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The Fokker-Planck equation, and its application in plasma physics

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In honour of Max Planck (1858–1947) on the occasion of his 150th birthday.

In a classic paper Max Planck derived an equation, now known as the Fokker-Planck equation, which plays a central role in the statistics description of many body problems. The equation is used in many branches of physics as well as chemistry and biology to describe a variety of different processes. In plasma physics, on which this paper concentrates, it forms the corner stone for the description of the Coulomb collisions. Furthermore, it is pointed out that also the so-called quasi-linear theory is closely related to the Fokker-Planck equation.

1 Introduction

In a classic paper [1] Max Planck derived the proof of an equation which plays a central role in the statistics description of many body problems. With reference to the work of Albert Einstein [2] and Adriaan Fokker [3], he speaks of the Einstein-Fokker equation ('Einstein-Fokkerscher Satz'), but the equation is now known as the Fokker-Planck equation.

In his work on the Brownian motion, Einstein had derived an evolution equation for a system subject to small perturbations. Fokker extended this equation to the case where the perturbations depend on the parameters that characterise the system, but did not present a formal derivation. This derivation was delivered by Max Planck who gave the formalism a solid physics basis, and also extended it to include more than one variable. It appears that R. A. Fisher derived a special form of the equation independently [4] for dealing with the distribution of mutations. And also Kolmogorov derived the equation, known as the 'Kolmogorov forward equation', independently in 1931 [5] for the limit of infinitely short time intervals. From 1934 and on, however, Kolmogorov refers to the equation as the Fokker-Planck equation.

Max Planck derived the equation with the problem of the brownian motion and radiation absorption / emission in mind. Indeed many of the early applications deal with the random motion of particles (see for instance [6]), but the formalism has now been applied in many branches of physics and chemistry. A simple search in the ISI database gives some 5000 hits of scientific publications in which the Fokker-Planck equation is explicit mentioned in the title or abstract. The applications cover a large range of subjects including, for instance: Nonlinear gluon evolution [7], stochastic resonance [8], the Kramers problem [9], Galactic cluster systems [10], liquid crystals [11], X-ray sources [12], and the turbulent cascades [13].

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This paper does not properly review the whole field of applications, which would be rather large indeed. It focuses on the use of the Fokker-Planck equation for fully ionised plasmas and serves as an example for its application. It will be clear from this example that the formalism is a powerful one.

2 The Fokker-Planck equation

In this section the Fokker-Planck equation is derived, following closely the original derivation of Max Planck [1]. Consider a collection of many subsystems whose state can be characterised by the vector $\mathbf{x}$. To develop a physical picture one can think of the special case in which the subsystem is a particle and $\mathbf{x}$ is its position in space. Without loss of generality this interpretation is adopted below, mostly to make the text more easily readable.

Assume there exists some process that leads to small changes ($\Delta\mathbf{x}$) in the position ($\mathbf{x}$), such that different particles at the same position can have a different displacement. This process can be conveniently described through the introduction of a transition probability $\psi(\mathbf{x}, \Delta\mathbf{x})$ which gives the probability of a change in the position $\Delta\mathbf{x}$ of a particle at $\mathbf{x}$ during a time interval Δt. If no particles leave the system then this transition probability satisfies

$$\int \mathrm{d}^3(\Delta\mathbf{x})\psi(\mathbf{x}, \Delta\mathbf{x}) = 1. \tag{1}$$

Although an implicit dependence of ψ on time can exist because for some processes ψ must be calculated self-consistently from the distribution of particles, it is assumed that ψ does not have an explicit time dependence. As far as the process that leads to the changes $\Delta\mathbf{x}$ is concerned there is, therefore, no 'history' effect, or put in other words, a change in position of a specific particle is independent of all its previous displacements. Such a process is know as a Markov process [14].

Because of the statistical nature of the changes in position, a distribution function (F) is introduced to describe the probability P of finding a particle in the volume $\mathrm{d}^3\mathbf{x}$ around the position $\mathbf{x}$

$$\mathrm{d}^3P = F(\mathbf{x}, t)\mathrm{d}^3\mathbf{x} \qquad \rightarrow \qquad \int \mathrm{d}^3\mathbf{x}\, F(\mathbf{x}, t) = 1. \tag{2}$$

Assuming that there are no physical processes that lead to a time evolution of F other than those described by the transition probability ψ, the distribution function at a time point t can be found by multiplying the probability of finding the particle at $\mathbf{x} - \Delta\mathbf{x}$ at $t - \Delta t$ (i.e. $F(\mathbf{x} - \Delta\mathbf{x}, t - \Delta t)$) with the probability of a change in position of $\Delta\mathbf{x}$ (i.e. $\psi(\mathbf{x} - \Delta\mathbf{x}, \Delta\mathbf{x})$), and integrating over all possible changes $\Delta\mathbf{x}$

$$F(\mathbf{x}, t) = \int \mathrm{d}^3(\Delta\mathbf{x})F(\mathbf{x} - \Delta\mathbf{x}, t - \Delta t)\psi(\mathbf{x} - \Delta\mathbf{x}, \Delta\mathbf{x}). \tag{3}$$

The Fokker-Planck equation is the evolution equation for the distribution function F under the action of the changes described by the function ψ with the additional assumption that the changes in the position $\Delta\mathbf{x}$ are small as well as that the time interval Δt for these changes is short. Here, small refers to the comparison with the length scales on which both F as well as ψ vary, i.e.

$$\Delta\mathbf{x} \cdot \frac{1}{F}\frac{\partial F}{\partial \mathbf{x}} \ll 1, \qquad \Delta\mathbf{x} \cdot \frac{1}{\psi}\frac{\partial \psi}{\partial \mathbf{x}} \ll 1. \tag{4}$$

