FIBRE BUNDLES ASSOCIATED WITH SPACE-TIME*)

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Contents

Few words have been abused by physicists more than relativity, symmetry, covariance, invariance and gauge or coordinate transformations. These notions used extensively since the advent of the theory of relativity, are hardly ever precisely defined in physical texts. This gives rise to many misunderstandings and controversies; the discussion on the significance of the "principle of general covariance" has been one of the best known among them. This polemic started around 1917 [10] and has been revived during the recent years (cf. Fock [11] and Anderson [12]).

Fibre bundles provide a convenient framework for discussing the concepts of relativity, invariance, and gauge transformations. They have been originally introduced in order to formulate and solve "global", topological problems. We shall not be concerned with these here. However, the notion of a fibre bundle is very appropriate also for local problems of differential geometry and field theory. The concept of induced representations of Lie groups may be most easily explained

*) This paper is an expanded and modified version of the notes by H.P. Künzle and P. Szekeres of the lectures given at King's College, London, in September, 1967.

using the language of bundles. The canonical formalism of classical mechanics assumes the cotangent bundle of the manifold of positions to be the underlying space. Classical electrodynamics may be interpreted as a theory of an infinitesimal connection in a principal fibre bundle with the structure group $U(1)$. A similar interpretation can be given to the Yang-Mills field and in general to all fields resulting from "gauge transformations of the second kind".

Every time anyone argues for the desirability of using new mathematical concepts or methods, people raise the question whether these new concepts are really necessary. Quite often the answer is *no*, in the sense that there are no practical problems that could not be solved without introducing the new methods. For example, at the beginning of the development of electromagnetic theory Maxwell's equations were written explicitly, component by component. Any problem of classical electrodynamics can be solved with the help of that system of equations, but today no one will deny the usefulness of vector calculus. A less trivial example is the following. With some skill, any result in special relativity may be obtained on the basis of the physical interpretation and of the form of the Lorentz transformations, as given by Einstein in his 1905 paper. It is hard to imagine however, that general relativity could have ever been invented without the four-dimensional, geometric picture of space-time, or that it could have been sensibly formulated without using the concepts of Riemannian geometry. It is our belief that fibre bundles may play a somewhat similar role: as they provide a natural framework for a number of physical theories, they can open ways to new, fruitful generalizations. For the moment, they help us to clarify a number of fundamental concepts and by doing so, leave us with more time to worry about the really difficult questions.

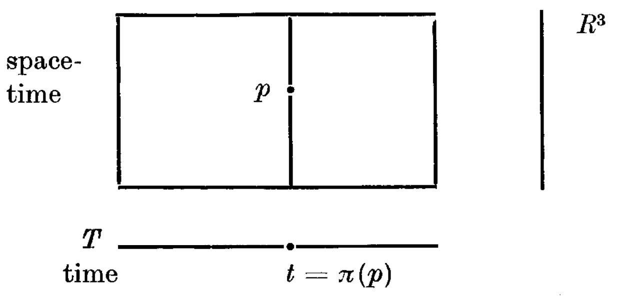
In this paper we present the basic information on the local structure of differentiable fibre bundles together with some of their applications to physics. In the Introduction we give a few examples of structures which are fibre bundles according to the precise definitions to be given later. We take advantage of these examples to introduce, in a loose way, a number of terms used in describing fibre bundles. The second chapter contains the construction of the space of quantities of type σ (e.g., tensors, densities) over a finite-dimensional vector space. The reason for presenting this construction is that it is analogous to the one that leads from a principal fibre bundle to an associated bundle (Section 5.6). Chapters 3-6 contain standard definitions of differentiable manifolds, Lie groups, G-manifolds, fibre bundles and connections. This material is presented for the convenience of a physicist who may not want to search for it in specialized mathematical literature. Chapter 7 is devoted to an analysis of fields admitting gauge transformations, such as the electromagnetic or the Yang-Mills field. An essential difference between these fields and the gravitational field in general relativity is stressed, in contradistinction to what is often asserted. The isomorphism between the Kaluza-Klein theory and the Utiyama approach to electrodynamics is exhibited and generalized to an arbitrary field whose gauge group admits an invariant metric. In Chapter 8 it is shown what meaning should be attributed to the general-relativistic principle of invariance. An application of fibre bundles to the derivation and formulation of conservation laws in physics may be found in another paper [13].

1. Introduction

Bundles are a generalization of the concept of Cartesian product. An example from the history of science will clarify the need for such a generalization.

In Aristotelian physics both, space and time, were absolute, every event being defined by an instant of time and a location in space (cf. Penrose [5]). This is equivalent to saying that space-time E is a Cartesian product $T \times S$, where T is the time axis and S is the three-dimensional space.

In Galilean physics time remains absolute but space is relative. This can be described by saying that there is a *projection* map $\pi : E \rightarrow T$ which associates to any event $p \in E$ the corresponding instant of time $t = \pi(p)$. T is called the *base space*. The inverse image of t, $\pi^{-1}(t)$, is called a *fibre*. Each fibre is isomorphic to



Euclidean 3-space R^3, which is therefore called a *typical fibre*. Such a triple (E, T, π), with $\pi : E \rightarrow T$ being a surjective projection map is called a *bundle* with base space base T and bundle space E. In this example all fibres are isomorphic and it is possible to represent the bundle as a Cartesian product. However this representation is frame dependent, that is to say there is no *natural isomorphism* between the fibres.

The best way of illustrating the concept of *naturality* which has been connected above to the physical concept of relativity, is to give an example from vector space theory. Let V be an n-dimensional vector space and V^* the dual space of linear functions on V. V^* is also an n-dimensional vector space and is therefore isomorphic to V, but there is no natural isomorphism. It is necessary to define a basis in V in order to construct the isomorphism. On the other hand, the space V^{**} (the dual of V^*) is again an n-dimensional vector space, but this time there is a natural isomorphism with V. A precise definition of natural equivalence is given in Chapter 8. Further examples of bundles:

1) *Tangent bundle*: Let E be an n-dimensional differentiable manifold and $\tau(E)$ the set of all tangent vectors at all points of E. Let $\pi : \tau(E) \to E$ be the mapping which maps tangent vectors onto the point of E to which they are attached. The triple $(\tau(E), E, \pi)$ is known as the *tangent bundle* of the manifold E. Each fibre $\pi^{-1}(p), p \in E$, is isomorphic to R^n, which is therefore a *typical fibre*. The isomorphism is not natural and there are as many isomorphisms from $\pi^{-1}(p)$ onto R^n as there are bases at p. Any two isomorphisms are obtained from each other by applying a member of the group $GL(n, R)$ which is called the *structure group* of the bundle.

If a natural isomorphism between fibres could be defined, e.g., by a parallel transport (integrable connection), there would be a natural isomorphism of the tangent bundle onto the product bundle $(E \times F, E, pr_1)$. In general relativity this is not possible: even if E admits of a global coordinate system $x = (x^i)$ then the mapping

$$\tau(E) \ni X \mapsto (\pi(X),\ (X^i)) \in E \times R^n,$$

where $X^i = X(x^i)$, is an isomorphism but not a natural one.

2) *Bundle of linear frames*: Let $P(E)$ be the set of all vector frames at all points of E. A bundle may be constructed as in (1), the typical fibre being $GL(n, R)$. For let (e_i) be a basis at $p \in E$ and (r_i) a second basis at p. Then $r_i = e_j a^j{}_i$ with $(a^j{}_i) = a \in GL(n, R)$. Thus, there is an isomorphism $r \mapsto a$ and $GL(n, R)$ is the typical fibre. If $\bar{e}_i$ is another basis, then $r_i = \bar{e}_j \bar{a}^j{}_i$. The new isomorphism $r \mapsto \bar{a}$ is connected to the previous one $r \mapsto a$ by a single matrix transformation. Hence the structure group is again $GL(n, R)$ and is a typical fibre at the same time. In this case the bundle is termed a *principal fibre bundle.*

In Greek physics space-time $E = T \times S$ has the structure of a product bundle. In Galilean physics and special relativity this is no longer so, but the bundle of linear frames $(P(E), E, \pi)$ is a product bundle. In general relativity there is no natural isomorphism of $(P(E), E, \pi)$ onto a product $E \times F$, but it will be shown that the bundle $(P(P(E)), P(E), \pi)$ is a product. In this way generalizations of general relativity may be conceived*).

2. Tensors and Tensor Densities

2.1. Let V be an n-dimensional real vector space and $P(V)$ the set of all frames of V; i.e., an element $r \in P(V)$ is a set (r_i), $i = 1, \ldots, n$, of n linearly independent vectors

*) D. D. Ivanenko suggested that this generalization be referred to as the "second relativization".

$r_i \in V$. It can also, with advantage, be regarded as an isomorphism

$$r: R^n \to V$$

such that

$$(q^1, ..., q^n) \mapsto q^i r_i = r(q).$$

Now let $a \in GL(n, R)$, then $r \circ a$ is another isomorphism $R^n \to V$ which again regarded as a frame in $P(V)$ will be denoted by $ra = (r_j a^j{}_i)$. This shows that $GL(n, R)$ acts to the right on $P(V)$: $(ra)b = r(ab)$.

Obviously, this action is transitive and free. (G acts *freely* on a space M iff $a \neq id_G$ implies $ra \neq r$, G acts *effectively* iff $xa = x$ for any x implies $a = id_G$, G acts *transitively* iff for any $x, y \in M$ there exists $a \in G$ such that $xa = y$).

2.2 If $\sigma: GL(n, R) \to GL(m, R)$ is a homomorphism, consider the mapping

$$P(V) \times R^m \to P(V) \times R^m$$

defined by

$$(r, q) \mapsto (ra, \sigma_{a^{-1}}(q))$$

for any $a \in GL(n, R)$, where $\sigma_a \equiv \sigma(a)$. This defines an action of $GL(n, R)$ on $P(V) \times R^m$ which, however, is no longer transitive. Introduce therefore, the quotient space

$$\sigma(V) \equiv (P(V) \times R^m)/GL(n,R)$$

and the canonical map $\iota: P(V) \times R^m \to \sigma(V)$ which maps elements of $P(V) \times R^m$ equivalent under the action of $GL(n, R)$ onto the same element of $\sigma(V)$: $\iota(r, q) = \iota(r', q') \Leftrightarrow$ there exists $a \in GL(n, R)$ such that $ra = r'$ and $\sigma_{a^{-1}}(q) = q'$. Finally define $\iota_r: R^m \to \sigma(V)$ by $\iota_r(q) = \iota(r, q)$. This map ι_r is bijective.

PROOF: (a) By definition any element of $\sigma(V)$ can be given in the form $\iota(r', q')$ for certain $r' \in P(V)$, $q' \in R^m$. But for any given $r \in P(V)$ there is an $a \in GL(n, R)$ such that $r = r'a$ [by the transitivity of $GL(n, R)$ on $P(V)$]. Choose $q = \sigma_{a^{-1}}(q')$, then $\iota(r', q') = \iota(r, q) = \iota_r(q)$, proving that ι_r is surjective.

(b) Assume $\iota_r(q) = \iota_r(q')$, then there exists a such that $r = ra$ and thus $a = id$ (since GL acts freely). Therefore $q' = \sigma_{id}(q) = q$, proving injectivity.

