## *Isoparametric Elements*

**Introduction**

In this chapter, we introduce the isoparametric formulation of the element stiffness matrices.

After considering the linear-strain triangular element (LST) in Chapter 8, we can see that the development of element matrices and equations expressed in terms of a global coordinate system becomes an enormously difficult task (if even possible) except for the simplest of elements such as the constant-strain triangle of Chapter 6.

Hence, the isoparametric formulation was developed.

## *Isoparametric Elements*

**Introduction**

The isoparametric method may appear somewhat tedious (and confusing initially), but it will lead to a simple computer program formulation, and it is generally applicable for two- and three-dimensional stress analysis and for nonstructural problems.

The isoparametric formulation allows elements to he created that are nonrectangular and have curved sides.

Numerous commercial computer programs (as described in Chapter 1) have adapted this formulation for their various libraries of elements.

## Isoparametric Elements

**Introduction**

First, we will illustrate the isoparametric formulation to develop the simple bar element stiffness matrix.

Use of the bar element makes it relatively easy to understand the method because simple expressions result.

Then, we will consider the development of the isoparametric formulation of the simple quadrilateral element stiffness matrix.

## Isoparametric Elements

**Introduction**

Next, we will introduce numerical integration methods for evaluating the quadrilateral element stiffness matrix.

Then, we will illustrate the adaptability of the isoparametric formulation to common numerical integration methods.

Finally, we will consider some higher-order elements and their associated shape functions.

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

The term *isoparametric* is derived from the use of the same shape functions (or interpolation functions) [N] to define the element's geometric shape as are used to define the displacements within the element.

Thus, when the interpolation function is $u = a_1 + a_2 s$ for the displacement, we use $x = a_1 + a_2 s$ for the description of the nodal coordinate of a point on the bar element and, hence, the physical shape of the element.

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

Isoparametric element equations are formulated using a natural (or intrinsic) coordinate system ***s*** that is defined by element geometry and not by the element orientation in the global-coordinate system.

In other words, axial coordinate *s* is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space.

There is a relationship (called a *transformation mapping)* between the natural coordinate systems and the global coordinate system *x* for each element of a specific structure.

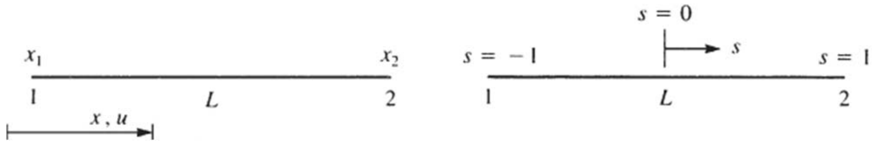
## Isoparametric Elements

**Isoparametric Formulation of the Bar Element**

First, the natural coordinate ***s*** is attached to the element, with the origin located at the center of the element.

The ***s*** axis need not be parallel to the $x$ axis-this is only for convenience.

Consider the bar element to have two degrees of freedom-axial displacements $u_1$ and $u_2$ at each node associated with the global $x$ axis.




## Isoparametric Elements

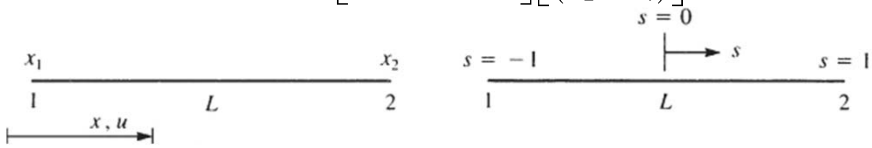
**Isoparametric Formulation of the Bar Element**

For the special case when the $s$ and $x$ axes are parallel to each other, the $s$ and $x$ coordinates can be related by:

$$x = x_c + \frac{L}{2}s$$

Using the global coordinates $x_1$ and $x_2$ with $x_c = (x_1 + x_2)/2$, we can express the natural coordinate $s$ in terms of the global coordinates as:

$$s = \left[\frac{x - (x_1 + x_2)}{2}\right]\left[\frac{2}{(x_2 - x_1)}\right]$$



## Isoparametric Elements

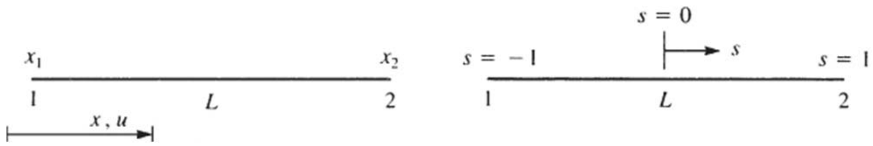
### Isoparametric Formulation of the Bar Element

The shape functions used to define a position within the bar are found in a manner similar to that used in Chapter 3 to define displacement within a bar (Section 3.1).

We begin by relating the natural coordinate to the global coordinate by:

$$x = a_1 + a_2 s$$

Note that $-1 \leq s \leq 1$.




## Isoparametric Elements

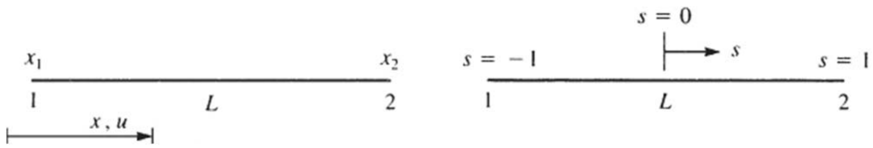
### Isoparametric Formulation of the Bar Element

Solving for the *a's* in terms of $x_1$ and $x_2$, we obtain:

$$x = \left(\frac{1}{2}\right)\left[(1-s)x_1 + (1+s)x_2\right]$$

In matrix form:

$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



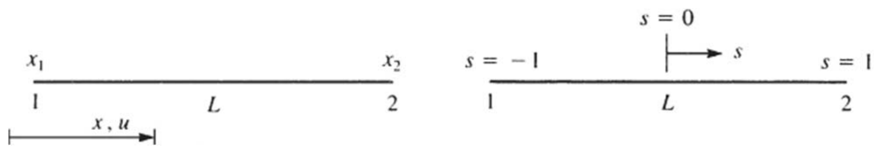
## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

The linear shape functions map the *s* coordinate of any point in the element to the *x* coordinate.

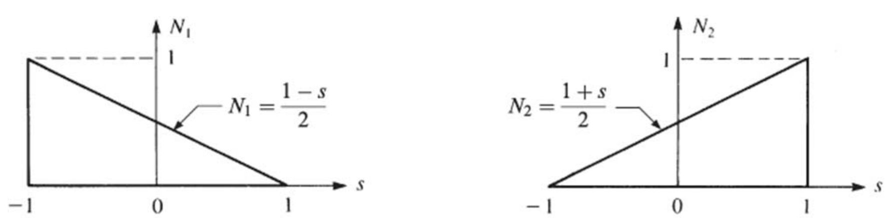
For instance, when $s = -1$, then $x = x_1$ and when $s = 1$, then $x = x_2$

$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

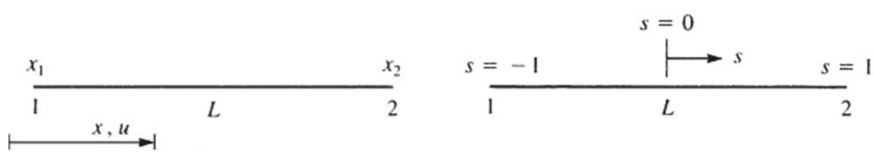



## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

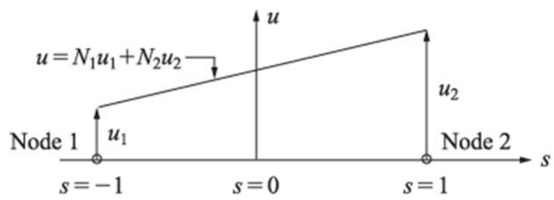



$$\{x\} = [N_1 \quad N_2]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$

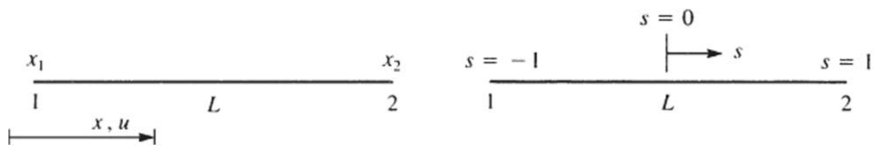


## Isoparametric Elements

### Isoparametric Formulation of the Bar Element




$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



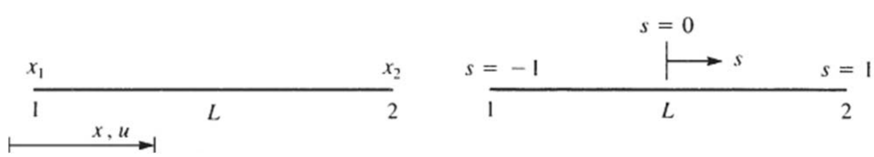

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

When a particular coordinate $s$ is substituted into [N] yields the displacement of a point on the bar in terms of the nodal degrees of freedom $u_1$ and $u_2$.