Note that this assumption must not hold for the dependence of ψ on $\Delta\mathbf{x}$, i.e. ψ is allowed to strongly vary when changing $\Delta\mathbf{x}$. Short refers to the timescale on which the distribution function evolves

$$\Delta t \frac{1}{F}\frac{\partial F}{\partial t} \ll 1. \tag{5}$$

The assumptions given above allow for a Taylor expansion of both ψ as well as F in Eq. (3)

$$F(\mathbf{x}) = \int \mathrm{d}^3(\Delta\mathbf{x}) \Bigg[\psi F - \Delta\mathbf{x} \cdot \left(\psi \frac{\partial F}{\partial \mathbf{x}} + F \frac{\partial \psi}{\partial \mathbf{x}} \right) - \Delta t \psi \frac{\partial F}{\partial t} + \frac{1}{2} \Delta\mathbf{x}\Delta\mathbf{x} : \left(\psi \frac{\partial^2 F}{\partial \mathbf{x}\partial \mathbf{x}} + 2 \frac{\partial \psi \partial F}{\partial \mathbf{x}\, \partial \mathbf{x}} + \frac{\partial^2 \psi}{\partial \mathbf{x}\partial \mathbf{x}} F \right) \Bigg], \tag{6}$$

where all quantities are evaluated at the position $\mathbf{x}$. To proceed the averages

$$\langle \Delta\mathbf{x} \rangle = \frac{1}{\Delta t} \int \mathrm{d}^3(\Delta\mathbf{x})\, \Delta\mathbf{x}\, \psi(\mathbf{x}, \Delta\mathbf{x}), \tag{7}$$

$$\langle \Delta\mathbf{x}\Delta\mathbf{x} \rangle = \frac{1}{\Delta t} \int \mathrm{d}^3(\Delta\mathbf{x})\, \Delta\mathbf{x}\Delta\mathbf{x}\, \psi(\mathbf{x}, \Delta\mathbf{x}) \tag{8}$$

are defined. These have the dimensions of a velocity and a diffusion tensor, respectively, and are determined by the transition probability ψ only. Using these definitions, the evolution equation of the distribution function can be written as

$$\frac{\partial F}{\partial t} = -\frac{\partial}{\partial \mathbf{x}} \cdot \Big[\langle \Delta\mathbf{x} \rangle F \Big] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{x}\partial \mathbf{x}} : \Big[\langle \Delta\mathbf{x}\Delta\mathbf{x} \rangle F \Big], \tag{9}$$

where the colon refers to a double contraction with the diffusion tensor. This is the famous Fokker-Planck equation. The evolution of the distribution function under the action of the process introduced through the function ψ is that of a convection (given by the first term on the right hand side) and a dispersion (given by the second term) in the position vector $\mathbf{x}$.

Several points are worth noting here. The Taylor expansion in $\Delta\mathbf{x}$ is up to second order. One might be lead to the conclusion that if $\Delta\mathbf{x}$ is small, a first order expansion will be sufficient. For many randomising processes, however, the mean direction is zero, i.e. $\langle \Delta\mathbf{x} \rangle = 0$, or is smaller by one order in the smallness parameter that characterises the random changes, i.e. $\mathbf{x}\langle \Delta\mathbf{x} \rangle \approx \langle \Delta\mathbf{x}\Delta\mathbf{x} \rangle$. In both cases one needs to go to second order in the Taylor expansion to obtain the correct evolution equation. Higher order terms (third or more) on the other hand are smaller by one order (or more) in the smallness parameter. This point will be encountered when discussing the collisions in a plasma.

Also, although the averages ($\langle \Delta\mathbf{x} \rangle$, $\langle \Delta\mathbf{x}\Delta\mathbf{x} \rangle$) are determined only by the transition probability ψ, it must be noted that the physics process that leads to the random changes can itself depend on the distribution of particles. If this is the case the Fokker Planck equation is nonlinear in the distribution function F.

It should be noted that the changes in position are introduced through an arbitrary transition probability ψ, which up to now has not been specified. Often the processes considered are 'micro-scopic' and a more formal treatment of such processes should allow for a more direct derivation of the equation for the evolution of the distribution function. The Fokker-Planck equation, however, is a very powerful tool to add the physics of micro-scopic processes by hand into the evolution equation of a 'macro-scopic' distribution function. The procedure is as follows: A physics model for the micro-scopic process is constructed. The changes in position obtained from such a model often depend on the initial conditions, which are generally unknown since there is, for instance, no information on the exact positions of all the particles in the system. Some assumption of the distribution over all initial conditions is made, which allows for the calculation of the transition probability ψ. Once this function is determined, the Fokker-Planck equation follows straight forwardly. Of course, for any many body problem it is imposible to describe the evolution of all the particles. The reduction of the problem through the Fokker-Planck equation is often the only reasonable alternative.

The Fokker-Planck equation is related to several other equations used in the description of random processes. It is a special form of the Markov chain, assuming only small displacements. It is also related to the Langevin equation, which can be written in an equivalent form if the Langevin force represents Gaussian noise (see [15]).

3 The Fokker-Planck collision term

The most well known application of the Fokker-Planck equation in plasma physics is the description of the collisions due to the Coulomb interaction between the particles. For an ideal plasma with many particles in the so-called Debye sphere (see below), small angle scattering collisions dominate and the collision operator can be described by the Fokker Planck equation. In fact this collision operator, first derived by Landau [16], is generally referred to as 'Fokker-Planck operator'.

Obviously, the system we want to describe is not that of a single particle for which the probability is given by the function F, but rather a system with many particles which can be characterised by a distribution function f which gives the density of particles in phase space

$$N = \int \mathrm{d}^3\mathbf{x}\mathrm{d}^3\mathbf{v}\, f(\mathbf{x},\mathbf{v}), \tag{10}$$

where N is the total number of particles. Here, not only the position ($\mathbf{x}$), but also the velocity ($\mathbf{v}$) of the particles is considered. In writing the distribution function in the form above, as a function of a 6D phase space, whereas the positions and the velocities of all the particles are characterised by 6N coordinates, it is silently assumed that the particles are uncorrelated, i.e. the probability of finding particle 1 at position $\mathbf{x}$ and with velocity $\mathbf{v}$ is independent of the positions and velocities of all the other particles. Furthermore the probability of finding particle 2 at the same position ($\mathbf{x}$) with the same velocity ($\mathbf{v}$) is equal to that of particle 1. In essence $f = NF$.

Before proceeding, one more concept essential to plasma physics needs to be introduced. In a plasma every charge is shielded by all the other charges, i.e. an ion will attract the electrons while repelling the ions. The negative cloud of charge formed in this way around the ion will shield its charge and the electric potential will fall off more rapidly with the distance to the ion when compared with the vacuum case. The scale length associated with this shielding is the Debye length (λ_D)

$$\lambda_D = \sqrt{\frac{k_B T}{4\pi n e^2}}, \tag{11}$$

where k_B is the Boltzmann constant, T is the temperature, n is the particle density, and e is the elementary charge (note that CGS unit system is used here). For distances larger than the Debye length one can, therefore, neglect the Coulomb interaction. Note that the shielding implies that the position of the particles can not be treated as independent, and the particles must be correlated. This point will be discussed to some extend in the next section.

