it holds that

$$v(H) - \sum_{K\in P(H\setminus\{i\})} v(K) = 0.$$

The definition of a restricted null player for a quasi-building system game depends not only on the set system but also on the choice function.

Definition 3.7. A value f on $\mathcal{V}$ satisfies the *restricted null player property* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ it holds that $f_i(v, \mathcal{Q}) = 0$ whenever i is a restricted null player for $(v, \mathcal{Q})$.

The restricted null player property says that a player who never contributes any worth to form a feasible coalition should receive zero payoff.

Proposition 3.8. *The AMV-value satisfies the restricted null player property.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$, and let i be a restricted null player for $(v, \mathcal{Q})$. Since the AMV-value of $(v, \mathcal{Q})$ is the average of all marginal vectors $m^T(v, \mathcal{Q})$, $T \in \mathcal{T}(\mathcal{Q})$, it suffices to show that $m_i^T(v, \mathcal{Q}) = 0$ for all $T \in \mathcal{T}(\mathcal{Q})$. Take any $T \in \mathcal{T}(\mathcal{Q})$. Let $H = \bar{F}^T(i)$, then $i \in U(H)$ and $H \in \mathcal{H}$. Since i is a restricted null player for $(v, \mathcal{Q})$, it follows that

$$\begin{aligned} m_i^T(v, \mathcal{Q}) &= v(\bar{F}^T(i)) - \sum_{j \in S^T(i)} v(\bar{F}^T(j)) \\ &= v(H) - \sum_{K \in P(H \setminus \{i\})} v(K) = 0. \quad \square \end{aligned}$$

Some of the feasible coalitions in a quasi-building system game are essential.

Definition 3.9. For a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$, a subset $H \in \mathcal{H}$ is *essential* if it is equal to $[n]$ or is a member of $P(H' \setminus \{h\})$ for some essential $H' \in \mathcal{H}$ and $h \in U(H')$.

This recursively defines, for a quasi-building system $\mathcal{Q}$ on a players set, the set of essential coalitions for $\mathcal{Q}$. The grand coalition is essential and also the members of a maximal partition induced by an essential coalition. A coalition of $\mathcal{H}$ is then inessential for $\mathcal{Q} = (\mathcal{H}, U)$, if it is not the grand coalition and not a member of the maximal partition $P(H \setminus \{h\})$ for any essential $H \in \mathcal{H}$ and $h \in U(H)$. For a quasi-building system $\mathcal{Q}$, the set of inessential coalitions is denoted by $\mathcal{I}(\mathcal{Q})$.

Example 3.10. Consider the quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on the set $[3]$, where $\mathcal{H} = 2^{[3]} \setminus \{\{2, 3\}\}$ and $U([3]) = \{1\}$, $U(\{1, 2\}) = \{1\}$, $U(\{1, 3\}) = \{1\}$. The coalitions $\{1, 2\}$, $\{1, 3\}$, and $\{1\}$ are inessential, because they are not a member of any maximal partition. Note that $U([3]) = \{1\}$ and $P(\{2, 3\}) = \{\{2\}, \{3\}\}$, which implies that both $\{2\}$ and $\{3\}$ are essential coalitions.

In a hierarchical structure, being a rooted tree, any connected subset of nodes not containing all the successors of every node is inessential. For a quasi-building system game an inessential coalition does not play any role in the formation of coalitions.

Lemma 3.11. *Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on $[n]$ and let $H \in \mathcal{H}$. Then $H \in \mathcal{I}(\mathcal{Q})$ if and only if $H \neq \bar{F}^T(i)$ for all $T \in \mathcal{T}(\mathcal{Q})$ and $i \in [n]$.*

Proof. Suppose $H \in \mathcal{I}(\mathcal{Q})$ and $H = \bar{F}^T(i)$ for some $T \in \mathcal{T}(\mathcal{Q})$ and $i \in [n]$. Since $H \in \mathcal{I}(\mathcal{Q})$, it holds that $H \neq [n]$. Therefore there exists $H_1 \in \mathcal{H}$ and $h_1 \in U(H_1)$ such that $H_1 = \bar{F}^T(h_1)$ and $H \in P(H_1 \setminus \{h_1\})$, which would be a contradiction unless $H_1 \in \mathcal{I}(\mathcal{Q})$. If $H_1 \in \mathcal{I}(\mathcal{Q})$, by following the same argument there exists $H_2 = \bar{F}^T(h_2)$ for some $h_2 \in U(H_2)$ satisfying $H_1 \in P(H_2 \setminus \{h_2\})$. Then it must hold that $H_2 \in \mathcal{I}(\mathcal{Q})$ to avoid a contradiction, and so on. Since the players set is finite and $[n] \notin \mathcal{I}(\mathcal{Q})$, we obtain a contradiction.

Next, suppose $H \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$. If $H = [n]$, then $H = \bar{F}^T(r(T))$ for every tree $T \in \mathcal{T}(\mathcal{Q})$. If $H \neq [n]$, then there exists $H_1 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H \in P(H_1 \setminus \{h_1\})$ for some $h_1 \in U(H_1)$. Since H_1 is essential, there exists $H_2 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ such that $H_1 \in P(H_2 \setminus \{h_2\})$ for some $h_2 \in U(H_2)$, and so on. Since the player set is finite, there is a finite sequence of players $(h_1, \ldots, h_m)$ and feasible sets $(H_0, H_1, \ldots, H_m)$ for some $m < n$ such that $H_m = [n]$, $H_0 = H$, and $h_j \in U(H_j)$ and $H_{j-1} \in P(H_j \setminus \{h_j\})$ for $j = 1, \ldots, m$. Then as in the proof of Theorem 2.3 we can construct a tree $T \in \mathcal{T}(\mathcal{Q})$ with $r(T) = h_m$ and containing $\{h_j, h_{j-1}\}$ for $j = 2, \ldots, m$ among its edges. For this T it holds that $H = \bar{F}^T(j)$ for some $j \in S^T(h_1)$. $\square$

Definition 3.12. A value f on $\mathcal{V}$ satisfies the *inessential coalition property* if for every $(v, \mathcal{Q}), (w, \mathcal{Q}) \in \mathcal{V}$ it holds that $f(v, \mathcal{Q}) = f(w, \mathcal{Q})$ whenever $v(H) = w(H)$ for all $H \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$.

A solution that satisfies the inessential coalition property is independent of the worths of feasible coalitions that do not play any role in the formation of coalitions. The AMV-value on the class of quasi-building system games satisfies this property, since according to Lemma 3.11 for a quasi-building system game the marginal vector corresponding to any compatible rooted tree only depends on the worths of essential coalitions.

Proposition 3.13. *The AMV-value satisfies the inessential coalition property.*

While an inessential coalition cannot be a maximal subset of successors of a node in any compatible rooted tree, there might also exist feasible coalitions that are maximal subsets of successors of nodes in every compatible rooted tree.