Since $u$ and $x$ are defined by the same shape functions at the same nodes, the element is called *isoparametric.*

$$\{x\} = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \qquad N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2}$$



## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 3 - Strain-Displacement and Stress-Strain Relationships**

We now want to formulate element matrix [B] to evaluate [k].

We use the isoparametric formulation to illustrate its manipulations.

For a simple bar element, no real advantage may appear evident.

However, for higher-order elements, the advantage will become clear because relatively simple computer program formulations will result.

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 3 - Strain-Displacement and Stress-Strain Relationships**

To construct the element stiffness matrix, determine the strain, which is defined in terms of the derivative of the displacement with respect to *x.*

The displacement *u,* however, is now a function of *s* so we must apply the chain rule of differentiation to the function *u* as follows:

$$\frac{du}{ds} = \frac{du}{dx}\frac{dx}{ds} \qquad \varepsilon_x = \frac{du}{dx} \quad \Rightarrow \quad \varepsilon_x = \frac{du}{dx} = \frac{du}{ds} \Big/ \frac{dx}{ds}$$

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

**Step 3 - Strain-Displacement and Stress-Strain Relationships**

The derivative of $u$ with respect to $s$ is: $\dfrac{du}{ds} = \dfrac{u_2 - u_1}{2}$

The derivative of $x$ with respect to $s$ is: $\dfrac{dx}{ds} = \dfrac{x_2 - x_1}{2} = \dfrac{L}{2}$

Therefore the strain is: $\{\varepsilon_x\} = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$

Since $\{\varepsilon\} = [B]\{d\}$, the strain-displacement matrix $[B]$ is:

$$[B] = \begin{bmatrix} -\dfrac{1}{L} & \dfrac{1}{L} \end{bmatrix}$$

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

**Step 3 - Strain-Displacement and Stress-Strain Relationships**

Recall that use of linear shape functions results in a constant [*B*] matrix, and hence, in a constant strain within the element.

For higher-order elements, such as the quadratic bar with three nodes, [*B*] becomes a function of natural coordinates *s.*

The stress matrix is again given by Hooke's law as:

$$\{\sigma\} = E\{\varepsilon\} = E[B]\{d\}$$

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

The stiffness matrix is: $[k] = \int_0^L [B]^T E [B] A\, dx$

However, in general, we must transform the coordinate *x* to *s* because [*B*] is, in general, a function of *s.*

$$\int_0^L f(x)dx = \int_{-1}^{1} f(s)\left|[J]\right| ds$$

where [*J*] is called the ***Jacobian*** matrix.

In the one-dimensional case, we have |[*J*]| = *J.*

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

For the simple bar element: $\left|[J]\right| = \dfrac{dx}{ds} = \dfrac{L}{2}$

The Jacobian determinant relates an element length (*dx*) in the global-coordinate system to an element length (*ds*) in the natural-coordinate system.

In general, |[J]| is a function of *s* and depends on the numerical values of the nodal coordinates.

This can be seen by looking at for the equations for a quadrilateral element.

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

#### Step 4 - Derive the Element Stiffness Matrix and Equations

The stiffness matrix in natural coordinates is:

$$[k] = \frac{L}{2}\int_{-1}^{1} [B]^T E [B] A\, ds$$

For the one-dimensional case, we have used the modulus of elasticity *E* = [*D*].

Performing the simple integration, we obtain:

$$[k] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

#### Step 4 - Derive the Element Stiffness Matrix and Equations

For higher-order one-dimensional elements, the integration in closed form becomes difficult if not impossible.

Even the simple rectangular element stiffness matrix is difficult to evaluate in closed form.

However, the use of numerical integration, as described in Section 10.3, illustrates the distinct advantage of the isoparametric formulation of the equations

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

Determine the body-force matrix using the natural coordinate system *s.* The body-force matrix is:

$$\{f_{\mathbf{b}}\} = \int_V [N]^{\mathrm{T}} \{X_{\mathbf{b}}\}\, dV \qquad \{f_{\mathbf{b}}\} = \int_V [N]^{\mathrm{T}} \{X_{\mathbf{b}}\}\, A\, dx$$

Substituting for $N_1$ and $N_2$ and using $dx = (L/2)ds$

$$\{f_{\mathbf{b}}\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_{\mathbf{b}}\} \frac{L}{2} ds \ = \frac{ALX_{\mathbf{b}}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

## *Isoparametric Elements*

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

The physical interpretation of the results for $\{f_b\}$ is that since *AL* represents the volume of the element and $X_b$ the body force per unit volume, then $ALX_b$ is the total body force acting on the element.

The factor ½ indicates that this body force is equally distributed to the two nodes of the element.

$$\{f_{\mathbf{b}}\} = A\int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{X_{\mathbf{b}}\} \frac{L}{2} ds \ = \frac{ALX_{\mathbf{b}}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

Determine the surface-force matrix using the natural coordinate system *s.* The surface-force matrix is:

$$\{f_s\} = \int_S [N_s]^T \{T_x\}\, dS$$

Assuming the cross section is constant and the traction is uniform over the perimeter and along the length of the element, we obtain:

$$\{f_s\} = \int_0^L [N_s]^T \{T_x\}\, dx$$

where we now assume {*Tx*} is in units of force per unit length.

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

**Step 4 - Derive the Element Stiffness Matrix and Equations**

Substituting for $N_1$ and $N_2$ and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \dfrac{1-s}{2} \\ \dfrac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Since $\{T_x\}$ is in force-per-unit-length $\{T_x\}L$ is now the total force.

The ½ indicates that the uniform surface traction is equally distributed to the two nodes of the element.

## Isoparametric Elements

### Isoparametric Formulation of the Bar Element

#### Step 4 - Derive the Element Stiffness Matrix and Equations

Substituting for $N_1$ and $N_2$ and using $dx = (L/2)ds$

$$\{f_s\} = \int_{-1}^{1} \begin{Bmatrix} \frac{1-s}{2} \\ \frac{1+s}{2} \end{Bmatrix} \{T_x\} \frac{L}{2} ds \quad = \{T_x\} \frac{L}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Note that if $\{T_x\}$ were a function of $x$ (or $s$), then the amounts of force allocated to each node would generally not be equal and would be found through integration.

## Isoparametric Elements

### Isoparametric Formulation of the Quadrilateral Element

Recall that the term *isoparametric* is derived from the use of the same interpolation functions to define the element shape as are used to define the displacements within the element.

The approximation for displacement is:

$$u = a_1 + a_2 s + a_3 t + a_4 st$$

The description of a coordinate point in the plane element is:

$$x = a_1 + a_2 s + a_3 t + a_4 st$$

The natural-coordinate systems *s-t* defined by element geometry and not by the element orientation in the global-coordinate system *x-y.*

## Isoparametric Elements

### Isoparametric Formulation of the Quadrilateral Element

Much as in the bar element example, there is a transformation mapping between the two coordinate systems for each element of a specific structure, and this relationship must be used in the element formulation.

We will now formulate the isoparametric formulation of the simple linear plane quadrilateral element stiffness matrix.

This formulation is general enough to be applied to more complicated (higher-order) elements such as a quadratic plane element with three nodes along an edge, which can have straight or quadratic curved sides.

## Isoparametric Elements

### Isoparametric Formulation of the Quadrilateral Element

Higher-order elements have additional nodes and use different shape functions as compared to the linear element, but the steps in the development of the stiffness matrices are the same.