It is assumed that the interaction of the particles inside the Debye sphere can be modelled by a sum of two body collisions. A two body coulomb interaction leads to a scattering over an angle $\Delta\theta$ in the centre of mass frame as sketched in Fig. 1. This angle is determined by the impact parameter b, which is the shortest distance between the particles which would be attained if no Coulomb interaction was present

$$\sin(\Delta\theta/2) = \frac{b_0}{\sqrt{b^2 + b_0^2}}, \tag{12}$$

where b_0 is the impact parameter for 90° angle scattering

$$b_0 = \left| \frac{e_1 e_2 (m_1 + m_2)}{m_1 m_2 |\mathbf{v}_1 - \mathbf{v}_2|^2} \right|. \tag{13}$$

The indices 1 and 2 denote the two colliding particles. For $b < b_0$ large angle scattering collisions occur. For $b_0 < b < \lambda_D$ the particles are scattered over small angles, and for $b > \lambda_D$ the interaction can be neglected. Since the Debye length is much smaller than the plasma size (one of the definitions of a plasma), the plasma can often be treated as approximately homogeneous over the scale length λ_D, and the

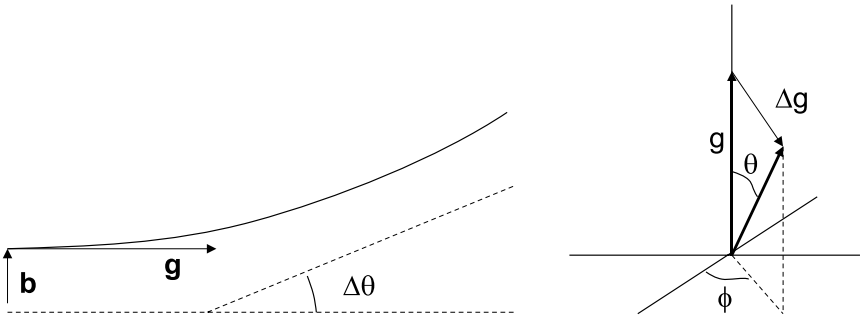

Fig. 1 Scattering of the particle in the centre of mass frame

collisions can be treated locally. The collision operator then depends on the local velocity distribution and does not involve either derivatives towards position or an integration over space.

The collisions are the micro-scopic process that needs to be captured in a transition probability. It is clear from the equations above that the change in velocity due to the collision depends on the initial conditions, essentially through the mutual distance between the particles. It is assumed that the particles are uniformly distributed in space before the collision. This assumption, which is consistent with the uncorrelated distribution, allows for a direct evaluation of the averages $\langle \Delta \mathbf{v} \rangle$ and $\langle \Delta \mathbf{v} \Delta \mathbf{v} \rangle$. The change in velocity of a single particle (index 1) with velocity ($\mathbf{v}_1$) due to collisions with all other particles (index 2) is considered

$$\langle \Delta \mathbf{v}_1 \rangle = \int \int \mathrm{d}b \mathrm{d}\phi \, b \int \mathrm{d}^3 \mathbf{v}_2 \Delta \mathbf{v}_1 |\mathbf{v}_1 - \mathbf{v}_2| f_2(\mathbf{v}_2), \tag{14}$$

with a similar equation for $\langle \Delta \mathbf{v}_1 \Delta \mathbf{v}_1 \rangle$. In the equation above also an index 2 is used for the distribution function since particle 2 might be of a different species than particle 1. Of course, the full collision operator contains a sum over all the species, but for simplicity this sum is not written explicitly here.

We can now address the question: Is a change in velocity more likely to occur due to one large angle scattering collision or many small angle scattering collisions? Given a particle with velocity $\mathbf{v}$, the timescale (τ_L) on which a large angle scattering collisions is expected to occur is

$$\frac{1}{\tau_L} \approx \pi \int \mathrm{d}^3 \mathbf{v}_2 \, b_0^2 |\mathbf{v}_1 - \mathbf{v}_2| f_2(\mathbf{v}_2). \tag{15}$$

This timescale is compared below with the typical time for a 90 degree angle scattering due to the accumulative effect of many small angle scattering collisions. The scatter angle for $b > b_0$ is roughly

$$\Delta\theta \approx 2\frac{b_0}{b}. \tag{16}$$

This small angle scattering process leads to a diffusion in θ with a diffusion coefficient

$$\begin{aligned} \langle \Delta\theta \Delta\theta \rangle = \int_{b_{\min}}^{b_{\max}} \mathrm{d}b \int \mathrm{d}^3 \mathbf{v}_2 \, |\mathbf{v}_1 - \mathbf{v}_2| 4\frac{b_0^2}{b^2} f_2(\mathbf{v}_2) 2\pi b = \\ 8\pi \ln\left(\frac{b_{\max}}{b_{\min}}\right) \int \mathrm{d}^3 \mathbf{v}_2 \, b_0^2 |\mathbf{v}_1 - \mathbf{v}_2| f_2(\mathbf{v}_2). \end{aligned} \tag{17}$$

Since no Coulomb interaction occurs for distances larger than the Debye length, the upper limit of the integration over b is $b_{\max} = \lambda_D$. The lower limit can be chosen to be the impact parameter for 90° scattering (b_0) averaged over the Maxwell distribution

$$b_{\min} = \left\langle \frac{e_1 e_2 (m_1 + m_2)}{m_1 m_2 |\mathbf{v}_1 - \mathbf{v}_2|^2} \right\rangle = \frac{e^2}{3 k_B T} = \frac{1}{12\pi n \lambda_D^2}, \tag{18}$$

where a Hydrogen plasma has been assumed for simplicity. With these assumptions one arrives at

$$\ln\Lambda = \ln\left(\frac{b_{\max}}{b_{\min}}\right) = \ln(12\pi n\lambda_D^3) \tag{19}$$

The logartihm above is known as the Coulomb logarithm and is roughly the logarithm of the number of particles in the Debye sphere. For many plasmas of interest $\ln\Lambda$ has a value larger than 10. It is clear that the integration limits are not accurately defined. The operator thus derived has some arbitraryness. The logartihm, however, makes that even a factor 2 in the integration limits does not strongly influence the result.