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Definition 3.14. For a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$, a subset $H \in \mathcal{H}$ is *closed* if for every $H' \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $H \subset H'$ it holds that $H \cap U(H') = \emptyset$.

A closed coalition of a quasi-building system on a players set is a feasible coalition of which no player belongs to the choice set of an essential coalition that contains it. Note that the grand coalition is by definition closed. In a hierarchical structure any player together with his successors in the hierarchy is a closed coalition.

Definition 3.15. A value f on $\mathcal{V}$ satisfies the *closed coalition property* if for every $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$ it holds that $\sum_{i\in H} f_i(v, \mathcal{Q}) = v(H)$ whenever $H \in \mathcal{H}$ is a closed coalition.

A solution that satisfies the closed coalition property allocates to the players of a feasible coalition whose members are not able to form bigger essential coalitions, as total payoff the worth of that coalition. Note that the closed coalition property implies efficiency.

Proposition 3.16. *The AMV-value satisfies the closed coalition property.*

Proof. Let $(v, \mathcal{Q}) \in \mathcal{V}$ with $\mathcal{Q} = (\mathcal{H}, U)$, and let $H \in \mathcal{H}$ be a closed coalition for $(v, \mathcal{Q})$. We first show that for every $T \in \mathcal{T}(\mathcal{Q})$ it holds that $H = \bar{F}^T(i)$ for some $i \in [n]$. Suppose there exists $T \in \mathcal{T}(\mathcal{Q})$ such that $H \neq \bar{F}^T(j)$ for all $j \in [n]$. Because of condition (Q2) there exists $i \in H$ such that $H \subset \bar{F}^T(i)$. This implies that $i \in H \cap U(\bar{F}^T(i))$, whereas $\bar{F}^T(i) \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, which contradicts that H is a closed coalition. Since the AMV-value of $(v, \mathcal{Q})$ is the average of the marginal vectors corresponding to by all rooted trees that are compatible with $\mathcal{Q}$, it suffices to show that $\sum_{j\in H} m_j^T(v, \mathcal{Q}) = v(H)$ for all $T \in \mathcal{T}(\mathcal{Q})$. Take any $T \in \mathcal{T}(\mathcal{Q})$. Let $i \in H$ be such that $\bar{F}^T(i) = H$, then it follows that

$$\sum_{j\in H} m_j^T(v, \mathcal{Q}) = \sum_{j\in \bar{F}^T(i)} m_j^T(v, \mathcal{Q}) = v(\bar{F}^T(i)) = v(H). \quad \square$$

From the proof of the proposition and from Lemma 3.11 it follows that every closed coalition is essential. Also note that in a hierarchical structure a connected subset of nodes is either closed or inessential.

4. Core

In this section we introduce the subclasses of union stable, intersection-closed, and chain quasi-building systems and give convexity-type conditions for each of these classes under which the AMV-value is an element of the core.

The core of a cooperative game is the set of efficient and coalitionally rational payoff vectors. On the class of quasi-building system games the core is defined as follows.

Definition 4.1. Let $\mathcal{Q} = (\mathcal{H}, U)$ be a quasi-building system on $[n]$. The *core* of a game $(v, \mathcal{Q}) \in \mathcal{V}$ is given by the set

$$C(v, \mathcal{Q}) = \left\{ x \in \mathbb{R}^n | \sum_{i=1}^n x_i = v([n]), \ \sum_{i\in H} x_i \geq v(H) \text{ for all } H \in \mathcal{H} \right\}.$$

The core of a quasi-building system game reflects that only coalitions that are feasible are able to block payoff vectors.

4.1. Union stable quasi-building systems

In this subsection we consider the class of union stable quasi-building systems.

Definition 4.2. A quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$ is *union stable* if the following condition holds:

(Q3) For every $H_1 \in \mathcal{H}$ and $H_2 \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $H_1 \cap U(H_2) \neq \emptyset$, it holds that $H_1 \cup H_2 \in \mathcal{H}$.

When $[n]$ is a set of players, condition (Q3) says that the union of two feasible coalitions, of which at least one is essential, is also feasible if their intersection contains an element in the choice set of the essential coalition. Note that this condition is weaker than the union stable condition for set systems. Union stable quasi-building system games cover games on any kind of communication graphs, building sets, augmenting systems, antimatroids, and distributive lattices, see Section 5 for details.

Definition 4.3. For a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$, a pair (A, B) is *union-closed* if $A \in \mathcal{H}$, $B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, and $A \cap U(B) \neq \emptyset$.

A feasible coalition A and an essential coalition B of a quasi-building system form a union-closed pair of coalitions if there exists a player in A which is in the choice set of B. In case of a union stable quasi-building system the union $A \cup B$ is feasible.

Definition 4.4. Let $\mathcal{Q} = (\mathcal{H}, U)$ be a union stable quasi-building system on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-supermodular* if for every union-closed pair (A, B), $i \in A \cap U(B)$, and maximal partition $\mathcal{D}$ of $A \cap B \setminus \{i\}$ into elements of $\mathcal{H}$ it holds that

$$f(A) + \sum_{K \in P(B\setminus\{i\})} f(K) \leq f(A \cup B) + \sum_{K \in \mathcal{D}} f(K).$$

Note that condition (Q2) implies that the set $B \setminus \{i\}$ has a unique maximal partition $P(B \setminus \{i\})$. Since the maximal partition of $A \cap B \setminus \{i\}$ into feasible coalitions may not be unique, the condition should hold for all such maximal partitions. The next example shows that, for a union-closed pair, a maximal partition of its intersection into feasible coalitions might not be unique.

Example 4.5. Consider a quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on the set [5], where $\mathcal{H} = \{[5], \{1, 2, 3, 4\}, \{1, 2, 3, 5\}, \{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ and $U([5]) = \{4, 5\}$, $U(\{1, 2, 3, 4\}) = \{1\}$, $U(\{1, 2, 3, 5\}) = \{3\}$, $U(\{1, 2\}) = \{1\}$, $U(\{2, 3\}) = \{3\}$. $\mathcal{Q}$ is union stable and has two compatible rooted trees: $T_1 = \{\{4, 3\}, \{3, 1\}, \{1, 2\}, \{3, 5\}\}$ with root 4 and $T_2 = \{\{5, 1\}, \{1, 3\}, \{3, 2\}, \{1, 4\}\}$ with root 5. The pair (A, B) with $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$ is union-closed with $A \cap U(B) = \{4\}$. The intersection $A \cap B \setminus \{4\} = \{1, 2, 3\}$ has two maximal partitions into feasible coalitions, namely $\{\{1\}, \{2, 3\}\}$ and $\{\{1, 2\}, \{3\}\}$. Both $\{1\}$ and $\{3\}$ are inessential coalitions.

In the next theorem, it is shown that $\mathcal{Q}$-supermodularity is a sufficient condition for the AMV-value to be in the core.

