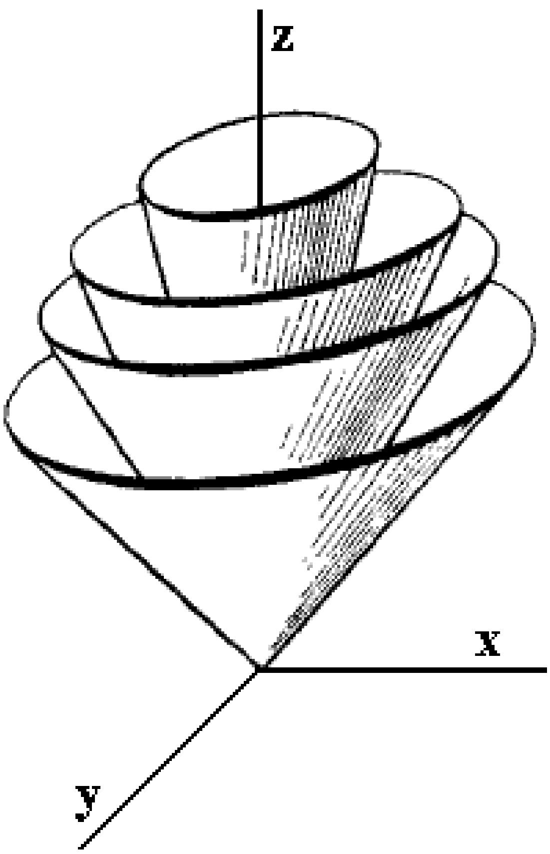
LIE GROUP ANALYSIS CLASSICAL HERITAGE

Edited by Nail H. Ibragimov



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The cover illustration:
An example of invariant surfaces
Sophus Lie, "Lectures on transformation groups", 1893, p. 414

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Preface

Today, acquaintance with modern group analysis and its classical foundations becomes an important part of mathematical culture of anyone constructing and investigating mathematical models. However, many of classical works in Lie group analysis, e.g. important papers of S.Lie and A.V.Bäcklund written in German and some fundamental papers of L.V. Ovsyannikov written in Russian have been not translated into English till now. The present small collection offers an English translation of four fundamental papers by these authors.

I have selected here some of my favorite papers containing profound results significant for modern group analysis. The first paper imparts not only Lie's interesting view on the development of the general theory of differential equations but also contains Lie's theory of group invariant solutions. His second paper is dedicated to group classification of second-order linear partial differential equations in two variables and can serve as a concise practical guide to the group analysis of partial differential equations even today. The translation of Bäcklund's fundamental paper on non-existence of finite-order tangent transformations higher than first-order contains roots of the modern theory of Lie–Bäcklund transformation groups. Finally, Ovsyannikov's paper contains an essential development of the group classification of hyperbolic equations given in Lie's second paper. Moreover, it contains two proper invariants for hyperbolic equations discovered by Ovsyannikov.

I am greatly indebted to Ms. Elena Ishmakova and Ms. Roza Yakushina for their excellent work during the translation.

Karlskrona, 17 December 2004 Nail H. Ibragimov

Contents

GENERAL THEORY OF PARTIAL DIFFERENTIAL EQUATIONS OF AN ARBITRARY ORDER

By Sophus Lie

Translated from German by
N.H.Ibragimov and E.D. Ishmakova

[Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebiger Ordnung. Von Sophus Lie. Leipz. Berichte, Heft I, 1895, S. 53–128. Reprinted in Collected Works of S. Lie [27], vol. 4, paper IX]

I reported about the new theories contained in this paper to the Royal Scientific Society of Saxony on 16 October 1893. Then I developed them in detail in my lectures at Leipzig University during the winter semester 1893-1894.

Sophus Lie

1. Theory of differential equations is the most important discipline in all modern mathematics.

It would be correct to say that the notions of derivative and integral whose origin goes back at least to Archimedes were in fact introduced to the science later in works of Kepler, Descartes, Cavalieri, Fermat and Wallis. However, these scientists can by no means be regarded as founders of the infinitesimal calculus for they did not realize that differentiation and integration are inverse operations.

Nowadays it is taken for granted that this fundamental discovery belongs to Newton and Leibnitz*. Moreover, they realized the immense significance of the above notions and simultaneously developed appropriate algorithms.

*The question whether Newton or Leibnitz was the first to note that differentiation and integration are inverse operations obviously was most important for the old priority dispute on foundation of the Infinitesimal Calculus. If I am not mistaken Mr.Zeuthen, who has made a great contribution to the history of mathematics, will consider the above question in his forthcoming work.

Particularly important were, on the one hand, Newton's discovery of the binominal formula showing how to obtain numerous differentials and integrals and, on the other hand, Leibnitz's introduction of the symbols dx, dy and finally his consideration of higher-order derivatives with applications.

2. The notions of differential and integral, introduced by Newton and Leibnitz were applied only to drawing tangents, defining an arc length, determining areas and volumes. Understanding both concepts independently of their applications gave these great mathematicians a possibility to tackle the problems which nowadays are formulated in terms of differential equations. Perhaps it would be correct to say that it is formulation and integration of differential equations that set up an epoch-making progress and characterize first of all the Newton–Leibnitz era and at the same time modern higher mathematics.

Newton's derivation of Kepler's laws for planetary motions from the gravitation law, also formulated by Newton, is particularly renowned and there are good reasons for that. This discovery that was epoch-making not only for mechanics was based in fact on integration of a system of differential equations. Namely, the gravitation law provides the differential equations governing the motion of planets and Kepler's laws are nothing else but the corresponding integral equations.

3. The brothers Jacob and Johann Bernoulli (1654–1705, 1667–1748) made a further contribution to the theory of differential equations; especially famous are their investigations of geodesic curves and isoperimetric problems that are considered to be the origin of variational calculus.

The Italian mathematician Riccati (1676–1754) paid attention to particular cases of the following equation which later became so popular:

$$\frac{dy}{dx} = X(x) + X_1(x)y + X_2(x)y^2.$$

This equation should certainly be considered as the simplest and the most significant among non-integrable differential equations. In particular, new group-theoretic investigations show that this equation can be interpreted as an analogue of the algebraic equation of the fifth degree.

Clairaut (1713–1765), who is particularly famous for the investigation of spiral curves, preoccupied himself with equations of the form

$$y - xy' - \varphi(y') = 0,$$

whose integration theory, as is well known, is connected with the notion of linear coordinates. Therefore, one can say that the origin of the notion of linear coordinates (and furthermore, of duality) goes back to Clairaut.

A further important contribution to the theory of differential equations was made by d'Alembert (1717–1783). By formulating the general mechanical principle,* bearing his name, he reduced all problems of dynamics to differential equations and furnished Newton's revolutionary mechanical ideas with a general and definite form. Furthermore, d'Alembert's elegant investigations of ordinary and linear partial differential equations were also very important.

4. The names of Euler, Lagrange and Laplace, Monge, Ampére and Pfaff determine a new epoch in the general theory of differential equations characterized in particular by the fact that partial differential equations were also included into the general theory.

In regard of the above I obviously can not dwell upon every contribution to the theory of differential equations made by these great mathematicians and their equally renowned followers: Gauss, Cauchy, Fourier, Abel, Jacobi and Riemann. However, I will at least try to outline in Chapter I several most important trends in recent studies on differential equations. I will dwell upon only those trends which are connected with my investigations in the field. Then I give in Chapter II a proper survey of some of my own theories that till now have only been sketched in the Norwegian language. In the next chapters I describe in detail important general integration theories also developed by myself.

*Recently Mr. Ostwald tried to formulate a general principle aimed at embracing all natural laws. However, as I already said before, the existing mathematical formulation of the principle is unfortunately so vague and unclear that mathematicians can not understand its meaning. Provided that the principle is properly formulated one can answer the question wether e.g. Hertz had similar ideas.

Chapter 1

Comparative review of new studies on differential equations

5. In the beginning of this chapter, a survey of investigations on differential equations made after Euler and Lagrange is given. Although these observations may seem incomplete and imperfect, nevertheless they enlighten the tendencies of our own endeavor. Particularly, we dwell on theory of characteristics developed by Monge, Laplace, Ampére and Darboux thus providing a basis for our own investigations represented in the next chapters.

§ 1. Different directions in theory of differential equations

In my opinion, the major part of papers on differential equations published within the last 120 years can be divided into four or five categories having much in common.

6. I assign to the first category first of all investigations on partial differential equations of the first order started by Euler, Lagrange and Monge and continued by Pfaff, Cauchy, Hamilton, Jacobi, A. Mayer and others. To the same category I refer research on partial differential equations of second and higher orders started by Monge and Laplace. Among followers of Laplace and Monge in this field are Ampére, Darboux and some other French mathematicians who ensured a considerable advance in the theory of differential equations. In all these works the notion of characteristics introduced by Monge played an important part implicitly or explicitly.

7. To the second direction I refer investigations started by d'Alembert, Fourier and Cauchy and continued by Riemann, Weierstrass, Méray, Schwarz, C. Newmann, Poincaré, Picard and many other outstanding mathematicians dealing with integrability conditions and solution of initial value problems.

8. The third direction comprises the most part of recent studies on linear differential equations based on the general theory of functions developed by Cauchy and Abel. It should be noted that Gauss and Riemann paved the way in this direction, but Fuchs (1866) is to be considered as its founder.

The notion of discrete groups introduced by Galois enlightened new horizons in this direction. It neither occurs explicitly in the works of Riemann* known to me nor in earlier papers of Fuchs. To the best of my knowledge, the first works where Galois groups are applied in the general theory of (linear) differential equations are publications of C. Jordan of the first half of 1874. Moreover, Fuchs made a further progress by applying (1875) Cayley's theory of invariants to linear differential equations. In addition it should be mentioned that the same results, as Klein demonstrated later, can be derived much simpler and even in a more complete form via combination of the above C. Jordan's investigations with F. Klein's determination† (1874) of all discrete projective groups on the line.

Numerous function-theoretic investigations of Schwarz, Hermite, Thomé, Frobenius, Fuchs, Klein, Poincaré, Picard, Appell, Painlevé and others belong to this direction. There is no need to dwell upon these results here. Nevertheless, we will consider further (although only in outline) investigations of Cockle, Laguerre and Halphen as well as Picard and Vessiot on linear differential equations.

9. To the fourth direction I refer those investigations where my general notion of continuous groups is applied in integration theory explicitly or implicitly. In works of this direction a fundamental role is also played by the notion of differential invariant which follows directly from the notion of group.

I do not consider it necessary to dwell upon the history of origin of these

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*It is sufficient that in Riemann's papers, as well as in many earlier works, the notion of group occurs implicitly. However, it is still unknown whether the notion of group belongs to Riemann. The fact that Riemann does not even quote Galois in his investigations of Abelian integrals does not show clearly whether he was aware of Galois' works and ideas.

†In November 1873 I informed Klein that I had found all continuous projective groups with one variable. This information, if I am not mistaken, stimulated Klein to determine all discrete projective groups with one variable in the spring of 1874.

It is also interesting to note that already at that time (letter of 30 April 1874) Klein occupied himself with the problem of all single-valued functions of a discrete projective group.

two notions. All these investigations are based on my discovery dating back to 1869–1870 that integration methods of earlier mathematicians which have been regarded as independent theories can be deduced from the general principle furnished by the notion of continuous groups. This observation lead me at once to a series of new, although simple, integration methods all having group-theoretic nature (Ges. d. Wiss., Christiania 1871, Math. Ann. Bd.V[(1)]).

10. In 1872 I outlined several integration theories based on the theory of invariants of the infinite group of all contact transformations as well as the theory of invariants of the infinite group of all point transformations (cf. short Rèsumè mehrerer neuer Theorien, April 1872; Zur Theorie der Differentialprobleme, October 1872; Zur Invariantentheorie der Berührungstransforma-tionen, December 1872; Ges. d. Wiss., Christiania[(2)]). I represent these theories in detail in the same journal in 1873–February 1875; cf. Math. Ann. Bd. VIII and XI[(3)].

There is no need to cite my numerous publications on the same subject that followed. I would like only to point out that the most important results are summarized in Math. Ann., Bd. XXIV and XXV[(4)].

It is typical for all my investigations that I do not restrict myself by deriving known integrations but I rather give in each case all possible reductions. It should be noted of course, that my proof of impossibility of further reductions is not yet exhaustive.

11. Further I refer to this direction remarkable works of Laguerre and Halphen on transformations of ordinary linear differential equations. This investigations in fact deal with the infinite group:

$$x_1 = \varphi(x), \quad y_1 = y\chi(x),$$

which is mentioned by neither of the authors of course[(5)]. I think that Laguerre (1879) and Halphen (1882) did not know my theory of invariants of contact transformations and my integration theories based on it, otherwise they would certainly mention many similarities between the two theories*.