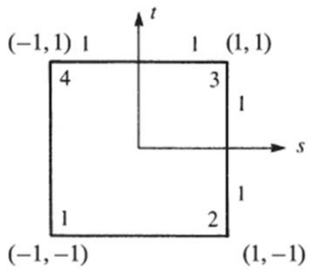
We will briefly discuss these elements after examining the linear plane element formulation.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

The natural ***s-t*** coordinates are attached to the element, with the origin at the center of the element.




The ***s*** and ***t*** axes need not be orthogonal, and neither has to be parallel to the *x* or *y* axis.

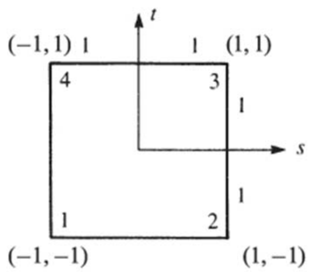
The orientation of ***s-t*** coordinates is such that the four corner nodes and the edges of the quadrilateral are bounded by +1 or -1

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

The natural ***s-t*** coordinates are attached to the element, with the origin at the center of the element.



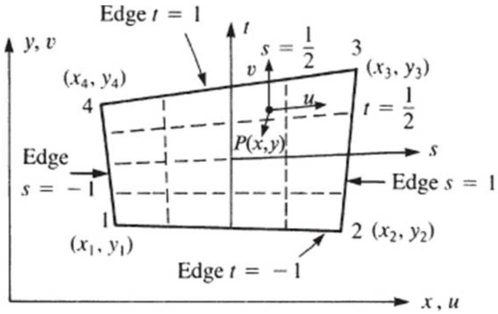

This orientation will later allow us to take advantage more fully of common numerical integration schemes.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

Consider the quadrilateral to have eight degrees of freedom $u_1$, $v_1$, ... , $u_4$, and $v_4$ associated with the global $x$ and $y$ directions. The element then has straight sides but is otherwise of arbitrary shape.




## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

For the special case when the distorted element becomes a rectangular element with sides parallel to the global ***x-y*** coordinates, the ***s-t*** coordinates can be related to the global element coordinates $x$ and $y$ by

$$x = x_c + bs \qquad y = y_c + ht$$

where $x_c$ and $y_c$ are the global coordinates of the element centroid.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

Assuming global coordinates $x$ and $y$ are related to the natural coordinates $s$ and $t$ as follows:

$$x = a_1 + a_2 s + a_3 t + a_4 st \qquad y = a_5 + a_6 s + a_7 t + a_8 st$$

Solving for the $a$'s in terms of $x_1$, $x_2$, $x_3$, $x_4$, $y_1$, $y_2$, $y_3$, $y_4$, we obtain

$$x = \frac{1}{4}\left[(1-s)(1-t)x_1 + (1+s)(1-t)x_2 + (1+s)(1+t)x_3 + (1-s)(1+t)x_4\right]$$

$$y = \frac{1}{4}\left[(1-s)(1-t)y_1 + (1+s)(1-t)y_2 + (1+s)(1+t)y_3 + (1-s)(1+t)y_4\right]$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

In matrix form:

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix}$$

where:

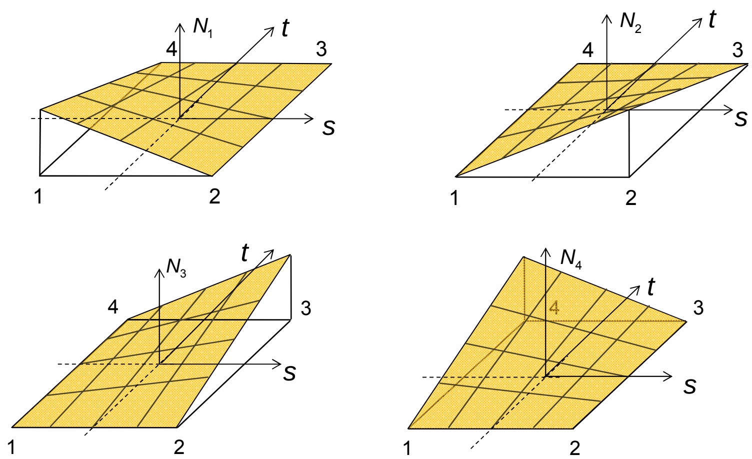
$$N_1 = \frac{(1-s)(1-t)}{4} \qquad N_2 = \frac{(1+s)(1-t)}{4}$$

$$N_3 = \frac{(1+s)(1+t)}{4} \qquad N_4 = \frac{(1-s)(1+t)}{4}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

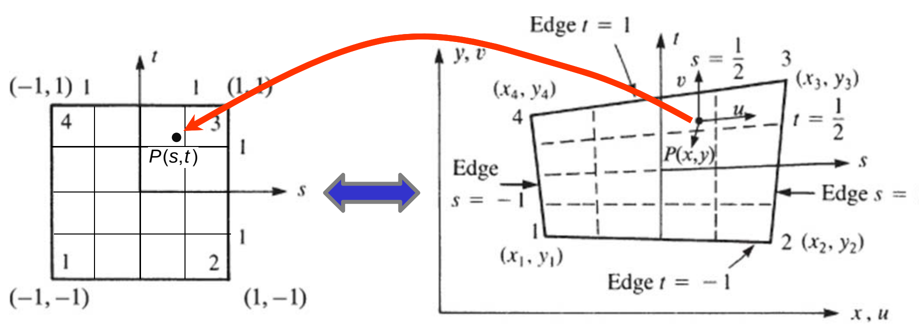


## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

These shape functions are seen to map the ***s*** and ***t*** coordinates of any point in the square element to those ***x*** and ***y*** coordinates in the quadrilateral element.

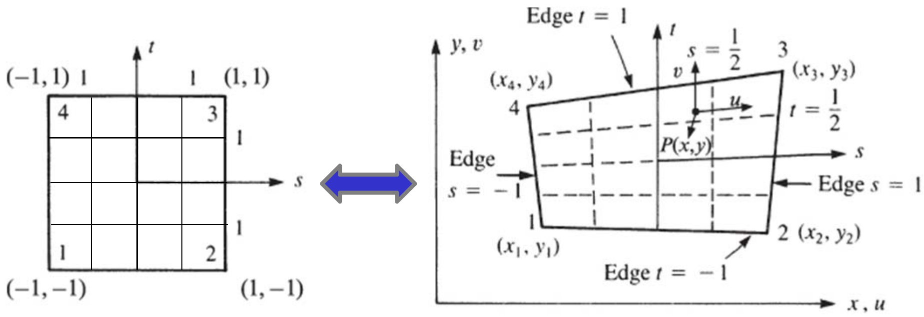


## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

Consider square element node 1 coordinates, where *s* = -1 and *t* = -1 then $x = x_1$ and $y = y_1$.

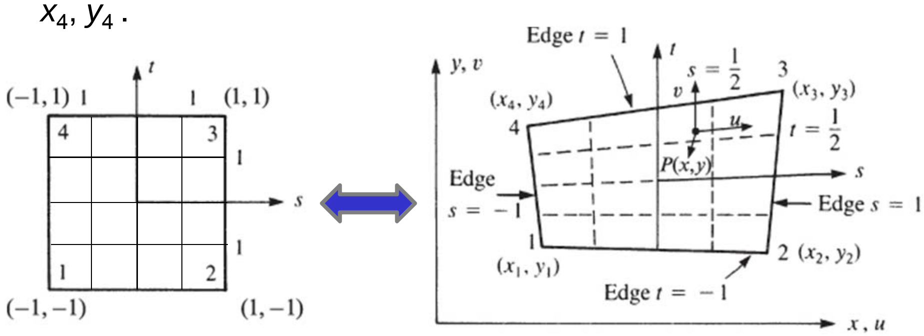



## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

Other local nodal coordinates at nodes 2, 3, and 4 on the square element in *s*-*t* isoparametric coordinates are mapped into a quadrilateral element in global coordinates $x_1$, $y_1$ through $x_4$, $y_4$ .