Theorem 4.6. *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a union stable quasi-building system game. If v is $\mathcal{Q}$-supermodular, then $AMV(v, \mathcal{Q}) \in C(v, \mathcal{Q})$.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for every $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) \geq v(H).$$

Take any $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$. Let $H_1, \dots, H_s$ be the maximally connected subsets of H in T. For $k = 1, \dots, s$ denote $H_k = \{i_1^k, \dots, i_{t_k}^k\}$ and let $h < l$ if $i_h^k \in F^T(i_l^k)$. For $k = 1, \dots, s$ denote $r_k = i_{t_k}^k$ and let $h < l$ if $r_h \in F^T(r_l)$. Since T is a tree, r_k is the root of the subtree of T on $\bar{F}^T(r_k)$ containing the set H_k, $k = 1, \dots, s$. Moreover, $\bar{F}^T(r_s)$ contains the sets H and $\bar{F}^T(r_k)$ for $k = 1, \dots, s - 1$. For $k = 1, \dots, s$ it holds that

$$H_k = \bar{F}^T(r_k) \setminus \Big(\bigcup_{h=1}^{t_k}\Big(\bigcup_{j \in S^T(i_h^k)\setminus H_k} \bar{F}^T(j)\Big)\Big),$$

which implies that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) = \sum_{k=1}^{s}\Big(v(\bar{F}^T(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k)\setminus H_k} v(\bar{F}^T(j))\Big).$$

To show that the latter expression is at least equal to $v(H)$, let $I^k = H \cup (\cup_{h=1}^k \bar{F}^T(r_h))$ for $k = 0, \dots, s$ and $I_h^k = I^{k-1} \cup (\cup_{j=1}^h \bar{F}^T(i_j^k))$ for $h = 0, \dots, t_k, k = 1, \dots, s$. Note that $I^0 = H$ and $I^s = \bar{F}^T(r_s)$ and that for $k = 1, \dots, s$ it holds that $I_0^k = I^{k-1}$ and $I_{t_k}^k = I^k$. We first show by induction that $I_h^k \in \mathcal{H}$ for all $h = 0, \dots, t_k$, $k = 1, \dots, s$. Since $I_0^1 = I^0 = H$ and $H \in \mathcal{H}$, it holds that $I_0^1 \in \mathcal{H}$. Suppose $I_h^1 \in \mathcal{H}$ for some $h < t_1$. Since $\bar{F}^T(i_h^1) \in \mathcal{H}$ and $i_h^1 \in H \cap U(\bar{F}^T(i_h^1)) \subseteq I_h^1 \cap U(\bar{F}^T(i_h^1))$, it follows from (Q3) that the union I_{h+1}^1 of the sets I_h^1 and $\bar{F}^T(i_{h+1}^1)$ is in $\mathcal{H}$. In particular, this implies for $h = t_1 - 1$ that $I_{t_1}^1$ is in $\mathcal{H}$. Since $I_{t_1}^1 = I^1 = I_0^2$, it also holds that $I_0^2 \in \mathcal{H}$. Continuing the same argument, we obtain by induction that $I_h^k \in \mathcal{H}$ for all k and h.

Let $A = I_{h-1}^k$ and $B = \bar{F}^T(i_h^k)$ for some $h = 1, \dots, t_k$, $k = 1, \dots, s$. Then $A \in \mathcal{H}$, $B \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, $i_h^k \in A \cap U(B)$, and $A \cup B = I_h^k \in \mathcal{H}$. Hence, the pair (A, B) is union-closed in $\mathcal{Q}$. Concerning the intersection of A and B without i_h^k, for $j \in S^T(i_h^k) \setminus H_k$ define

$$D_h^k(j) = \{r \mid \bar{F}^T(r) \subset \bar{F}^T(j),\ \nexists\, l < k \text{ with } \bar{F}^T(r) \subset \bar{F}^T(r_l) \subset \bar{F}^T(j)\}.$$

Then $A \cap B \setminus \{i_h^k\}$ is maximally partitioned into elements of $\mathcal{H}$ by the collection

$$\mathcal{D} = \{\bar{F}^T(r) \mid r \in D_h^k(j),\ j \in S^T(i_h^k) \setminus H_k\} \cup \{\bar{F}^T(j) \mid j \in S^T(i_h^k) \cap H_k\}.$$

Since v is $\mathcal{Q}$-supermodular, $A = I_{h-1}^k$, and $A \cup B = I_h^k$, this implies that

$$v(I_{h-1}^k) + \sum_{K \in P(B\setminus\{i_h^k\})} v(K) \leq v(I_h^k) + \sum_{K \in \mathcal{D}} v(K).$$

Since $P(B \setminus \{i_h^k\}) \cap \mathcal{D} = \{\bar{F}^T(j) \mid j \in S^T(i_h^k) \cap H_k\}$, the terms indexed by these sets cancel on both sides and we obtain

$$v(I_{h-1}^k) + \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \le v(I_h^k) + \sum_{j \in S^T(i_h^k) \setminus H_k} \sum_{r \in D_h^k(j)} v(\bar{F}^T(r)).$$

Applying this inequality successively for $h = 1, \ldots, t_k$, $k = 1, \ldots, s$, we obtain that

$$v(I_0^1) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \le v(I_{t_s}^s) + \sum_{k=1}^{s} \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} \sum_{r \in D_h^k(j)} v(\bar{F}^T(r)).$$

Since $I_0^1 = H$, $I_{t_s}^s = \bar{F}^T(r_s)$, and each r_i, $i = 1, \ldots, s-1$, belongs to precisely one $D_h^k(j)$ for some $j \in S^T(i_h^k) \setminus H_k$, $h \in \{1, \ldots, t_k\}$, $k \in \{2, \ldots, s\}$, it follows that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(\bar{F}^T(r_k)) - \sum_{h=1}^{t_k} \sum_{j \in S^T(i_h^k) \setminus H_k} v(\bar{F}^T(j)) \Big) \ge v(H). \quad \square$$

4.2. Intersection-closed quasi-building systems

In this subsection we consider the class of intersection-closed quasi-building systems.

Definition 4.7. A quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$ is *intersection-closed* if the following conditions hold:

(Q4) If $H_1, H_2 \in \mathcal{H}$, then $H_1 \cap H_2 \in \mathcal{H}$.
(Q5) If $H_1, H_2 \in \mathcal{H}$, $H_1 \subset H_2$, and $i \in U(H_2) \cap H_1$, then $i \in U(H_1)$.

Condition (Q4) reflects the name of this subclass, as it is known as intersection-closedness, a property convex geometries possess, see [11]. When $[n]$ is a players set, condition (Q5) states that if a player is in the choice set of a feasible coalition, then he must also be in the choice set of any feasible subcoalition that contains this player. This is in line with the property called independence of irrelevant alternatives (IIA), Chernoff's Postulate 4 [7], or Sen's Condition α [19]. This property may not be compatible with union stable quasi-building systems. The second example in Example 2.4 is a union stable quasi-building system, but it does not satisfy condition (Q5), which can be seen if we take $H_1 = \{1, 2\}$ and $H_2 = [3]$. Intersection-closed quasi-building system games cover games on convex geometries and cycle-free graphical quasi-building systems, see Section 5 for details.