I should also mention that already in 1870 the English mathematician Cockle developed ideas on linear differential equations having, in particular, connection with Laguerre's ideas.

Finally, I refer to this direction a series of new investigations by Picard and Vessiot, importance of which I already pointed out several times.

*The fact that Halphen was acquainted with my earlier works on curves and surfaces with infinitely many projective transformations follows from quotes in his dissertation (1879). However, the fact that at first he was not aware of significance of my integration theories follows from publication in 1879–1881 of his integration theories which have a very special character. Moreover, his theories are incomplete because he does not minimize the order and number of operations necessary for integration.

12. If the historical exposition given here is correct, I can claim that I have been the first to use the concept of groups in the integration theory of differential equations.

Within Fuchs' direction the concept of groups, namely discrete groups, was used for the first time by C. Jordan in 1874. On the other hand my theory of groups of functions goes back to 1872, whereas the origin of my integration theory of a complete system with a known finite or infinite group can be traced even earlier.

13. Of course I am well aware of many important investigations on differential equations which can be referred to neither of the four above directions. They involve Briot and Bouquet's function-theoretic investigations, further works by Darboux on algebraic differential equations as well as valuable investigations initiated by Bruns and Poincaré on the three body problem and several other function-theoretic investigations.

The above historical comments do not lay claim to be complete. However since the number of investigations on differential equations, especially during the last decades, increases so rapidly it seems reasonable enlighten the relations between all these investigations. I will consider this issue in detail elsewhere when I finish studying the necessary bibliography.

14. These diverse directions have numerous points of contact of a highest interest. My own main endeavor is to apply the notion of continuous groups to the first three directions.

I have already shown in my earlier papers that it is natural to consider all the theory of first-order partial differential equations from a group-theoretic point of view.

In the present paper I am trying to reduce the theory of partial differential equations of an arbitrary order to the theory of first-order partial differential equations as much as possible and thus to make the general theory open for the group-theoretic approach. I investigate this direction further in my next paper, reported to the Scientific Society of Christiania in the beginning of the last year.

§ 2. Theory of characteristics of partial differential equations

15. It follows from Lagrange's theory of complete integrals of a partial differential equation of the first order

$$F(x, y, z, p, q) = 0$$

that, as Monge pointed out, all integral surfaces of the equation which contact each other in a point, have a general curve and contact each other along this curve. Monge called these curves characteristics*. He discovered that every equation $F = 0$ has (at most) ∞^3 characteristics, and that every integral surface contains ∞^1 characteristics. Whence, he made a conclusion that one can draw only one integral surface through every non-characteristic curve.

Monge extended the notion of characteristics to partial differential equations in x, y, z of an arbitrary order, although not in a precise form.

16. This extension for equations of the second order

$$F(x, y, z, p, q, r, s, t) = 0$$

looks approximately as follows.

Monge designated the curves satisfying the equation

$$\frac{\partial F}{\partial r} dy^2 - \frac{\partial F}{\partial s} dy dx + \frac{\partial F}{\partial t} dx^2 = 0$$

as characteristics. Thus, every integral surface contains ∞^1 Monge's characteristics which split into two different families, and hence cover the surface twice, provided that the expression

$$4\frac{\partial F}{\partial r}\frac{\partial F}{\partial t} - (\frac{\partial F}{\partial s})^2$$

does not vanish.

17. These curves are most readily defined indirectly. Namely, if one draws a non-characteristic curve on an arbitrary integral surface, then there is no other integral surface osculating the given surface along this curve.

Likewise the notion of characteristics extends to arbitrary partial differential equations in the variables x, y, z.

18. This approach lead Monge and Ampére to integration theories which reduce any partial differential equation of the form

$$A(rt - s^2) + Br + Cs + Dt + E = 0 \tag{1}$$

to an ordinary differential equation.

Namely, Monge and Ampére discovered that it is possible to determine three linear differential equations

$$a_k dx + b_k dy + c_k dz + d_k dp + e_k dq = 0 \qquad (k = 1, 2, 3) \tag{1'}$$

*Monge's chief contribution to the theory of differential equations is that he made this discipline approachable from a conceptual viewpoint by introducing elementary notions. He and his contemporaries lacked free treatment of the imaginaries, the concept of n-dimensional spaces as well as some function-theoretic rigor.

which hold for every characteristic of one family. The similar total system, satisfying the characteristics of the second family, is not considered here.

One can encounter several different significant cases. Together with Monge and Ampére we consider particularly the case when the total system $(1')$ is integrable, i.e. we assume that there exist two and only two independent integrals

$$u(x, y, z, p, q), \quad v(x, y, z, p, q).$$

Both u and v have constant values on every integral surface along a characteristic. However, the values of u and v, in general, vary while passing to another characteristic on the same surface. Consequently, u and v are connected, on every individual integral surface, by the relation

$$v - \varphi(u) = 0.$$

The form of the relation is naturally not always the same for different integral surfaces.

Since u and v are given functions of x, y, z, p, q, the equation

$$v - \varphi(u) = 0$$

is a partial differential equation of the first order. Specifically, the equation

$$v - \varphi(u) = 0$$

represents infinitely many first-order partial differential equations for φ is an arbitrary function.

19. Consequently, the common integral surfaces of the above Monge–Ampére equation of the second order are found by determining the solutions of all first-order equations

$$v - \varphi(u) = 0$$

and singling out those solutions that satisfy the above second-order equation.

It is readily seen now that there exists one and only one second-order partial differential equation which is satisfied by integral surfaces of all equations

$$v - \varphi(u) = 0.$$

Indeed, both equations

$$\frac{dv}{dx} - \varphi'(u)\frac{du}{dx} = 0, \quad \frac{dv}{dy} - \varphi'(u)\frac{du}{dy} = 0$$

obtained by differentiation yield upon eliminating $\varphi'(u)$ the single equation

$$\frac{dv}{dx}\frac{du}{dy} - \frac{dv}{dy}\frac{du}{dx} = 0,$$

which is obviously a second-order partial differential equation identical to the above Monge–Ampére equation.

20. Thus, one comes to the following beautiful result discovered by Monge and Ampére.

Theorem. *Suppose that the characteristic strips of the first order (belonging to one family) satisfy the equations of the form*

$$u(x, y, z, p, q) = a = \text{const.}, \quad v(x, y, z, p, q) = b = \text{const.}$$

on integral surfaces of the Monge–Ampére equation

$$Ar + Bs + Ct + D + E(rt - s^2) = 0.$$

Then all integral surfaces are obtained by determining the general intermediate integral:

$$v - \varphi(u) = 0$$

and defining the corresponding integral surfaces of these first order partial differential equations.

21. Let us consider the Monge–Ampére equation (1) again and find its integral surface containing a given curve $x = X(\tau), y = Y(\tau), z = Z(\tau)$ and a given developable surface along this curve. To solve this problem we first express both quantities $u(x, y, z, p, q)$ and $v(x, y, z, p, q)$ as functions of τ by substituting there the equations of the curve and the corresponding relations $p = P(\tau), q = Q(\tau)$. Then we eliminate τ and consider the resulting relation $v - \varphi(u) = 0$ as a first-order partial differential equation.[(6)] Finally, we find the integral surface of this first-order equation containing the given curve.

We do not dwell here on simplification of the integration procedure possible in particular cases.

22. Let us take now an arbitrary partial differential equation of the second order

$$F(x, y, z, p, q, r, s, t) = 0.$$

The characteristic strips of the second order* satisfy a total system of six linear differential equations of the form

$$\left\{ \begin{aligned} a_k dx + b_k dy + c_k dz + d_k dp + e_k dq + \\ + f_k dr + g_k ds + h_k dt = 0 \end{aligned} \right. \quad (k = 1, 2, 3, \ldots, 6) \tag{2}$$

*I call a characteristic a characteristic strip of the first or second, … order when I take into account not only the values of x, y, z along the characteristic but also the values of the derivatives $p, q, r, s, t \ldots$.

for all integral surfaces.

Now let us suppose together with Darboux that this total system is integrable and admits precisely two independent integrals

$$u(x, y, z, \dots, r, s, t), \quad v(x, y, z, \dots, r, s, t).$$

Then, the values of u and v on every integral surface remain unaltered along a characteristic (of one family), whereas they vary, in general, when shifting from one characteristic to another on the same integral surface. Whence Darboux makes a conclusion that every integral surface of $F = 0$ satisfies a definite equation of the form

$$v - \varphi(u) = 0.$$

Since the form of the function φ is given, this new equation is a second-order partial differential equation which has common integral surfaces with $F = 0$.

23. Although it was not Darboux's concern, we would like to find an integral surface of $F = 0$, that contains a given curve

$$y = Y(x), \quad z = Z(x),$$

and contacts a given developable surface along this curve. In other words, given a strip

$$y = Y(x), \quad z = Z(x), \quad p = P(x), \quad q = Q(x),$$

we have to find the corresponding integral surface $F = 0$.

In order to determine the quantities of r, s, t as functions of x for all points of the given curve, it suffices, in general, to use the equations

$$P' = r + sY', \quad Q' = s + tY'$$

together with $dF = 0$. As it has already been noted by Monge and Cauchy this procedure is ineffective if the determinant

$$\begin{vmatrix} 1 & Y' & 0 \\ 0 & 1 & Y' \\ F_r & F_s & F_t \end{vmatrix} = F_r Y'^2 - F_s Y' + F_t$$

vanishes in all points of the given curve, i.e. when the curve is a characteristic. Bearing in mind this exception, and using the above reasoning one can determine not only r, s, t, but also third- and higher-order derivatives along the curve as functions of x. Using the equations

$$y = Y(x), \quad z = Z(x), \quad P = P(x), \quad q = Q(x),$$

$$r = R(x), \quad s = S(x), \quad t = T(x),$$

we express $u(x, \ldots, t)$ and $v(x, \ldots, t)$ as functions of x :

$$u = U(x), \quad v = V(x)$$

along the curve. Whence eliminating x we obtain a certain relation between u and v :

$$v - \varphi(u) = 0.$$

Since firstly, as it is known from Cauchy's general theory that given initial conditions determine an integral surface of $F = 0$, secondly, there certainly exists one and only one equation $v - \psi(u) = 0$ which holds for all points of our integral surface, and finally we already know that the equation $v - \varphi(u) = 0$ obtained above holds for ∞^1 points of our integral surface, we conclude that the equation $v - \varphi(u) = 0$ holds for all points of our surface. We will show later how this considerations allow one to determine the integral surface.

24. Darboux also noticed that the linear differential equations (2) possess a square root and therefore they represent two total systems in fact. He restricted himself to the case when both of these total systems have two independent integrals

$$u_1, v_1 \quad \text{and} \quad u_2, v_2.$$

Consequently, every integral surface of $F = 0$ satisfies definite equations $v_1 - \varphi_1(u_1) = 0$ and $v_2 - \varphi_2(u_2) = 0$ simultaneously.

Conversely, Darboux showed that for any given functions φ_1 and φ_2 the following three second-order partial differential equations:

$$F = 0, \quad v_1 - \varphi_1(u_1) = 0, \quad v_2 - \varphi_2(u_2) = 0$$

furnish an unboundedly integrable system[(7)]; its ∞^3 common integral surfaces are determined by means of ordinary differential equations.

25. This is the crux of the Darboux theory, which as he mentioned himself can be extended to several directions.

He paid a special attention to the case of a second-order equation when both of its total systems of characteristic strips of the first order have two integrals, u_1, v_1 and u_2, v_2, respectively. He pointed out that in this case the system of equations

$$F = 0, \quad v_1 - \varphi_1(u_1) = 0, \quad v_2 - \varphi_2(u_2) = 0$$

is always unboundedly integrable. In this manner Darboux integrates by means of ordinary differential equations all partial differential equations common integral of which belongs to Ampére's first class.

26. Unfortunately, Darboux gave only an outline of his valuable and far-reaching investigations. Therefore, his publications by no means contain the full utilization of his new ideas.

M. Lévy* amplified Darboux's theory considerably by an important although obvious remark.

Lévy treated a second-order partial differential equation $F = 0$ which has common integral surfaces with an equation of the second or higher order. The analytic representation of these surfaces does not depend on arbitrary constants alone. He discovered that these common integral surfaces possess characteristic strips which are determined by a simultaneous system of ordinary differential equations.

Specifically, Lévy discovered that if there are two such second-order equations, then all common integral surfaces osculating each other in one point always osculate each other along a curve which represents a characteristic for both differential equations.