Observe that

$$\iota_{ra} \circ \sigma_{a^{-1}}(q) = \iota(ra, \sigma_{a^{-1}}(q)) = \iota(r, q) = \iota_r(q),$$

i.e.,

$$\iota_{ra} \circ \sigma_{a^{-1}} = \iota_r. \tag{1}$$

Define addition and multiplication with $\alpha \in R$ by

$$\iota_r(q_1) + \iota_r(q_2) \equiv \iota_r(q_1 + q_2), \tag{2}$$

$$\alpha \iota_r(q) \equiv \iota_r(\alpha q). \tag{3}$$

These definitions are independent of the choice of r because of (1). Therefore $\sigma(V)$

becomes an m-dimensional vector space. Now let $u \in \sigma(V)$ and define

$$\tilde{u} : P(V) \to R^m$$

by

$$\tilde{u}(r) = \iota_r^{-1}(u).$$

Denoting the action of $a \in GL(n, R)$ in $P(V)$ by δ_a (i.e. $\delta_a(r) \equiv ra$) one infers

$$\tilde{u} \circ \delta_a(r) = \tilde{u}(ra) = \iota_{ra}^{-1}(u) = \sigma_a^{-1} \circ \iota_r^{-1}(u) = \sigma_a^{-1} \circ \tilde{u}(r),$$

i.e.,

$$\tilde{u} \circ \delta_a = \sigma_a^{-1} \circ \tilde{u} \tag{4}$$

(Note that $\sigma_{a^{-1}} = \sigma(a^{-1}) = [\sigma(a)]^{-1} = \sigma_a^{-1}$ because σ is a homomorphism and that since ι_r has an inverse it follows from (1) that

$$\iota_{ra} = \iota_r \circ (\sigma_{a^{-1}})^{-1} \qquad \text{or} \qquad \iota_{ra}^{-1} = \sigma_{a^{-1}} \circ \iota_r^{-1}).$$

Equation (4) is nothing else but the transformation law of the tensor density u: Under a change "δ_a" of the basis the "components" $\tilde{u}$ of u transform according to a certain representation σ of the linear group. Such quantities u are called *quantities of type* σ. For example:

(i) $\sigma_a \equiv \tau_a = a$, the identity representation. It induces a natural isomorphism between V and $\tau(V)$. Relation (1) becomes $\iota_{ra}^{-1} = a^{-1}\iota_r^{-1}$ showing that the map $r \circ \iota_r^{-1} : \tau(V) \to V$ is an isomorphism independent of r, since

$$r' \circ \iota_{r'}^{-1} = ra \circ \iota_{ar}^{-1} = ra\, a^{-1}\, \iota_r^{-1} = r\iota_r^{-1}.$$

$$\begin{tikzcd} R^n \arrow[r, "r"] \arrow[d, "id"'] & V \arrow[d] \\ R^n \arrow[r, "\iota_r"] & \tau(V) \end{tikzcd}$$

(ii) If σ is any representation so is σ^*, defined by $\sigma_a^* = {}^t\sigma_{a^{-1}}$ (where ${}^tA$ is the transpose matrix of A). $\sigma^*(V)$ is called the space of *quantities contragredient to* $\sigma(V)$.

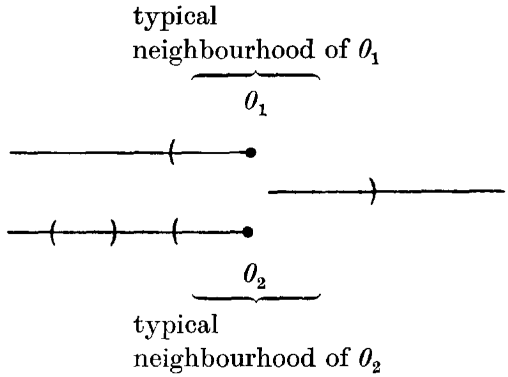
(iii) $\sigma_a = (\det a)^w$ gives rise to *scalar densities of weight* w.

3. Differentiable Manifolds

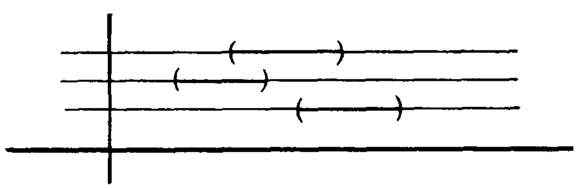
3.1. A *manifold* is a topological space E which is

(i) locally Euclidean, i.e., for any $p \in E$ there exists an open neighbourhood U of p and a homeomorphism x of U onto an open subset of R^n. (U, x) is called a *chart* of E.

(ii) E is Hausdorff, i.e., for any pair of points $p \neq q$ there exist disjoint open sets $U \ni p$, $V \ni q$. This does not follow from (i); for consider the topological space consisting of two closed halflines $(-\infty, 0]$ and an open halfline $(0, +\infty)$, in which the basis of open neighbourhoods of 0_1 or 0_2 consists of all sets of the form $(-a, 0] \cup (0, b)$. This topology satisfies (i) but any two open sets containing 0_1 and 0_2 will contain a segment $(0, c)$



(iii) E has a countable basis. Without this condition the topology of R^2 considered as $\bigcup_{a \in R} (a, R)$ satisfies (i) and (ii) but does not have a countable basis. This would



induce a topology on R^2 making it essentially 1-dimensional.

(iv) One assumes either that E is connected, or at least that the dimension n is the same at all points (implied by connectedness).

A *differentiable atlas* for E is a collection of charts (U_i, x_i) such that $\bigcup_i U_i = M$ and wherever $U_i \cap U_j \neq \emptyset$, the mapping $x_j \circ x_i^{-1}: x_i(U_i \cap U_j) \to x_j(U_i \cap U_j)$ is differentiable, of class C^∞, say. Given a differentiable atlas A, we adjoin all charts (U, x) such that $x_i \circ x^{-1}$: $x(U \cap U_i) \to x_i(U \cap U_i)$ is differentiable, to form a new atlas $\tilde{A}$ which is *maximal* or *complete*. This defines what is called a *differentiable structure* on E. A manifold E together with a differentiable structure is called a *differentiable manifold*. In general, there isn't any unique differentiable structure for a given manifold E.

Let $f: E \to F$ be a mapping from a differentiable manifold E to a differentiable manifold F (in future we will drop the adjective "differentiable" which will be assumed). The map f is then said to be *differentiable* if for any two charts (U, x),

(V, y) of E and F, respectively, the mapping

$$y \circ f \circ x^{-1}: x(U) \to y(V)$$

is differentiable. We shall denote by $C(E)$ the set of all differentiable functions on E (differentiable maps from E to R).

3.2. A *category* $\mathcal{A}$ consists of a class of sets $E, F, \ldots$ called the *objects* of $\mathcal{A}$ and a class of mappings between these sets Mor (E, F), ... called the *morphisms* of $\mathcal{A}$. If $f: E \to F$ and $g: F \to G$ are a pair of morphisms it is always possible to construct the morphism $g \circ f: E \to G$. This composition is necessarily associative. It is further postulated that Mor (E, E) is non-empty and always contains the identity morphism $\mathrm{id}_E: E \to E$.

Example: The category of differentiable manifolds. The objects are differentiable manifolds, the morphisms are differentiable mappings between manifolds. Isomorphism in this category is called a diffeomorphism.

Suppose we have two categories $\mathcal{A}$ and $\mathcal{B}$ and a correspondence C between their objects and morphism. C is called a *covariant functor* if the following situation arises

$$\begin{array}{lllll} \mathcal{A}: & E & \xrightarrow{\quad f \quad} F & \xrightarrow{\quad g \quad} G \\ \big\downarrow C & & & \\ \mathcal{B}: & C(E) & \xrightarrow{\quad C(f) \quad} C(F) & \xrightarrow{\quad C(g) \quad} C(G) \end{array}$$

$$C(g) \circ C(f) = C(g \circ f), \qquad C(\mathrm{id}_E) = \mathrm{id}_{C(E)}.$$

C is a *contravariant functor* if we have the following

$$\begin{array}{lllll} \mathcal{A}: & E & \xrightarrow{\quad f \quad} F & \xrightarrow{\quad g \quad} G \\ \big\downarrow C & & & \\ \mathcal{B}: & C(E) & \xleftarrow{\quad C(f) \quad} C(F) & \xleftarrow{\quad C(g) \quad} C(G) \end{array}$$

$$C(f) \circ C(g) = C(g \circ f), \qquad C(\mathrm{id}_E) = \mathrm{id}_{C(E)}.$$

Example: Let $\mathcal{A}$ be the category of differentiable manifolds and $\mathcal{B}$ the category of vector spaces and vector-space homomorphisms. Clearly $C(E)$ is a vector space. If $h: E \to F$ is a differentiable mapping between manifolds, define $h^*: C(F) \to$ $\to C(E)$ by $h^* f = f \circ h$ $(f \in C(F))$. Clearly then $(k \circ h)^* = h^* \circ k^*$ and $(C, {}^*)$ is a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$.

3.3. A *tangent vector* at a point $p \in E$ is defined as a linear mapping $A: C(E) \to R$ which satisfies the Leibnitz rule at p:

$$A(f \cdot g) = f(p) A(g) + g(p) A(f).$$

Example: A *differentiable curve* through p is a differentiable mapping a: $(-1, 1) \to E$ such that $a(0) = p$. If f is a function differentiable in a neighbourhood of p, then the *vector X tangent to the curve* at p is defined by

$$X(f) = \frac{d}{dt}\,(f \circ a)\,(t)\,\Big|_{t=0}.$$

Conversely, given a tangent vector X at p it is always possible to find a differentiable curve through p to which X is tangent. Thus it is possible to identify tangent vectors at p with classes of differentiable curves at p. If (U, x) is a chart at $p \in U$, we define a *natural basis* associated with this chart as $\frac{\partial}{\partial x^i}$ $(i = 1, ..., n)$ defined by

$$\frac{\partial}{\partial x^i}\,(x^j) = \delta_i^j.$$

If A is any tangent vector p and $A(x^i) = A^i$, then it may be shown that $A = A^i \frac{\partial}{\partial x^i}$. Hence the tangent vectors at p form an n-dimensional vector space denoted $\tau_p(E)$, the *tangent space* at p. The dual space $\tau_p^*(E)$ of linear functions $w : \tau_p(E) \to R$ is called the space of *forms* or *cotangent space* at p.

If $h : E \to F$ is a differentiable mapping from a manifold E to a manifold F we define $h'A \in \tau_{h(p)}(F)$ for any $A \in \tau_p(E)$ by

$$(h'A)\,(f) = A\,(h^*f), \qquad f \in C(F)$$

(the prime is omitted on h where there is no danger of ambiguity) and if $w \in \tau_{h(p)}^*(F)$ we define $h^*w \in \tau_p^*(E)$ by

$$(h^*w)\,(A) = w(h'A),$$

h' corresponds to a covariant functor from the category of pointed differentiable manifolds to the category of vector spaces and h^* corresponds to a contravariant functor to the same category.

$$E \xrightarrow{\quad h \quad} F$$

$$\tau_p(E) \xrightarrow{\quad h' \quad} \tau_{h(p)}(F) \qquad \text{Covariant functor}$$

$$\tau_p^*(E) \xleftarrow{\quad h^* \quad} \tau_{h(p)}^*(F) \qquad \text{Contravariant functor}$$

3.4. *Product manifolds*

Given two manifolds E, F and an atlas of charts (U, x), (V, y) for E and F respectively, we can define an atlas in the product topological space $E \times F$ consisting of charts $(U \times V,\ x \times y)$, where

$$x \times y : U \times V \to R^{n+m}$$

is defined by $(x\times y)(p, q) = (x(p), y(q))$. This makes $E\times F$ into a differentiable manifold.