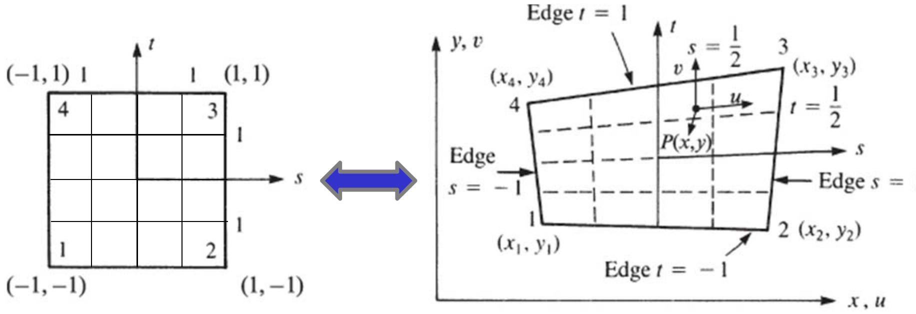


## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

Also observe the property that $N_1 + N_2 + N_3 + N_4 = 1$ for all values of $s$ and $t$.




## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 1** Select Element Type

We have always developed the element interpolation functions either by assuming some relationship between the natural and global coordinates in terms of the generalized coordinates ***a***'s or, similarly, by assuming a displacement function in terms of the ***a***'*s.*

However, physical intuition can often guide us in directly expressing shape functions based on the following two criteria set forth in Section 3.2 and used on numerous occasions:

$$\sum_{i=1}^{n} N_i = 1 \qquad\qquad i = 1, 2, ,\ldots, n$$

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 2** Select of Displacement Functions

The displacement functions within an element are now similarly defined by the same shape functions as are used to define the element geometric shape:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

We now want to formulate element matrix [B] to evaluate [k].

However, because it becomes tedious and difficult (if not impossible) to write the shape functions in terms of the ***x*** and ***y*** coordinates, as seen in Chapter 8, we will carry out the formulation in terms of the isoparametric coordinates ***s*** and ***t***.

This may appear tedious, but it is easier to use the ***s***- and ***t***-coordinate expressions.

This approach also leads to a simple computer program formulation.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

To construct an element stiffness matrix, we must determine the strains, which are defined in terms of the derivatives of the displacements with respect to the $x$ and $y$ coordinates.

The displacements, however, are now functions of the $s$ and $t$ coordinates.

The derivatives $\partial u/\partial x$ and $\partial v/\partial y$ are now expressed in terms of $s$ and $t$.

Therefore, we need to apply the chain rule of differentiation.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

The chain rule yields:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

The strains can then be found; for example, $\varepsilon_x = \partial u/\partial x$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

Using Cramer's rule, which involves the determinants of matrices, we can obtain:

$$\frac{\partial f}{\partial x} = \frac{\begin{vmatrix} \frac{\partial f}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial f}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}} \qquad \frac{\partial f}{\partial y} = \frac{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial f}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial f}{\partial t} \end{vmatrix}}{\begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

The determinant in the denominator is the determinant of the *Jacobian* matrix [J].

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}$$

We now want to express the element strains as: $\{\varepsilon\} = [B]\{d\}$

Where [B] must now be expressed as a function of ***s*** and ***t***.

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

The usual relationship between strains and displacements given in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial (\ )}{\partial x} & 0 \\ 0 & \dfrac{\partial (\ )}{\partial y} \\ \dfrac{\partial (\ )}{\partial y} & \dfrac{\partial (\ )}{\partial x} \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Where the rectangular matrix on the right side is an *operator matrix;* that is, $\partial$**( )**/$\partial$***x*** and $\partial$**( )**/$\partial$***y*** represent the partial derivatives of any variable we put inside the parentheses.

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

Evaluating the determinant in the numerators, we have

$$\frac{\partial (\ )}{\partial x} = \frac{1}{\left|[J]\right|}\left[\frac{\partial y}{\partial t}\frac{\partial (\ )}{\partial s} - \frac{\partial y}{\partial s}\frac{\partial (\ )}{\partial t}\right]$$

$$\frac{\partial (\ )}{\partial y} = \frac{1}{\left|[J]\right|}\left[\frac{\partial x}{\partial s}\frac{\partial (\ )}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial (\ )}{\partial s}\right]$$

Where |[***J***]| is the determinant of [***J***].

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

We can obtain the strains expressed in terms of the natural coordinates (***s-t***) as:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{\|[J]\|} \begin{Bmatrix} \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} & 0 \\ 0 & \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} \\ \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} & \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} \end{Bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

We can express the previous equation in terms of the shape functions and global coordinates in compact matrix form as:

$$\{\varepsilon\} = [D'][N]\{d\}$$

$$[D'] = \frac{1}{\|[J]\|} \begin{Bmatrix} \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} & 0 \\ 0 & \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} \\ \dfrac{\partial x}{\partial s}\dfrac{\partial(\ )}{\partial t} - \dfrac{\partial x}{\partial t}\dfrac{\partial(\ )}{\partial s} & \dfrac{\partial y}{\partial t}\dfrac{\partial(\ )}{\partial s} - \dfrac{\partial y}{\partial s}\dfrac{\partial(\ )}{\partial t} \end{Bmatrix}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

The shape function matrix [N] is the 2 x 8 {d} is the column matrix.

$$\underset{3\times 8}{[B]} = \underset{3\times 2}{[D']} \underset{2\times 8}{[N]}$$

The matrix multiplications yield

$$[B(s,t)] = \frac{1}{|[J]|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

Here *i* is a dummy variable equal to 1, 2, 3, and 4, and

$$a = \frac{1}{4}\left[y_1(s-1) + y_2(-s-1) + y_3(1+s) + y_4(1-s)\right]$$

$$b = \frac{1}{4}\left[y_1(t-1) + y_2(1-t) + y_3(1+t) + y_4(-1-t)\right]$$

$$c = \frac{1}{4}\left[x_1(t-1) + x_2(1-t) + x_3(1+t) + x_4(-1-t)\right]$$

$$d = \frac{1}{4}\left[x_1(s-1) + x_2(-s-1) + x_3(1+s) + x_4(1-s)\right]$$

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

Using the shape functions, we have

$$N_{1,s} = \frac{1}{4}(t-1) \qquad N_{1,t} = \frac{1}{4}(s-1)$$

where the comma followed by the variable *s* or *t* indicates differentiation with respect to that variable; that is, $N_{1,s} = \partial N_1/\partial \mathbf{s}$ and so on.

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

The determinant |[J]| is a polynomial in *s* and *t* and is tedious to evaluate even for the simplest case of the linear plane quadrilateral element.

However, we can evaluate |[J]| as

$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \qquad \{Y_c\}^T = [y_1 \quad y_2 \quad y_3 \quad y_4]$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 3** Strain-Displacement and Stress-Strain Relationships

We observe that $|[J]|$ is a function of ***s*** and ***t*** and the known global coordinates $x_1$, $x_2$, ... , $y_4$.

Hence, [**B**] is a function of ***s*** and ***t*** in both the numerator and the denominator and of the known global coordinates $x_1$ through $y_4$.

The stress-strain relationship is a function of ***s*** and ***t***.

$$\{\sigma\} = [D][B]\{d\}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

We now want to express the stiffness matrix in terms of ***s-t*** coordinates.

For an element with a constant thickness *h*, we have

$$[k] = \iint_A [B]^T [D][B]\, h\, dx\, dy$$

However, [*B*] is now a function of ***s*** and ***t***, we must integrate with respect to ***s*** and ***t***.

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

Once again, to transform the variables and the region from ***x*** and ***y*** to ***s*** and ***t***, we must have a standard procedure that involves the determinant of [*J*].

$$\int\int_A f(x,y)\,dx\,dy = \int\int_A f(s,t)\left|[J]\right|ds\,dt$$

where the inclusion of |[J]| in the integrand on the right side of equation results from a theorem of integral calculus.

## Isoparametric Elements

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

We also observe that the Jacobian (the determinant of the Jacobian matrix) relates an element area (*dx dy*) in the global coordinate system to an elemental area (*ds dt*) in the natural coordinate system.

For rectangles and parallelograms, *J* is the constant value *J* = *A*/4, where *A* represents the physical surface area of the element.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\,h\left|[J]\right|ds\,dt$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

The |[*J*]| and [*B*] are complicated expressions within the integral.