Whence, he made a conclusion that a second-order equation $F = 0$ can be dealt with by integration of a simultaneous system of ordinary differential equations, provided that at least one of the two total systems for characteristic strips, e.g. of the m-th order, possesses two independent integrals u, v. In this way Lévy deals with second–order partial differential equations, general integral of which does not belong to Ampére's first class.

Lévy indicated that one has to integrate *three* systems of ordinary differential equations.

27. However, in 1874 I noticed that it is always sufficient to integrate *two* simultaneous systems of ordinary differential equations under Lévy's assumptions. Namely, upon obtaining two quantities u and v by integrating the repeatedly mentioned total system, one can pose oneself the problem of finding such an integral surface of $F = 0$ which contains a given curve and contacts a given developable surface along it. These initial conditions provide four relations (cf. p. 10 and further)

$$y = Y(x), \quad z = Z(x), \quad p = P(x), \quad q = Q(x),$$

which determine the values of y, z, p, q as functions of x along the curve. Furthermore, it follows from the above procedure that the functions Y, Z, P, Q satisfy the relation

$$Z' - P - QY' = 0.$$

Let us denote the values of r, s, t along the curve by $R(x), S(x), T(x)$. They are determined by the relations

$$P' = R - SY' = 0, \quad Q' - S - TY' = 0,$$

*Comptes Rendus 1872.

$$F(x, Y, Z, P, Q, R, S, T) = 0,$$

which, in general, are solvable with respect to R, S and T. If we assume for the sake of simplicity that u and v contain only derivatives of the first and second order, we readily define u and v as functions of x

$$u = U(x), \quad v = V(x),$$

whence, upon eliminating x we obtain a relation

$$v - \varphi(u) = 0.$$

According to Darboux, this equation has common integral surfaces with the equation $F = 0$, that do not depend on arbitrary constants only. The corresponding characteristic strips can be obtained, as Lévy pointed out, by means of integration of a simultaneous system. One singles out among these strips those ∞^1 which have a common system of values x, y, z, p, q, r, s, t with the surface-strips

$$y = Y, \quad z = Z, \quad p = P, \quad q = Q, \quad r = R(x), \quad s = S(x), \quad t = T(x).$$

This completes the determination of the desired integral surface* for $F = 0$.

28. I present here the considerations that assured me initially that Lévy's statements are correct.

First let us take a first-order partial differential equation $F(x, y, z, p, q) = 0$. If we consider all integral surfaces of this equation that have a common element x, y, z, p, q, then the corresponding values of r, s, t satisfy two linear equations. Now one takes the equation

$$0 = \frac{1}{2}r(X - x)^2 + s(X - x)(Y - y) + \frac{1}{2}t(Y - y)^2 + \dots,$$

which represents the Dupin indicatrix of all these surfaces. These second-order curves furnish a bundle comprising ∞^1 concentric conic sections that contact each other in two points.

This observation leads directly to the result fundamental for the Laplace and Monge integration theory of the equation $F(x, y, z, p, q) = 0$, namely to the following theorem.

If two integral surfaces of an equation $F(x, y, z, p, q) = 0$ have a common element x, y, z, p, q, then they also have a common neighboring element.

On the other hand, let us assume together with Darboux and Lévy that there exist two second-order partial differential equations

$$r - f(x, y, z, p, q, s) = 0, \quad t - \varphi(x, y, z, p, q, s) = 0$$

*Lie, Verhandl. der Ges. der Wissensch. zu Christiania 1874, p. 247[(8)].

which have ∞^∞ common integral surfaces and hence meet the condition $f'\varphi' - 1 = 0$, then one immediately arrives at the conclusion that the third-order indicatrix-curves of these surfaces:

$$\alpha(X-x)^3 + 3\beta(X-x)^2(Y-y) + 3\gamma(X-x)(Y-y)^2 + \delta(Y-y)^3 + \ldots = 0$$

are ∞^1 osculating each other curves of the third-order.

Whence it follows immediately that all common integral surfaces osculating each other at one point, osculate each other along a curve.

Chapter 2

Addition to the Monge's theory of characteristics

29. Understanding the above Monge, Amperé, Darboux and Lévy's theories required a lot of independent work due to their short representation, especially of Lévy's investigations. Then I immediately noticed that Darboux's new ideas did not enjoy a full application in Lévy's works. Subsequently, I outlined several methods and completed the theory of characteristics of my predecessors considerably in my report to the Scientific Society of Christiania of February 1880[(9)].

Since in this chapter I am going give a detailed representation of my old theories I consider it reasonable to explain my usual terminology.

We say that a system of two second-order partial differential equations

$$F_1(x, y, z, p, q, r, s, t) = 0, \quad F_2(\ldots) = 0 \tag{3}$$

is *unboundedly integrable* if these equations possess common integral surfaces which do not satisfy any other differential equations of the second or first order.

However, according to Darboux's general theory, two and only two cases are conceivable, namely when the equations possess ∞^4 or ∞^∞ common integral surfaces, respectively. In the latter case I say that our unboundedly integrable system of second-order partial differential equations constitutes an *involutory system*, or a *Darboux system*.

Accordingly, I say that a system of partial differential equations of the m-th order is *unboundedly integrable* if the equations possess common integral manifolds which satisfy no other equation of the m-th or lower order. Further, I call an unboundedly integrable system of differential equations a *Darboux system* if the common integral manifolds do not depend on arbitrary constants only. In particular, I call a Darboux system an *involutory* *system* if the number of common integral manifolds has its maximum value.

In the special case (3) that we considered first every Darboux system is an involutory one. However, this is not always true. As a rule there are several different classes of unboundedly integrable systems that are called Darboux systems in my terminology. Among these classes the involutory systems represent a separate class which is to be considered as the most important one.

30. Now let us obtain Lévy's result again and consider the unboundedly integrable system of the second order:

$$F_1(x, y, z, p, q, r, s, t), \quad F_2(\ldots) = 0.$$

If we select ∞^3 surfaces among all integral surfaces of the system, we can arrange these ∞^3 surfaces in infinitely many ways into ∞^1 families approximately defined by the equation

$$\Phi(x, y, z, a, b) = c.$$

Here a, b, c are arbitrary constants, c having a fixed value for all surfaces of one family.

Eliminating the parameters a and b from the equation $\Phi = c$ and its differential consequences:

$$\Phi_x + \Phi_z p = 0, \quad \Phi_y + \Phi_z q = 0,$$

one obtains a first-order partial differential equation

$$V(x, y, z, p, q) = c.$$

For every value of the parameter c, the latter equation has at least ∞^2 common integral surfaces with the equations $F_1 = 0, F_2 = 0$. Thus one arrives at the following problem. Given an unboundedly integrable system $F_1 = 0, F_2 = 0$, find all equations $V(x, y, z, p, q) = c$ having at least ∞^2 common integral surfaces with $F_1 = 0$ and $F_2 = 0$ for every value of c. This requirement is entailed by the fact that the four second-order equations

$$F_1 = 0, \quad F_2 = 0,$$

$$V_x + V_z p + V_p r + V_q s = 0, \quad V_y + V_z q + V_p s + F_q t = 0$$

must have at least ∞^3 common integral surfaces.

Eliminating r, s and t from these equations, we obtain one or sometimes two equations of the form

$$\Omega(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0$$

homogeneous with respect to derivatives of V.

31. Furthermore, we can certainly assume that the equations $F_1 = 0$ and $F_2 = 0$ have the following form*:

$$r + R(x, y, z, p, q, s) = 0, \quad t + T(\ldots) = 0.$$

Differentiation of both equations with respect to x and y leads to the relations

$$\begin{aligned} \alpha + R_s\beta + \qquad\qquad\qquad & \ldots = 0,\\ \beta + \quad R_s\gamma + \qquad & \ldots = 0,\\ T_s\beta + \qquad \gamma + \qquad & \ldots = 0,\\ T_s\gamma + \quad \delta + & \ldots = 0. \end{aligned}$$

for determining the third-order derivatives[10] $\alpha, \beta, \gamma, \delta$.

If the determinant

$$\begin{vmatrix} 1 & R_s & 0 & 0 \\ 0 & 1 & R_s & 0 \\ 0 & T_s & 1 & 0 \\ 0 & 0 & T_s & 1 \end{vmatrix} = 1 - T_s R_s \equiv 1 - T'R'$$

does not vanish, our unboundedly integrable system has precisely ∞^4 integral surfaces. If the determinant vanishes identically, the system $F_1 = 0, F_2 = 0$ is a Darboux system or obviously an involutory system.

32. Elimination of r, s and t from the four equations

$$r + R = 0, \quad t + T = 0,$$

$$V_x + V_z\, p + V_p\, r + V_q\, s = 0, \quad V_y + V_z\, q + V_p\, s + V_q\, t = 0$$

yields one or two first-order partial differential equations determining V as a function of the five variables x, y, z, p, q. The number of the equations depends on whether the third-order determinants

$$V_q - R'V_p, \quad T'V_q - V_p, \quad V_q(V_q - R'V_p), \quad V_p(T'V_q - V_p)$$

of the matrix

$$\begin{Vmatrix} 1 & R' & 0 \\ 0 & T' & 1 \\ V_p & V_q & 0 \\ 0 & V_p & V_q \end{Vmatrix}$$

*Our system can always be reduced to this form by means of an appropriate contact transformation.

vanish identically or not. One can verify that the equations $V_q - R'V_p = 0$ and $T'V_q - V_p = 0$ imply $1 - R'T' = 0$. Hence, the system of equations $r + R = 0, t + T = 0$ is involutory.

However, in this case the quantities

$$R' = \frac{V_q}{V_p}, \quad T' = \frac{V_p}{V_q}$$

are functions of x, y, z, p, q, and R and T have the form

$$R = s\frac{V_q}{V_p} + m(x, y, z, p, q) = as + m$$

$$T = s\frac{V_p}{V_q} + n(x, y, z, p, q) = \frac{1}{a}s + n.$$

Consequently we obtain the equations[11]

$$V_x + V_z p - V_p\left(s\frac{V_q}{V_p} + m\right) + V_q s = 0,$$

$$V_y + V_z q + V_p s - V_q\left(s\frac{V_p}{V_q} + n\right) = 0.$$

Since the terms with s in these equations cancel out, V must satisfy the first-order partial differential equations

$$V_x + V_z p - mV_p = 0, \quad V_y + V_z q - nV_q = 0,$$

$$V_q - aV_p = 0.$$

It immediately follows that $V = c$ is a common intermediate integral of equations $F_1 = 0, F_2 = 0$ and we can disregard this case here.

33. Thus, let us consider an unboundedly integrable system of two second-order partial differential equations

$$F_1(x, y, z, p, q, r, s, t) = 0, \quad F_2(\ldots) = 0.$$

Here the following three cases are possible.

Our equations can have a common intermediate integral

$$W(x, y, z, p, q) = c$$

obtained by integration of a first-order ordinary differential equation. In this case the system $F_1 = 0, F_2 = 0$ is equivalent to the equations

$$\frac{dW}{dx} = 0, \qquad \frac{dW}{dy} = 0.$$

Furthermore, it is possible that the equations $F_1 = 0, F_2 = 0$ have exactly ∞^4 integral surfaces defined by integration of a fourth-order ordinary differential equation.

Finally, one can consider the case when $F_1 = 0, F_2 = 0$ have ∞^∞ common integral surfaces among which, however, there exist no more than ∞^2 surfaces satisfying one and the same first-order partial differential equation.

34. These three cases are distinguished by the following criterion.

One brings the equations $F_1 = 0, F_2 = 0$ to the form

$$r + R(x, y, z, p, q, s) = 0, \quad t + T(\ldots) = 0$$

and considers the equations

$$V_x + V_z p - V_p R + V_q s = 0, \quad V_y + V_z q + V_p s - V_q T = 0. \tag{4}$$

If s falls out from these two linear partial differential equations, then this is the first case. In this case R and S have the forms

$$R = a(x, y, z, p, q,)s + m(x, y, z, p, q), \quad S = \frac{1}{a}s + n(x, y, z, p, q),$$

and three linear partial differential equations

$$V_q - aV_p = 0, \quad V_x + pV_z - mV_p = 0, \quad V_y + qV_z - nV_q = 0$$

together with the fourth equation, obtained via the Poisson bracket, define a complete system. This system has one and only one solution $V = W(x, y, z, p, q)$. It follows that $W = c$ is the desired intermediate integral.

If the system (4) does not meet the above criterion, then elimination of s from (4) leads to a single first-order partial differential equation for $V(x, y, z, p, q)$:

$$\Omega(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0,$$

homogeneous with respect to the derivatives $V_x, \ldots$.