There is a *natural isomorphism* $\varkappa$:

$$\tau_p(E)\times\tau_q(F) \xrightarrow{\varkappa} \tau_{(p,q)}(E\times F), \qquad p\in E, \qquad q\in F.$$

Let A be the tangent to the curve $t\mapsto a(t)$ at $p = a(0)$ and B the tangent to the curve $t\mapsto b(t)$ at $q = b(0)$. Then define $\varkappa(A, B)$ as tangent to the curve $t\mapsto (a(t), b(t))$ at (p, q).

Suppose now there is given a differentiable map $h: E\times F\to H$. Define $h_p: F\to H$ by $h_p(q') = h(p, q')$ and $h_q: E\to H$ by $h_q(p') = h(p', q)$; then h_p and h_q are also differentiable.

LEMMA 1: $h\circ\varkappa(A, B) = h_qA+h_pB$.

PROOF: Let $f\in C(H)$, then

$$\begin{aligned}((h\circ\varkappa)(A, B))(f) &= \frac{d}{dt} f\circ h(a(t), b(t))\Big|_{t=0}\\ &= \frac{d}{dt} f\circ h(p, b(t))\Big|_{t=0} + \frac{d}{dt} f\circ h(a(t), q)\Big|_{t=0}\\ &= \frac{d}{dt} f\circ h_p\circ b(t)\Big|_{t=0} + \frac{d}{dt} f\circ h_q\circ a(t)\Big|_{t=0}\\ &= (h_pB)(f)+(h_qA)(f).\end{aligned}$$

3.5. *Vector fields*

There are two equivalent definitions:

1) If $\tau(E) = \bigcup_{p\in E}\tau_p(E)$, a *vector field* X is a mapping $X: E\to\tau(E)$ such that $X(p) = X_p\in\tau_p(E)$. Then for any $f\in C(E)$ define $Xf\in C(E)$ by $(Xf)(p) = X_p(f)$.

2) A vector field X is defined as a linear mapping

$$X: C(E)\to C(E)$$

such that $X(f, g) = f(X(g))+g(X(f))$ for any $f, g\in C(E)$.

Let $\chi(E)$ be the set of all vector fields on E. It is a Lie algebra with respect to the bracket defined by

$$[X, Y]f = X(Yf)-Y(Xf).$$

If $h: E\to F$ is a differentiable mapping and X is a vector field on E, it is not in general possible to transport it into a vector field on F. If X and Y are vector fields in E and F, the pair (X, Y) is called h-related if for any $p\in E$, $h(X_p) = Y_{h(p)}$, i.e., if for any $f\in C(F)$ and $p\in E$ we have $Y_{h(p)}(f) = (hX_p)(f) = X_p(f\circ h)$ or equivalently

$Yf) \circ h = X(f \circ h)$ for any f, or

$$h^*(Yf) = X(h^*f).$$

This can be expressed by the following commutative diagram:

X and Y are h-related iff
$$\begin{array}{ccc} C(E) & \xleftarrow{h^*} & C(F) \\ X \downarrow & & \downarrow Y \\ C(E) & \xleftarrow{h^*} & C(F) \end{array}$$
is commutative.

Lemma 2: If (X_1, Y_1) and (X_2, Y_2) are h-related, then $([X_1, X_2], [Y_1, Y_2])$ is h-related.

Proof:
$$\begin{aligned}([Y_1, Y_2]f) \circ h &= (Y_1(Y_2f) - Y_2(Y_1f)) \circ h \\ &= X_1((Y_2f) \circ h) - X_2((Y_1f) \circ h) \\ &= X_1X_2(f \circ h) - X_2X_1(f \circ h) \\ &= [X_1, X_2]\,(f \circ h).\end{aligned}$$

Lemma 3: Let $h: E \to F$ be a surjective differentiable map and X a vector field on E. Then there exists a vector field Y on F such that (X, Y) are h-related if and only if

$$X(h^*C(F)) \subset h^*(C(F)).$$

Proof: *If*, $X(h^*f) = h^*(Yf) \in h^*(C(F))$.

Only if: If $X(h^*C(F)) \subset h^*(C(F))$, then for any $f \in C(F)$ there exists $g \in C(F)$ such that $X(f \circ h) = g \circ h$. The function g is unique because h is surjective: $g \circ h = g' \circ h \Rightarrow g = g'$.

Define $Y: C(F) \to C(F)$ by $Yf = g$. Y is a vector field since

$$\begin{aligned}X(f_1f_2 \circ h) &= (f_1 \circ h) X(f_2 \circ h) + (f_2 \circ h) X(f_1 \circ h) \\ &= (f_1 \circ h)\,(Yf_2) \circ h + (f_2 \circ h)\,(Yf_1) \circ h \\ &= (f_1Yf_2 + f_2Yf_1) \circ h.\end{aligned}$$

Hence $Y(f_1f_2) = f_1Yf_2 + f_2Yf_1$. It is clear that (X, Y) is h-related.

3.6. *A one-parameter group of transformations* of a differentiable manifold E is a differentiable mapping $\Phi: R \times E \to E$ denoted by $\Phi(t, p) = \Phi_t(p)$, such that $\Phi_{t+s}\,(p) = \Phi_t \circ \Phi_s(p)$, $\Phi_o = \mathrm{id}_E$. Clearly, for any t, Φ_t is a differentiable automorphism, or transformation of E. The inverse transformation of Φ_t is then $\Phi_t^{-1} = \Phi_{-t}$. Hence the transformations $\{\Phi_t, t \in R\}$ form a group.

If p is fixed the mapping $t \mapsto \Phi_t(p)$ defines a curve through p called a *trajectory*. The trajectories define a tangent vector field in E by

$$Xf = \frac{d}{dt} f \circ \Phi_t \Big|_{t=0},$$

i.e., consisting of the tangent vector to the trajectory through each point of E.

Conversely we may ask, given a vector field X can we define a one-parameter group where trajectories are everywhere tangent to this vector field. The answer is that it is possible to do so locally in the neighbourhood of any point, i.e., for any $p \in E$ there is a neighbourhood $U \ni p$, an $\varepsilon > 0$ and a differentiable map $\Phi: (-\varepsilon, \varepsilon) \times \times U \to E$ such that $\Phi_t \circ \Phi_s = \Phi_{s+t}$ whenever both sides are defined, $\Phi_0 = \mathrm{id}_U$ and X is tangent to the curves $t \to \Phi_t(q)$.

LEMMA 4: Let $h: E \to E$ be a transformation of the manifold E and $X \in \chi(E)$ which generates the local group of local transformations Φ_t. Then X is invariant with respect to h (i.e., X is h-related to itself) if and only if $h \circ \Phi_t = \Phi_t \circ h$.

PROOF: If X generates the curve $t \mapsto \Phi_t(p)$ then $(hX)_{h(p)}$ is tangent to the curve

$$t \mapsto h \circ \Phi_t(p) = h \circ \Phi_t \circ h^{-1} \circ h(p)$$

at $h(p)$; hence hX generates $h \circ \Phi_t \circ h^{-1}$. But X is invariant under h if and only if $hX = X$ or equivalently $\Phi_t = h \circ \Phi_t \circ h^{-1}$.

4. Lie Groups

4.1. *Definition, Lie algebra*

A *Lie group* G is a group which is at the same time a differentiable manifold such that the group operation

$$G \times G \to G,$$
$$(a, b) \mapsto a^{-1}b$$

is a differentiable map.

Denote by $\gamma_a(\delta_a)$ the left (right) translation of G by the element $a \in G$, i.e.,

$$\left.\begin{aligned} \gamma_a(b) &= ab \\ \delta_a(b) &= ba \end{aligned}\right\} \text{ for all } b \in G. \tag{1}$$

A vector field $X \in \chi(G)$ is called left (right) invariant iff

$$\gamma_a X_b = X_{ab} \quad (\delta_a X_b = X_{ba}) \quad \text{for all } a, b \in G.$$

In order for X to be left invariant it is sufficient that

$$\gamma_a X_e = X_a \quad \text{for all } a \in G. \tag{2}$$

Then for any $b \in G$,

$$\gamma_a X_b = \gamma_a \gamma_b X_e = \gamma_{ab} X_e = X_{ab}.$$

This proves the existence of a left invariant vector field which is moreover uniquely determined by its value at e. Therefore the set G' of all left invariant vector fields forms a vector space isomorphic to $\tau_e(G)$.

Since for $X, Y \in G'$

$$[X, Y] = [\gamma_a X, \gamma_a Y] = \gamma_a[X, Y] \tag{3}$$

(by Lemma 2) G' is a subalgebra of $\chi(G)$, called the *Lie algebra* of G. It follows that the category of Lie groups is related by a functor to the category of Lie algebras.

$$\begin{array}{ccccl} G & \xrightarrow{h} & H & & \text{Lie group homomorphisms} \\ \downarrow & & \downarrow & & \\ G' & \xrightarrow{h'} & H' & & \text{induced Lie algebra homomorphism.} \end{array}$$

For if X is a left invariant vector field on G and $h: G \to H$ a homomorphism, then $h' X$ is again left invariant.

Example: If V is an n-dimensional vector space, $G = GL(V)$ the group of automorphisms of V, then $G' = \text{End } V$, the Lie algebra of all endomorphisms of V with

$$[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha \text{ for all } \alpha, \beta \in \text{End } V.$$

4.2. *1-dimensional subgroups of Lie groups*

Every $X \in G'$ generates a global 1-parameter group of transformations Φ_t. In fact, assume Φ_t is defined for $|t| < \varepsilon$. Since X is left invariant $\gamma_a \circ \Phi_t = \Phi_t \circ \gamma_a$ for all a (by Lemma 4).

Now $\Phi_{t+s}(e) = \Phi_t \circ \Phi_s(e) = \Phi_t(\Phi_s(e)e) = \Phi_t \circ \gamma_{\Phi_s(e)}(e) = \gamma_{\Phi_s(e)} \circ \Phi_t(e) = \Phi_s(e)\Phi_t(e)$.

It follows that if Φ_s and Φ_t are defined then so is Φ_{t+s} which shows that Φ_t is defined for all $t \in R$.

By $\exp X = \Phi_1(e)$ one defines the *exponential map* $\exp: G' \to G$ which satisfies

$$\begin{array}{ccc} G & \xrightarrow{h} & H \\ \uparrow \exp & & \uparrow \exp \\ G' & \xrightarrow{h'} & H' \end{array}$$

From $\Phi_t(a) = \Phi_t \circ \gamma_a(e) = \gamma_a \circ \Phi_t(e) = a\Phi_t(e)$ and $\Phi_t(e) = \exp tX$ it follows that $\Phi_t(a) = a \exp tX$ or

$$\Phi_t = \delta_{\exp tX}. \tag{4}$$

Entirely analogous relations hold also for right invariant vector fields.