For the class of intersection-closed quasi-building system games, a convexity-type condition is defined as follows.

Definition 4.8. Let $\mathcal{Q} = (\mathcal{H}, U)$ be an intersection-closed quasi-building system on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-convex* if for all $S \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, $T \in \mathcal{H}$, $T \subset S$, and $i \in U(S) \cap T$ it holds that

$$f(T) - \sum_{K \in P(T \setminus \{i\})} f(K) \le f(S) - \sum_{K \in P(S \setminus \{i\})} f(K).$$

A game on an intersection-closed quasi-building system $\mathcal{Q}$ is $\mathcal{Q}$-convex if the marginal loss caused by a player is greater whenever he is removed from a larger essential coalition. This condition is similar to a convexity condition introduced in [5] on the class of games on convex geometries to guarantee that the marginal vectors corresponding to all maximal chains are in the core.

Theorem 4.9. *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be an intersection-closed quasi-building system game. If v is $\mathcal{Q}$-convex, then $AMV(v, \mathcal{Q}) \in C(v, \mathcal{Q})$.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for every $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) \ge v(H).$$

Take any $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$. Denote $H = \{i_1, \ldots, i_s\}$ and let $h < l$ if $i_h \in F^T(i_l)$. Since T is a rooted tree, i_s is uniquely determined. This implies that

$$\sum_{j \in H} m_j^T(v, \mathcal{Q}) = \sum_{k=1}^{s} \Big(v(\bar{F}^T(i_k)) - \sum_{K \in P(F^T(i_k))} v(K) \Big).$$

Let $Q_k = \bar{F}^T(i_k) \cap H$, $k = 1, \ldots, s$, then $Q_s = H$ and $Q_k \in \mathcal{H}$ for $k = 1, \ldots, s-1$, since $\bar{F}^T(i_k) \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ for $k = 1, \ldots, s-1$ and $H \in \mathcal{H}$. For $k = 1, \ldots, s$, since $i_k \in U(\bar{F}^T(i_k)) \cap Q_k$ and $Q_k \subseteq \bar{F}^T(i_k)$, it follows from (Q5) that $i_k \in U(Q_k)$

and thus $Q_k \setminus \{i_k\}$ has a unique maximal partition $P(Q_k \setminus \{i_k\})$. Clearly, each member of $P(Q_k \setminus \{i_k\})$, $k \in \{1, \ldots, s\}$, is equal to Q_j for some $j < k$. Since the game is $\mathcal{Q}$-convex, we obtain

$$m^T_{i_k}(v, \mathcal{Q}) = v(\bar{F}^T(i_k)) - \sum_{K \in P(F^T(i_k))} v(K) \geq v(Q_k) - \sum_{K \in P(Q_k \setminus \{i_k\})} v(K),$$

for $k = 1, \ldots, s$. Adding up this inequality for $k = 1, \ldots, s$, we get

$$\sum_{j \in H} m^T_j(v, \mathcal{Q}) \geq \sum_{k=1}^{s} \Big(v(Q_k) - \sum_{K \in P(Q_k \setminus \{i_k\})} v(K) \Big).$$

Since $Q_s = H$, this inequality becomes

$$\sum_{j \in H} m^T_j(v, \mathcal{Q}) \geq v(H) + \sum_{k=1}^{s-1} v(Q_k) - \sum_{k=1}^{s} \sum_{K \in P(Q_k \setminus \{i_k\})} v(K).$$

The last two terms cancel out since it holds that $\{Q_1, \ldots, Q_{s-1}\} = \bigcup_{k=1}^{s} P(Q_k \setminus \{i_k\})$, and the desired result follows. □

The next example is a game on a quasi-building system $\mathcal{Q}$ which is both union stable and intersection-closed. The game is $\mathcal{Q}$-convex, but not $\mathcal{Q}$-supermodular.

Example 4.10. Consider a quasi-building system game $(v, \mathcal{Q})$ with $\mathcal{Q} = (\mathcal{H}, U)$ on the set [5], where $\mathcal{H} = \{[5], \{1, 2, 3, 4\}, \{1, 2, 3\}, \{2, 4\}, \{1\}, \{2\}, \{3\}, \{5\}\}$, U is such that $U([5]) = U(\{1, 2, 3, 4\}) = \{4\}$, $U(\{1, 2, 3\}) = \{2\}$, $U(\{2, 4\}) = \{4\}$, and v is such that $v([5]) = 5$, $v(\{1, 2, 3, 4\}) = 2$, $v(\{1, 2, 3\}) = 2$, $v(\{2, 4\}) = 3$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{5\}) = 0$. $\mathcal{Q}$ is both union stable and intersection-closed, with $\{1, 2, 3, 4\}$, $\{2, 4\}$ and $\{2\}$ being inessential coalitions. There is one compatible tree, $T = \{\{4, 2\}, \{4, 5\}, \{2, 1\}, \{2, 3\}\}$ with root 4. Its corresponding marginal vector is therefore the AMV-value, $m^T(v, \mathcal{Q}) = AMV(v, \mathcal{Q}) = (0, 2, 0, 3, 0)$, and is an element of the core. This game is $\mathcal{Q}$-convex, but not $\mathcal{Q}$-supermodular (take the pair (A, B) with $A = \{2, 4\}$ and $B = \{1, 2, 3\}$).

4.3. Chain quasi-building systems

In this subsection, we consider the class of chain quasi-building systems.

Definition 4.11. A quasi-building system $\mathcal{Q} = (\mathcal{H}, U)$ on $[n]$ is a *chain quasi-building system* if the following condition holds:

(Q6) For every $H \in \mathcal{H}$ and $h \in U(H)$ it holds that $H \setminus \{h\} \in \mathcal{H}$.

Condition (Q6) is a combination of condition (Q2) and the one-point extension property, i.e., if $H \in \mathcal{H}$, $H \neq [n]$, then there exists $i \in [n] \setminus H$ such that $H \cup \{i\} \in \mathcal{H}$ and $i \in U(H \cup \{i\})$. The next lemma immediately follows from condition (Q6).

Lemma 4.12. *Let $\mathcal{Q} = (\mathcal{H}, U)$ be a chain quasi-building system on $[n]$. Then every rooted tree compatible with $\mathcal{Q}$ is a line-tree.*

This subclass contains distributive lattices and more general the sets of coalitions induced by dominance structures [16], augmentation systems containing the grand coalition, and convex geometries, see Section 5 for details.

Regarding a relationship between the AMV-value and the core on the class of chain quasi-building system games, we introduce the following condition.

Definition 4.13. Let $\mathcal{Q} = (\mathcal{H}, U)$ be a chain quasi-building system on $[n]$. A function $f : \mathcal{H} \to \mathbb{R}$ is *$\mathcal{Q}$-increasing* if

$$\sum_{i=1}^{k} (f(S^i) - f(S^i \setminus H_i)) \geq f(H)$$

holds for every $H \in \mathcal{H}$ and $S^1, \ldots, S^k \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ satisfying $S^1 \subset \cdots \subset S^k$, $H \subset S^k$, and $S^i \setminus H_i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, where $H_i = (S^i \setminus S^{i-1}) \cap H \neq \emptyset$, for all $i = 1, \ldots, k$.