Given an arbitrary solution $V(x, y, z, p, q)$ of $\Omega = 0$, the equations

$$F_1 = 0, \quad F_2 = 0, \quad V = c$$

provide an unboundedly integrable system with ∞^2 common integral surfaces for every value of the constant c. Obviously, there always exist ∞^∞ different equations $V = c$ defining ∞^2 integral surfaces for the system $F_1 = 0, F_2 = 0$.

However it should be noted, that one can by no means make a conclusion from the above that the mentioned equations $F_1 = 0$ and $F_2 = 0$ have ∞^∞ common integral surfaces. This happens if and only if the quantities $R_s T_s - 1$

vanish identically. Otherwise the equations $F_1 = 0, F_2 = 0$ have only ∞^4 common integral surfaces.

35. Thus, given an unboundedly integrable second-order system:

$$F_1(x, y, z, p, q, r, s, t) = 0, \quad F_2(\ldots) = 0 \tag{5}$$

it is always reasonable to look for a first-order partial differential equation

$$V(x, y, z, p, q) = c$$

satisfied by at least ∞^2 common integral surfaces of the system. Eliminating r, s, t from the equations $F_1 = 0, F_2 = 0$ and

$$V_x + V_z p + V_p r + V_q s = 0, \quad V_y + V_z q + V_p s + V_q t = 0$$

one arrives at a single first-order partial differential equation

$$\Omega(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0,$$

for determining V as a function of x, y, z, p, q. This is possible if we ignore the case when $F_1 = 0$ and $F_2 = 0$ have a common intermediate integral.

We intend to demonstrate that the partial differential equation $\Omega = 0$ is always *semilinear**. We show further that if

$$z = f(x, y)$$

is a common integral surface for both equations $F_1 = 0, F_2 = 0$, then the three equations

$$z = f(x, y), \quad p = f_x, \quad q = f_y$$

define a two-dimensional manifold in the five-dimensional space x, y, z, p, q, the manifold being in my terminology a proper integral manifold of the first-order partial differential equation $\Omega = 0$.

36. Indeed, let

$$z = f(x, y, a, b, c)$$

be the equation for ∞^3 common integral surfaces of both equations $F_1 = 0$ and $F_2 = 0$. Then there exist infinitely many functions $V(x, y, z, p, q)$ satisfying two second-order partial differential equations

$$V_x + V_z p + V_p r + V_q s = 0, \quad V_y + V_z q + V_p s + V_q t = 0$$

upon substitution $z = f(x, y, a, b, c)$ (cf. p. 17 and further).

*Cf. my paper in Göt. Nachrichten, October 1872, p. 480 and further[(12)].

Let us solve the equations $z = f, p = f_x, q = f_y$ with respect to a, b, c :

$$a = A(x, y, z, p, q), \quad b = B(x, y, z, p, q), \quad c = C(x, y, z, p, q)$$

and introduce the notation

$$\Phi(x, y, A, B, C) \equiv \Phi(x, y, a, b, c).$$

One can see now that the functions V and f satisfy the following equations identically:

$$\begin{cases} V_x + V_z p + V_p f_{xx} + V_q f_{xy} \equiv 0, \\ V_y + V_z q + V_p f_{xy} + V_q f_{yy} \equiv 0. \end{cases} \tag{6}$$

However the equations

$$F_1(x, y, z, p, q, f_{xx}, f_{xy}, f_{yy}) = 0, \quad F_2(\ldots) = 0 \tag{7}$$

are also satisfied identically. Eliminating f_{xx}, f_{xy}, f_{yy} from the equations (6) and (7) one obtains the following equation:

$$\Omega(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0.$$

The only assumption regrading the function V is that each of the ∞^1 equations $V = c$ has at least ∞^2 common integral surfaces with the equations $F_1 = 0$ and $F_2 = 0$.

37. Let us assume that the equation $z = f(x, y, a, b, c)$ provides ∞^3 common integral surfaces for the equations $F_1 = 0, F_2 = 0$ and use the notation

$$\Psi(x, y, A, B, C) \equiv \Psi.$$

Analytically, our assumption means that the equations

$$F_1(x, y, z, p, q, f_{xx}, f_{xy}, f_{yy}) = 0, \quad F_2(\ldots) = 0 \tag{7}$$

are satisfied identically.

The equations

$$z = f(x, y, a, b, c), \quad p = f_x, \quad q = f_y$$

define ∞^3 two-dimensional point manifolds in the five-dimensional space x, y, z, p, q, each of them having ∞^4 elements with the coordinates*

$$x, y, z, p, q, V_x : V_y : V_z : V_p : V_q$$

*Math. Ann. Bd. IX, pp. 250–251[(13)].

determined by the following five equations:

$$z = f, \quad p = f_x, \quad q = f_y,$$

$$V_x + V_z f_x + V_p f_{xx} + V_q f_{xy} = 0, \quad V_y + V_z f_y + V_p f_{xy} + V_q f_{yy} = 0.$$

These ∞^3 point manifolds possess $\infty^3 \cdot \infty^4 = \infty^7$ elements which are singled out from ∞^9 elements of the five-dimensional space by the equations

$$V_x + V_z p + V_p f_{xx} + V_q f_{xy} = 0, \quad V_y + V_z p + V_p f_{xy} + V_q f_{yy} = 0. \qquad (6)$$

38. Invoking our previous considerations, let the partial differential equations $F_1(x, \ldots, t) = 0, F_2 = 0$ have more than ∞^3 common integral surfaces. Then both equations (6) may have ∞^∞ different forms. However, it is readily seen, that our ∞^7 elements in a five-dimensional space satisfy an equation having a definite form, namely the equation

$$\Omega(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0$$

obtained by elimination of f_{xx}, f_{xy}, f_{yy} from the equations (7) and(6).

This proves our previous statement that the first-order partial differential equation $\Omega(\ldots) = 0$, considered in the space x, y, z, p, q, is satisfied by every two-dimensional point manifold $z = f, p = f_x, q = f_y$ provided that the equation $z = f$ represents in the three-dimensional space x, y, z a surface satisfying both equations $F_1 = 0$ and $F_2 = 0$.

39. If the unboundedly integrable system $F_1 = 0, F_2 = 0$ in the space x, y, z has only ∞^4 integral surfaces $z = \varphi(x, y, a, b, c, d)$ the result is trivial. Namely, the equations

$$z = \varphi(x, y, a, b, c, d), \quad p = \varphi_x, \quad q = \varphi_y$$

determine in the space x, y, z, p, q exactly ∞^4 two-dimensional point manifolds, whose $\infty^4 \cdot \infty^4 = \infty^8$ elements

$$x, y, z, p, q, V_x : V_y : V_z : V_p : V_q$$

obviously satisfy one and only one equation

$$\Theta(x, y, z, p, q, V_x, V_y, V_z, V_p, V_q) = 0.$$

Thus, every ∞^4 point manifold provides, according to my terminology, a complete solution of the first-order partial differential equation $\Theta = 0$.

40. The obtained result is of special interest since two partial differential equations $F_1 = 0, F_2 = 0$ compose an involutory system of equations if they have ∞^∞ common integral surfaces $z = f$.

Then, as it is known from the theory of first-order partial differential equations, every proper integral manifold, i.e. every integral manifold that has the maximum possible number of elements, is generated by characteristic strips. In particular, it is known that integral manifolds of the equation $\Omega = 0$ are generated by the available ∞^7 characteristic strips and that every integral manifold contains ∞^3 characteristic strips.

We claim that the *locus* of a strip in a five-dimensional space is a curve, not a point. In other words, we claim that the quantities x, y, z, p, q can have constant values not for every characteristic strip. This immediately follows from the fact that at least some of the five values V_x, V_y, V_z, V_p, V_q appear in $\Omega = 0$.

Moreover, the three values x, y, z can also have constant values not for every characteristic strip. This is provided by the fact that the number of existing two-dimensional integral manifolds is at least ∞^4 and that the set of all two-dimensional manifolds does not satisfy any first-order partial differential equation other than $\Omega = 0$ so that every characteristic strip belongs at least to one two-dimensional manifold $z = f, p = f_x, q = f_y$. It follows that if x, y, z or only x, y have constant values for a characteristic strip, then z, p, q are also constant for the same strip.

Then, ∞^7 characteristic strips in five-dimensional spaces are represented by $\Omega = 0$ and by seven additional equations involving the quantities

$$x, y, z, p, q, V_x : V_y : V_y : V_z : V_p : V_q,$$

and containing *seven* arbitrary constants. These equations yield two and only two equations in x, y, z with arbitrary constants:

$$\varphi_1(x, y, z, c_1, \ldots, c_7) = 0, \quad \varphi_2(\ldots) = 0.$$

However, it follows from the above that in the three-dimensional space all the surfaces $z = f$ are generated by infinitely many curves belonging to the family $\varphi_1 = 0, \varphi_2 = 0$.

41. It remains only to prove that the family of curves $\varphi_1 = 0, \varphi_2 = 0$ contains not ∞^7, but ∞^5 curves only and hence the number of characteristic curves in five-dimensional spaces is also exactly ∞^5, although the number of characteristic strips in the same spaces is ∞^7*.

*I paid attention to first-order partial differential equations with the number of characteristic strips larger than that of characteristic curves already in my first work on the general theory of partial differential equations (Scientific Society of Christiania 1872, Kurzes Résumé... (14)).

I will dwell upon important relations between the notions of *Darboux system* and *involutory system* elsewhere.

Namely, since the number of available two-dimensional integral manifolds of the equation $\Omega = 0$ is expressed by the symbol ∞^∞, then every integral manifold can contain only ∞^1 characteristic curves. It follows that every characteristic curve in five-dimensional spaces represents the locus of ∞^2 characteristic strips.

Further, it follows that only ∞^1 characteristic curves pass through every point of a five-dimensional space. These curves satisfy three non-linear equations of the form

$$\varphi(x, y, z, p, q, dx, dy, dz, dp, dq) = 0.$$

Given an arbitrary curve satisfying these three equations one constructs ∞^1 contacting characteristics and obtains a two-dimensional proper integral manifold of the equation $\Omega = 0$ and simultaneously an integral surface of the involutory system $F_1 = 0, F_2 = 0$.

This, I believe, provides a proper generalization of Lévy's theory and discloses its very essence.

42. Consider now an unboundedly integrable system of three first-order equations:

$$\mathcal{F}_1(x, y, z_1, z_2, p_1, q_1, p_2, q_2) = 0, \quad \mathcal{F}_2 = 0, \quad \mathcal{F}_3 = 0, \tag{A}$$

where z_1 and z_2 are functions of x, y. Any solution

$$z_1 = f(x, y), \quad z_2 = \varphi(x, y)$$

to this system defines a two-dimensional manifold in the four-dimensional space x, y, z_1, z_2.

Since the system $\mathcal{F}_1 = 0, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0$ is unboundedly integrable, there exist at least ∞^3 such two-dimensional integral manifolds. If the number of these manifolds is precisely ∞^3, then it follows immediately from my general theory that there exists a definite *semilinear* first-order partial differential equation,

$$\Phi\left(x, y, z_1, z_2, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z_1}, \frac{\partial V}{\partial z_2}\right) = 0,$$

for which these ∞^3 manifolds are solutions in my sense and moreover they provide a complete solution. However, it seems unexpected at first that our integral manifolds $z_1 = f, z_2 = \varphi$ furnish solutions of a partial differential equation $\Phi = 0$ if their number is more than ∞^3.

43. Let us assume that the involutory system $\mathcal{F}_1 = 0, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0$ contains, inter alia, ∞^2 integral manifolds not satisfying any parameter-free finite relation $\psi(x, y, z_1, z_2,) = 0$. These manifolds satisfy two finite equations in x, y, z_1, z_2 with two parameters a and b. Let us bring one of these relations to the form

$$V(x, y, z_1, z_2, a) - b = 0$$

and differentiate it with respect to x and y :

$$V_x + V_{z_1} p_1 + V_{z_2} p_2 = 0, \quad V_y + V_{z_1} q_1 + V_{z_2} q_2 = 0.$$

Elimination of p_1, p_2, q_1, q_2 from these equations and $\mathcal{F}_1 = 0, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0$ yields a relation

$$\Omega(x, y, z_1, z_2, V_x, V_y, V_{z_1}, V_{z_2}) = 0,$$

i.e. a first-order partial differential equation. We assume here without further discussion that the above relation is the only one.

Let us prove now that this partial differential equation is *semilinear* and that *every* integral manifold $z_1 = f(x, y), z_2 = \varphi(x, y)$ of our involutory system (A) solves the equation $\Omega = 0$.