The typical time-scale (τ_s) for ninety degree angle scattering is

$$\tau_s = \frac{\pi^2}{4\langle\Delta\theta\Delta\theta\rangle} \tag{20}$$

and, therefore, the ratio of time scales is

$$\frac{\tau_L}{\tau_s} = \frac{32}{\pi^2}\ln\Lambda, \tag{21}$$

and the cummulative effect of many small angle scattering collisions is much larger than that of one single large angle scattering collision. The Fokker-Planck operator describes only the former type.

Having established under what conditions the small angle scattering collisions dominate, the collision operator will be derived below. We choose a coordinate system ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$), with $\mathbf{e}_3$ aligned with the relative velocity $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$. The scattering over the angle $\Delta\theta$ leads to a change in the relative velocity (see Fig. 1)

$$\Delta\mathbf{g} = g(\cos\phi\sin\Delta\theta, \sin\phi\sin\Delta\theta, 1-\cos\Delta\theta) \approx g(\Delta\theta\cos\phi, \Delta\theta\sin\phi, \frac{1}{2}(\Delta\theta)^2), \tag{22}$$

where terms of the order $(\Delta\theta)^3$ have been neglected. The average over the initial conditions implies an integration over b, but also over the angle ϕ. It then follows that there is no contribution to first order in $\Delta\theta$ to the averages $\langle\Delta\mathbf{v}_1\rangle$ and $\langle\Delta\mathbf{v}_1\Delta\mathbf{v}_1\rangle$. In second order the change in the $\mathbf{e}_3$ direction of a particle 1 colliding with particle 2 is

$$\Delta\mathbf{v}_1 = \frac{m_2}{m_1+m_2}\Delta\mathbf{g} \approx -\frac{m_2}{m_1+m_2}\mathbf{g}\frac{(\Delta\theta)^2}{2} = -\frac{2e_1^2e_2^2(m_1+m_2)}{m_1^2m_2}\frac{\mathbf{g}}{g^4b^2}\frac{1}{b^2}. \tag{23}$$

Averaging as before yields

$$\begin{aligned}\langle\Delta\mathbf{v}_1\rangle &= -\frac{2e_1^2e_2^2(m_1+m_2)}{m_1^2m_2}\int \mathrm{d}^3\mathbf{v}_2\int_{b_{\min}}^{b_{\max}}\mathrm{d}b\,2\pi b\frac{\mathbf{g}}{g^3b^2}\frac{1}{b^2}f_2(\mathbf{v}_2)\\ &= -\frac{4\pi e_1^2e_2^2(m_1+m_2)\ln\Lambda}{m_1^2m_2}\int \mathrm{d}^3\mathbf{v}_2\frac{\mathbf{g}}{g^3}f_2(\mathbf{v}_2).\end{aligned} \tag{24}$$

The change $\langle\Delta\mathbf{v}_1\rangle$ in the direction perpendicular to $\mathbf{g}$ is zero to all orders because of the random angle ϕ. However, $\Delta\mathbf{v}\Delta\mathbf{v}$ has a nonzero average. In this direction there is, therefore, a diffusion but no convection.

$$\begin{aligned}\Delta\mathbf{v}_1\Delta\mathbf{v}_1 &\approx \left(\frac{m_2}{m_1+m_2}\right)^2 g^2\frac{(\Delta\theta)^2}{2}(\mathbf{e}_1\mathbf{e}_1+\mathbf{e}_2\mathbf{e}_2) = \frac{(\Delta\theta)^2}{2}g^2\left(\frac{g^2\overline{\mathbf{I}}-\mathbf{gg}}{g^2}\right)\\ &= \frac{2e_1^2e_2^2}{m_1^2}\left(\frac{g^2\overline{\mathbf{I}}-\mathbf{gg}}{g^4}\right)\cdot\frac{1}{b^2}\end{aligned} \tag{25}$$

And the average is

$$\langle \Delta \mathbf{v}_1 \Delta \mathbf{v}_1 \rangle = \frac{4\pi e_1^2 e_2^2 \ln \Lambda}{m_1^2} \int \mathrm{d}^3 \mathbf{v}_2 \left(\frac{g^2 \overline{\mathbf{I}} - \mathbf{gg}}{g^3} \right) f_2(\mathbf{v}_2). \tag{26}$$

These averages can be directly used in the Fokker-Planck equation

$$\frac{\partial f_1}{\partial t} = -\frac{\partial}{\partial \mathbf{v}_1} \cdot \left[\langle \Delta \mathbf{v}_1 \rangle f_1(\mathbf{v}_1) \right] + \frac{1}{2} \frac{\partial^2}{\partial \mathbf{v}_1 \partial \mathbf{v}_1} : \left[\langle \Delta \mathbf{v}_1 \Delta \mathbf{v}_1 \rangle f_1(\mathbf{v}_1) \right], \tag{27}$$

using the relation

$$\frac{\partial}{\partial \mathbf{v}_1} \cdot \left(\frac{g^2 \overline{\mathbf{I}} - \mathbf{gg}}{g^3} \right) = -\frac{2\mathbf{g}}{g^3} = -\frac{\partial}{\partial \mathbf{v}_2} \cdot \left(\frac{g^2 \overline{\mathbf{I}} - \mathbf{gg}}{g^3} \right), \tag{28}$$

one can derive

$$\begin{aligned} \langle \Delta \mathbf{v}_1 \rangle &= \frac{m_1 + m_2}{2 m_2} \frac{\partial}{\partial \mathbf{v}_1} \left[\langle \Delta \mathbf{v}_1 \Delta \mathbf{v}_1 \rangle \right] \\ &= \frac{2\pi e_1^2 e_2^2 (m_1 + m_2) \ln \Lambda}{m_1^2 m_2} \int \mathrm{d}^3 \mathbf{v}_2 \left(\frac{g^2 \overline{\mathbf{I}} - \mathbf{gg}}{g^3} \right) \frac{\partial f_2(\mathbf{v}_2)}{\partial \mathbf{v}_2}. \end{aligned} \tag{29}$$

With the help of these relations the collision operator can be written in the form

$$\frac{\partial f_1(\mathbf{v}_1)}{\partial t} = \frac{2\pi e_1^2 e_2^2 \ln \Lambda}{m_1} \frac{\partial}{\partial \mathbf{v}_1} \cdot \int \mathrm{d}^3 \mathbf{v}_2 \left(\frac{g^2 \overline{\mathbf{I}} - \mathbf{gg}}{g^3} \right) \cdot \left[\frac{f_2(\mathbf{v}_2) \partial f_1(\mathbf{v}_1)}{m_1 \quad \partial \mathbf{v}_1} - \frac{f_1(\mathbf{v}_1) \partial f_2(\mathbf{v}_2)}{m_2 \quad \partial \mathbf{v}_2} \right] \tag{30}$$

This is the Fokker-Planck collision operator in Landau form. Note that the collision operator is a local operator, i.e. all quantities are evaluated in the same position. Furthermore, the convection and dispersion coefficient of the Fokker-Planck equation are of equal magnitude.

