WILCOXON SIGNED-RANK TEST

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The Wilcoxon signed-rank test, due to Wilcoxon (9), is a nonparametric test procedure used for the analysis of matched-pair data or for the one-sample problem. In the matched-pair setting it is used to test the hypothesis that the probability distribution of the first sample is equal to the probability distribution of the second sample. This hypothesis can be tested from statistics calculated on the intrapair differences. The hypothesis commonly tested is that these differences come from a distribution centered at zero.

Consider the following example. A study was conducted in which nine people of varying weights were put on a particular exercise regimen to determine the program's effect on the resting heart rate of the subjects. Given that a low resting heart rate is beneficial in reducing blood pressure and increasing overall cardiovascular fitness, this exercise regimen was developed to help people lower their resting heart rate. To test the effectiveness of the regimen, the resting heart rate measurement for each subject was taken before the induction of the regimen, and at six months after beginning the regimen. Table 1 presents the data from this study.

Because this study involves before and after measurements of the same individuals, an independent sample test procedure cannot be executed. The **null hypothesis** in the Wilcoxon signed-rank test is that the set of pairwise differences have a probability distribution centered at zero. A key assumption is that the differences arise from a continuous, symmetric distribution. In the example, the null hypothesis would be that there is no resting heart rate difference before and after the exercise regimen ($H_0 : \mu_d = 0$). In this instance, μ_d represents the location parameter for the distribution of differences. One **alternative hypothesis** is that the resting heart rate before the exercise regimen is higher than the resting heart rate after the exercise regimen ($H_1 : \mu_d > 0$).

To execute the test, the absolute values of the differences, $|d_i|$, are computed. These values also are given in Table 1. After computing the absolute values, one must order them from smallest to largest disregarding any zeros. In the case of absolute differences being tied for the same ranks, the mean rank (mid-rank) is calculated and assigned to each tied value. The rankings for the absolute differences are also given in Table 1. Test statistics for the Wilcoxon signed-rank test are calculated by either summing the ranks assigned to the positive differences (T_+) or by summing the ranks assigned to the negative differences (T_-). If there are n differences, then the two sums are related through

$$T_- = \left\{ \frac{[n(n+1)]}{2} \right\} - T_+. \qquad (1)$$

In the example, the sum of the ranks of the positive differences is

$$T_+ = 5 + 3 + 9 + 7 + 4 + 6 + 8 = 42.$$

Note that $\{[n(n+1)]/2\} = 45$ so

$$T_- = 45 - 42 = 3.$$

To test the null hypothesis, a rejection region can be determined for the test statistic, T_+. This rejection region can be determined from the exact null hypothesis distribution of T_+. This null distribution is easily derived from a permutational argument, as each of the possible configuration of signs ($+$ or $-$) is equally likely under the null hypothesis. Tables of this exact null distribution are available in standard nonparametric texts such as (3,4), or (7). This null distribution depends only on n, hence the test procedure is nonparametric, i.e. distribution-free.

Table 1. Resting Heart Rate of Nine People Before and after Initiation of an Exercise Regimen

Subject	Heart Rate at Baseline (y_i)	Heart Rate at 6 Months (x_i)	Difference $d_i = y_i - x_i$	Absolute Value of Difference $\lvert d_i \rvert$	Rank of Absolute Difference (sign)
1	80	72	+8	8	5(+)
2	76	70	+6	6	3(+)
3	78	82	−4	4	2(−)
4	90	76	+14	14	9(+)
5	84	86	−2	2	1(−)
6	86	76	+10	10	7(+)
7	81	74	+7	7	4(+)
8	84	75	+9	9	6(+)
9	88	76	+12	12	8(+)

For **large samples** the standard normal distribution Z, can be used as an approximation to test hypotheses. For this situation a two-tailed rejection region for the null hypothesis based on T_+, is given:

$$Z_+ = \frac{\left\{\dfrac{[T_+ - n(n+1)]}{4}\right\}}{\left\{\dfrac{[n(n+1)(2n+1)]}{24}\right\}^{1/2}} > Z_{1-\alpha/2} \quad (2)$$

or

$$Z_- = \frac{\left\{\dfrac{[T_- - n(n+1)]}{4}\right\}}{\left\{\dfrac{[n(n+1)(2n+1)]}{24}\right\}^{1/2}} > Z_{1-\alpha/2}. \quad (3)$$

A one-tailed test is conducted in a similar fashion with the comparison made to $Z_{1-\alpha}$.

Other issues regarding the Wilcoxon signed-rank test include its testing efficiency and the construction of estimators. The **asymptotic relative efficiency** (ARE) of this test relative to the paired t test is never less than 0.864 in the entire class of continuous symmetric distributions, and is 0.955 if the underlying distribution of differences is normal, see (2). The handling of ties and zeros is discussed by Pratt (5) and Cureton (1). Point and **confidence interval** estimators are easily derived from the test procedure, and details are described in Lehmann (4). Lehmann (4) also describes power properties for the test procedure when shift alternatives are of interest. Extensions to **censored data** are discussed by Woolson & Lachenbruch (10) and Schemper (8). References for other aspects of the Wilcoxon signed-rank test are given by Randles & Wolfe (7) and Randles (6).

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