44. In order to prove this let us take ∞^2 arbitrary integral manifolds

$$z_1 = f(x, y, a, b), \quad z_2 = \varphi(x, y, a, b)$$

of the involutory system (A) and note that every such manifold possesses ∞^3 *elements* with coordinates

$$x, y, z_1, z_2, V_x : V_y : V_{z_1} : V_{z_2}$$

determined, according to my general theory, by $z_1 = f, z_2 = \varphi$ together with the equations

$$V_x + V_{z_1} f_x + V_{z_2} \varphi_x = 0, \quad V_y + V_{z_1} f_y + V_{z_2} \varphi_y = 0.$$

Solving $z_1 = f, z_2 = \varphi$ with respect to a, b :

$$a = A(x, y, z_1, z_2) \quad b = B(x, y, z_1, z_2),$$

and denoting for the sake of brevity

$$\psi(x, y, A, B) \equiv \psi,$$

one obtains two first-order partial differential equations from

$$V_x + V_{z_1} f_x + V_{z_2} \varphi_x = 0, \quad V_y + V_{z_1} f_y + V_{z_2} \varphi_y = 0. \qquad \text{(B)}$$

Their solutions represent the ∞^2 above chosen integral manifolds $z_1 = f,\ z_2 = \varphi$ of the involutory system $\mathcal{F}_1 = 0, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0$.

The form of the equations (B) depends more or less upon choice of integral manifolds of the involutory system. However, it is possible to find a first-order partial differential equation of an absolutely definite form, satisfied by all integral manifolds of the involutory system.

Indeed, there exist three equations

$$\mathcal{F}_k(x, y, f, \varphi, f_x, \varphi_x, f_y, \varphi_y) = 0,$$

satisfied identically in x, y, a, and b, for which we have already selected ∞^2 integral manifolds of the involutory system. Accordingly, there exist three equations

$$\mathcal{F}_k(x, y, z_1, z_2, f_x, \varphi_x, f_y, \varphi_y) = 0$$

satisfied identically in x, y, z_1, z_2.

Let us eliminate $f_x, \varphi_x, f_y, \varphi_y$ from the system of five equations comprising the above three equations and two first-order partial differential equations (B). The resulting first-order partial differential equation

$$\Omega(x, y, z_1, z_2, V_x, V_y, V_{z_1}, V_{z_2}) = 0,$$

obviously has a definite form and is satisfied by every integral manifold of the involutory system $\mathcal{F}_1 = 0, \ldots, \mathcal{F}_3 = 0$. This proves my earlier statement.

45. We arrive at the fundamental conclusion that every integral manifold of the involutory system is generated by characteristics.

Indeed, our partial differential equation $\Omega = 0$ has ∞^5 characteristic strips in the four-dimensional space x, y, z_1, z_2. Every integral manifold of the involutory system (A) interpreted as an integral manifold of $\Omega = 0$, contains ∞^2 characteristic strips. This however does not imply that every integral manifold contains ∞^2 different characteristic curves. The fact that there can not exist ∞^∞ different two-dimensional integral manifolds of the involutory system (A) also makes it impossible.

Thus, every two-dimensional integral manifold $z_1 = f, z_2 = \varphi$ of our involutory system $\mathcal{F}_1 = 0, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0$ contains ∞^2 different characteristic strips and only ∞^1 different characteristic curves. Every such curve is the locus of ∞^1 characteristic strips.

46. Thus, the semilinear partial differential equation $\Omega = 0$ in a four-dimensional space has ∞^5 characteristic strips, but only ∞^4 characteristic *curves,* among which ∞^1 always pass through a point x, y, z_1, z_2, in general position.

It follows from the above that every point x, y, z_1, z_2 is the vertex of an elementary cone containing only ∞^1 directions of motion and thus being defined not by one, but by two Monge equations

$$\Phi_1(x, y, z_1, z_2, dx_1, dy_1, dz_1, dz_2) = 0, \quad \Phi_2 = 0.$$

We conclude further that the *locus* of all ∞^2 characteristic strips that pass through one point is a two-dimensional manifold which is eo ipso an integral manifold of the involutory system.

47. It should be noted that the above properties of the equation $\Omega = 0$ by no means result directly from the fact that this first-order partial differential equation is *semilinear.*

Given a semilinear equation in four variables x_1, x_2, x_3, x_4 :

$$\Pi(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0,$$

let us consider x as given constants and p as homogeneous plain coordinates in a three-dimensional space. Then, in general, $\Pi = 0$ represents a *ruled surface* in this three-dimensional space*. Namely, in the given case, the ruled surface reduces to a curve in a three-dimensional space.

48. Let us proceed to further conclusions.

In the x, y, z_1, z_2 space there are obviously infinitely many curves, namely ∞^∞, satisfying both Monge equations $\Phi_1 = 0, \Phi_2 = 0$. If we take an arbitrary integral curve of the equations $\Phi_1 = 0, \Phi_2 = 0$, it will contact a certain characteristic curve of the equation $\Omega = 0$ in every point. The ∞^1 characteristic curves thus obtained generate a two-dimensional point manifold which furnishes an integral manifold of the equation $\Omega = 0$ as well as of the involutory system.

This reduces integration of the involutory system

$$\mathcal{F}_1(x, y, z_1, z_2, p_1, q_1, p_2, q_2) = 0, \quad \mathcal{F}_2 = 0, \quad \mathcal{F}_3 = 0$$

to the simplest operations.

49. Consider now an arbitrary unboundedly integrable system of differential equations. We can assume that the system is of the first-order without loss of generality. Here the independent and dependent variables are $x_1, \ldots, x_n$ and $z_1, \ldots, z_m$, respectively. Further we set

$$\frac{\partial z_i}{\partial x_k} = p_{ik}.$$

Thus, the equations of our unboundedly integrable system have the form

$$F_j(x_1, \ldots, x_n, z_1, \ldots, z_m, p_{11}, \ldots, p_{mn}) = 0 \quad (j = 1, 2, \ldots, q). \tag{a}$$

Let us take n equations

$$V_{x_k} + V_{z_1} \cdot p_{1k} + V_{z_2} \cdot p_{2k} + \ldots + V_{z_m} \cdot p_{mk} = 0 \quad (k = 1, 2, \ldots, n) \tag{b}$$

*Göttinger Nachrichten. October 1872 (15).

and assume that the number q is so large that one can eliminate the mn quantities p_{ik} from $q+n$ equations (a) and (b). This yields a system of *first-order* partial differential equations for the single unknown function V :

$$\Omega_k(x_1,\ldots,x_n,z_1,\ldots,z_m,V_{x_1},\ldots,V_{x_n},V_{z_1},\ldots,V_{z_m})=0.$$

We claim that this system of first-order partial differential equations is semi-linear. Moreover we assert that a solution

$$z_1=\varphi_1(x_1,\ldots,x_n),\quad z_2=\varphi_2,\ldots,z_m=\varphi_m$$

of the original unboundedly integrable system $F_1=0,\ldots,F_q=0$ provides, in my sense, a solution of the system $\Omega_1=0,\Omega_2=0,\ldots.$

50. For the proof, we note that the equations $z_1=\varphi_1,\ldots,z_m=\varphi_m$ represent in the space $x_1,\ldots,x_n,z_1,\ldots,z_m$ an n-dimensional manifold. According to my general theory, the elements

$$x_1,\ldots,x_n,z_1,\ldots,z_m,V_{x_1},\ldots,V_{x_n},V_{z_1},\ldots,V_{z_m}$$

of this manifold are determined by n equations

$$V_{x_i}+V_{z_1}\frac{\partial\varphi_1}{\partial x_i}+\cdots+V_{x_m}\frac{\partial\varphi_m}{\partial x_i}=0.$$

We can assume that the equations $z_1=\varphi_1,\ldots,z_m=\varphi_m$ contain m arbitrary parameters $a_1,\ldots,a_m$ and that they are solvable with respect to this parameters:

$$a_i=A_i(x_1,\ldots,x_n,z_1,\ldots,z_m)=A_i(x,z).$$

Let us use the notation

$$\psi(x_1,\ldots,x_n,A_1,\ldots,A_m)\equiv[\psi(x_1,\ldots,x_n,a_1,\ldots,a_m)].$$

Then, n equations

$$V_{x_i}+V_{z_1}\left[\frac{\partial\varphi_1}{\partial x_i}\right]+\cdots+V_{z_m}\left[\frac{\partial\varphi_m}{\partial x_i}\right]=0\quad(i=1,\ldots,n)\tag{c}$$

provide a linear system of first-order partial differential equations. The equations

$$z_1=\varphi_1(x_1,\ldots,x_n,a_1,\ldots,a_m),\ldots z_m=\varphi_m$$

furnish a complete solution of this system.

51. Note that the form of equations (c) is determined not only by the form of the original equations $F_j=0$. Taking different equations $z_1=$

$\varphi_1(x,a), \ldots, z_m = \varphi_m(x,a)$ as a basis, one obtains different systems (c). However, it is possible to determine a system of first-order partial differential equations with independent variables $x_1, \ldots, x_n, z_1, \ldots, z_m$ and with one unknown function V satisfied by all manifolds $z_1 = \varphi_1, \ldots, z_m = \varphi_m$.

Indeed, the equations

$$F_j\left(x_1, \ldots, x_n, z_1, \ldots, z_m, \left[\frac{\partial\varphi_1}{\partial x_1}\right], \ldots, \left[\frac{\partial\varphi_m}{\partial x_n}\right]\right) \equiv 0$$

hold identically for all manifolds $z_1 = \varphi_1, \ldots, z_m = \varphi_m$. Eliminating $[\partial\varphi_i : \partial x_k]$ from the latter equations and from Equation (c), one obtains a system of first-order partial differential equations

$$\Omega_k(x_1, \ldots, x_n, z_1, z_m, V_{x_1}, \ldots, V_{x_n}, V_{z_1}, \ldots V_{z_m}) = 0$$

satisfied by all manifolds $z_1 = \varphi_1, \ldots, z_m = \varphi_m$. The resulting system, *if it exists at all,* has a definite form; moreover, it can be found simply by elimination.

52. Thus, the desired result is obtained and it provides the following theorem which also holds for partial differential equations of an arbitrary higher order for they can always be reduced to first-order equations.

Theorem. *Given an unboundedly integrable system of q first-order partial differential equations with n independent and m dependent variables:*

$$F_j(x_1, \ldots, x_n, z_1, \ldots, z_m, p_{11}, \ldots, p_{mn}) = 0 \qquad \left(p_{ik} = \frac{\partial z_i}{\partial x_k}\right),$$

one considers the equations

$$V_{x_i} + V_{z_1}p_{1i} + \cdots + V_{z_m}p_{mi} = 0 \quad (i = 1, \ldots, n).$$

Elimination of nm quantities p_{ik} from the above $q + n$ equations (if it is possible) yields a system of first-order partial differential equations with the unknown function V. The resulting system

$$\Omega_k(x_1, \ldots, x_n, z_1, \ldots, z_m, V_{x_1}, \ldots, V_{x_n}, V_{z_1}, \ldots, V_{z_m}) = 0$$

is satisfied by all integral manifolds $z_1 = \varphi_1, \ldots, z_m = \varphi_m$ of the equations $F_1 = 0, \ldots, F_q = 0$.

53. The question immediately arises of what is the practical significance of this theorem.

In order to answer this question we note first of all that if the above elimination is possible and the equations $\Omega_k = 0$ are obtained, then we can always find a family of point manifolds generating all integral manifolds.

The procedure requires only integration of a system of ordinary differential equations.

The precise* answer to the question is that every system $F_j = 0$ can be reduced to another unboundedly integrable system

$$F'_j(x'_1, x'_2, \ldots, x'_{n-1}, z'_1, z'_2, \ldots, p'_{11}, \ldots) = 0$$

with *less than n independent variables* provided the system $\Omega_k = 0$ is really available.

For example, if the number n of the independent variables in the original system $F_j = 0$ equals two, then integration of the system $F_j = 0$ can be reduced to integration of ordinary differential equations provided that the system $\Omega_k = 0$ exists.

Reduction of the number of independent variables is to be considered, in general, more effective than reduction of number of dependent variables. The latter, in its turn, is to be compared with reduction of order of a system.

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*All the considerations represented in this chapter can be found, although in a short form, in my above-mentioned work of 1880[(16)].

The following significant conclusion was not at the time so clearly formulated.

Chapter 3

Every infinitesimal contact transformation of a partial differential equation generates a special integral manifold

54. An infinitesimal point or contact transformation admitted by a given partial differential equation maps, in certain cases, all integral manifolds of the equation into themselves.