The collision operator conserves particles, momentum and energy

$$\int \mathrm{d}^3 \mathbf{v}_1 \, C_{12} = 0, \tag{31}$$

$$\int \mathrm{d}^3 \mathbf{v}_1 \, m_1 \mathbf{v}_1 C_{12} + \int \mathrm{d}^3 \mathbf{v}_2 m_2 \mathbf{v}_2 C_{21} = 0, \tag{32}$$

$$\int \mathrm{d}^3 \mathbf{v}_1 \, m_1 v_1^2 C_{12} + \int \mathrm{d}^3 \mathbf{v}_2 m_2 v_2^2 C_{21} = 0, \tag{33}$$

where C_{12} represents the right hand side of Eq. (30), i.e. the effect of collision on the distribution of species 1 due to collisions with species 2. These properties can readily be verified through the substitution of the Fokker Planck collision term into the integrals and performing partial integrations. The conservation of particles, momentum and energy is no surprise since the elastic Coulomb collisions satisfy these conservation laws. The Fokker-Planck collision operator, however, has another important property: it satisfies an H-theorem. The change in the entropy density $s = \sum_i \int \mathrm{d}^3 \mathbf{v} \, f_i \ln f_i$ due to collisions can be shown to be larger than or equal to zero

$$\frac{\partial s}{\partial t} = -\sum_i \sum_j \int \mathrm{d}^3 \mathbf{v} \, C_{ij} \ln f_i \geq 0. \tag{34}$$

The entropy production is zero if and only if the distribution function of all species is Maxwellian with a common temperature and mean velocity. Coulomb collisions therefore drive the distribution towards thermodynamic equilibrium.

In a rather straight forward way, the Fokker-Planck formalism allowed for the derivation of a nonlinear integral-differential operator which describes the collisional processes in a plasma. This operator, furthermore, has all the desired features connected with conserved quantities and entropy production. Above, we have used the non-relativistic limit, but an equivalent operator can be derived for particles with relativistic velocities [17].

4 The Vlasov-Fokker-Planck equation

The power of the Fokker-Planck formalism becomes especially clear when one considers the alternative, which will be discussed very briefly at the end of this section. Before entering this discussion, first the effects of the electro-magnetic field are introduced in the formalism. The derivation given below is an heuristic one.

The particles at position $\mathbf{x}$ and with velocity $\mathbf{v}$ move through phase space according to the equations of motion

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}, \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{e_i}{m}\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right], \tag{35}$$

where $\mathbf{E}$ is the electric and $\mathbf{B}$ is the magnetic field. When no particles are lost, the distribution function must satisfy a continuity equation

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\mathbf{x},\mathbf{v}}\left[(\dot{\mathbf{x}},\dot{\mathbf{v}})f\right] = 0, \tag{36}$$

where "div" is the divergence operator in the six dimensional phase space. Our kinetic equation thus can be written in the form

$$\frac{\partial f}{\partial t} + \mathrm{div}_{\mathbf{x},\mathbf{v}}\left[\left(\mathbf{v}, \frac{e}{m}\left(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right)\right) f\right] = 0. \tag{37}$$

Because $\mathbf{x}$ and $\mathbf{v}$ are independent in this equation, and because

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{v}\times\mathbf{B} = 0, \tag{38}$$

this equation can be written in the form

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}\left[\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\right]\cdot\frac{\partial f}{\partial \mathbf{v}} = \left.\frac{\partial f}{\partial t}\right|_{\mathrm{col}} \tag{39}$$

where on the right hand side we have added the effect of collisions described by the Fokker-Planck equation.

The change in velocity of the particles is determined by the electric and magnetic fields. These fields consist of the fluctuating fields that are due to the discreteness of the particles acting within the Debye sphere as well as the more smeared out fields in accordance with the smoothed distribution function. The latter electro-magnetic fields follow from the Maxwell equations with as source the charge density and current, both calculated from the uncorrelated distribution

$$\rho = \sum_i e_i \int \mathrm{d}^3\mathbf{v}\, f_i, \quad \mathbf{J} = \sum_i e_i \int \mathrm{d}^3\mathbf{v}\, \mathbf{v} f_i \tag{40}$$

The electric and magnetic fields on the left hand side of the equation are calculated using the charge density and current given above, whereas the effect of the rapidly fluctuating electric field due to the discreteness

of the particles is entirely contained in the Fokker-Planck collision operator. In a rather straight forward way a consistent description of the plasma dynamics is, therefore, obtained. When collisions are neglected, i.e. when the right hand side of the equation above is set to zero, the equation is known as the Vlasov equation. This equation is already rich in its structure since it is nonlinear in the velocity distribution through the electric and magnetic fields which depend on the sources of the Maxwell equations.

We are now in a position to discuss the alternative and more formal derivation of the Vlasov-Fokker-Planck equation. In a formal derivation one starts with a distribution function that depends on the coordinates of every individual particle

$$f = f_N(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N). \tag{41}$$

Of course the evolution equation of this distribution is unpratical for any realistic number of particles. One, therefore, reduces the size of the problem through the integration over phase space

$$f_s(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_s, \mathbf{v}_s) = \frac{1}{(N-s)!} \int \Pi_{i=s+1}^{N} \mathrm{d}^3\mathbf{x}_i \mathrm{d}^3\mathbf{v}_i \, f_N(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N). \tag{42}$$

It can be shown that the evolution equation for f_s depends only on f_{s+1}. A proper hierachy of equations, known as the BBGKY hierachy after Bogoliubov, Born, Green, Kirkwood, and Yvon, is therefore obtained. To proceed a cluster expansion is introduced

$$f_1(\mathbf{x}_1, \mathbf{v}_1) = f(\mathbf{x}_1, \mathbf{v}_1) \qquad f_2(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2) = f(\mathbf{x}_1, \mathbf{v}_1) f(\mathbf{x}_2, \mathbf{v}_2) + P(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2). \tag{43}$$