Integration to determine the element stiffness matrix is usually done numerically.

The stiffness matrix is of the order 8 x 8.

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

***Body Forces*** - The element body-force matrix will now be determined from

$$\underset{(8 \times 1)}{\{f_b\}} = \int_{-1}^{1}\int_{-1}^{1} \underset{(8 \times 2)}{[N]^T} \underset{(2 \times 1)}{\{X_b\}}\, h \left|[J]\right| ds\, dt$$

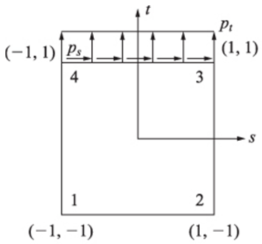
Like the stiffness matrix, the body-force matrix has to be evaluated by numerical integration.

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

***Surface Forces*** - The surface-force matrix, say, along edge *t* = 1 with overall length *L,* is




$$\underset{(4\times 1)}{\{f_s\}} = \frac{L}{2}\int_{-1}^{1} \underset{(4\times 2)}{[N_s]^T} \; \underset{(2\times 1)}{\{T\}}\, h\, ds$$

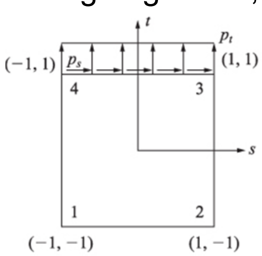
$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \frac{hL}{2}\int_{-1}^{1} \begin{bmatrix} N_3 & 0 & N_4 & 0 \\ 0 & N_3 & 0 & N_4 \end{bmatrix}^T_{along\ t=1} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} ds$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Step 4** Derive the Element Stiffness Matrix and Equations

***Surface Forces*** - For the case of uniform (constant) $p_s$, and $p_t$, along edge *t* = 1, the total surface-force matrix is




$$\underset{(4\times 1)}{\{f_s\}} = \frac{L}{2}\int_{-1}^{1}\int_{-1}^{1} \underset{(4\times 2)}{[N_s]^T} \; \underset{(2\times 1)}{\{T\}}\, h\, ds$$

$$\begin{Bmatrix} f_{s3s} \\ f_{s3t} \\ f_{s4s} \\ f_{s4t} \end{Bmatrix} = \frac{hL}{2}\begin{Bmatrix} p_s \\ p_t \\ p_s \\ p_t \end{Bmatrix}$$

## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Steps 5 - 7**

Steps 5 through 7, which involve assembling the global stiffness matrix and equations, determining the unknown nodal displacements, and calculating the stress, are identical to those in presented in previous chapters.

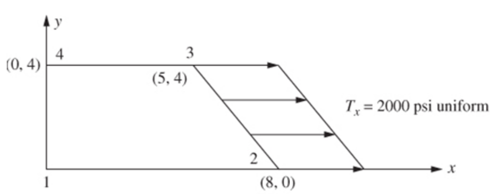
## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Example 1**

For the four-noded linear plane quadrilateral element shown below with a uniform surface traction along side 2-3, evaluate the force matrix by using the energy equivalent nodal forces.

Let the thickness of the element be $h$ = 0.1 in.




$$\{f_s\} = \frac{hL}{2}\int_{-1}^{1}[N_s]^T\{T\}\,ds$$

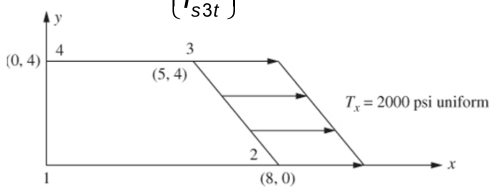
## Isoparametric Elements

### Isoparametric Formulation of the Quadrilateral Element

### Example 1

With length of side 2-3 given by: $L = \sqrt{(5-8)^2 + (4-0)^2} = 5$

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = \frac{hL}{2} \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T_{along\ s=1} \begin{Bmatrix} p_s \\ p_t \end{Bmatrix} dt$$



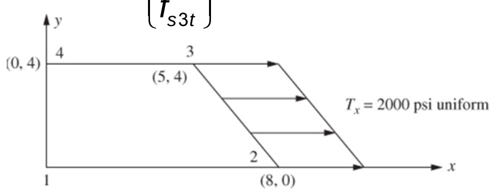

## Isoparametric Elements

### Isoparametric Formulation of the Quadrilateral Element

### Example 1

Substituting for ***L***, the surface traction matrix, and the thickness *h* = 0.1 we obtain

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = \frac{(0.1\text{in.})5\text{in.}}{2} \int_{-1}^{1} \begin{bmatrix} N_2 & 0 & N_3 & 0 \\ 0 & N_2 & 0 & N_3 \end{bmatrix}^T_{along\ s=1} \begin{Bmatrix} 2{,}000 \\ 0 \end{Bmatrix} dt$$



## Isoparametric Elements

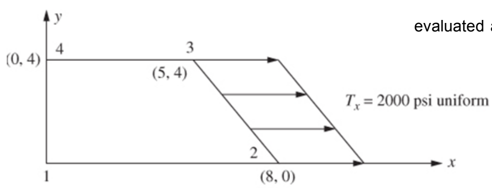
### Isoparametric Formulation of the Quadrilateral Element

**Example 1**

Simplifying gives:

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 0.25\,\text{in.}^2 \int_{-1}^{1} \begin{bmatrix} 2{,}000\,N_2 \\ 0 \\ 2{,}000\,N_3 \\ 0 \end{bmatrix} dt \qquad = 500\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} N_2 \\ 0 \\ N_3 \\ 0 \end{bmatrix} dt$$

evaluated along $s = 1$ evaluated along $s = 1$




## Isoparametric Elements

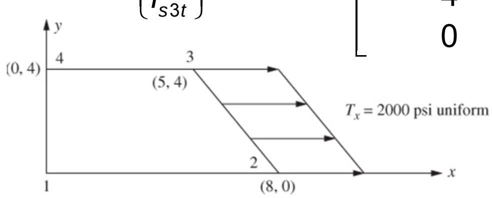
### Isoparametric Formulation of the Quadrilateral Element

**Example 1**

Substituting the shape functions, we have

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 500\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} \dfrac{s-t-st+1}{4} \\ 0 \\ \dfrac{s+t+st+1}{4} \\ 0 \end{bmatrix} dt \qquad = 250\,\text{lb.} \int_{-1}^{1} \begin{bmatrix} 1-t \\ 0 \\ t+1 \\ 0 \end{bmatrix} dt$$

evaluated along $s = 1$



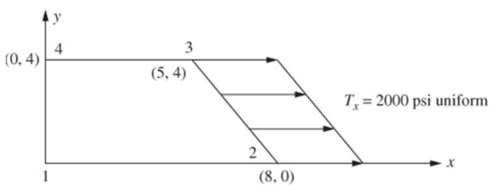
## *Isoparametric Elements*

**Isoparametric Formulation of the Quadrilateral Element**

**Example 1**

Performing the integration gives:

$$\begin{Bmatrix} f_{s2s} \\ f_{s2t} \\ f_{s3s} \\ f_{s3t} \end{Bmatrix} = 250\,\text{lb.}\int_{-1}^{1} \begin{bmatrix} 1-t \\ 0 \\ t+1 \\ 0 \end{bmatrix} dt \quad = 500\,\text{lb.} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad = \begin{bmatrix} 500 \\ 0 \\ 500 \\ 0 \end{bmatrix} \text{lb.}$$




## *Isoparametric Elements*

**Newton-Cotes and Gaussian Quadrature**

In this section, we will describe two methods for numerical evaluation of definite integrals, because it has proven most useful for finite element work.

The Newton-Cotes methods for one and two intervals of integration are the well known trapezoid and Simpson's one-third rule, respectively.

We will then describe Gauss' method for numerical evaluation of definite integrals.

After describing both methods, we will then understand why the Gaussian quadrature method is used in finite element work.

## Isoparametric Elements

### Newton-Cotes and Gaussian Quadrature

The Newton-Cotes method is a common technique for evaluation of definite integrals.

To evaluate the integral $\mathrm{I} = \int_{-1}^{1} y\, dx$

we assume the sampling points of *y(x)* are spaced at equal intervals.