A game on a chain quasi-building system $\mathcal{Q}$ is $\mathcal{Q}$-increasing if the worth of a feasible coalition is at most equal to the sum of its marginal contributions to any chain of essential coalitions.

Theorem 4.14. *Let $(v, \mathcal{Q}) \in \mathcal{V}$ be a chain quasi-building system game. If v is $\mathcal{Q}$-increasing, then $AMV(v, \mathcal{Q}) \in C(v, \mathcal{Q})$.*

Proof. Let $\mathcal{Q} = (\mathcal{H}, U)$. Since the AMV-value is efficient and the solution is the average of the marginal vectors corresponding to all compatible rooted trees, it suffices to show that for every $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$ it holds that

$$\sum_{j\in H} m_j^T(v, \mathcal{Q}) \geq v(H).$$

Take any $T \in \mathcal{T}(\mathcal{Q})$ and $H \in \mathcal{H}$. Let $H_1, \ldots, H_k$ be the maximal connected subsets of H in the tree T. By Lemma 4.12, T is a line-tree. Then, in the collection $\{\bar{F}^T(h) \mid h \in [n]\}$, for $i = 1, \ldots, k$, there exists unique minimal element $\bar{S}^i = \bar{F}^T(\bar{r}_i)$ such that $H_i \subseteq \bar{S}^i$ and there exists unique maximal $\underline{S}^i = \bar{F}^T(\underline{r}_i)$ such that $H_i \cap \underline{S}^i = \emptyset$. Note that $\bar{r}_i \in H_i$ and $\bar{S}^i \setminus H_i = \underline{S}^i$ for each $i = 1, \ldots, k$. The sets $\bar{S}^i$, $i = 1, \ldots, k$, can be ordered such that $\bar{S}^1 \subset \cdots \subset \bar{S}^k$. Note that $H \subset \bar{S}^k$. For $i = 1, \ldots, k$ it holds that $\bar{S}^i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$ since $\bar{S}^i = \bar{F}^T(\bar{r}_i)$. Also for $i = 1, \ldots, k$ it holds that $\bar{S}^i \setminus H_i \in \mathcal{H} \setminus \mathcal{I}(\mathcal{Q})$, since $\bar{S}^i \setminus H_i = \underline{S}^i$ and $\underline{S}^i = \bar{F}^T(\underline{r}_i)$. Because T is a line-tree and $H_i = \bar{F}^T(\bar{r}_i) \setminus \bar{F}^T(\underline{r}_i)$ for $i = 1, \ldots, k$, we have

$$\begin{aligned}
\sum_{j\in H} m_j^T(v, \mathcal{Q}) &= \sum_{i=1}^{k} \sum_{j\in H_i} m_j^T(v, \mathcal{Q}) \\
&= \sum_{i=1}^{k} (v(\bar{F}^T(\bar{r}_i)) - v(\bar{F}^T(\underline{r}_i))) \\
&= \sum_{i=1}^{k} (v(\bar{S}^i) - v(\bar{S}^i \setminus H_i)) \geq v(H),
\end{aligned}$$

where the last inequality holds since v is $\mathcal{Q}$-increasing. □

5. Examples of quasi-building systems

In this section we discuss how a quasi-building system is induced when the underlying structure for cooperation has some specific properties, such as a collection of connected subsets in a graph or hierarchical coalitions in a dominance structure, or a combinatorial structure such as augmenting system, antimatroid, building set, distributive lattice, and convex geometry.

5.1. Union stable quasi-building systems

In this subsection, we give examples of union stable quasi-building systems. We start with quasi-building systems that are induced by graphs. A (simple) directed graph or digraph $G = (V, E)$ consists of a set of nodes V and a set E of directed edges on V, i.e., $E \subseteq \{(i, j) \in V \times V \mid j \neq i\}$. Nodes $i, j \in V$ are linked in G if $(i, j) \in E$ or $(j, i) \in E$. A path in G between nodes i and j is a sequence of nodes $(i_1 = i, \ldots, i_k = j)$ such that i_h and i_{h+1}, $h = 1, \ldots, k-1$, are linked in G. Given a digraph $G = (V, E)$, the subgraph $G(V')$ on $V' \subseteq V$ has as set of edges the directed edges in E restricted to $V' \times V'$. $V' \subseteq V$ is connected in G if for any two nodes in V' there exists a path in $G(V')$ between these nodes. A path $(i_1, \ldots, i_k)$ in G is directed from i_1 to i_k if $(i_h, i_{h+1}) \in E$ for $h = 1, \ldots, k-1$. Node j is a successor of node i in G if there is a directed path in G from i to j. The set of successors of node i in G is denoted by $F^G(i)$.

Definition 5.1. For a digraph $G = ([n], E)$, the *graphical quasi-building system* $\mathcal{Q}(G) = (\mathcal{H}, U)$ consists of a set system $\mathcal{H}$ and mapping $U : \mathcal{H} \to 2^{[n]}$ satisfying the following conditions:

- $\mathcal{H}$ consists of all subsets of $[n]$ that are connected in G;
- U assigns to every $H \in \mathcal{H}$ the set of all nodes which are undominated in the subgraph $G(H)$, that is, $h \in U(H)$ if, for every $j \in H$ such that $h \in F^{G(H)}(j)$, it holds that $j \in F^{G(H)}(h)$.

Note that in any digraph on $[n]$ every subgraph contains at least one undominated node.

A directed graph $G = (V, E)$ satisfying that $(i, j) \in E$ whenever $(j, i) \in E$ is identified with the undirected graph (V, L), where $\{i, j\} \in L$ if $(i, j) \in E$. For a graphical quasi-building system induced by an undirected graph it holds that the choice function is the identical function.

Lemma 5.2. *For any connected digraph G on $[n]$ it holds that $\mathcal{Q}(G)$ is a union stable quasi-building system on $[n]$.*

Proof. Let $\mathcal{Q}(G) = (\mathcal{H}, U)$. Since G is connected, it holds that $[n] \in \mathcal{H}$. Take any nonempty $H \in \mathcal{H}$. Then H is connected in G and there exists an undominated node in $G(H)$. This implies that $U(H)$ is a nonempty subset of H, which proves condition (Q1). Since G is a digraph, for every $h \in U(H)$ there exists a unique maximal partition of $H \setminus \{h\}$ into elements of $\mathcal{H}$. Also because G is a digraph, any feasible subset of $H \setminus \{h\}$ is a subset of one of the partition members. Consequently, condition (Q2) is fulfilled. Since for every $H_1, H_2 \in \mathcal{H}$ satisfying $H_1 \cap H_2 \neq \emptyset$ it holds that $H_1 \cup H_2 \in \mathcal{H}$, condition (Q3) is also fulfilled. □

From this lemma and Theorem 4.6 we get the following corollary.