This expansion is motivated by the expectation that the particles are largely uncorrelated, i.e. it is expected that the pair correlation function P is much smaller than the product of the uncorrelated distribution functions, i.e. $P \ll ff$. Using the cluster expansion in the evolution equation of the lowest order equation of the BBGKY hierachy yields the Vlasov equation (39) with the electric field determined by the charge density of Eq. (40), and the collision term given by

$$\left.\frac{\partial f}{\partial t}\right|_{\mathrm{col}} = \frac{1}{m} \int \mathrm{d}^3\mathbf{x}_2 \mathrm{d}^3\mathbf{v}_2 \frac{\partial}{\partial \mathbf{x}_1} \left[\frac{e^2}{|\mathbf{x}_1 - \mathbf{x}_2|}\right] \frac{\partial P(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2)}{\partial \mathbf{v}_1}. \tag{44}$$

Thus, so far, the result is completely consistent with the description given earlier. Not only is the Vlasov equation recovered, the rapid fluctuating fields due to the discreteness of the particles can indeed be split off and modelled by a seperate term. This term is proportional to the pair correlation function rather than the uncorrelated distribution function. It can, furthermore, be shown that P is significant only when the distance between the particles is smaller than the Debye length.

To obtain a collision operator in closed form, one has to solve for the pair correlation function. A solution can be obtained from the second equation of the BBGKY hierachy again using the cluster expansion, if 3 particle correlations are neglected, and if it is assumed that the pair correlation function relaxes on a short time-scale. The result is the Balescu-Lenard collision operator [18, 19], which we will not reproduce here since it is hardly ever used in the description of a plasma. The Balescu-Lernard operator is not equivalent with the Fokker-Planck operator derived above, but it can be shown that the Landau form is a good approximation of the more complex Balescu-Lenard operator. The latter is so complex that it is considered unpratical by many researchers in the field.

5 Rosenbluth potentials

The equation derived by Landau is convenient for analytic treatment, but is not the most efficient form for numerical evaluation. A convenient formalism that reduces the computational costs was derived by M.N. Rosenbluth [20]. M.N. Rosenbluth formulated the problem defining two potentials

$$\phi_2 = -\frac{1}{4\pi}\int \mathrm{d}^3\mathbf{v}_2 \frac{f_2(\mathbf{v}_2)}{|\mathbf{v}-\mathbf{v}_2|}, \tag{45}$$

$$\psi_2 = -\frac{1}{8\pi}\int \mathrm{d}^3\mathbf{v}_2 f_2(\mathbf{v}_2)|\mathbf{v}-\mathbf{v}_2|. \tag{46}$$

These are called potentials since they satisfy the equations

$$\nabla_v^2\phi_2 = f_2 \qquad \nabla_v^2\psi_2 = \phi_2, \tag{47}$$

where the v indicates that the ∇^2 is an operator in velocity space ($\mathbf{v}$). Writing the collision operator explicitly as the divergence of a flux in velocity space

$$C_{12} = -\nabla_{v_1}\cdot\mathbf{S}_{12} \quad \text{with} \quad \mathbf{S}_{12} = -\overline{\mathbf{D}}\cdot\nabla_{v_1}f_1 + \mathbf{F}f_1, \tag{48}$$

where $\overline{\mathbf{D}}$ is the diffusion tensor and $\mathbf{F}$ is the convection velocity, one can derive that

$$\overline{\mathbf{D}} = -\frac{16\pi^2 e_1^2 e_2^2 \ln\Lambda}{m_1^2}\nabla_{v_1}\nabla_{v_1}\psi_2(v_1), \qquad \mathbf{F} = -\frac{16\pi^2 e_1^2 e_2^2 \ln\Lambda}{m_1 m_2}\nabla_{v_1}\phi_2(v_1). \tag{49}$$

The formulation given above is convenient since the potentials can be efficiently calculated through an expansion in spherical harmonics, similar to the techniques used in electro-statics [21]. For simplicity a cylindric symmetric distribution function is assumed here. Such a distribution function is often found to exist for plasmas with a strong magnetic field. The fast gyration of the particles around the field line then makes that the distribution function depends only on the magnitude of the perpendicular (to the magnetic field) velocity direction, not on its direction. In this case

$$f_2(v,\theta) = \sum_{l=0}^{\infty} f_2^l(v)P_l(\cos(\theta)) \quad \rightarrow \quad f^l(v) = \frac{2l+1}{2}\int_0^{\pi}\mathrm{d}\theta\, f(v,\theta)P_l(\cos(\theta))\sin\theta, \tag{50}$$

where v is the magnitude of the velocity, θ is the angle between the velocity and the magnetic field, and P_l is the Legendre function. A similar equation applies for ϕ_2 and ψ_2. The coefficients $\psi_2^l(v)$ and $\phi_2^l(v)$ can readily be calculated by first determining the Legendre decomposition f_2^l of f_2 and then integrating f_2^l

$$\phi_2^2(v) = -\frac{1}{2l+1}\left[\int_0^v \mathrm{d}v' \frac{(v')^{l+2}}{v^{l+1}}f_2^l(v') + \int_v^{\infty}\mathrm{d}v'\frac{v^l}{(v')^{l-1}}f_2^l\right] \tag{51}$$

and a different, but in its structure similar equation for ψ_2. (For the mathematical details of this decomposition the reader is referred to [21]).

Suppose that the distribution is represented on a N times N grid in (v,θ) coordinates. The Landau form of the collision operator requires an integral over velocity space (N^2 operation) for every grid point, and the computational costs therefore scale as N^4. The formulation using the Rosenbluth potentials requires LN^2 operations to calculate the coefficients f_2^l, where L is the number of Legendre harmonics (upper limit $L = N$). The integrals of the type give above require LN operations, and the reconstruction of the potentials $\psi_2(v,\theta)$ and $\phi_2(v,\theta)$ require again LN^2 operations. The evaluation of the collision operator then scales as LN^2. This scaling is always more benificial than the direct evaluation of the Landau integral, but the gain is often even larger due to the fact that only a few Legendre polynomals must be kept (often $L \leq 2$ is used).