Since the limits of integration are from -1 to 1 using the isoparametric formulation, the Newton-Cotes formula is given by

$$\mathrm{I} = \int_{-1}^{1} y\, dx = h\sum_{i=0}^{n} C_i y_i \; = h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

## Isoparametric Elements

### Newton-Cotes and Gaussian Quadrature

The constants $C_i$ are the Newton-Cotes constants for numerical integration with *i* intervals.

The number of intervals will be one less than the number of sampling points, *n.*

The term *h* is the interval between the limits of integration (for limits of integration between -1 and 1 this makes *h* = 2).

$$\mathrm{I} = \int_{-1}^{1} y\, dx = h\sum_{i=0}^{n} C_i y_i \; = h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

## *Isoparametric Elements*

### Newton-Cotes and Gaussian Quadrature

The Newton-Cotes constants have been published and are summarized in the table below for *i* = 1 to 6.

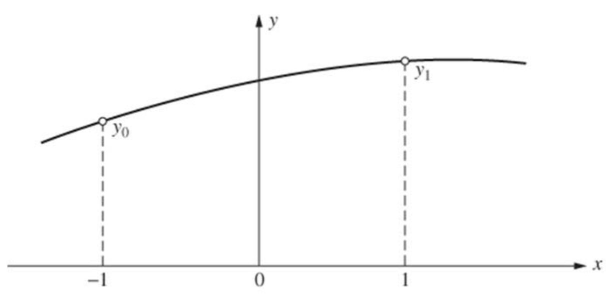
| Intervals, *i* | No. of Points, *n* | $C_0$ | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 1/2 | 1/2 | | | (trapezoid rule) | | |
| 2 | 3 | 1/6 | 4/6 | 1/6 | | (Simpson's 1/3 rule) | | |
| 3 | 4 | 1/8 | 3/8 | 3/8 | 1/8 | (Simpson's 3/8 rule) | | |
| 4 | 5 | 7/90 | 32/90 | 12/90 | 32/90 | 7/90 | | |
| 5 | 6 | 19/288 | 75/288 | 50/288 | 50/288 | 75/288 | 19/288 | |
| 6 | 7 | 41/840 | 216/840 | 27/840 | 272/840 | 27/840 | 216/840 | 41/840 |

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;\; = h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

## *Isoparametric Elements*

### Newton-Cotes and Gaussian Quadrature

The case *i* = 1 corresponds to the well known trapezoid rule illustrated below.




$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;\; = \left[y_0 + y_1\right]$$

## *Isoparametric Elements*

**Newton-Cotes and Gaussian Quadrature**

The case $i$ = 2 corresponds to the well-known Simpson one-third rule.

It has been shown that the formulas for $i$ = 3 and $i$ = 5 have the same accuracy as the formulas for $i$ = 2 and $i$ = 4, respectively.

Therefore, it is recommended that the even formulas with $i$ = 2 and $i$ = 4 be used in practice.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

## *Isoparametric Elements*

**Newton-Cotes and Gaussian Quadrature**

To obtain greater accuracy one can then use a smaller interval (include more evaluations of the function to be integrated).

This can be accomplished by using a higher-order Newton-Cotes formula, thus increasing the number of intervals $i$.

It has been shown that we need to use $n$ equally spaced sampling points to integrate exactly a polynomial of order at most $n$ - 1.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i \;= h\left[C_0 y_0 + C_1 y_1 + C_2 y_2 + \cdots + C_n y_n\right]$$

## Isoparametric Elements

### Newton-Cotes and Gaussian Quadrature

On the other hand, using Gaussian quadrature we will show that we use unequally spaced sampling points $n$ and integrate exactly a polynomial of order at most $2n$ - 1.

For instance, using the Newton-Cotes formula with $n$ = 2 sampling points, the highest order polynomial we can integrate exactly is a linear one.

However, using Gaussian quadrature, we can integrate a cubic polynomial exactly.

## Isoparametric Elements

### Newton-Cotes and Gaussian Quadrature

Gaussian quadrature is then more accurate with fewer sampling points than Newton-Cotes quadrature

This is because Gaussian quadrature is based on optimizing the position of the sampling points (not making them equally spaced as in the Newton-Cotes method) and also optimizing the weights $W_i$ (see the table below).

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=0}^{n} W_i y(x_i)$$

| Order $N$ | Points $u_i$ | Weights $w_i$ |
|---|---|---|
| 1 | 0.000000000 | 2.00000000 |
| 2 | ±0.577350269 | 1.00000000 |
| 3 | 0.000000000 | 0.88888889 |
| | ±0.774596669 | 0.55555556 |
| 4 | ±0.339981044 | 0.65214515 |
| | ±0.861136312 | 0.34785485 |

## *Isoparametric Elements*

### Newton-Cotes Example

Using the Newton-Cotes method with $i$ = 2 intervals ($n$ = 3 sampling points), evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx \qquad \mathrm{I} = \int_{-1}^{1}\left[3^x - x\right]dx$$

Using three sampling points means we evaluate the function inside the integrand at $x$ = -1, $x$ = 0, and $x$ = 1, and multiply each evaluated function by the respective Newton-Cotes numbers.

$$\mathrm{I} = \int_{-1}^{1} y\,dx = h\sum_{i=0}^{n} C_i y_i = 2\left[\tfrac{1}{6}y_0 + \tfrac{4}{6}y_1 + \tfrac{1}{6}y_2\right]$$

## *Isoparametric Elements*

### Newton-Cotes Example

Using the Newton-Cotes method with $i$ = 2 intervals ($n$ = 3 sampling points), evaluate the integrals:

$$\boxed{\mathrm{I} = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx} \qquad \mathrm{I} = \int_{-1}^{1}\left[3^x - x\right]dx$$

$$\mathrm{I} = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx \approx 2\left[\tfrac{1}{6}(1.8775826) + \tfrac{4}{6}(1) + \tfrac{1}{6}(1.8775826)\right]$$

$$= 2.5850550 \qquad \boxed{0.027\%\text{ error}}$$

$$\mathrm{I} = \int_{-1}^{1}\left[x^2 + \cos\left(\frac{x}{2}\right)\right]dx = 2.5843688$$

## Isoparametric Elements

### Newton-Cotes Example

Using the Newton-Cotes method with $i$ = 2 intervals ($n$ = 3 sampling points), evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1} \left[ x^2 + \cos\left(\frac{x}{2}\right) \right] dx \qquad \boxed{\mathrm{I} = \int_{-1}^{1} \left[ 3^x - x \right] dx}$$

$$\mathrm{I} = \int_{-1}^{1} \left[ 3^x - x \right] dx \approx 2\left[ \tfrac{1}{6}(1.3333333) + \tfrac{4}{6}(1) + \tfrac{1}{6}(2) \right]$$

$$= 2.4444444 \qquad \boxed{0.706\% \text{ error}}$$

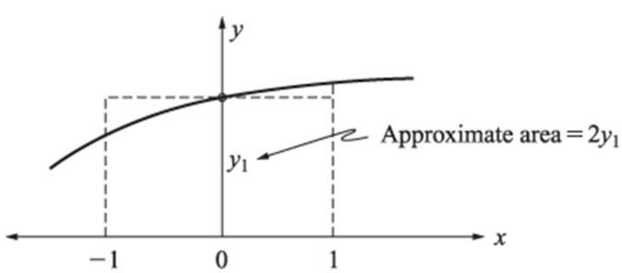
$$\mathrm{I} = \int_{-1}^{1} \left[ 3^x - x \right] dx = 2.427305$$

## Isoparametric Elements

### Gaussian Quadrature

To evaluate the integral: $\mathrm{I} = \int_{-1}^{1} y\, dx$

where $y = y(x)$, we might choose (sample or evaluate) $y$ at the midpoint $y(0) = y_1$ and multiply by the length of the interval, as shown below to arrive at $\mathrm{I} = 2y_1$, a result that is exact if the curve happens to be a straight line.



## *Isoparametric Elements*

**Gaussian Quadrature**

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\, dx = \sum_{i=1}^{n} W_i y(x_i)$$

That is, to approximate the integral, we evaluate the function at several sampling points *n,* multiply each value $y_i$ by the appropriate weight $W_i$ , and add the terms.