Corollary 5.3. *Let G be a connected digraph on $[n]$ and let $(v, \mathcal{Q}(G)) \in \mathcal{V}$. If v is $\mathcal{Q}(G)$-supermodular, then $AMV(v, \mathcal{Q}(G)) \in C(v, \mathcal{Q}(G))$.*

For the class of cooperative games with directed communication situation, the Average Covering Tree value, introduced in [15], is the average of the marginal vectors that correspond to the set of all covering trees induced from the graph. For this case the AMV-value coincides with the Average Covering Tree value, since for any digraph G the set of rooted trees being compatible with graphical quasi-building system $\mathcal{Q}(G)$ coincides with the collection of covering trees on G.

A quasi-building system is a generalization of a building set [17]. A set system $\mathcal{H}$ is a building set on $[n]$ if the following conditions are satisfied:

(B1) $\mathcal{H}$ is a set system on $[n]$ containing both $\emptyset$ and $[n]$.
(B2) If $S, T \in \mathcal{H}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{H}$.
(B3) For all $i \in [n]$, $\{i\} \in \mathcal{H}$.

Proposition 5.4. *For a set system $\mathcal{H}$ on $[n]$ and function $U : \mathcal{H} \to 2^{[n]}$ satisfying $U(H) = H$ for all $H \in \mathcal{H}$, it holds that $(\mathcal{H}, U)$ is a union stable quasi-building system if and only if $\mathcal{H}$ is a building set.*

Proof. Suppose $(\mathcal{H}, U)$ is a union stable quasi-building system on $[n]$ with $U(H) = H$ for all $H \in \mathcal{H}$. Condition (B1) obviously holds. Let $S, T \in \mathcal{H}$ with $S \cap T \neq \emptyset$. If $S \cup T = [n]$, then (B2) is verified by (Q1). Suppose $S \cup T \neq [n]$. Take any $j \in [n] \setminus (S \cup T)$, then $j \in U([n])$ since $U([n]) = [n]$. Because of (Q2) and since $S \cap T \neq \emptyset$, $S \cup T$ is contained in some single member of the partition $P([n] \setminus \{j\})$. Let R be this set, then $R \in \mathcal{H}$. If $S \cup T = R$, then (B2) is verified. Otherwise, take any $j' \in R \setminus (S \cup T)$. Again, by (Q2) and, since $U(R) = R$ and $S \cap T \neq \emptyset$, we get that $S \cup T$ belongs to a single member of the partition $P(R \setminus \{j'\})$, and so on. At some step, we get $S \cup T \in \mathcal{H}$ and (B2) is verified. For verifying (B3), take any $i \in [n]$ and $j \in [n] \setminus \{i\}$. Then there is a unique $S \in P([n] \setminus \{j\})$ in $\mathcal{H}$ containing i. If $S \neq \{i\}$, take any $j' \in S \setminus \{i\}$ and the unique member of $P(S \setminus \{j'\})$ in $\mathcal{H}$ containing i, and so on, until we get $\{i\} \in \mathcal{H}$.

For the reverse implication, let the set system $\mathcal{H}$ be a building set and consider $(\mathcal{H}, U)$ where $U(H) = H$ for all $H \in \mathcal{H}$. Condition (Q1) follows from condition (B1) and the supposition $U(H) = H$ for all $H \in \mathcal{H}$. For condition (Q2), we first show that there exists a unique maximal feasible partition of $H \setminus \{h\}$ for every $H \in \mathcal{H}$ and $h \in H$. Take any $H \in \mathcal{H}$ and $h \in H$. Due to (B3), there exists a feasible partition of $H \setminus \{h\}$, and therefore there is at least one maximal partition $\mathcal{S}$. Suppose there is another maximal feasible partition of $H \setminus \{h\}$, say, $\mathcal{T}$. Since $\mathcal{S} \neq \mathcal{T}$, there exists $S \in \mathcal{S}$ such that $S \not\subseteq T$ for all $T \in \mathcal{T}$, otherwise $\mathcal{S}$ cannot be a maximal partition of $H \setminus \{h\}$. Let $\mathcal{T}^S = \{T \in \mathcal{T} \mid T \cap S \neq \emptyset\}$. Then $|\mathcal{T}^S| \geq 2$ and from (B2) it follows that $\bigcup_{T \in \mathcal{T}^S}(T \cup S) = \bigcup_{T \in \mathcal{T}^S} T \in \mathcal{H}$, which contradicts that $\mathcal{T}$ is a maximal partition of $H \setminus \{h\}$. To show the second part of condition (Q2), take any $H \in \mathcal{H}$ and $h \in H$. Let $\{S_1, \dots, S_k\}$ be the unique maximal partition of $H \setminus \{h\}$ and suppose there exists $T \subseteq H \setminus \{h\}$ such that $T \in \mathcal{H}$ and T is a not a subset of S_h for $h = 1, \dots, k$. Let $J = \{j \in \{1, \dots, k\} | \ T \cap S_j \neq \emptyset\}$. Then $|J| \geq 2$, and from (B2) it follows that $\bigcup_{j \in J}(T \cup S_j) = \bigcup_{j \in J} S_j \in \mathcal{H}$, which contradicts that $\{S_1, \dots, S_k\}$ is a maximal partition of $H \setminus \{h\}$. Finally, condition (Q3) follows from (B2). $\square$

For a given set system several choice functions may exist such that each of them together with the set system forms a quasi-building system. Here we define a maximal quasi-building system amongst such.

For a set system $\mathcal{F}$ on $[n]$, let us define the pair $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$ as follows:

- $\mathcal{H} = \mathcal{F}$;
- $U : \mathcal{H} \to 2^{[n]}$ is such that for every $H \in \mathcal{F}$ the set $U(H)$ consists of all $h \in H$ such that H and h satisfy condition (Q2).

Note that $\mathcal{Q}(\mathcal{F})$ is a quasi-building system on $[n]$ if and only if $\emptyset, [n] \in \mathcal{F}$ and $U(H) \neq \emptyset$ for every nonempty $H \in \mathcal{F}$.

Proposition 5.4 implies that, for any building set $\mathcal{F}$ on $[n]$, the pair $\mathcal{Q}(\mathcal{F})$ is a union stable quasi-building system. The AMV-value of a game $(v, \mathcal{Q}(\mathcal{F})) \in \mathcal{V}$ is then equal to the Gravity Center solution of the game v on the building set $\mathcal{F}$, introduced in [17]. This is because the collection of maximal strictly nested sets of a building set $\mathcal{F}$ defined in [17] corresponds one-to-one to the collection of rooted trees compatible with $\mathcal{Q}(\mathcal{F})$. In [17] it is shown that, for a building set, the Gravity Center solution coincides with the Shapley value defined in [12] using the Monge algorithm.[3]

In [3], augmenting systems are introduced as set systems in cooperative games. A set system $\mathcal{F}$ on $[n]$ is an augmenting system on $[n]$ if it satisfies the following conditions:

(S1) $\emptyset \in \mathcal{F}$.
(S2) If $S, T \in \mathcal{F}$ with $S \cap T \neq \emptyset$, then $S \cup T \in \mathcal{F}$.
(S3) If $S, T \in \mathcal{F}$ and $S \subset T$, then there exists $i \in T \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

An augmenting system on $[n]$ may not contain $[n]$ as a feasible coalition.[4]

3 For a set system $\mathcal{F}$, that is not a building set, the Monge algorithm gives also marginal vectors which correspond to rooted trees which are incompatible with $\mathcal{Q}(\mathcal{F})$.