4.3. *The canonical form* $\tilde{\omega}$ on a Lie group G is a 1-form with values in G' defined by

$$\tilde{\omega}_a(X) = Y \in G' \quad \text{for any } a \in G \text{ and } X \in \tau_a(G)$$

such that Y regarded as a vector $Y_e \in \tau_e(G)$ is given by

$$Y_e = {\gamma_a}^{-1} X.$$

Clearly, $\tilde{\omega}$ is left invariant, i.e., $\gamma_a^* \tilde{\omega} = \tilde{\omega}$ and satisfies $\delta_a^* \tilde{\omega} = ad'_{a^{-1}} \circ \tilde{\omega}$.

4.4. *Lie groups of transformations* (G-*manifolds*)

A G-manifold is a triple (E, G, Ψ) where E is a manifold, G a Lie group and $\Psi: G \times E \to E$ a differentiable map such that

$$\Psi_a \circ \Psi_b = \Psi_{ba}, \qquad \Psi = \mathrm{id}_E, \tag{5}$$

where

$$\Psi_a(p) \equiv \Psi(a, p). \tag{6}$$

G is then also called a Lie group of transformations of E. If $A \in G'$, a vector field $\tilde{A}$ on $G \times E$ can be defined by $\tilde{A}_{(a,p)} = (A_a, 0)$.

The map Ψ is clearly surjective. It follows by Lemma 4 that there is a unique X on E such that $\tilde{A}$ and X are Ψ-related. X is called the *Killing vector field corresponding to* A.

It can be shown that the vector field X generates $\Phi_t = \Psi_{\exp tA}$ and could also be defined by this relation. Then

$$\Psi' : G' \to \chi(E)$$

defined by $\Psi'(A) = X$ is clearly a Lie algebra homomorphism and, moreover, if G acts effectively, Ψ' is injective, if G acts freely then $A \neq 0$ implies $(\Psi'(A))_p \neq 0$ for all $p \in E$ (cf. [4], p. 42). Again, Lie groups of transformations form a category, the objects of which are triples (E_i, G_i, Ψ_i), whereas the morphisms are couples (h, g), where

$h: E_1 \to E_2$ a differentiable map,
$g: G_1 \to G_2$ a Lie group homomorphism

such that the diagram

$$\begin{tikzcd} E_1 \times G_1 \arrow[r, "\Psi_1"] \arrow[d, "{(h, g)}"'] & E_1 \arrow[d, "h"] \\ E_2 \times G_2 \arrow[r, "\Psi_2"] & E_2 \end{tikzcd} \tag{7}$$

is commutative.

Lemma 5: If (h, g) is a morphism from (E_1, G_1, Ψ_1) to (E_2, G_2, Ψ_2) and $X = \Psi_1' A$ and $Y = \Psi_2' \, gA$ then X and Y are h-related.

Proof: The morphism property of (h, g) implies according to (7) that $\Psi_2 \circ (h, g) = h \circ \Psi_1$ and therefore $h'(X) = h' \circ \Psi_1' A = \Psi_2' \circ (h', g')A = \Psi_2' gA = Y$ by the definition of the Killing vector fields.

Examples: (1) If U is a differentiable manifold and G a Lie group a trivial G-manifold is defined by $(U \times G, G, \delta)$, where

$$\delta : (U \times G) \times G \to U \times G$$

is defined by

$$((p, b), a) \mapsto (p, ba).$$

(2) $(\Psi_{a^{-1}}, \operatorname{ad} a)$ is an example of an automorphism of (E, G, Ψ), where $\operatorname{ad} a(b) = aba^{-1}$. In fact, the diagram (7) is commutative since by (5) and (6)

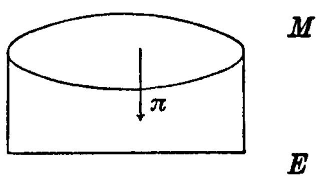
$$\begin{aligned} \Psi \circ (\operatorname{ad} a, \Psi_{a^{-1}})\,(b, p) &= \Psi(\operatorname{ad} a(b),\ \Psi_{a^{-1}}(p)) = \Psi_{aba^{-1}} \circ \Psi_{a^{-1}}(p) = \Psi_{ba^{-1}}(p) \\ &= \Psi_{a^{-1}} \circ \Psi_b(p) = \Psi_{a^{-1}} \circ \Psi(b, p), \end{aligned}$$

i.e.,

$$\Psi \circ (\operatorname{ad} a, \Psi_{a^{-1}}) = \Psi_{a^{-1}} \circ \Psi.$$

5. Bundles

5.1. *A bundle* is a triple (M, E, π) consisting of two manifolds M and E and a surjective map $\pi : M \to E$.



The category of bundles consists of such triples and morphisms (f, h) called bundle homomorphisms which are couples of maps

$$f : M_1 \to M_2,$$
$$h : E_1 \to E_2$$

such that the diagram

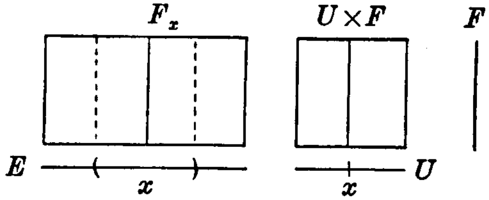
$$\begin{tikzcd} M_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\ E_1 \arrow[r, "h"] & E_2 \end{tikzcd}$$

is commutative.

If $U \subset E$ is an open subset, $(\pi^{-1}(U),\ U,\ \pi|_{\pi^{-1}(U)})$ is called a *sub-bundle* of (M, E, π). The simplest example is the *cartesian product bundle* $(E \times F,\ E,\ \pi)$ of two manifolds E and F with $\pi = pr_1$ (first projection) defined by $\pi(x, y) = x$ for all $x \in E$, $y \in F$.

5.2. In general the counter images $\pi^{-1}(x)$ of points $x \in E$ need not be isomorphic. If they are, one speaks of a fibre bundle. More precisely, a *fibre bundle* is a *locally trivial bundle*, i.e., there exists a manifold F, called the *typical fibre*, such that for each $x \in E$ there exists a neighbourhood U of x such that the sub-bundle $(\pi^{-1}(U),\ U,\ \pi|_{\pi^{-1}(U)})$ is isomorphic to the product bundle $(U \times F,\ U,\ pr_1)$.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "h"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$



The simplest non trivial example of a fibre bundle is probably the Möbius strip, a bundle over the circle T.

A *cross section* of the bundle (M, E, π) is a differentiable map $\Phi: E \to M$ such that $\pi \circ \Phi = \mathrm{id}_E$. It could also be defined as an E-bundle morphism from the trivial bundle (E, E, id_E) into (M, E, π).

By a *local cross section* we mean a cross section of a sub-bundle $(\pi^{-1}(U),\ U,\ \pi)$. A local cross section exists in every fibre bundle.

5.3. *Principal fibre bundles*

Definition: Let P and E be manifolds and G a Lie group. Then (P, E, G, π, ψ) is called a *principal fibre bundle* if

(i) (P, E, π) is a fibre bundle with typical fibre G,

(ii) (P, G, ψ) is a G-manifold (G acting on P to the right),

(iii) there exist local bundle isomorphisms which are at the same time G-manifold isomorphisms, i.e. for any $x \in E$ there is an open neighbourhood $U \ni x$ and a diffeomorphism h such that

(a) h is a bundle isomophism, i.e., the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

is commutative;

(b) h is a G-manifold isomorphism (cf. Section 4.4); the diagram

$$\begin{array}{ccc}
\pi^{-1}(U)\times G & \xrightarrow{h\times \mathrm{id}} & (U\times G)\times G \\
\psi \downarrow & & \downarrow \delta \\
\pi^{-1}(U) & \xrightarrow{h} & U\times G
\end{array}$$

or

$$\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{h} & U\times G \\
\psi_a \downarrow & & \downarrow (\mathrm{id},\ \delta_a) \\
\pi^{-1}(U) & \xrightarrow{h} & U\times G
\end{array}$$

commutes for any $a \in G$.

The general requirement for such composite structures is that the morphisms with respect to one category must also be morphisms in the other category.

From (iii) it follows that (ψ, π) is a bundle homomorphism of $(P\times G,\ P,\ pr_1)$ into $(P,\ E,\ \pi)$ and that G acts freely and transitively on the fibres of P. In fact, let $r\in\pi^{-1}(x)\subset P$ for $x\in U$ and assume that $h(r)=(x,\ b)$, then $\pi(r)=pr_1\circ h(r)$ $= pr_1\ (x,\ b)\ =\ x$ by (a), but with the help of (b) it follows that

$$\begin{aligned}\pi(\psi_a(r)) = pr_1 \circ h \circ \psi_a(r) &= pr_1 \circ (id,\ \delta_a) \circ h(r) \\ &= pr_1(p,\ \delta_a b) = x;\end{aligned}$$

so G leaves the fibres $\pi^{-1}(x)$ invariant. Now G acts transitively and freely by right translations on itself and therefore on $\pi^{-1}(x)$ since every fibre is isomorphic to G by (i).

A principle fibre bundle is *trivial* iff it has a cross section.

Proof: If $f:P\to E\times G$ is an isomorphism, then $\Phi:E\to P$ defined by $\Phi(x)$ $=f^{-1}(x,\ \chi(x))$ is clearly a cross section for an arbitrary function $\chi:E\to G$. Conversely, if $\Phi:E\to P$ is a cross section, then for any $r\in\pi^{-1}(x)$ there is a unique $a\in G$ such that $\psi_a(\Phi(x))=r$ since the action of G on $\pi^{-1}(x)$ is transitive and free. Define $f(r)$ $=(p,\ a)\in E\times G$. This is differentiable and bijective since it is so on each fibre separately.

5.4. It is clear how to define *morphisms in the category of principal fibre bundles*: a triple (h, g, f) is a morphism of $(P_1,\ E_1,\ G_1,\ \pi_1,\ \psi_1)$ into $(P_2,\ E_2,\ G_2,\ \pi_2,\ \psi_2)$ if (h, g) is a G-manifold homomorphism of $(P_1G_1\psi_1)$ into $(P_2G_2\psi_2)$ and (h, f) a fibre bundle homomorphism of $(P_1E_1\pi_1)$ into $(P_2E_2\pi_2)$ and the diagram

$$\begin{array}{ccccc}
P_1\times G_1 & \xrightarrow{\psi_1} & P_1 & \xrightarrow{\pi_1} & E_1 \\
h\times g \downarrow & & \downarrow h & & \downarrow f \\
P_2\times G_2 & \xrightarrow{\psi_2} & P_2 & \xrightarrow{\pi_2} & E_2
\end{array}$$

is commutative.

In many cases the base manifolds of several bundles are the same; then it is convenient to restrict oneself to the category of bundles with $f = \mathrm{id}_E$.