Gauss's method chooses the sampling points so that for a given number of points, the best possible accuracy is obtained.

Sampling points are located symmetrically with respect to the center of the interval.

## *Isoparametric Elements*

**Gaussian Quadrature**

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\, dx = \sum_{i=1}^{n} W_i y(x_i)$$

In general, Gaussian quadrature using *n* points (Gauss points) is exact if the integrand is a polynomial of degree *2n* - 1 or less.

In using *n* points, we effectively replace the given function $y = f(x)$ by a polynomial of degree *2n*- 1.

The accuracy of the numerical integration depends on how well the polynomial fits the given curve.

## *Isoparametric Elements*

**Gaussian Quadrature**

Generalization of the formula leads to:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = \sum_{i=1}^{n} W_i y\left(x_i\right)$$

If the function $f(x)$ is not a polynomial, Gaussian quadrature is inexact, but it becomes more accurate as more Gauss points are used.

Also, it is important to understand that the ratio of two polynomials is, in general, not a polynomial; therefore, Gaussian quadrature will not yield exact integration of the ratio.

## *Isoparametric Elements*

**Gaussian Quadrature -** Two-Point Formula

To illustrate the derivation of a two-point ($n$ = 2) consider:

$$\mathrm{I} = \int_{-1}^{1} y\,dx = W_1 y\left(x_1\right) + W_2 y\left(x_2\right)$$

There are four unknown parameters to determine: $W_1$, $W_2$, $x_1$, and $x_2$.

Therefore, we assume a cubic function for $y$ as follows:

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

## *Isoparametric Elements*

**Gaussian Quadrature -** Two-Point Formula

In general, with four parameters in the two-point formula, we would expect the Gauss formula to exactly predict the area under the curve.

$$A = \int_{-1}^{1} \left(C_0 + C_1 x + C_2 x^2 + C_3 x^3\right) dx = 2C_0 + \frac{2}{3} C_2$$

However, we will assume, based on Gauss's method, that $W_1 = W_2$ and that $x_1 = x_2$ as we use two symmetrically located Gauss points at $x = \pm a$ with equal weights.

The area predicted by Gauss's formula is

$$A_G = W\, y(-a) + W\, y(a) = 2W\left(C_0 + C_2 a^2\right)$$

## *Isoparametric Elements*

**Gaussian Quadrature -** Two-Point Formula

If the error, $e = A - A_G$, is to vanish for any $C_0$ and $C_2$, we must have, in the error expression:

$$\frac{\partial e}{\partial C_0} = 0 = 2 - 2W \qquad \Rightarrow \qquad W = 1$$

$$\frac{\partial e}{\partial C_2} = 0 = \frac{2}{3} - 2a^2 W \qquad \Rightarrow \qquad a = \sqrt{1/3} = 0.5773....$$

Now $W = 1$ and $a = 0.5773$ ... are the $W_i$'s and $a_i$'s ($x_i$'s) for the two-point Gaussian quadrature as given in the table.

## Isoparametric Elements

### Gaussian Quadrature Example

Use three-point Gaussian Quadrature evaluate the integrals:

$$\boxed{\mathrm{I} = \int_{-1}^{1} \left[ x^2 + \cos\left(\frac{x}{2}\right) \right] dx}$$

$$\mathrm{I} = \int_{-1}^{1} \left[ 3^x - x \right] dx$$

$$\mathrm{I} \approx \sum_{i=1}^{3} W_i \left[ x_i^{\,2} + \cos\left(\frac{x_i}{2}\right) \right]$$

$$\approx \frac{5}{9}(1.5259328)$$

$$+\frac{8}{9}(1.0)$$

$$+\frac{5}{9}(1.5259328) \quad = 2.5843698$$

| Order $N$ | Points $u_i$ | Weights $w_i$ |
|---|---|---|
| 1 | 0.000000000 | 2.00000000 |
| 2 | ±0.577350269 | 1.00000000 |
| 3 | 0.000000000 | 0.88888889 |
| | ±0.774596669 | 0.55555556 |
| 4 | ±0.339981044 | 0.65214515 |
| | ±0.861136312 | 0.34785485 |

0.00004% error

## Isoparametric Elements

### Gaussian Quadrature Example

Use three-point Gaussian Quadrature evaluate the integrals:

$$\mathrm{I} = \int_{-1}^{1} \left[ x^2 + \cos\left(\frac{x}{2}\right) \right] dx$$

$$\boxed{\mathrm{I} = \int_{-1}^{1} \left[ 3^x - x \right] dx}$$

$$\mathrm{I} \approx \sum_{i=1}^{3} W_i \left[ 3^{x_i} - x_i \right]$$

$$\approx \frac{5}{9}(1.2015923)$$

$$+\frac{8}{9}(1.0)$$

$$+\frac{5}{9}(1.5673475) \quad = 2.4271888$$

| Order $N$ | Points $u_i$ | Weights $w_i$ |
|---|---|---|
| 1 | 0.000000000 | 2.00000000 |
| 2 | ±0.577350269 | 1.00000000 |
| 3 | 0.000000000 | 0.88888889 |
| | ±0.774596669 | 0.55555556 |
| 4 | ±0.339981044 | 0.65214515 |
| | ±0.861136312 | 0.34785485 |

0.00477% error

## *Isoparametric Elements*

### Gaussian Quadrature Example

In two dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other as

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,ds\,dt \approx \int_{-1}^{1}\left[\sum_{i=1}^{n} W_i f(s_i,t)\right]dt$$

$$\approx \sum_{j=1}^{n} W_j \left[\sum_{i=1}^{n} W_i f(s_i,t_j)\right]$$

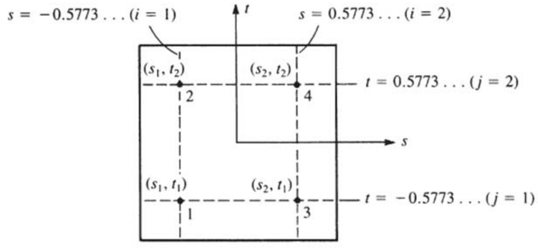
$$\approx \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j f(s_i,t_j)$$

## *Isoparametric Elements*

### Gaussian Quadrature Example

For example, a four-point Gauss rule (often described as a 2 x 2 rule) is shown below with *i* = 1, 2 and *j* = 1, 2 yields

$$\mathrm{I} \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_i W_j f(s_i,t_j) \approx W_1 W_1 f(s_1,t_1) + W_1 W_2 f(s_1,t_2) + W_2 W_1 f(s_2,t_1) + W_2 W_2 f(s_2,t_2)$$




The four sampling points are at $s_i$ *and* $t_i$ = ±0.5773... and $W_i$ = 1.0

## Isoparametric Elements

### Gaussian Quadrature Example

In three dimensions, we obtain the quadrature formula by integrating first with respect to one coordinate and then with respect to the other two as

$$\mathrm{I} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(s,t,z)\,ds\,dt\,dz \approx \sum_{i=1}\sum_{j=1}\sum_{k=1} W_i W_j W_k f\left(s_i, t_j, z_k\right)$$

## Isoparametric Elements

### Evaluation of the Stiffness Matrix by Gaussian Quadrature

For the two-dimensional element, we have shown in previous chapters that

$$[k] = \iint_A [B]^T [D][B]\, h\, dx\, dy$$

where, in general, the integrand is a function of *x* and *y* and nodal coordinate values.

## *Isoparametric Elements*

### Evaluation of the Stiffness Matrix by Gaussian Quadrature

We have shown that [k] for a quadrilateral element can be evaluated in terms of a local set of coordinates ***s***-***t***, with limits from -1 to 1within the element.

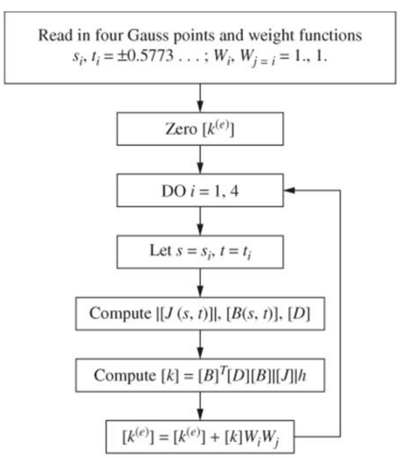
$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B]\, h \left|[J]\right| ds\, dt$$

Each coefficient of the integrand $[B]^T$ $[D]$ $[B]$ $|[J]|$ evaluated by numerical integration in the same manner as $f(s, t)$ was integrated.