Consider, for example, a linear partial differential equation with constant coefficients[17]

$$ap + bq - c = 0,$$

which, as is well known, can be rewritten as follows:

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z} = 0.$$

Its integral surfaces

$$\frac{z}{c} - \frac{x}{a} = \Omega\left(\frac{x}{a} - \frac{y}{b}\right)$$

are cylindrical surfaces with parallel generators. Each of these surfaces is transformed into itself by means of the infinitesimal translation[18]

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}\cdot$$

Furthermore, given an arbitrary linear partial differential equation[19]

$$\xi(x, y, z)p + \eta(x, y, z)q - \zeta(x, y, z) = 0,$$

every integral surface admits the infinitesimal transformation with the symbol

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}+\zeta\frac{\partial f}{\partial z}.$$

This statement is a group-theoretic formulation of Lagrange's famous theory of linear partial differential equations with three variables.

55. Turning to a non-linear partial differential equation of the first order,

$$W(x,y,z,p,q)=0,$$

one can also find infinitesimal transformations mapping its every integral surface into itself. Namely, this property holds for every infinitesimal contact transformation with the characteristic function of the form

$$\varrho(x,y,z,p,q)\cdot W,$$

where $\varrho(x,y,z,p,q)$ is supposed to be regular for the general solution x, y, z, p, q of the equation $W(x,y,z,p,q)=0$.

This observation extends to all first-order partial differential equations with an *arbitrary number of variables,* and provides a basis for all integration theories that reduce such equations to *ordinary* differential equations*.

56. However, there are infinitesimal contact transformations, which although map a partial differential equation (of the first order) into itself, and hence convert every integral manifold into an integral manifold, still do not leave all integral manifolds invariant.

This chapter deals with similar cases and in particular discusses the question of how one can benefit from existence of such transformations for integrating partial differential equations. We shall prove that such transformations can always be used to find the well-known solutions by means of a relatively simple integration procedure. These solutions are characterized by being invariant with respect to the above infinitesimal transformations(20).

The next chapter demonstrates that such transformations can be used in many other ways.

To provide a better understanding of my theory I begin with several simple but instructive examples.

57. Consider a linear partial differential equation(21)

$$\alpha(y,z)p+\beta(y,z)q-\gamma(y,z)=0, \tag{8}$$

*I have proved earlier that an infinitesimal contact transformation never maps all integral manifolds of a partial differential equation of the second or higher order into themselves.

with coefficients independent of x. If new variables x_1, y_1, z_1 are introduced by a transformation

$$x_1 = x + a, \quad y_1 = y, \quad z_1 = z \quad (a = \text{const.})$$

the form of the equation remains unaltered. Geometrically it means that every translation along the x-axis transforms every integral surface into an integral surface. In other words, the *infinitesimal* translation

$$\frac{\partial f}{\partial x}$$

converts every integral surface into an integral surface.

In this particular case our general theory allows to find the integral surfaces of equation (8) that are mapped into themselves by the infinitesimal translation $\partial f/\partial x$.

Indeed, let us make the substitution

$$z = Y(y)$$

in the linear partial differential equation (8) and obtain the following first-order ordinary differential equation:

$$\beta(y, Y)\frac{dY}{dy} - \gamma(y, Y) = 0$$

with the solution $Y(y)$ depending on one arbitrary constant.

Hence, among ∞^∞ integral surfaces of every linear partial differential equation in x, y, z admitting an infinitesimal transformation along the x-axis and thus having the form

$$\alpha(y, z)p + \beta(y, z)q - \gamma(y, z) = 0$$

there are ∞^1 cylindrical surfaces mapped into themselves by any infinitesimal translation.

58. Consider an arbitrary linear partial differential equation, e.g. with three independent variables:

$$Af = 0 = \alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y} + \gamma\frac{\partial f}{\partial z}$$

and assume that an infinitesimal transformation

$$Xf = 0 = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z}$$

mapping our equation $Af = 0$ into itself is known.

According to my general theory this assumption is represented analytically in the form

$$X(A(f)) - A(X(f)) = \varrho \cdot Af.$$

Furthermore, invoking *Jacobi* and *Bours'* general theories, we deduce form the latter equation that the following two equations

$$Af = 0, \quad Xf = 0$$

have a common solution $\varphi(x, y, z)$. If we set φ equal to an arbitrary constant we obtain ∞^1 surfaces

$$\varphi(x, y, z) = a,$$

which are generated by path curves of the infinitesimal transformation Xf and, moreover, satisfy the partial differential equation

$$\alpha p + \beta q - \gamma = 0.$$

Thus we arrive at the following theorem.

Theorem. *If an infinitesimal transformation*

$$Xf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z}$$

converts every integral surface of the linear partial differential equation

$$\alpha p + \beta q - \gamma = 0 = Af$$

into an integral surface, then there are ∞^1 integral surfaces of $Af = 0$, which are mapped into themselves by the infinitesimal transformation(22).

59. One can easily extend this theorem to *arbitrary partial differential equations* admitting a known or unknown infinitesimal *contact* transformation.

Consider a first-order partial differential equation with three variables

$$F(x, y, z, p, q) = a = \text{const.}$$

and assume that $F = a$ (for every value of a) admits the infinitesimal contact transformation

$$[Wf] - W \frac{\partial f}{\partial z} \cdot$$

The notation

$$x = x_1, \quad y = x_2, \quad z = x_3, \quad -p = p_1 : p_3, \quad -q = p_2 : p_3,$$

brings our partial differential equation to the form

$$N(x_1, x_2, x_3, p_1, p_2, p_3) = a,$$

whereas the symbol of the infinitesimal contact transformation becomes the *Poisson* bracket (Hf), where

$$H = -p_3 W(x_1, x_2, x_3, -p_1 : p_3, -p_2 : p_3).$$

Here N and H are homogeneous in p. More specifically, N is homogeneous of order *zero* and H of the *first* order.

The condition that every of ∞^1 partial differential equations $N = a$ is invariant under the infinitesimal contact transformation (Hf) is expressed analytically by the equation

$$(HN) = 0$$

holding identically. Whence, introducing the notation

$$H : p_3 = M(x_1, x_2, x_3, p_1, p_2, p_3) = -W$$

one concludes that two equations

$$N = a, \quad M = 0$$

have ∞^1 common integral surfaces for every value of a.

Indeed, the relation $(HN) = 0$ immediately provides the equation

$$(p_3 M, N) = 0 = p_3(MN) + \frac{\partial N}{\partial x_3} M,$$

that shows that the expression in brackets (MN) vanishes when $M = 0$.

The resulting integral surfaces of each equation $N = a$ are indeed mapped into themselves by the infinitesimal contact transformation (Hf).

60. The proof can be simplified in a certain sense if we retain the original coordinates x, y, z, p, q. Then the invariance of every of ∞^1 partial differential equations

$$F(x, y, z, p, q) = a$$

with respect to the infinitesimal contact transformation

$$[Wf] - W\frac{\partial f}{\partial z}$$

is equivalent to the condition that the expression

$$[WF] - W\frac{\partial F}{\partial z}$$

vanishes identically. It follows that the bracket

$$[WF]$$

vanishes for $W = 0$. This demonstrates again that the equations

$$F = a, \quad W = 0$$

have ∞^1 common integral surfaces which are mapped into themselves by our infinitesimal contact transformation.

61. The theory extends to all first-order partial differential equations with n variables and offers the clue to all investigations on first-order partial differential equations by Lagrange and his followers till Jacobi inclusively. The discovery of the transformation theory of first-order partial differential equations outlined here initiated my general investigations in the field*.

I do not consider it necessary to dwell here on these considerations dating back to 1871–1872, although they lead me to the general theory of contact transformations as well as to my theory of transformation groups. Instead, I consider it reasonable to concentrate on extension of my previous developments† to arbitrary partial differential equations of the second and higher orders.

In the next chapter I will show that one can gain even more benefit for integrating partial differential equations from infinitesimal contact transformations. Although on cursory examination developments in the next chapter may seem to be based on absolutely different principles as compared to the present chapter, both theories actually have a common source.

62. Let us assume that a second order partial differential equation to be integrated:

$$F(x, y, z, p, q, r, s, t) = 0$$

admits a known infinitesimal contact transformation

$$[Wf] - W\frac{\partial f}{\partial z} \cdot$$

Then the first-order partial differential equation

$$W(x, y, z, p, q) = 0,$$

has ∞^2 common integral surfaces with $F = 0$.

In order to prove this statement let us introduce new variables via a contact transformation

$$x' = X(x, y, z, p, q), \quad y' = Y, \quad z' = Z, \quad p' = P, \quad q' = Q$$

*Cf. Kurzes Résumé …, Scientific Society of Christiania, Aprill 1872(23).

†Math. Annalen Bd. XI, p. 490, footnote(24).

such that the known infinitesimal contact transformation takes the form

$$\frac{\partial f}{\partial x'}\,.$$

In the new variables the equation $F = 0$ turns into a second-order partial differential equation, admitting the translations along the x'-axis, i.e. it does not contain x' and hence has the form

$$\Phi(y', z', p', q', r', s', t') = 0.$$

The substitution

$$z' = Y(y')$$

yields the second-order ordinary differential equation determining $Y(y')$:

$$\Phi\left(y', Y, 0, \frac{dY}{dy'}, 0, 0, \frac{d^2Y}{dy'^2}\right) = 0.$$

Its solution $Y(y')$ depends on two arbitrary parameters.

This completes the proof of our statement provided that both W and F are regular on points x, y, z, p, q satisfying $W = 0$.

63. The above important, although particular, result can be formulated as follows.

Let a second-order partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0$$

admit the infinitesimal contact transformation

$$[Wf] - W\frac{\partial f}{\partial z}$$

then the partial differential equations

$$F = 0, \quad W = 0$$

have ∞^2 common integral surfaces provided that both W and F behave regularly on points x, y, z, p, q satisfying the equation $W = 0$. In other words, the equation $F = 0$ has, in general, ∞^2 integral surfaces which are mapped into themselves by the infinitesimal transformation*.

64. Although this theorem is very simple, it reveals a real source of numerous well known but isolated from each other results.

*Under some conditions all integral surfaces of $W = 0$ may simultaneously satisfy the equation $F = 0$.

If a second-order partial differential equation admits all motions, it represents simply a relation between two radii of curvature:

$$\Omega(R_1, R_2) = 0.$$

Integral surfaces of such equation include known cylindrical surfaces, surfaces of revolution and helical surfaces. In each case one obtains at least ∞^7 integral surfaces admitting an infinitesimal motion.

For example, there are surfaces of revolution and helical surfaces which are simultaneously minimal surfaces, otherwise they have a constant curvature or a constant mean curvature.

Some partial differential equations

$$\Omega(R_1, R_2,) = 0$$

admit not only all ∞^6 motions but also all ∞^7 similarity transformations. This is the case with the equation

$$R_1 : R_2 = a = \text{const.}$$

It follows that there are surfaces with constant ratio of radii of curvature that either represent spiral surfaces or admit an infinitesimal transformation

$$c_1(xp+yq+zr)+c_2(yp-xq)+c_3(zq-yr)+c_4(xr-zp)+c_5p+c_6q+c_7r.$$

This observation led me in due time to discover minimal surfaces which are spiral surfaces.

65. The above theorem admits further generalizations, which are of great interest and can be considered as a natural consequence of my general theory.

Let us consider partial differential equations in x, y, z. Repeating the above speculations almost word for word we arrive at the following theorem.

Theorem. *If an m-th order partial differential equation*

$$F\left(x, y, z, p, q, r, \dots, \frac{\partial^m z}{\partial y^m}\right) = 0$$

admits an infinitesimal contact transformation

$$[Wf] - W\frac{\partial f}{\partial z},$$

then both partial differential equations

$$F = 0 \quad W = 0$$

have, in general, ∞^m common integral surfaces obtained by integration of an m-th order ordinary differential equation only. This auxiliary equation contains arbitrary constants, which disappear upon integration of the first-order equation $W = 0$.

66. Consider as a first example to the above theorem the fourth order partial differential equation defining all isothermic surfaces. This differential equation admits obviously the ten-parameter group of all conformal point transformations. Our theory provides ∞^{9+4} isothermic surfaces each of them admitting an infinitesimal conformal transformation.

As a second example we consider the fourth-order partial differential equation defining all translation surfaces. This equation admits all ∞^{12} linear point transformations of the space. It follows that there are

$$\infty^{11+4}$$

translation surfaces admitting an infinitesimal linear transformation.

These translation surfaces are interesting by themselves and they can be really determined. The surfaces fall into two categories depending on whether the infinitesimal linear transformation leave invariant or permute their points at infinity.

However, we can neglect the first category, for it is clear from the above that it includes only developable surfaces or more specifically only cylindrical surfaces.