If then, in particular, g and h are surjective maps, P_1 is called an *extension* of P_2. (An important example from physics is the spin bundle which is an extension of the bundle of Lorentz frames over the spacetime manifold; here $g: SL(2, C) \to L_+^\uparrow$). On the other hand P_1 is called a *reduction* of P_2 if $E_1 = E_2$, $f = \mathrm{id}_E$, h and g are injective and h is moreover regular (i.e., $h: \tau_p(P_1) \to \tau_{h(p)}(P_2)$ is injective for all $p \in P_1$; the regularity of g follows already from the group structure.)

5.5. Example: *Bundle of linear frames* of an n-dimensional differential manifold E.

Let $P(E) = \bigcup_{x \in E} P(\tau_x(E))$ (cf. Section 2.1). There is an obvious projection $\pi: P(E) \to E$ which associates to a frame the point of E where it is attached. A natural differentiable structure on $P(E)$ which changes the latter into a differentiable manifold and π into a differentiable map can be defined as follows: Let (U, x) be a chart in E and $e: U \to P(E)$ the map associating to $y \in U$ the natural basis $e(y) = (e_1(y), \ldots, e_n(y))$ with respect to the coordinate system x. If now $r = (r_1, \ldots, r_n) \in \pi^{-1}(x)$, then there exists a unique $a(r) \in GL(n, R)$ such that $r = e(x)a(r)$ (namely the matrix $a = (a^j_{\ i})$ such that $r_i = e_j(x) a^j_{\ i}$). Now require that the map

$$\pi^{-1}(U) \to U \times GL(n, R),$$
$$r \mapsto (\pi(r), a(r)),$$

which is clearly bijective, be a diffeomorphism. ($GL(n, R)$ has a natural manifold structure as an open subset of R^{n^2}.) It can be checked that all charts $\pi^{-1}(U) \to U \times GL(n, R) \to R^{n+n^2}$ form an atlas which changes $P(E)$ into a differentiable manifold (of dimension $n+n^2$). It has been already established that $P(E)$ is a locally trivial bundle. The remaining properties (ii) and (iii) concerning the action of $GL(n, R)$ on $P(E)$ are also easily verified. Thus $P(E)$ is a principal bundle over E with structure group $GL(n, R)$.

5.6. *Associated fibre bundles*

Let (P, E, G, ψ, π) be a principal fibre bundle and F a manifold on which G acts to the left, i.e.,

$$\sigma: G \times F \to F, \qquad \sigma_a(q) = \sigma(a, q)$$

with $a \in G$, $q \in F$, such that $\sigma_a \circ \sigma_b = \sigma_{ab}$, $\sigma_e = \mathrm{id}_F$.

Let $\check{\sigma}_a = \sigma_{a^{-1}}$ for all $a \in G$, then $(FG\check{\sigma})$ is a G-manifold (where G acts on F to the right). Now consider the action of G in $P \times F$ defined by

$$\psi: (P \times F) \times G \to P \times F,$$

where

$$(r, q, a) \mapsto (\psi_a(r), \sigma_{a^{-1}}(q))$$

and form the quotient set $M = (P \times F)/G$ with the canonical map $\iota: P \times F \to M$.

A map ϱ such that

$$\begin{tikzcd} P\times F \arrow[rr, "\iota"] \arrow[dr, "\pi\circ pr_1"'] & & M \arrow[dl, "\varrho"] \\ & E & \end{tikzcd}$$

is then well defined because $\iota(r, q) = \iota(r', q')$ iff there exists $a \in G$ such that $r' = \psi_a(r)$, $q' = \sigma_{a^{-1}}(q)$.

To introduce a differentiable structure on M let h be a local isomorphism from $\pi^{-1}(U) \subset P$ onto $U \times G$ for any open $U \subset E$ and consider the diagram

$$\begin{tikzcd} \pi^{-1}(U)\times F \arrow[rr, "h\times \mathrm{id}"] \arrow[d, "\iota"'] & & (U\times G)\times F \cong U\times(G\times F) \arrow[d, "\mathrm{id}\times\sigma"] \\ M \supset \varrho^{-1}(U) \arrow[rr, dashed, "k"] \arrow[dr, "\varrho"'] & & U\times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

There follows the existence of a bijection k which makes this diagram commutative. (The construction is entirely analogous to the one in Section 2.2: let $\iota_r(q) \equiv \iota(r, q)$ and show that ι_r is bijective, noting that G acts transitively and freely on the fibres of P.) A topology and differentiable structure on M can then be defined by requiring that k be a diffeomorphism for any choice of the local automorphism h. So $(ME\varrho)$ becomes a fibre bundle called the *bundle associated with* P, *with standard fibre* F.

5.7. Example: Let $(P(E), E, Gl(n, R), \pi, \delta)$ be the bundle of linear frames over E and $\sigma: GL(n, R) \to GL(m, R)$ a differentiable homomorphism. Then $(R^m GL(n, R)\sigma)$ is a left $GL(n, R)$ manifold. The associated bundle with standard fibre R^m can be interpreted as follows

$$\begin{aligned}(P(E)\times R^m)_{/GL(n,R)} &= \Big(\bigcup_{x\in E} P(\tau_x(E))\times R^m\Big)_{/GL(n,R)} \\ &= \bigcup_{x\in E}(P(\tau_x(E))\times R^m)_{/GL(n,R)} = \bigcup_{x\in E}\sigma(\tau_x(E)) \overset{\text{def}}{=} \sigma(E).\end{aligned}$$

As each fibre is the space of quantities of type σ over $\tau_x(E)$, a cross section of $\sigma(E)$ is called a *field of quantities of type* σ over the manifold E. If in particular $m = n$ and $\sigma \equiv \tau = \mathrm{id}_{GL(n, R)}$, then $\tau(E)$ is called the *tangent bundle*. Clearly, τ is a covariant functor from the category of differentiable manifolds into the category of (vector) bundles.

5.8. The "*transformation law*" *of cross sections* in a general associated bundle can be derived in complete analogy to those of tensor densities over a vector space in Section 2.2. Let (P, E, G, π, Ψ) be a principal fibre bundle, $(F, G, \check{\sigma})$ a G-manifold and (M, E, ϱ) the corresponding associated bundle. Denote by $C(M)$ the set of all cross sections of M. If $f \in C(M)$, a map

$$\tilde{f}: P \to F \tag{1}$$

can be defined so that

$$\begin{tikzcd} P \arrow[r, "\mathrm{id}\times\tilde{f}"] \arrow[d, "\pi"'] & P\times F \arrow[d, "\iota"] \\ E \arrow[r, "f"] & M \end{tikzcd}$$

is commutative. (Recall that $\iota_r: F \to F_{\pi(r)} \equiv \varrho^{-1}(\pi(r))$ is bijective). Then it follows for any $r \in P$ and $a \in G$ that

$$\tilde{f} \circ \psi_a(r) = \iota_{\psi_a(r)}{}^{-1} \circ f \circ \pi \circ \psi_a(r)$$

$$= \sigma_{a^{-1}} \circ \iota_r{}^{-1} \circ f \circ \pi(r) = \sigma_{a^{-1}} \circ \tilde{f}(r),$$

i.e.,

$$\tilde{f} \circ \psi_a = \sigma_{a^{-1}} \circ \tilde{f} = \check{\sigma}_a \circ \tilde{f}. \tag{2}$$

This means that

$$\begin{tikzcd} P\times G \arrow[r, "\psi"] \arrow[d, "\tilde{f}\times\mathrm{id}"'] & P \arrow[d, "\tilde{f}"] \\ F\times G \arrow[r, "\check{\sigma}"] & F \end{tikzcd} \tag{3}$$

commutes, expressing the fact that $\tilde{f}$ is a morphism in the category of G-manifolds mapping P into F. Conversely, if such a morphism $\tilde{f}: P \to F$ is given, a cross section $f \in C(M)$ can be defined by

$$f(x) \overset{\mathrm{def}}{=} \iota_r \circ \tilde{f}(r), \quad \text{where} \quad r \in \pi^{-1}(x).$$

The cross section f is well defined because in view of (2) for any other choice $r' = \psi_a(r) \in \pi^{-1}(x)$, one obtains

$$\iota_{r'} \circ \tilde{f}(r') = \iota_{\psi_a(r)} \circ \tilde{f} \circ \psi_a(r) = \iota_{\psi_a(r)} \circ \sigma_{a^{-1}} \circ \tilde{f}(r) = \iota_r \circ \tilde{f}(r).$$

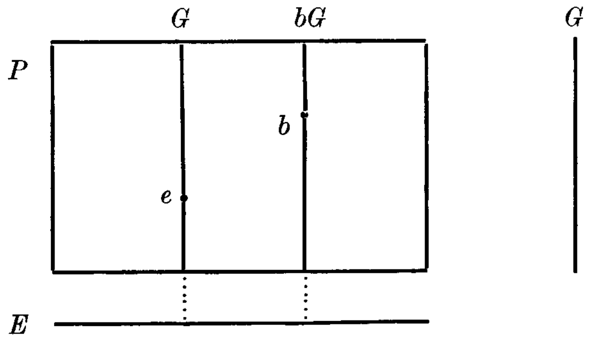
This establishes a 1-1-correspondence between the cross sections of M and the G-manifold homomorphisms from P into F:

$$C(M) \leftrightarrow \mathrm{Hom}_G(P, F).$$

5.9. If the manifold F is a vector space on which G acts as a group of vector space automorphisms the associated bundle is called a *vector bundle*; $C(M)$ and $\mathrm{Hom}_G(P, F)$ can then be given a natural vector space structure and $f \mapsto \tilde{f}$ becomes an isomorphism of the vector spaces $C(M)$ and $\mathrm{Hom}_G(P, F)$.

Example: *Induced representation*

Let P be a Lie group, G a closed Lie subgroup of P, then (P, G, δ) with $\delta_a(b) = ba$ for all $a \in G$, $b \in P$ is a G-manifold. The set $E = P/G$ of left cosets can be given the structure of a differentiable manifold, and (P, E, G, π, δ) becomes a principal fibre bundle.



If now F is any vector space and $\sigma: G \to GL(F)$ a Lie homomorphism (a representation of G in F), then $(F\ G\ \check{\sigma})$ becomes a G-manifold. Let $M = (P \times F)/G$ be the corresponding associated bundle, then there exists a vector space isomorphism

$$\sim\ : C(M) \to \mathrm{Hom}_G(P, F).$$

Let $\tilde{f} \in \mathrm{Hom}_G(P, F)$, i.e., $\tilde{f}: P \to F$ such that $\tilde{f} \circ \delta_a = \sigma_{a^{-1}} \circ \tilde{f}$ for all $a \in G$, then if $b \in P$ it follows that $\tilde{f} \circ \gamma_{b^{-1}} \in \mathrm{Hom}_G(P, F)$ because $\gamma_{b^{-1}} \circ \delta_a = \delta_a \circ \gamma_{b^{-1}}$. Since

$$\begin{aligned}(\alpha\tilde{f} + \beta\tilde{g}) \circ \gamma_{b^{-1}}(c) &= (\alpha\tilde{f} + \beta\tilde{g})\,(b^{-1}c) \\ &= \alpha\tilde{f}(b^{-1}c) + \beta\tilde{g}(b^{-1}c) \\ &= \alpha\tilde{f} \circ \gamma_{b^{-1}}(c) + \beta\tilde{g} \circ \gamma_{b^{-1}}(c) \qquad \text{for all } c \in P,\end{aligned}$$

the map $\omega_b: \mathrm{Hom}_G(P, F) \to \mathrm{Hom}_G(P, F)$, where $\omega_b(\tilde{f}) = \tilde{f} \circ \gamma_{b^{-1}}$ is linear, and from $(\tilde{f} \circ \gamma_{b^{-1}}) \circ \gamma_{a^{-1}} = \tilde{f} \circ \gamma_{b^{-1}a^{-1}}$ it follows that $\omega_a \circ \omega_b = \omega_{ab}$ for any $a, b \in P$.