4 If for an augmenting system on $[n]$ it holds that $[n]$ is not a member, but every $i \in [n]$ belongs to at least one member, then there exists a unique maximal partition of $[n]$ into feasible coalitions satisfying that every feasible coalition is a subset of one of the partition members. The analysis can then be applied separately to every partition member.

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Lemma 5.5. *For any augmenting system $\mathcal{F}$ on $[n]$ with $[n] \in \mathcal{F}$, $\mathcal{Q}(\mathcal{F})$ is a union stable quasi-building system on $[n]$.*

Proof. Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$, then $\mathcal{H} = \mathcal{F}$. The empty set belongs to $\mathcal{H}$ by condition (S1) and $[n]$ belongs to $\mathcal{H}$ by assumption. By definition $U(H) \subseteq H$ for all $H \in \mathcal{H}$. From condition (S1) and repeated application of condition (S3) starting with $S = \emptyset$, it follows that for every nonempty $H \in \mathcal{H}$ there exists $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$, which implies that $U(H)$ is nonempty. Condition (Q1) is, therefore, satisfied. Condition (Q2) is satisfied by construction and condition (Q3) follows from (S2). □

From this lemma and Theorem 4.6 we get the following corollary.

Corollary 5.6. *Let $\mathcal{F}$ be an augmenting system on $[n]$ with $[n] \in \mathcal{F}$ and let $(v, \mathcal{Q}(\mathcal{F})) \in \mathcal{V}$. If v is $\mathcal{Q}(\mathcal{F})$-supermodular, then $AMV(v, \mathcal{Q}(\mathcal{F})) \in C(v, \mathcal{Q}(\mathcal{F}))$.*

Antimatroids, another class of set systems, introduced in [10] and studied in [1] as set systems for cooperative games, form a subclass of the class of augmenting systems. In [13], a restriction among players is expressed as precedence constraints, represented by a partially ordered set, see also [14]. For the set system of ideals of a partially ordered set (distributive lattice) the corresponding quasi-building system is union stable.

5.2. Intersection-closed quasi-building systems

In this subsection, we construct intersection-closed quasi-building systems from convex geometries. A set system $\mathcal{F}$ is a convex geometry on $[n]$ if it satisfies the following conditions (e.g., see [11]):

(C1) $\emptyset \in \mathcal{F}$.
(C2) If $S, T \in \mathcal{F}$, then $S \cap T \in \mathcal{F}$.
(C3) If $S \in \mathcal{F}$, $S \neq [n]$, then there exists $i \in [n] \setminus S$ such that $S \cup \{i\} \in \mathcal{F}$.

The connected subsets in a digraph may not form a convex geometry, because condition (C2) may not be satisfied. In a convex geometry the grand coalition $[n]$ is necessarily feasible. Since union stability does not imply intersection-closedness and vice versa, there is no inclusion relationship between the class of convex geometries and the class of augmenting systems.

Lemma 5.7. *For any convex geometry $\mathcal{F}$ on $[n]$, $\mathcal{Q}(\mathcal{F})$ is an intersection-closed quasi-building system on $[n]$.*

Proof. Let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$, then $\mathcal{H} = \mathcal{F}$. The empty set and $[n]$ belong to $\mathcal{H}$ by (C1) and (C3). Suppose that for some nonempty $H \in \mathcal{H}$ there is no $h \in H$ such that $H \setminus \{h\} \in \mathcal{H}$. By (C1), H cannot be a singleton and therefore $|H| \geq 2$. From (C1) and (C3) it follows that there exists a sequence of n sets $S_1, \ldots, S_n$, with $|S_k| = k$, $S_k \in \mathcal{H}$, $k = 1, \ldots, n$, and $S_1 \subset S_2 \subset \cdots \subset S_n = [n]$. Consider S_{n-1} and denote it as $[n] \setminus \{i_1\}$. From (C2) it follows that $i_1 \notin H$, otherwise $H \cap ([n] \setminus \{i_1\}) = H \setminus \{i_1\} \in \mathcal{H}$, which contradicts the supposition. Next, consider S_{n-2} and denote it as $[n] \setminus \{i_1, i_2\}$. Similarly, it holds that $i_2 \notin H$, and so on. Now consider $S_{|H|}$ and let $T = [n] \setminus S_{|H|}$. $H \cap T = \emptyset$ and therefore $S_{|H|} = H$. Then $S_{|H|-1} \in \mathcal{H}$ and there exists $h \in H$ such that $H \setminus \{h\} = S_{|H|-1}$ is feasible, which again is a contradiction. This proves condition (Q1). Condition (Q2) is satisfied by construction, condition (Q4) is the same as (C2). To show that condition (Q5) holds, take any $S, T \in \mathcal{F}$, $T \subset S$, and $i \in U(S) \cap T$. From the unique maximal partition $P(S \setminus \{i\})$ of $S \setminus \{i\}$ into feasible coalitions, take the minimum set of feasible coalitions $S_1, \ldots, S_l$ which covers $T \setminus \{i\}$, i.e., $S_k \cap (T \setminus \{i\}) \neq \emptyset$ for any $k = 1, \ldots, l$ and $T \setminus \{i\} \subset \bigcup_{k=1}^{l} S_k$. From intersection-closedness between T and $S_1, \ldots, S_l$, there is a feasible partition $\{T_1, \ldots, T_l\}$ of $T \setminus \{i\}$, where $T_k = T \cap S_k$, $k = 1, \ldots, l$. Furthermore, $\{T_1, \ldots, T_l\}$ is the unique maximal partition of $T \setminus \{i\}$ since $P(S \setminus \{i\})$ is the unique maximal partition of $S \setminus \{i\}$ satisfying (Q2). Therefore $P(T \setminus \{i\})$ exists and satisfies (Q2), which implies that $i \in U(T)$. □

From this lemma and Theorem 4.14 we get the following corollary.