## *Isoparametric Elements*

### Evaluation of the Stiffness Matrix by Gaussian Quadrature

A flowchart to evaluate [*k*] for an element using four-point Gaussian quadrature is shown here.



## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

The explicit form for four-point Gaussian quadrature (now using the single summation notation with $i$ = 1, 2, 3, 4), we have

$$[k] = \int_{-1}^{1}\int_{-1}^{1} [B]^T [D][B] h \left|[J]\right| ds\, dt$$

$$= \left[B(s_1,t_1)\right]^T [D]\left[B(s_1,t_1)\right]\left|\left[J(s_1,t_1)\right]\right| W_1 W_1$$

$$+ \left[B(s_2,t_2)\right]^T [D]\left[B(s_2,t_2)\right]\left|\left[J(s_2,t_2)\right]\right| W_2 W_2$$

$$+ \left[B(s_3,t_3)\right]^T [D]\left[B(s_3,t_3)\right]\left|\left[J(s_3,t_3)\right]\right| W_3 W_3$$

$$+ \left[B(s_4,t_4)\right]^T [D]\left[B(s_4,t_4)\right]\left|\left[J(s_4,t_4)\right]\right| W_4 W_4$$

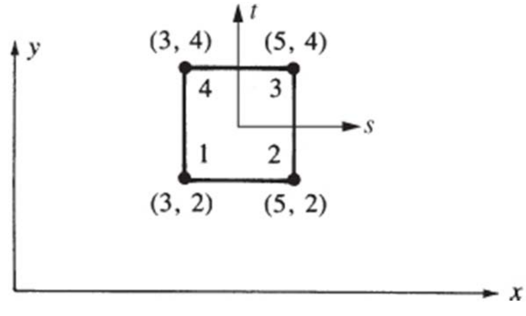
where $s_1$=$t_1$= -0.5773, $s_2$=-0.5773, $t_2$=0.5773, $s_3$=0.5773, $t_3$=-0.5773, and $s_4$=$t_4$=0.5773 and $W_1$=$W_2$=$W_3$=$W_4$=1.0

## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Evaluate the stiffness matrix for the quadrilateral element shown below using the four-point Gaussian quadrature rule.

Let $E$ = 30 x $10^6$ psi and $\upsilon$ = 0.25. The global coordinates are shown in inches. Assume $h$ = 1 in.



## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Using the four-point rule, the four points are:

$$
\begin{aligned}
(s_1,t_1) &= (-0.5773,\ -0.5773)\\
(s_2,t_2) &= (-0.5773,\ \ \ 0.5773)\\
(s_3,t_3) &= (\ \ \ 0.5773,\ -0.5773)\\
(s_4,t_4) &= (\ \ \ 0.5773,\ \ \ 0.5773)
\end{aligned}
$$

With $W_1 = W_2 = W_3 = W_4 = 1.0$

## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

$$
\begin{aligned}
[k] = {} & \left[B(s_1,t_1)\right]^T [D]\left[B(s_1,t_1)\right]\left|\left[J(s_1,t_1)\right]\right| \\
& + \left[B(s_2,t_2)\right]^T [D]\left[B(s_2,t_2)\right]\left|\left[J(s_2,t_2)\right]\right| \\
& + \left[B(s_3,t_3)\right]^T [D]\left[B(s_3,t_3)\right]\left|\left[J(s_3,t_3)\right]\right| \\
& + \left[B(s_4,t_4)\right]^T [D]\left[B(s_4,t_4)\right]\left|\left[J(s_4,t_4)\right]\right|
\end{aligned}
$$

First evaluate $|[J]|$ at each Gauss, for example:

$$\left|\left[J(-0.5773,-0.5773)\right]\right|$$

## Isoparametric Elements

### Evaluation of the Stiffness Matrix by Gaussian Quadrature

Recall:

$$\left|[J]\right| = \frac{1}{8}\{X_c\}^T \begin{bmatrix} 0 & 1-t & t-s & s-1 \\ t-1 & 0 & s+1 & -s-t \\ s-t & -s-1 & 0 & t+1 \\ 1-s & s+t & -t-1 & 0 \end{bmatrix} \{Y_c\}$$

$$\{X_c\}^T = [x_1 \quad x_2 \quad x_3 \quad x_4] \qquad \{Y_c\}^T = [y_1 \quad y_2 \quad y_3 \quad y_4]$$

For this example:

$$\{X_c\}^T = [3 \quad 5 \quad 5 \quad 3] \qquad \{Y_c\}^T = [2 \quad 2 \quad 4 \quad 4]$$

## Isoparametric Elements

### Evaluation of the Stiffness Matrix by Gaussian Quadrature

Recall:

$$\left|\left[J(-0.5773, -0.5773)\right]\right|$$

$$= \frac{1}{8}[3 \quad 5 \quad 5 \quad 3] \begin{bmatrix} 0 & 1-(-0.5773) & (-0.5773)-(-0.5773) & (-0.5773)-1 \\ (-0.5773)-1 & 0 & (-0.5773)+1 & -(-0.5773)-(-0.5773) \\ (-0.5773)-(-0.5773) & -(-0.5773)-1 & 0 & (-0.5773)+1 \\ 1-(-0.5773) & (-0.5773)+(-0.5773) & -(-0.5773)-1 & 0 \end{bmatrix} \begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$= 1.000$$

Similarly:

$$\left|\left[J(-0.5773, 0.5773)\right]\right| = 1.000$$

$$\left|\left[J(0.5773, -0.5773)\right]\right| = 1.000$$

$$\left|\left[J(0.5773, 0.5773)\right]\right| = 1.000$$

## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

To evaluate [*B*] consider:

$$\left[B(-0.5773, -0.5773)\right]$$

$$= \frac{1}{\left|\left[J(-0.5773, -0.5773)\right]\right|}\left[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]\right]$$

where

$$[B_i] = \begin{bmatrix} a(N_{i,s}) - b(N_{i,t}) & 0 \\ 0 & c(N_{i,t}) - d(N_{i,s}) \\ c(N_{i,t}) - d(N_{i,s}) & a(N_{i,s}) - b(N_{i,t}) \end{bmatrix}$$

## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

For this example:

$$a = \frac{1}{4}\left[y_1(s-1) + y_2(-s-1) + y_3(1+s) + y_4(1-s)\right]$$

$$= \frac{1}{4}\left[2\left((-0.5773)-1\right) + 2\left(-(-0.5773)-1\right) + 4\left(1+(-0.5773)\right) + 4\left(1-(-0.5773)\right)\right]$$

$$= 1.000$$

Similar computations are used to obtain *b, c,* and *d.*

## *Isoparametric Elements*

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

The shape functions are computed as:

$$N_{1,s} = \frac{1}{4}(t-1) = \frac{1}{4}((-0.5773)-1) = -0.3943$$

$$N_{1,t} = \frac{1}{4}(s-1) = \frac{1}{4}((-0.5773)-1) = -0.3943$$

Similarly, [$B_2$], [$B_3$], and [$B_4$] must be evaluated like [$B_1$] at (-0.5773, -0.5773).

We then repeat the calculations to evaluate [*B*] at the other Gauss points.

## *Isoparametric Elements*

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

Using a computer program written specifically to evaluate [*B*], at each Gauss point and then [*k*], we obtain the final form of [*B*(-0.5773, -0.5773)], as

$$\left[B(-0.5773, -0.5773)\right]$$

$$= \begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & -0.1057 & -0.3943 \end{bmatrix}$$

With similar expressions for [*B*(-0.5773, 0.5773)], and so on.

## Isoparametric Elements

**Evaluation of the Stiffness Matrix by Gaussian Quadrature**

The matrix [D] is:

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 0.5(1-\nu) \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \times 10^6\ psi$$

Finally, [*k*] is:

$$[k] = 10^4 \begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

## Axisymmetric Elements

**Problems**

25. To be assigned from your textbook “A First Course in the Finite Element Method” by D. Logan.

# End of Chapter 10