Therefore

$$\omega: P \to GL(\mathrm{Hom}_G(P, F))$$

is a representation of P, said to be *induced* by the representation of the Lie subgroup $G \subset P$.

5.10. A transformation $\Phi : E \to E$ of the base manifold can be extended to an automorphism $(\bar{\Phi}, \mathrm{id}_G, \Phi)$ of the bundle of linear frames $P(E)$ as follows: Identify $r \in P(E)$ with the isomorphism

$$r : R^n \to \tau_{\pi(r)}(E) \tag{4}$$

defined as in Section 2.1 and put $\bar{\Phi}(r) = \tau_{\pi(r)}(\Phi) \circ r$, where $\tau_{\pi(r)}(\Phi)$ is the isomorphism the space tangent to E at $\pi(r)$ onto the tangent space at $\Phi(\pi(r))$ induced by Φ of (cf. Section 3.3):

$$\begin{tikzcd} R^n \arrow[r, "r"] \arrow[rr, bend right, "\bar{\Phi}(r)"'] & r_{\pi(r)}(E) \arrow[r, "\tau_{\pi(r)}(\Phi)"] & \tau_{\Phi(\pi(r))}(E) \end{tikzcd}$$

The morphism properties of $(\bar{\Phi}, id_G, \Phi)$ according to section 5.4 are now easily verified:

$$\bar{\Phi} \circ \delta_a = \delta_a \circ \bar{\Phi}, \tag{5}$$

$$\pi \circ \bar{\Phi} = \Phi \circ \pi, \tag{6}$$

also

$$\overline{\Phi \circ \Psi} = \bar{\Phi} \circ \bar{\Psi}, \tag{7}$$

which follows from the covariant nature of the functor τ.

Now suppose that X is a given vector field on E, Φ_t the generated (possibly local) group of transformations. Then the extension $\bar{\Phi}_t$ of Φ_t is by (7) also a group of transformations and thus induces a vector field $\bar{X}$ on $P(E)$. It is clear that $\bar{X}$ and X are π-related. Moreover, $\bar{X}$ is invariant with respect to δ_a (by Lemma 4).

Let f be a tensor field on E, i.e., a cross section $f \in C(\sigma(E))$, where $\sigma(E)$ is a bundle associated to $P(E)$ (cf. Section 5.7). An $\tilde{f} \in \mathrm{Hom}_{GL(n,R)}(P(E), R^m)$ corresponds to f that is a set of m real functions on $P(E)$. Therefore

$$\bar{X}\tilde{f} : P(E) \to R^m$$

makes sense and since moreover $\bar{X}$ is invariant under δ_a it follows that

$$(\bar{X}\tilde{f}) \circ \delta_a = \sigma_{a^{-1}} \circ \bar{X}\tilde{f},$$

so that $\bar{X}\tilde{f} \in \mathrm{Hom}_{GL(n,R)}(P(E), R^m)$. This finally implies that there exists an element $\mathcal{L}_X f \in C(\sigma(E))$ such that $\widetilde{\mathcal{L}_X f} = \bar{X}\tilde{f}$. $\mathcal{L}_X f$ is called the *Lie derivative* of f with respect to X.

6. Connections

6.1. *Affine spaces*

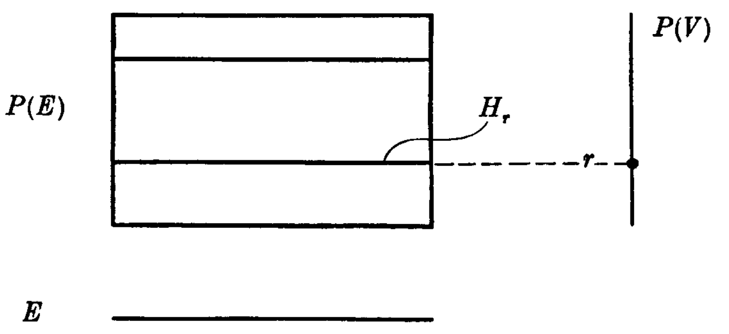
Let E be a set and $(V, +)$ an n-dimensional vector space regarded as abelian group. The triple $(E, V, +)$ is then called an *affine space* if V acts on E freely and transitively, i.e., if $p, q \in E$ there exists a unique vector $u \in V$ such that $q = u + p$. If $e = (e_1 \dots e_n)$ is a basis of V and $o \in E$, then (o, e) is a *basis of* E in the sense that any $p \in E$ can be written as

$$p = x^i(p)e_i + o.$$

The map $x: E \to R^n$ s induced in such a way by a basis (o, e) is obviously bijective and can be used to introduce an (n-dimensional) differentiable structure on E. The structure does not depend on the choice of (o, e) and (E, x) is a global chart.

If $e(p)$ is the natural basis of $\tau_p(E)$ associated with the chart (E, x), then the linear map $\tau_p(E) \to V$ defined by the extension of $e_i(p) \mapsto e_i$ is a natural isomorphism (i.e., is independent of the particular choice of (o, e)). All tangent spaces can therefore be naturally identified with V or there exists a teleparallelism over E. In other words, the bundle $P(E)$ of an affine space E may be identified with the product $E \times P(V)$.

Given $r \in P(V)$, define $H_r \subset P(E)$ as the set of all bases parallel to r.



Clearly, H_r is a submanifold of $P(E)$,

$$\pi: H_r \to E \text{ is a diffeomorphism}$$

and for any $a \in GL(n, R)$

$$\delta_a(H_r) = H_{\delta_a(r)}.$$

Such a "global slicing" of $P(E)$ can not be introduced over an arbitrary manifold E but generalizations are possible in two ways:

(a) an "infinitesimal parallelism" can be defined over an arbitrary manifold E;

(b) a connection can also be defined on an arbitrary principal fibre bundle P.

6.2. *Connections on a principal fibre bundle*

If (P, E, G, π, ψ) is a principal bundle, the (differentiable) assignment to each $r \in P$ of a subspace H_r of $\tau_r(P)$ such that

$$\begin{aligned} &\text{(i) } \pi: H_r \to \tau_{\pi(r)}(E) \text{ is a (vector space) isomorphism,} \\ &\text{(ii) } \psi_a(H_r) = H_{\psi_a(r)} \end{aligned} \tag{1}$$

is called a *connection on P*.

In addition to the "horizontal subspace" H_r which has dimension $n = \dim E$, a "vertical subspace" V_r may be defined as the set of all vertical vectors, i.e., all vectors tangent to the fibre through r.

Since the fibres are diffeomorphic to G it follows that $\dim V_r = \dim G$, so that $\dim \tau_r(P) = \dim H_r + \dim V_r$.

Assume now that $X \in V_r \cap H_r$, then $\pi X = 0$ since X is vertical and therefore $X = 0$ since π is an isomorphism. This proves that $\tau_r(P)$ is a direct sum of H_r and V_r so that any $X \in \tau_r(P)$ decomposes uniquely:

$$X = \mathrm{ver} X + \mathrm{hor} X.$$

Since G acts freely and transitively on the fibres of P it follows that the map

$$\psi'_r : G' \to \tau_r(P)$$

defined by

$$\psi'_r A = (\psi' A)_r,$$

where $\psi' A$ is the Killing vector field corresponding to A, is a bijection onto V_r (cf. Section 4.4).

6.3. *Connection form*

Given a connection on P, a 1-form ω with values in G' can be defined by $\omega_r(X) = \psi_r'^{-1}(\mathrm{ver} X)$ for all $X \in \tau_r(P)$ which completely describes the connection. It is called the *connection form* and has the properties

(A) $X \in H_r \Leftrightarrow \omega_r(X) = 0$,
(B) $\omega_r(\psi' A)_r = A$ for any $A \in G'$,
(C) $\psi_a^* \omega = ad'_{a^{-1}} \omega$ for any $a \in G$.

PROOF: (A) and (B) are immediate consequences of the definition. From $\pi \circ \psi_a = \pi$ and (1) it follows that $\psi_a \circ \mathrm{ver} = \mathrm{ver} \circ \psi_a$ and $\psi_a \circ \mathrm{hor} = \mathrm{hor} \circ \psi_a$. Thus

$$\psi'_a X = \psi'_a\ (\mathrm{ver} X + \mathrm{hor} X) = \psi'_a\ (\mathrm{ver} X) + \psi'_a(\mathrm{hor} X)$$

is a decomposition into vertical and horizontal components.

If $X \in H_r$, then $\psi_a^* \omega(X) = \omega(\psi'_a X) = 0 = ad'_{a^{-1}}(\omega(X))$ by (1), so that (C) holds trivially in this case.

Now recall from Section 4.4. that $(\psi_a, \mathrm{ad}_{a^{-1}})$ is an automorphism of (P, G, ψ). Therefore the diagram

$$\begin{tikzcd}[column sep=large, row sep=large] G' \arrow[r, "\mathrm{ad}'_{a^{-1}}"] \arrow[d, shift left, "\psi'"] & G' \arrow[d, shift right, "\psi'"'] \\ \text{Kill. vect.} \arrow[r, "\psi'_a"] \arrow[u, shift left, "\omega"] & \text{Kill. vect.} \arrow[u, shift right, "\omega"'] \end{tikzcd}$$

commutes (by Lemma 5); moreover (B) means that $\omega \circ \psi' = \mathrm{id}$; now $(\psi_a^* \omega)(\psi' A) = \omega(\psi'_a \circ \psi' A) = \omega \circ \psi' \circ \mathrm{ad}'_{a^{-1}} A = \mathrm{ad}'_{a^{-1}} A$ or $\psi_a^* \omega \circ \psi' = \mathrm{ad}'_{a^{-1}}$ or $\psi_a^* \omega = \mathrm{ad}'_{a^{-1}} \circ \psi'^{-1} = \mathrm{ad}'^{-1}_a \omega$.

Conversely, any given G'-valued 1-form on P satisfying (B) and (C) is the connection form of a unique connection on P. (Define: $H_r = \{X \in \tau_r(P) : \omega_r(X) = 0\}$, then properties (i) and (ii) of the definition in 6.1 are easily checked.)

Principal bundles with connections form a category with morphisms defined in a natural way: in addition to the requirements related to the principal bundle structure one demands commutativity of the diagram

$$\begin{tikzcd}[column sep=huge, row sep=large] \tau(P_1) \arrow[r, "\omega_1"] \arrow[d, "h"'] & G'_1 \arrow[d, "g'"] \\ \tau(P_2) \arrow[r, "\omega_2"] & G'_2, \end{tikzcd}$$

where the notation is like that of Section 5.4.