Corollary 5.8. *Let $\mathcal{F}$ be a convex geometry on $[n]$ and let $(v, \mathcal{Q}(\mathcal{F})) \in \mathcal{V}$. If v is $\mathcal{Q}(\mathcal{F})$-convex, then $AMV(v, \mathcal{Q}(\mathcal{F})) \in C(v, \mathcal{Q}(\mathcal{F}))$.*

5.3. Chain quasi-building systems

When a set system $\mathcal{F}$ forms a distributive lattice on $[n]$, $\mathcal{Q}(\mathcal{F})$ is a chain quasi-building system. In this case the AMV-value coincides with the Shapley value in [13]. The set of hierarchical coalitions in a digraph describes a dominance structure in [16] and is a generalization of a distributive lattice. Given a digraph $G = (V, E)$, a subset $S \subseteq V$ is a hierarchical coalition in G if $i \in S$, $(i, j) \in E$, and $i \notin F^G(j)$ imply $\bar{F}^G(j) \subset S$. For a hierarchical coalition S in G node $i \in S$ is undominated in the subgraph $G(S)$ if $i \in F^{G(S)}(j)$ implies $j \in F^{G(S)}(i)$. For a digraph G on $[n]$ the collection of hierarchical coalitions with their undominated players as choice sets forms a chain quasi-building system on $[n]$ and the AMV-value of a game on such a quasi-building system coincides with the Shapley value of the corresponding digraph game in [16].

Some set systems on $[n]$ always have compatible rooted trees that are line-trees. For a set system $\mathcal{F}$ on $[n]$, let us define the pair $\mathcal{Q}^c(\mathcal{F}) = (\mathcal{H}, U^c)$ as follows:

- $\mathcal{H} = \mathcal{F}$;
- $U^c : \mathcal{H} \to 2^{[n]}$ is such that $U^c(H) = \{h \in H \mid H \setminus \{h\} \in \mathcal{F}\}$ for all $H \in \mathcal{F}$.

By definition, any rooted tree that is compatible with $\mathcal{Q}^c(\mathcal{F})$ is a line-tree on $[n]$.

Lemma 5.9. *If a set system $\mathcal{F}$ is an augmenting system with $[n] \in \mathcal{F}$ or convex geometry on $[n]$, then $\mathcal{Q}^c(\mathcal{F})$ is a chain quasi-building system on $[n]$.*

Proof. For both set systems the one-point extension conditions (S3) and (C3) ensure that for every nonempty $H \in \mathcal{F}$ it holds that $U^c(H) \neq \emptyset$. Therefore, condition (Q1) is satisfied. Conditions (Q2) and (Q6) are satisfied because for every $H \in \mathcal{F}$ and $h \in U^c(H)$ it holds that $H \setminus \{h\} \in \mathcal{F}$. □

For an augmenting system containing $[n]$ or convex geometry $\mathcal{F}$ on $[n]$, the AMV-value of the game $(v, \mathcal{Q}^c(\mathcal{F}))$ coincides with the Shapley value in [6] and the Shapley value in [4], respectively.

For a set system $\mathcal{F}$ on $[n]$, let $\mathcal{Q}(\mathcal{F}) = (\mathcal{H}, U)$ and $\mathcal{Q}^c(\mathcal{F}) = (\mathcal{H}, U^c)$, then for any $H \in \mathcal{H}$ it holds that $U^c(H) \subseteq U(H)$. Therefore, any rooted tree that is compatible with $\mathcal{Q}^c(\mathcal{F})$ is also compatible with $\mathcal{Q}(\mathcal{F})$. From this, Lemma 5.9, and Theorems 4.6, 4.9 and 4.14 we obtain the following corollary.

Corollary 5.10. *For any augmenting system containing $[n]$ (convex geometry) $\mathcal{F}$ on $[n]$ and game $(v, \mathcal{Q}^c(\mathcal{F})) \in \mathcal{V}$, it holds that if v is $\mathcal{Q}^c(\mathcal{F})$-increasing or $\mathcal{Q}(\mathcal{F})$-supermodular ($\mathcal{Q}(\mathcal{F})$-convex), then $AMV(v, \mathcal{Q}^c(\mathcal{F})) \in C(v, \mathcal{Q}^c(\mathcal{F}))$.*

Note that if a set system $\mathcal{F}$ is a distributive lattice on $[n]$, then $\mathcal{Q}^c(\mathcal{F}) = \mathcal{Q}(\mathcal{F})$.

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References

[1] E. Algaba, J.M. Bilbao, R. van den Brink, A. Jiménez-Losada, Cooperative games on antimatroids, Discrete Math. 282 (2000) 1–15.
[2] S. Ambec, Y. Sprumont, Sharing a river, J. Econom. Theory 107 (2002) 453–462.
[3] J.M. Bilbao, Cooperative games under augmenting systems, SIAM J. Discrete Math. 17 (2003) 122–133.
[4] J.M. Bilbao, P.H. Edelmann, The Shapley value on convex geometries, Discrete Appl. Math. 103 (2000) 33–40.
[5] J.M. Bilbao, E. Lebrón, N. Jiménez, The core of games on convex geometries, European J. Oper. Res. 119 (1999) 365–372.
[6] J.M. Bilbao, M. Ordóñez, Axiomatizations of the Shapley value for games on augmenting systems, European J. Oper. Res. 196 (2009) 1008–1014.
[7] H. Chernoff, Rational selection of decision functions, Econometrica 22 (1954) 422–443.
[8] G. Demange, On group stability in hierarchies and networks, J. Polit. Econ. 112 (2004) 754–778.
[9] J. Derks, R.P. Gilles, Hierarchical organization structures and constraints in coalition formation, Internat. J. Game Theory 24 (1995) 147–163.
[10] R.P. Dilworth, Lattices with unique irreducible decompositions, Ann. of Math. 41 (1940) 771–777.
[11] P.H. Edelman, R.E. Jamison, The theory of convex geometries, Geom. Dedicata 19 (1985) 247–270.
[12] U. Faigle, M. Grabisch, M. Heyne, Monge extensions of cooperation and communication structures, European J. Oper. Res. 206 (2010) 104–110.
[13] U. Faigle, W. Kern, The Shapley value for cooperative games under precedence constraints, Internat. J. Game Theory 21 (1992) 249–266.
[14] M. Grabisch, L. Xie, The restricted core of games with distributive lattices: how to share benefits in a hierarchy, Math. Methods Oper. Res. 73 (2011) 189–208.
[15] A.B. Khmelnitskaya, Ö. Selçuk and, A.J.J. Talman, 2012. The average covering tree value for directed graph games, CentER Discussion Paper 2012–037, Tilburg University, Tilburg.
[16] A.B. Khmelnitskaya, Ö. Selçuk and, A.J.J. Talman, The Shapley value for directed graph games, Oper. Res. Lett. 44 (2016) 143–147.
[17] G.A. Koshevoy, A.J.J. Talman, Solution concepts for games with general coalitional structure, Math. Social Sci. 68 (2014) 19–30.
[18] R.B. Myerson, Graphs and cooperation in games, Math. Oper. Res. 2 (1977) 225–229.
[19] A.K. Sen, Choice functions and revealed preference, Rev. Econom. Stud. 38 (1971) 307–317.
[20] L. Shapley, A value for n-person games, in: H.W. Kuhn, A.W. Tucker (Eds.), Contributions to the Theory of Games, Vol. II, Princeton University Press, Princeton, 1953, pp. 307–317.