Furthermore, energy conservation can be obtained keeping only the lowest order Legendre polynomal, while momentum conservation is obtained when keeping the first order polynomial, i.e. physical properties can be linked to the use of the low order Legendre harmonics.

There exist several convenient approximations for the Collision operator. They involve linearised operators for when the distribution is close to the Maxwell, and approximated forms which are valid when the particle is much faster than the particle it collides with (which is generally true for and electron-ion collisions). A rather complete discussion of the different forms can be found in [22]. The decomposition in Spherical harmonics can also be made for the relativistic operator [23].

6 Quasi-linear theory

The Fokker-Planck equation is know in plasma physics to describe the collisional processes. What isn't widely known is that the equation essentially describes many other phenomena. Below we will discuss the Landau damping in terms of the same physics. This approach is rather unusual. Landau derived his damping mechanism from the collisionless Vlasov equation which might at first appear not to include any random perturbations. Most text books stress the collective motion of the particles and the deterministic interaction with the wave fields. The books are correct, of course, but there is a randomness hidden in the assumptions which is clearly brought out starting from the Fokker-Planck equation.

Landau damping is one of the most important kinetic effects. This damping can be obtained from the Vlasov equation and, therefore, it is a collisionless damping mechanism. An electro static wave is assumed. From the Maxwell equations one finds

$$\mathbf{k} \times \mathbf{E} = 0 \quad \text{or} \quad \mathbf{E} \parallel \mathbf{k}, \tag{52}$$

i.e. the electric field is parallel to the wave vector ($\mathbf{k}$). Following Stix [24] an electric field of the form

$$\mathbf{E} = \tilde{E} \cos[kx - \omega t]\mathbf{e}_x \tag{53}$$

is assumed. The electric field is, furthermore, assumed sufficient small, such that it only generates a small perturbation in the velocity of the particles. This assumptions allows for an integration of the relevant equation of motion,

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = e\tilde{E}\cos[kx - \omega t], \tag{54}$$

along an unperturbed orbit

$$x = x_0 + v_0 t. \tag{55}$$

The result is

$$\begin{aligned}
\Delta v &= \frac{e\tilde{E}}{m}\int_0^t \mathrm{d}t' \ \cos[kx_0 + (kv_0 - \omega)t'] \\
&= \frac{e\tilde{E}}{m}\frac{1}{kv_0 - \omega}\left[\sin[kx_0 + (kv_0 - \omega)t] - \sin[kx_0]\right] \\
&= \frac{e\tilde{E}}{m}\frac{2}{kv_0 - \omega}\sin\left[\frac{(kv_0 - \omega)t}{2}\right]\cos\left[kx_0 + \frac{(kv_0 - \omega)t}{2}\right].
\end{aligned} \tag{56}$$

It is clear from this expression that Δv will be large when $kv_0 - \omega = 0$, or

$$v_0 = \frac{\omega}{k} \tag{57}$$

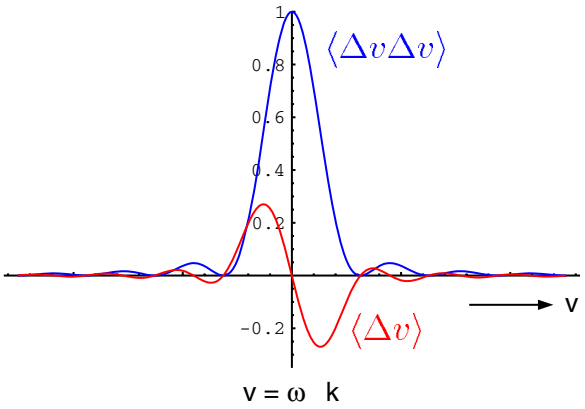

Fig. 2 (online colour at: www.ann-phys.org) The averages $\langle \Delta v \rangle$ and $\langle \Delta v \Delta v \rangle$ as a function of v.

which is the resonance condition. When the particles have a velocity equal to the phase velocity of the wave they will 'see' a constant electric field in their frame of motion, and can be efficiently accelerated or decelerated. Whether the change in velocity is positive of negative depends on the initial phase of the particle with respect to the wave (kx_0). It is assumed that there is some random process that destroys the phase relation between the particles and the wave on a typical time-scale τ. This essentially means that interaction of the particle with the wave is a Markov process. Then averaging over kx_0 one obtains

$$\langle \Delta v \rangle = \frac{1}{2\pi\tau} \int_0^{2\pi} \mathrm{d}(kx_0) \Delta v \Big|_{t\to\tau} = 0, \tag{58}$$

$$\langle \Delta v \Delta v \rangle = \frac{1}{2\pi\tau} \int_0^{2\pi} \mathrm{d}(kx_0) \Delta v \Delta v \Big|_{t\to\tau} = \frac{e^2 \tilde{E}^2}{m^2} \frac{2}{(kv_0 - \omega)^2 \tau} \sin^2 \left[\frac{(kv_0 - \omega)\tau}{2} \right]. \tag{59}$$

The function $\langle \Delta v \Delta v \rangle$ is shown in Fig. 2. This function is peaked around the resonance condition with a finite width (δv) due to the finite coherence time τ,

$$k\delta v \approx \frac{\pi}{\tau}. \tag{60}$$

If the difference of the particle and the phase velocity is small, such that the particle does not appreciably change its phase during the coherence time τ, it can still efficiently exchange energy with the wave. If one increases τ the maximum at $v = \omega/k$ increases, but also the width of the function decreases. The net exchange of energy between the particles and the wave is, therefore, independent of τ.