Thus, we can assume that our infinitesimal transformation permutes the points at infinity among themselves. Then, the plane at infinity will undergo projective transformation and will contain ∞^1 curves which remain invariant. These curves are defined by the equation

$$\omega\left(\frac{dy}{dx}, \frac{dz}{dx}\right) = a = \text{const.}$$

Assigning the parameter a two definite values a_1 and a_2 we choose two curves. My general theory of translation surfaces provides a partial differential equation

$$R(p, q)r + S(p, q)s + T(p, q)t = 0.$$

The integral surfaces of the latter equation are translation surfaces. Their generating curves always have ∞^1 tangents intersecting every of the two above chosen curves at infinity.

If

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \zeta\frac{\partial f}{\partial z} \tag{D}$$

is the symbol of the given infinitesimal transformation, the desired translation surfaces are given by the partial differential equations

$$\xi p + \eta q - \zeta = 0, \tag{E}$$

$$Rr + Ss + Tt = 0, \tag{F}$$

where the first equation contains eleven essential constants and in the second equation the parameters a_1, a_2 occur. Then the equations (E) and (F) have ∞^2 common integral surfaces for any values of the above thirteen parameters. This follows from the fact that the second-order partial differential equation (F) admits the infinitesimal transformation (D) as it is evident from geometric considerations.

67. In order to find the finite equations of these surfaces, one can first determine the path curves

$$\bar{x} = a, \quad \bar{y} = b$$

of the infinitesimal transformation (D) and write an ordinary differential equation in $\bar{x}, \bar{y}$:

$$W(\bar{x}, \bar{y}, \bar{y}', \bar{y}'') = 0.$$

Then the integral of the above equation:

$$\Pi(\bar{x}, \bar{y}, \alpha, \beta) = 0$$

represents the desired surfaces.

The above operations can be carried out effectively. In this regard we make the following observations.

Since a linear transformation maps any parallelogram to a parallelogram and hence carries congruent curves to similar curves, it maps a translation surface into a translation surface. If a translation surface admits a linear infinitesimal transformation one has two possibilities, namely, ∞^1 congruent similar curves of our surface are either permuted among themselves or mapped to other families of congruent curves. However, there is no need to consider the second possibility since it can occur only for the surfaces which I defined earlier and which can be interpreted as translation surfaces in infinitely many ways. Thus, it remains to consider only the possibility when the given infinitesimal linear transformation leaves every family of congruent curves invariant. In this case every individual curve of such family admits an infinitesimal linear transformation, because two independent linear [infinitesimal] transformations carry every curve of the family to a similar neighboring curve. It follows that one can determine both families of congruent curves lying on our surface, and hence the surface itself.

In particular, there are rectilinear minimal surfaces among the translation surfaces defined above which admit an infinitesimal linear transformation. We mention in passing that these helical surfaces are the only rectilinear translation surfaces which are not cylinders at the same time.

68. We would like to indicate how one can find all these surfaces by means of simplest possible calculations.

In earlier papers I demonstrated that one can bring every linear homogeneous infinitesimal transformations to a canonical form. If the canonical form is

$$\sum_{i,k}^{1\ldots3} c_{ik} x_i p_k$$

one takes a transformation

$$\sum c_{ik} x_i p_k + d_1 p_1 + d_2 p_2 + d_3 p_3$$

and tries to eliminate all the constants d_1, d_2, d_3, or at least some of them by means of the change of variables

$$x_k' = x_k + \alpha_k.$$

In particular, if the determinant

$$|c_{ik}|$$

does not vanish, one can simply set all d_k equal to zero.

Then one has to find all translation surfaces admitting the infinitesimal transformation

$$\sum_{ik} c_{ik} x_{ik} p_k + \sum_k d_k p_k \tag{G}$$

and note at the same time that every obtained curve admits, according to the above, a known infinitesimal transformation

$$\sum c_{ik} x_i p_k + \sum e_k p_k.$$

One defines the ∞^2 path curves of the latter transformation and chooses one of them, for example C; then all ∞^1 path curves of the first infinitesimal transformation that intersect C generate a surface possessing the required property.

These simple calculations provide the desired translation surfaces.

69. For example, let us take the infinitesimal transformation

$$axp + byq + czr \quad (abc \neq 0),$$

add a zero-order term to it and obtain the transformation

$$(ax+l)p+(by+m)q+(cz+n)r.$$

The path curves of the latter transformation are given by the equations

$$ax+l=(ax_0+l)e^{at},$$

$$by+m=(by_0+m)e^{bt},$$

$$cz+n=(cz_0+n)e^{ct}.$$

Accordingly, the path curves of the first transformation are defined by the equations

$$\bar{x}=xe^{a\tau}, \quad \bar{y}=ye^{b\tau}, \quad \bar{z}=ze^{c\tau}.$$

If one eliminates x, y, z from these six equations, assigning x_0, y_0, z_0 definite values and regarding t and τ as parameters, one obtains the following three equations:

$$a\bar{x}=-le^{a\tau}+(ax_0+l)e^{a(\tau+t)}$$

$$b\bar{y}=-me^{b\tau}+(by_0+m)e^{b(\tau+t)}$$

$$c\bar{z}=-ne^{c\tau}+(cz_0+n)e^{c(\tau+t)}$$

determining translation surfaces with the required properties.

70. Consider now a partial differential equation of order m with $n+1$ variables:

$$F\left(z, x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial^m z}{\partial x_n^m}\right)=0$$

admitting q known infinitesimal contact transformations

$$B_k f=[W_k f]-W_k\frac{\partial f}{\partial z}\cdot$$

We assume that the above transformations are pairwise permutable:

$$B_iB_kf-B_kB_if\equiv 0,$$

and that their characteristic functions do not satisfy any homogeneous relation

$$\Phi(W_1, \ldots, W_q)=0.$$

Then the equations

$$W_1=0, \ldots, \quad W_q=0, \quad F=0$$

have common solutions provided that the equation $F = 0$ behaves regularly for a system of values $z, x_1, \dots, x_n, p_1, \dots, p_n$ satisfying the equations

$$W_1 = 0, \dots, \quad W_q = 0.$$

In order to prove this theorem and simultaneously to specify it, we introduce new variables

$$Z, X_1, \dots, X_n, P_1, \dots, P_n$$

so that the characteristic functions of our infinitesimal transformations take the form

$$P_n, P_{n-1}, \dots, P_{n-q+1}.$$

In these variables the equation $F = 0$ takes the form

$$\Phi\left(Z, X_1, \dots, X_{n-q}, P_1, \dots, P_n, \frac{\partial^2 Z}{\partial X_1^2}, \dots\right) = 0$$

independent of $X_{n-q+1}, \dots, X_n$. Consequently, we set

$$Z = \Theta(X_1, \dots, X_{n-q}),$$

and obtain Θ from an obviously integrable partial differential equation of order m with the variables

$$Z, X_1, \dots, X_{n-q}.$$

This proves our statement and simultaneously provides an integration method often appearing efficient in practice, although it requires more integration operations than (strictly speaking) necessary.

71. The assumption made in the previous example that all the brackets $B_i B_k f - B_k B_i f$ vanish can be replaced by the following general supposition.

Let us suppose, for example, that the equation

$$F\left(z, x_1, \dots, x_n, \dots, \frac{\partial^m z}{\partial x_n^m}\right) = 0$$

admits two infinitesimal contact transformations generating a two-parameter group. If these transformations are not commutative, we can assume that the variables have already been chosen so that our infinitesimal transformations have the form

$$\frac{\partial f}{\partial x_1}, \qquad x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} + z \frac{\partial f}{\partial z}\,.$$

According to this supposition the equation $F = 0$ does not contain x_1 and is homogeneous in

$$x_2, \dots, x_n, z, dx_1, dx_2, \dots, dx_n, dz,$$

and hence has the form

$$\Phi\left(x_2, \dots, x_n, z, \frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}, z\frac{\partial^2 z}{\partial x_1^2}, \dots\right) = 0.$$

We set now

$$z = Z(x_2, \dots, x_n) = x_n W(x_2 : x_n, \dots, x_{n-1} : x_n)$$

and introduce the variables

$$x_2 : x_n = y_2, \dots, x_{n-1} : x_n = y_{n-1}.$$

Then we have:

$$\frac{\partial Z}{\partial x_1} = 0, \quad \frac{\partial Z}{\partial x_2} = \frac{\partial W}{\partial y_2}, \dots, \frac{\partial Z}{\partial x_{n-1}} = \frac{\partial W}{\partial y_{n-1}},$$

$$\frac{\partial Z}{\partial x_n} = W - y_2\frac{\partial W}{\partial y_2} - \dots - y_{n-1}\frac{\partial W}{\partial y_{n-1}}, \dots.$$

In cosequence, our partial differential equation takes the following form containing only the quantities $y_2, \dots, y_{n-1}, W$ and the corresponding derivatives up to order m :

$$\Pi\left(W, y_2, \dots, y_{n-1}, \frac{\partial W}{\partial y_2}, \dots\right) = 0.$$

71.* *Let a partial differential equation $F = 0$ of order m with variables $z, x_1, x_2, \dots, x_n (n > 2)$ admit a two-parameter group of contact transformations generated by two infinitesimal transformations*

$$[W_k f] - W_k\frac{\partial f}{\partial z} \qquad (k = 1, 2).$$

Then the partial differential equations

$$F = 0, \quad W_1 = 0, \quad W_2 = 0$$

have as many common solutions as possible.

72. In general, let a partial differential equation of order m with variables $z, x_1, \dots, x_n$ admit a q-parameter group of contact transformations

$$[W_k f] - W_k\frac{\partial f}{\partial z} \qquad (k = 1, 2, \dots, q)$$

where W does not satisfy any relation of the form

$$\Omega(W_1 : W_q, W_2 : W_q, \dots, W_{q-1} : W_q) = 0.$$

Then the equations

$$F = 0,\ W_1 = 0, \dots, W_q = 0$$

have as many common solutions as possible and every such solution is invariant under the q-parameter group*.

73. Let us prove the above statement.

In general, the set of all ∞^{2n+1} elements $z, x_1, \dots, x_n, p_1, \dots, p_n$ contains ∞^{2n+1-q} elements satisfying q equations

$$W_1 = 0, \dots, W_q = 0.$$

Furthermore, every element satisfying the above equations remains invariant under every infinitesimal transformation of our group and hence under every finite transformation of the group. In consequence the family of elements defined by the equations $W_1 = 0, \dots, W_q = 0$ remains invariant under all group transformations.

The $\infty^{2n+1-q+\varepsilon}$ elements of our invariant family make subdomains which individually remain invariant. The smallest invariant subdomains contain at most ∞^q elements every one of which is joined with all neighboring elements of this domain.

The $\infty^{\mu+\nu}$ elements of the family defined by $W_1 = 0, \dots, W_q = 0$ make about ∞^μ minimal invariant subdomains E_ν. Every subdomain E_ν contains ∞^ν elements that form a union of elements$^{(25)}$.

Neighboring complexes E_ν have peculiar relations, namely if two neighboring elements belonging to two different complexes E_ν are joined, then two arbitrary neighboring elements of the same two complexes E_ν are also joined. Therefore, ∞^μ complexes of elements E_ν have exactly the same relations with each other as characteristic strips of a first order partial differential equation.

74. It is not difficult to trace the internal reason leading to the above result.

Our assumption that q infinitesimal contact transformations

$$[W_k f] - W_k \frac{\partial f}{\partial z} \quad (k = 1, 2, \dots, q)$$

*We assume implicitly that the functions under consideration behave regularly for some values of $z, x_1, \dots,$ satisfying our system of equations.

generate a q-parameter group means that q characteristic functions $W_1, \ldots, W_q$ satisfy the relations

$$[W_iW_k] - W_i\frac{\partial W_k}{\partial z} + W_k\frac{\partial W_i}{\partial z} = \sum_s c_{iks}W_s.$$

It follows immediately that first-order differential equations

$$W_1 = 0, \ldots, W_q = 0$$

furnish, in my sense, an involutory system. This involutory system has characteristic manifolds which are precisely complexes of elements E_ν introduced above.

Our earlier supposition that W_k do not satisfy any homogeneous relation $\Omega = 0$ can obviously be replaced by general assumptions.

75. I proved earlier a general statement that there exists a one-to-one (i.e. not infinite-valued) correspondence between the characteristic manifolds of a first-order involutory system and the first-order elements of a properly selected space of points $\zeta, \xi_1, \ldots, \xi_{n'}$. The correspondence is established so that two neighboring characteristic manifolds, whose neighboring elements are always joined, are always reproduced in the space $\zeta, \xi_1, \ldots, \xi_{n'}$ as two joined elements (ζ, ξ, π).