6.4. *Curvature form*

If α is a k-form on a principal bundle P, the horizontal part of α is defined by

$$(\mathrm{hor}\,\alpha)_r(X_1, \ldots, X_k) = \alpha_r(\mathrm{hor}X_1, \ldots, \mathrm{hor}X_k) \tag{2}$$

for all $X_1, \ldots, X_k \in \tau_r(P)$. For the connection form by the definition $\mathrm{hor}\,\omega = 0$, but $\mathrm{hor}\,d\omega \neq 0$, in general. The 2-form

$$\Omega = \mathrm{hor}\,d\omega \tag{3}$$

is called the *curvature form* of the connection on P. *It vanishes if and only if the connection is trivial*, i.e., completely integrable or in other words, there is a complete parallelism. Outline of proof: connection integrable $\overset{\text{def}}{\Longleftrightarrow}$ horizontal subspaces are surface forming $\overset{\text{(Frobenius)}}{\Longleftrightarrow}$ (X, Y horizontal $\Rightarrow$ $[X, Y]$ horizontal).

Assume that this holds, then

$$\Omega(X, Y) = d\omega(\mathrm{hor}\,X, \mathrm{hor}\,Y)$$
$$= \frac{1}{2}\Big[(\mathrm{hor}\,X)(\omega(\mathrm{hor}\,Y)) - (\mathrm{hor}\,Y)(\omega(\mathrm{hor}\,X)) - \omega([\mathrm{hor}\,X, \mathrm{hor}\,Y])\Big]\ ^{*)}.$$

The last term vanishes according to the assumption that in the first two terms the G'-valued function $\omega(\mathrm{hor}\,Y)$, resp. $\omega(\mathrm{hor}\,X)$ vanishes everywhere along the stream-lines of $\mathrm{hor}\,X$, resp. $\mathrm{hor}\,Y$, thus the derivatives also vanish and $\Omega = 0$. Conversely, if $\Omega = 0$, then in particular, for horizontal X and Y

$$0 = 2\Omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([XY])$$
$$= -\omega([XY]).$$

Thus $[XY]$ must be horizontal.

The proof of *Bianchi's identity*

$$\mathrm{hor}\,d\Omega = 0 \tag{4}$$

is similar.

6.5. *Linear connections*

A linear connection over a manifold E is a connection on the bundle of linear frames $P(E)$. The structure group is $GL(n, R)$, its Lie algebra $GL'(n, R)$ the Lie algebra of endomorphisms of R^n diffeomorphic to R^{n^2}. Thus the connection form ω is essentially a collection of n^2 real valued 1-forms.

Another 1-form with values in R^n can be naturally defined on $P(E)$ by

$$\Theta_r(X) = r^{-1}(\pi X) \quad \text{for any } Y \in \tau_r(E), \tag{5}$$

where $r \in P(E)$ is regarded as a vector space isomorphism as in Section 5.10. This 1-form is called the *canonical form of* $P(E)$ and has the (obvious) property

$$\Theta(X) = 0 \Leftrightarrow \pi(X) = 0, \tag{6}$$

whereas for $X \in V_r$

$$\omega(X) = 0 \Leftrightarrow X = 0.$$

This means that

$$\Theta_r : H_r \to R^n,$$
$$\omega_r : V_r \to R^{n^2}$$

are both isomorphisms and hence (ω_r, Θ_r) is a basis of $\tau_r^*(P(E))$. Therefore a complete parallelism is defined on $P(E)$ or in other words, the bundle $P(P(E))$ is a product bundle.

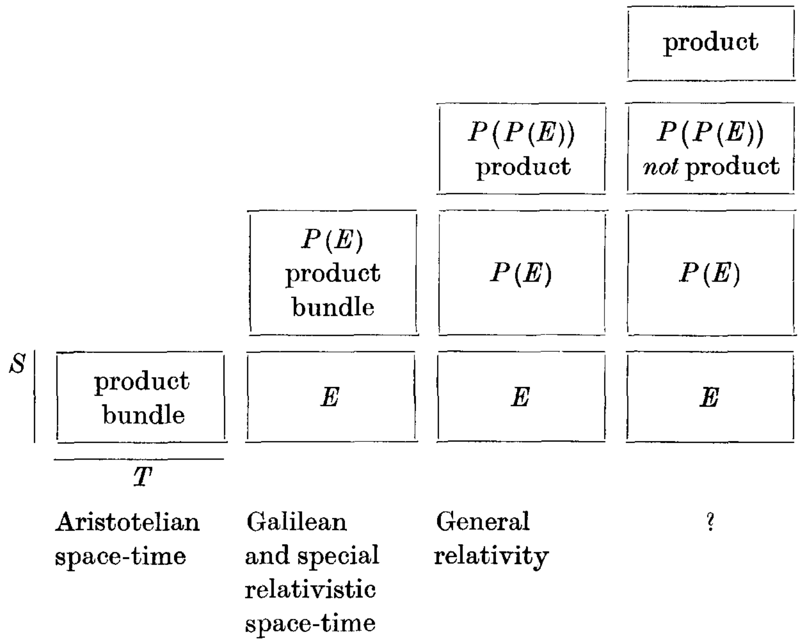
---

*) For the proof of the general relation

$$d\omega(X, Y) = \frac{1}{2}\big(X(\omega(Y)) - Y(\omega(X)) - \omega([XY])\big)$$

see e.g. [4], p. 36 or [6], p. 103.

Example: Increasing complexity of space-time structure:



6.6. *Covariant differentiation*

Let M be a vector bundle associated with (P, E, G, π, ψ). Suppose $f{:}E \to M$ is a cross section and $\tilde{f}{:}P \to F$ the corresponding element of $\mathrm{Hom}_G(P, F)$. Since F is a finite dimensional vector space, the F-valued 1-form $d\tilde{f}$ on P is well defined. It satisfies $\psi_a^* d\tilde{f} = \sigma_{a^{-1}} d\tilde{f}$ (cf. (5.2)).

In analogy to the isomorphism between $C(M)$ and $\mathrm{Hom}_G(P, F)$, established in Section 5.8, there is a *bijective correspondence* between k-forms $\alpha{:}\bigwedge^k(E) \to M$*) (vector bundle homomorphism) and horizontal k-forms of type σ on P, $\tilde{\alpha}{:}\bigwedge^k(P) \to F$ (vector bundle homomorphism and G-manifold homomorphism, $\tilde{\alpha}(Y_1 \wedge \ldots \wedge Y_k) = 0$ if any $\mathrm{hor}\, Y_i = 0$) such that the diagram

$$\begin{tikzcd} \bigwedge^k(P) \arrow[r, "pr \times \tilde{\alpha}"] \arrow[d, "\bigwedge^k \pi"'] & P \times F \arrow[d, "\iota"] \\ \bigwedge^k(E) \arrow[r, "\alpha"] & M \end{tikzcd}$$

is commutative.

*) $\bigwedge^k(E)$ is the bundle of k-vectors over E.

The *covariant derivative* ∇f of any $f \in C(M)$ is now defined as the unique 1-form field

$$\nabla f : \tau(E) \to M$$

corresponding to

$$\widetilde{\nabla f} = \operatorname{hor} d\tilde{f} : \tau(P) \to F. \tag{7}$$

To calculate $\widetilde{\nabla f}$ let $Y \in \tau_r(P)$; then

$$\operatorname{hor} d\tilde{f}(Y) = d\tilde{f}(\operatorname{hor} Y) = d\tilde{f}(Y) - d\tilde{f}(\operatorname{ver} Y).$$

There exists an $A \in G'$ such that $A = \omega_r(Y)$ and $\operatorname{ver} Y = \psi_r' A$. Now regard A as element $A_e \in \tau_e(G)$, then

$$d\tilde{f}(\operatorname{ver} Y) = d\tilde{f}(\psi_r' A_e) = (\psi_r' A_e)\tilde{f} = A_e(\tilde{f} \circ \psi_r),$$

but $\tilde{f} \circ \psi_r = \check{\sigma} \cdot \tilde{f}(r)$, where $\check{\sigma}(a) = \sigma_{a^{-1}}$ and "$\cdot$" denotes the action of $GL(F)$ in F; thus

$$d\tilde{f}(\operatorname{ver} Y) = A_e(\check{\sigma}) \cdot \tilde{f}(r).$$

But $\check{\sigma} \cdot \sigma = \mathrm{id}_F$, therefore

$$0 = A_e(\check{\sigma} \cdot \sigma) = \check{\sigma}(e) A_e(\sigma) + A_e(\check{\sigma}) \cdot \sigma(e) = A_e(\sigma) + A_e(\check{\sigma}),$$

and moreover, $A_e(\sigma) = \dfrac{d}{dt} \sigma(\exp tA)\big|_{t=0} = \sigma'(A)$, so that

$$d\tilde{f}(\operatorname{ver} Y) = -A_e(\sigma) \cdot \tilde{f}(r) = -\sigma' \circ \omega_r(Y) \cdot \tilde{f}(r),$$

or

$$\widetilde{\nabla f} = d\tilde{f} - (\sigma' \circ \omega) \cdot \tilde{f}. \tag{8}$$

Usually one expresses everything in terms of a local cross section $e : E \to P$ of P. For given e let

$$f_e = \tilde{f} \circ e : E \to F \tag{9}$$

$$\Gamma_e = e^*(\sigma' \circ \omega) : \tau(E) \to GL'(F) \tag{10}$$

$$\widetilde{\nabla_e f_e} = e^* \widetilde{\nabla f}; \tag{11}$$

then

$$\nabla_e f_e = df_e - \Gamma_e \cdot f_e. \tag{12}$$

If $e' : E \to P$ is another cross section, there exists a function $a : E \to G$ such that the diagram

$$\begin{tikzcd} E \arrow[r, "{(e,a)}"] \arrow[dr, "e'"'] & P \times G \arrow[d, "\psi"] \\ & P \end{tikzcd}$$

commutes.

Then the transformation law for $f \in C(M)$ becomes

$$f_{e'}(p) = \tilde{f} \circ e'(p) = \tilde{f} \circ \psi_{e(p)}(a(p)) = \check{\sigma}_a \cdot \tilde{f}(e(p)) = \sigma_{a^{-1}} \cdot \tilde{f} \circ e(p)$$
$$= \sigma_{a^{-1}} \cdot f_e(p), \quad \text{i.e.} \quad f_{e'} = \sigma_{a^{-1}} \cdot f_e. \tag{13}$$

For Γ we have $\Gamma_{e'} = e'^*(\sigma' \circ \omega) = \sigma' \circ (e, a)^* \psi^* \omega$; let $X \in \tau_p(E)$, then

$$\Gamma_{e'}(X) = \sigma' \circ \psi^* \omega(eX, aX) = \sigma' \circ (\mathrm{ad}'_{a(p)^{-1}} \omega(eX) + \tilde{\omega}_{a(p)}(aX)),$$

where $\tilde{\omega}$ is the canonical form of $GL(n, R)$ (cf. Section 4.3).