For the integration along an unperturbed orbit $\langle \Delta v \rangle = 0$, and $\langle \Delta v \Delta v \rangle \neq 0$. However, $\langle \Delta v \Delta v \rangle \propto \tilde{E}^2$, and therefore it is of second order in the small electric field. For consistency also $\langle \Delta v \rangle$ has to be calculated up to second order. This calculation requires first to construct the (linearly) perturbed orbit of the particle

$$\Delta x(t) = \int_0^t \mathrm{d}t' \, \Delta v(t'), \tag{61}$$

where Δv is the lowest order change in the velocity v, and then again integrating the force due to the electric field along this orbit . If one assumes that τ is sufficiently small ,such that $k\Delta x(\tau) < 1$ is satisfied, one can show that

$$\langle \Delta v \rangle = \frac{1}{2} \frac{\partial}{\partial v} \left[\langle \Delta v \Delta v \rangle \right], \tag{62}$$

where $\langle \Delta v \Delta v \rangle$ is given by the expression derived above. The dependence of Δv on v is shown in Fig. 2. This contribution can be understood as follows. If a particle is slightly slower than the wave and its phase

relation is such that it looses energy it will become even slower and will then be out of resonance with the wave. However, if its phase relation is such that the energy is increased it will be brought into resonance with the wave and it can absorb energy for a longer time. Therefore, on average, the particles with velocities slightly smaller than the wave will be accelerated. Similar arguments apply to particles that are faster than the wave. These particles will have a net decrease in velocity. It is essentially this mechanism that leads to the absorbtion of wave energy when the (velocity) gradient in the distribution is negative at the position of the resonance, i.e. when there are more particles that are slower than the wave compared with particles that are faster than the wave. It can be shown (see [24]) that the formalism above recovers the Landau result in the limit of a very small electric field and $\tau \to \infty$. Note that in the particle picture one has to go to second order, i.e. introduce an explicit nonlinearity, to arrive at the correct equation for the energy exchange, whereas the original Landau problem is based on a linearization of the Vlasov equation. Note too that, similar to the collision operator, $\langle \Delta v \rangle$ and $\langle \Delta v \Delta v \rangle$ are of the same order in the smallness parameter (in this case E). For this reason one has to go to second order in the Taylor expansion used to derive the Fokker-Planck equation.

Using the relation above the Fokker-Planck equation for the evolution of the distribution function can be written in the form

$$\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial}{\partial v}\left[\langle \Delta v \Delta v \rangle \frac{\partial f}{\partial v}\right], \tag{63}$$

i.e. the evolution is a diffusion equation in velocity space. Energy is absorbed from the wave when the (velocity) gradient is negative, whereas a positive gradient leads to a growing wave solution. The above equation, however, goes beyond the Landau problem which only gives the initial damping or amplification of the wave, it can describe the time evolution of the distribution function. For an initial distribution with a negative velocity gradient, the absorption of wave energy will lead to a diffusion of particles in velocity space around the resonance condition. This diffusion will lead to a reduction of the absolute value of the gradient and, therefore, lead to a reduction in the absorption of wave energy. This process is known as the quasi-linear plateau formation [25, 26], In fact, the formulation using the Fokker-Planck equation is equivalent to that of quasi-linear theory as is briefly outlined below.

The quasi-linear theory is directly derived from the Vlasov equation. The distribution function is split in a slowly varying f_0, and a rapidly fluctuating part f_1. It is assumed that the slow evolution of the distribution function can be neglected for the rapid fluctuating part. This leads to a set of coupled equations

$$\frac{\partial f_1}{\partial t} + v\frac{\partial f_1}{\partial x} + \frac{e}{m}E\frac{\partial f_0}{\partial v} = 0, \tag{64}$$

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} + \frac{e}{m}\left\langle E\frac{\partial f_1}{\partial v}\right\rangle = 0. \tag{65}$$

Since both the electric field as well as the pertubation f_1 are varying rapidly in time, the slow time evolution of f_0 is determined by the time average of the product.

To obtain a closed equation for the slowly varying distribution, the equation (64) is integrated toward time along the characteristics of the unperturbed (by the electric field) motion

$$f_1(x - vt, t) = -\frac{e}{m}\int_{-\infty}^{t} \mathrm{d}t' E(\mathbf{x}(t'), t')\frac{\partial f_0}{\partial v} \qquad \text{with} \qquad x(t) = x_0 + vt, \tag{66}$$

which after substitution yields

$$\frac{\partial f_0}{\partial t} + v\frac{\partial f_0}{\partial x} = \frac{\partial}{\partial v}\left[\left\langle \frac{e^2}{m^2}E(x,t)\int_{-\infty}^{t} \mathrm{d}t' E(\mathbf{x}(t'), t')\right\rangle \frac{\partial f_0}{\partial v}\right]. \tag{67}$$

It is clear from the expression above that the quasi-linear theory has the same structure as the Fokker-Planck equation. One can think of quasi-linear theory as a special case of the Fokker-Planck formalism for

which the relation given by Eq. (62) is satisfied. Its underlying physics assumptions satisfy all the criteria needed for the Fokker-Planck equation to apply.

7 Use of the Fokker-Planck equation and quasi-linear theory in plasma physics

Through the two examples from the area of high temperature plasma physics the strength of the Fokker-Planck formalism has been shown. We end this paper with a brief summary of where these two examples find their application in the area of magnetic confinement nuclear fusion research (which is admittedly more narrow than the area of high temperature plasma physics). This to illustrate that they deal with key physics processes in the description of such plasmas

In nuclear fusion research the collision time is generally shorter than the energy confinement time, and the plasma is close to thermodynamic equilibrium. The relaxation towards such an equilibrium and the extend to which it can be reached is determined by the Coulomb collisions. They should provided for the plasma heating through the slowing down of the fusion products. Collisions lead to a friction between the electron and ion fluids leading to a resistivity [27] which determines the efficiency by which a current can be driven in the plasma. Collisions also lead to the transport of particles and energy which is denoted as neo-classical transport, due to the influence of the specific particle orbits on the fluxes. Although transport is mostly dominated by small scale turbulence, neo-classical transport is nevertheless of importance for some transport channels, most dominantely the impurity transport [28–30] and in some cases the ion heat transport. Neo-classical theory furthermore predicts the existence of a current driven by the presure gradient known as the bootstrap current [31, 32], which could play an important role in sustaining the plasma equilibrium [33].

Quasi-linear theory has been extensively used for the description of all forms of wave heating. A consistent description requires that the effects of the collisions are kept in the description, since the extend to which the quasi-linear plateau is formed depends on the competition between the quasi-linear diffusion and the Coulomb collisions with the latter driving the distribution back to the Maxwellian. Besides plasma heating such calculations are of interest again for the possibility of current drive through waves. It is noted that quasi-linear theory is also used for the estimation of turbulent fluxes (see for instance [34]). In this case the structure of the theory is, however, somewhat different since the electo-magnetic field must be calculated self-consistently from the perturbed distribution.

Several numerical tools have been developed for calculating the evolution of the distribution function under the action of Coulomb collisions, the electric field, different forms of wave heating and high velocity neutral particle injection [35–42]. These numerical tools are frequently used to make predictions for various quantities related with the operation of a current experiment of a future reactor.

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