Denote $S = \sigma \circ a : E \to GL(F) \subset GL'(F)$, then

$$\sigma' \circ \mathrm{ad}'_{a(p)^{-1}} = (\sigma \circ \mathrm{ad}_{a(p)^{-1}})' = \mathrm{ad}'_{\sigma \circ a(p)^{-1}} \circ \sigma' = \mathrm{ad}'_{S(p)^{-1}} \circ \sigma'$$

because $\sigma \circ \mathrm{ad}_a = \mathrm{ad}_{\sigma(a)} \circ \sigma$, and

$$\sigma'(\tilde{\omega}_{a(p)}(aX)) = \tilde{\omega}_{a(p)}(aX)|_e(\sigma) = (\gamma_{a(p)^{-1}} aX)(\sigma)$$
$$= X(\sigma \circ \gamma_{\circ(p)^{-1}} \circ a) = S(p)^{-1} X(S) = S(p)^{-1} dS(X),$$

where $\sigma'(A) = A_e(\sigma)$ and the definition of $\tilde{\omega}$ in Section 4.3 has been used.

This finally gives us the transformation law of Γ, the Christoffel symbols for quantities of type σ:

$$\Gamma_{e'} = S^{-1} \Gamma_e S + S^{-1} dS. \tag{14}$$

The curvature

$$R_e = e^*(\sigma' \circ \Omega), \tag{15}$$

satisfies the simpler transformation law

$$R_{e'} = S^{-1} R_e S, \tag{16}$$

where S now denotes the operation of $\sigma \circ a$ on 2-forms. (The second term vanishes as a consequence of the horizontal character of Ω.)

7. Utiyama Theory of Gauge-Invariant Classical Fields

7.1. Consider a principal fibre bundle P with structure group $SO(2)$, over (Minkowski or Riemannian) space-time E. Assume that there is a connection on P characterized by a connection form ω. Consider the representation (homomorphism) $\sigma : SO(2) \to U(1)$ given by

$$\sigma \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = e^{i\alpha}.$$

(This requires an obvious generalization of the preceding considerations to complex spaces.) The Lie algebra of $SO(2)$ consists of matrices of the form $\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$ with $a \in R$, and $\sigma'\left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right) = ia$. Therefore if we construct the associated bundle M with standard fibre C defined by σ and wish to use real forms, we have to introduce A and F by

$$iA = \Gamma = e^*(\sigma' \circ \omega); \qquad iF = R = e^*(\sigma' \circ \Omega) \tag{1}$$

for any local cross section $e \in C(P)$.

If

$$S: E \to U(1), \qquad S(p) = e^{i\chi(p)},$$

describes a change from the local cross section $e \in C(P)$ to $\bar{e}$, then a cross section $f \in C(M)$ transforms according to (6.13) which becomes

$$\bar{f}(p) = e^{-i\chi(p)} f(p), \tag{2}$$

whereas the transformation laws of A and F follow from (1) and (6.14) and (6.16):

$$\bar{A} = A + d\chi; \qquad \bar{F} = F. \tag{3}$$

If we interpret f as a field describing charged particles, A as the *electromagnetic* potential and F as the electromagnetic field, then (3) is the gauge transformation law. The structure group being abelian, the relation between the connection and curvature and the Bianchi identity become respectively

$$F = dA \quad \text{and} \quad dF = 0\,^{*)} \tag{4}$$

with obvious interpretations in the electromagnetic theory. Moreover, the covariant derivative in the (complex) bundle associated to P by σ may be written as

$$\nabla_k f = (\partial_k - iA_k) f \tag{5}$$

(by (6.12)) and this formula may be used as a basis for introducing interactions between charged particles and the electromagnetic field. A representation $\sigma_n : SO(2) \to U(1)$ of the form $\sigma_n\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = e^{in\alpha}$ will lead to another associated bundle. Its cross sections may be interpreted as fields describing particles whose charge is n times that of particles described by the field f. The relation between the gauge transformations of both kinds is now clear; the structure group $SO(2)$ is the group of gauge transformations of the first kind, whereas the group of gauge transformations of the second kind acts, in an obvious way, in the set of all cross sections of P.

According to R. Utiyama [7] one can consider more general fields, admitting gauge transformations corresponding to changes of cross sections in a principal fibre bundle with structure group G. For example, if $G = SO(3)$, one obtains the Yang--Mills field [8]. It is easy to write the analogues of equations (3), (4), (5) for this general case.

It is sometimes asserted that the general theory of relativity may also be obtained in this way, by taking G to be the Lorentz group (or the Poincaré group, according to some authors; cf. T. W. B. Kibble [3]). This is not quite the case. The structure group of the bundle of linear bases of a Riemannian space-time may be

*) This follows from the structure equation if it is noticed that all Lie brackets and structure constants vanish for an abelian group (cf. [4], p. 77/78).

reduced to the Lorentz group L; therefore the Lorentz bundle $L(E)$ (= set of all "tetrads" in space-time E) is a principal bundle over E, with structure group L. The usual affine connection considered in GRT is indeed a connection in $L(E)$. However, in addition to its general bundle structure, the bundle of linear bases $P(E)$ (and also $L(E)$, as its reduction) has a property that is not shared by other principal fibre bundles. Namely, the following is true: a principal fibre bundle P over an n-dimensional differentiable manifold E and with structure group $GL(n)$ is (bundle) isomorphic to the bundle $P(E)$ of linear bases of E if and only if there exists an R^n-valued 1-form Θ on P of type τ, and such that $\Theta(X) = 0 \Leftrightarrow \pi' X = 0$. (Exercise: prove this theorem. Hint: on $P(E)$, Θ is the canonical form defined by (6.5), sometimes also called the "soldering form" of $P(E)$; the bundle $P(E)$ is "soldered" to E rather than being loosely connected to E, as general principal bundles are). Note also that for any manifold E, one can introduce the product bundle $E \times GL(n)$. In general, there not only isn't any natural isomorphism of $P(E)$ on $E \times GL(n)$, but no (global) isomorphism whatsoever (e.g. if E is a 2-sphere). It is possible, and this has been done by Utiyama, to consider principal fibre bundles over space-time with Lorentz structure group, completely unrelated to $P(E)$ or $L(E)$.

7.2. *Isomorphism between the Kaluza-Klein space and the Utiyama phase space*

A disadvantage of the Utiyama approach to electrodynamics is that it does not provide a natural method of deriving the other half of Maxwell equations (i.e., other than $dF = 0$). The full set of Maxwell equations is known to follow from a simple action principle in the Kaluza-Klein theory, or one of its modifications (cf. T. Kaluza [2], A. Einstein, and P.G. Bergmann [1]). It is interesting to know that, in fact, there is a definite isomorphism between the Utiyama theory and the Kaluza-Klein five-dimensional theory*). This isomorphism may be extended to a large class of theories with gauge invariant fields. In other words, for any such theory it is possible to construct a multidimensional Riemannian space which bears the same relation to that theory as the Kaluza-Klein space to electrodynamics.

Let G be a Lie group possessing an *invariant metric* h, i.e., a symmetric non-degenerate covariant tensor field of second order defined on G and invariant with respect to both left and right translations. For example, if G is semi-simple, then one can define h by $h_e(A, B) = \mathrm{Tr}(\mathrm{Ad}_A \circ \mathrm{Ad}_B)$, where $\mathrm{Ad}_A(C) = [A, C]$, A, B, C belong to the Lie algebra of G and e is the unit of G. An abelian group such as T, also has an invariant metric. Given a principal fibre bundle P with structure group G over a base manifold E (space-time) with a Riemannian metric g, one can define a Riemannian metric γ on P as follows. Let $X \in \tau_r(P)$ and write $\gamma(X)$, $h(A)$, etc., instead of $\gamma(X, X)$, $h(A, A)$, etc. We put

$$\gamma_r(X) = g_{\pi(r)}(\pi(X)) + h_e(\omega(X)).$$

*) The existence of such an isomorphism has been independently recognized by Engelbert Schucking and Włodzimierz Tulczyjew.

It follows from the properties of h that γ is non-singular and invariant with respect to G,

$$\psi_a^*\gamma = \gamma \quad \text{for any } a \in G.$$

To any $A \in G'$ there corresponds a Killing vector field $\psi' A$ of the metric γ. In particular, if $G = SO(2)$ so that G' can be identified with R and h may be taken to be the Euclidean metric on R (possibly with a numerical coefficient), then γ on P is the Riemannian metric of the five-dimensional Kaluza-Klein theory. It is also clear how one can construct a principal fibre bundle from the Kaluza-Klein space.

This construction, when applied to the theory of a general field arising from gauge invariance, leads to the following possibility. One can formulate an action integral of the form $\int_\Omega R$ where R is the Ricci scalar density calculated from the metric γ and $\Omega \subset P$. By varying this action with due care not to spoil the invariance of γ with respect to G, one can obtain a set of field equations, analogous to the Einstein-Maxwell set that one gets in the Kaluza-Klein theory [9].

8. Relativity and Naturality

We shall now say more precisely what is the meaning of the statement that the bundle of frames $P(E)$ in special relativity is a product bundle whereas in general relativity it is not.

To do this we need the notion of *natural equivalence*. Let $\mathcal{A}$ and $\mathcal{B}$ be categories and C_1, C_2 — two (covariant) functors from $\mathcal{A}$ to $\mathcal{B}$. The functors may be thought of as "general constructions" performed on objects of $\mathcal{A}$ and yielding certain objects of $\mathcal{B}$. In certain cases these constructions may be equivalent to each other. Namely, one says that there is natural equivalence between C_1 and C_2 if, for any object E of $\mathcal{A}$, there exists an isomorphism $T(E): C_1(E) \to C_2(E)$ such that the following diagram is commutative:

$$\begin{tikzcd}
C_1(E) \arrow[r, "C_1(h)"] \arrow[d, "T(E)"'] & C_1(F) \arrow[d, "T(F)"] \\
C_2(E) \arrow[r, "C_2(h)"'] & C_2(F)
\end{tikzcd},$$

where $h: E \to F$ is any morphism in $\mathcal{A}$.

An example of a natural equivalence was mentioned already in the Introduction: if $\mathcal{A} = \mathcal{B}$ is the category of finite-dimensional vector spaces, C_1 — the identity functor and C_2 — the functor of taking the double dual, then the usual embedding of V into V^{**} establishes a natural equivalence of C_1 and C_2.

Now, any physical theory of space and time has a starting point a as certain category. Its objects are models of space-time in that theory and the morphisms are mappings preserving the structure inherent in the theory and based on physicaf hypotheses. For example in special relativity the category in question is that ol affine spaces Aff, whereas in general relativity it is that of differentiable manifolds Diff. For the purposes of physics it is usually enough to consider only the invertible ones as morphisms, i.e., isomorphisms. When this is done, P (cf. Sections 5.5 and 5.10) becomes a functor from Diff into the category of fibre bundles, Bun. In the category Aff (which is a subcategory of Diff, if one forgets the vector space associated to an affine space) in addition to P, there is the functor P' defined by

$$P'(E, V, +) = (E \times P(V), E, pr_1).$$

It is quite easy to see that the functors P and P' from Aff to Bun are naturally equivalent and that nothing of the sort is true if Aff is replaced by Diff. The natural isomorphism

$$T(E): P(E) \rightarrow E \times P(V)$$

allows one to identify $P(E)$ with $E \times P(V)$ and so $P(E)$ for an affine space E becomes a Cartesian product.

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