Hence, the original partial differential equation of order m and the first-order equations $W_1 = 0, \ldots, W_q = 0$, i.e. the system of equations

$$F = 0, \quad W_1 = 0, \ldots, W_q = 0 \tag{L}$$

can be replaced by a single partial differential equation of order m :

$$\Phi\left(\zeta, \xi_1, \ldots, \xi_{n'}, \pi_1, \ldots, \pi_{n'}, \ldots, \frac{\partial^m \zeta}{\partial \xi_{n'}^m}\right) = 0.$$

Since the latter equation always has solutions, this is also true for the system of equations (L).

76. This proves the earlier announced result. Moreover, it becomes obvious that the above conclusions hold not only for an equation of order m but also for an unboundedly integrable system of order m. Thus, we can formulate the following theorem.

Theorem. *If an unboundedly integrable system of partial differential equations of order m :*

$$F_k\left(z, x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial^2 z}{\partial x_1^2}, \ldots, \frac{\partial^m z}{\partial x_n^m}\right) = 0 \quad (k = 1, 2, \ldots),$$

admits a continuous group of contact transformations

$$[W_k f] - W_k \frac{\partial f}{\partial z} \quad (k = 1, 2, \ldots),$$

then one finds all solutions of the system $F_1 = 0, F_2 = 0, \ldots,$ *which are invariant under the group, by means of adding* q *equations:*

$$W_1 = 0, \ldots, W_q = 0$$

to the original equations $F_k = 0$. *If the above* q *equations do not contradict each other, they furnish an involutory system of the first order. The characteristic manifolds of the latter system compose integral manifolds of the system* $F_1 = 0, F_2 = 0, \ldots$ *provided that they are sufficient in number.*

77. Consider a system of partial differential equations of the first order determining m quantities $z_1, z_2, \ldots, z_m$ as functions of $x_1, \ldots, x_n$. Let us denote

$$p_k^{(i)} = \frac{\partial z_i}{\partial x_k} \cdot$$

and write the system in the form

$$F_\nu(x_1, \ldots, x_n, z_1, \ldots, z_m, p_1^{(1)}, \ldots, p_n^{(m)}) = 0 \quad (\nu = 1, 2, \ldots).$$

Every system of quantities $x_1, \ldots, x_n, z_1, \ldots, z_m, p_1^{(l)}, \ldots, p_n^{(m)}$ is termed an *element.* Two neighboring elements[(26)] are said to be *united* if they satisfy the system of equations

$$dz_i - p_1^{(i)} dx_1 - \ldots - p_n^{(i)} dx_n = 0 \qquad (i = 1, \ldots, m).$$

A set of united elements is termed a *union of elements*[(27)]. A union of elements contains maximum ∞^n and minimum ∞^1 elements.

We refer to a union of elements as a *union of integrals* for a system of differential equations $F_1 = 0, F_2 = 0, \ldots,$ if all elements of the union satisfy the equations $F_k = 0$. We distinguish between unions of integrals $V_1, V_2 \ldots, V_n$, depending on the dimension of the corresponding union of integrals.

We say that the system of differential equations $F_1 = 0, F_2 = 0, \ldots,$ *is unboundedly integrable if every element* $x_1, \ldots, x_n, z_1, \ldots, z_m, p_1^{(l)}, \ldots, p_n^{(m)}$ *belongs at least to one union of integrals* V_n *of our system of equations.*

Furthermore, we call the system of equations $F_1 = 0, F_2 = 0, \ldots$ *an involutory system if every its union of integrals* V_q *in general position belongs at least to one union of integrals* V_n.

78. Let us assume that a first-order involutory system

$$F_1 = 0, \quad F_2 = 0, \ldots,$$

with the independent variables $x_1, \ldots, x_n$ and the dependent variables $z_1, \ldots, z_m$ admits given infinitesimal transformations

$$U_k f = \sum \xi_{ki}(x, z) \frac{\partial f}{\partial x_i} + \sum \zeta_{ki}(x, z) \frac{\partial f}{\partial z_i} \quad (k = 1, 2, \ldots, r)$$

Furthermore, we assume that the latter transformations generate an r-parameter group, i.e. that the largest determinants of the matrix

$$\|\xi_{k1} \ldots, \xi_{kn} \zeta_{k1}, \ldots, \zeta_{km}\|$$

do not vanish identically.

If sufficiently many elements in general position satisfy the extended system of first-order equations

$$F_1 = 0 \quad F_2 = 0, \ldots,$$

$$\zeta_{ki} - p_1^{(i)} \xi_{k1} - p_2^{(i)} \xi_{k2} - \ldots - p_n^{(i)} \xi_{kn} = 0 \quad (k = 1, \ldots, r;\ i = 1, \ldots, m)$$

then the original system of differential equations is unboundedly integrable.

The proof is exactly the same as in the previous example.

Chapter 4

Partial differential equations admitting an infinite group

79. The previous chapter was devoted to partial differential equations admitting a continuous group. We demonstrated that an admitted infinitesimal transformation furnishes special integral manifolds that are mapped by the transformation into themselves rather than into new integral manifolds. In consequence, if a given equation admits finite or infinite continuous group, the above theories provide methods for obtaining certain solutions. In case of infinite groups the solutions may depend, in general, not only on arbitrary constants alone.

This chapter illustrates by simple examples that the above speculations offer even more advantages. The development of new approaches which are of the utmost importance is based on our general theory of differential invariants. Our approach to partial differential equations admitting an infinite continuous group is based on introduction of new variables, namely of a complete system of differential invariants. According to our general theory the latter system of invariants exists for every continuous group.

80. Example 1. Let us first consider the infinite continuous group with the infinitesimal transformations

$$Z(z)\frac{\partial f}{\partial z}$$

where Z is an arbitrary function of z.

If we use the usual notation $Z', Z'', \ldots,$ for the derivatives of Z, the infinitesimal transformations of the corresponding extended groups are written

$$Z\frac{\partial f}{\partial z} + Z'\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} + \dots\right)$$
$$+Z''\left(p^2\frac{\partial f}{\partial r} + pq\frac{\partial f}{\partial s} + q^2\frac{\partial f}{\partial t} + \dots\right) + \dots.$$

In order to find all the differential invariants up to the second order we solve the equations

$$\frac{\partial f}{\partial z} = 0, \quad p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + r\frac{\partial f}{\partial r} + s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} = 0,$$
$$p^2\frac{\partial f}{\partial r} + pq\frac{\partial f}{\partial s} + q^2\frac{\partial f}{\partial t} = 0$$

which constitute a complete system according to our general theory. These equations provide two zero-order invariants

$$x \quad \text{and} \quad y,$$

one first-order differential invariant

$$u = p : q,$$

and two second-order differential invariants

$$\frac{qr - ps}{q^2} = \frac{\partial u}{\partial x}, \quad \frac{qs - pt}{q^2} = \frac{\partial u}{\partial y}\cdot$$

According to our general theory every differential invariant is a function of

$$x, y, u$$

and the successive derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial x^2}, \dots.$$

It means in my terminology that x, y and u provide a *complete system of differential invariants.*

81. Every relation of the form

$$\Omega\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \dots\right) = 0$$

yields an invariant differential equation

$$W(x, y, z, p, q, r, s, t, \dots) = 0.$$

If there exists an invariant differential equation of the second order which is not reducible to the form

$$\Omega\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

then all 3×3 determinants of the matrix

$$\begin{Vmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & p & q & r & s & t \\ 0 & 0 & 0 & 0 & 0 & p^2 & pq & q^2 \end{Vmatrix}$$

vanish whenever the above second-order equation holds. Since such an equation is not available we make the following conclusion.

The infinite group with the infinitesimal transformations

$$Z(z)\frac{\partial f}{\partial z}$$

has a complete system of differential invariants provided by

$$x, \ y, \ u = p : q.$$

Any second-order partial differential equation admitting all transformations of this infinite group has the form

$$\Omega\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

reducible to ordinary deferential equations.

82. Similar considerations lead to the general theorem.

If an m-th order partial differential equation with variables x, y, z admits all transformations of the infinite group

$$Z(z)\frac{\partial f}{\partial z}\,,$$

then it can be reduced without integration to an equation of order $m-1$ which is expressed in a general form via x, y, u. If one can integrate this auxiliary equation of order $m-1$, then one has to integrate only an ordinary differential equation of the first order:

$$p - \alpha(x, y)q = 0.$$

It appears that the given group

$$Z(z)\frac{\partial f}{\partial z}$$

can be by no means applied when dealing with our auxiliary equations.

83. Example 2. Consider the infinite group

$$\xi(x)\frac{\partial f}{\partial x} - \xi' z\frac{\partial f}{\partial z},$$

which I defined as a canonical form while determining all infinite groups with two variables in 1883*. Using the common notation $\delta x, \delta y, \delta z$ for the increments of x, y, z and setting

$$\delta x = \xi\delta\tau; \quad \delta y = 0, \quad \delta z = -\xi' z\delta\tau,$$

we obtain

$$\delta p = (-2\xi' p - \xi'' z)\delta\tau, \quad \delta q = -\xi' q\delta\tau,$$
$$\delta r = (-3\xi' r - 3\xi'' p - \xi''' z)\delta\tau,$$
$$\delta s = (-2\xi' s - \xi'' q)\delta\tau, \quad \delta t = -\delta' t\delta\tau.$$

Hence, the symbol of the twice extended group has the form

$$X'' f = \xi\frac{\partial f}{\partial x} - \xi'\left(z\frac{\partial f}{\partial z} + 2p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + 3r\frac{\partial f}{\partial r} + 2s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t}\right) -$$
$$-\xi''\left(z\frac{\partial f}{\partial p} + 3p\frac{\partial f}{\partial r} + q\frac{\partial f}{\partial s}\right) - \xi''' z\frac{\partial f}{\partial r}\cdot$$

The corresponding differential invariants of the zero, first and second orders are defined as solutions of the complete system

$$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial r} = 0, \quad z\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial s} = 0,$$
$$z\frac{\partial f}{\partial z} + 2p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + 2s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} = 0.$$

This yields one differential invariant of order zero, namely y which we denote by μ, one differential invariant of the first order $\nu = q : z$, and two differential invariants of the second order:

$$u = \frac{zs - pq}{z^3}, \qquad v = \frac{t}{z}\cdot$$

Hence, the general form of an invariant differential equation of the second order is

$$\Omega(\mu, \nu, u, v) = 0.$$

Indeed, the above equation provides all invariant second-order partial differential equations. This follows immediately from the fact that among the 4×4 determinants of the matrix

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*Verhandl. der Gesellschaft der Wissenschaften zu Christiania, 1883(28).

$$\left\| \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z & 2p & q & 3r & 2s & t \\ 0 & 0 & 0 & z & 0 & 3p & q & 0 \\ 0 & 0 & 0 & 0 & 0 & z & 0 & 0 \end{matrix} \right\|$$

there are non-vanishing ones. For example,

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & z & 2p & 3r \\ 0 & 0 & z & 3p \\ 0 & 0 & 0 & z \end{vmatrix} = z^3,$$

vanishes neither identically nor due to a second-order partial differential equation.

84. In the given case one can readily see that all differential invariants of the third and higher orders are obtained from μ, ν, u, v by differentiation and hence these four quantities furnish a complete system of differential invariants.

In particular, let us consider an arbitrary surface not satisfying any relation of the form

$$W(y, q : z) = 0 = W(\mu, \nu).$$

Hence, we can use μ, ν as Gaussian coordinates for the points of this surface or, in other words, we can take μ, ν as independent variables instead of x, y. Then, we can prove that four values

$$\frac{\partial u}{\partial \mu}, \quad \frac{\partial u}{\partial \nu}, \quad \frac{\partial v}{\partial \mu}, \quad \frac{\partial v}{\partial \nu} \tag{M}$$

are differential invariants (obviously of the third order) of our group.

In order to prove this let us choose two arbitrary m-th order differential invariants for example, U and V and consider the equation

$$\Phi(U, V) = 0$$

with an arbitrary function Φ. Selecting all possible Φ we obtain infinitely many m-th order differential equations every one of which represents an intermediate integral, namely the most general intermediate integral of the differential equation of the $(m+1)$-th order:

$$\begin{vmatrix} \dfrac{dU}{dx} & \dfrac{dU}{dy} \\ \dfrac{dV}{dx} & \dfrac{dV}{dy} \end{vmatrix} = 0 = \Delta.$$

Since the general expression $\Phi(U, V)$ is a differential invariant, it is obvious that $\Delta = 0$ is an invariant differential equation.

If U, V, W are three differential invariants, then

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dV}{dx} - c\frac{dW}{dx} \\ \frac{dU}{dy} & \frac{dV}{dy} - c\frac{dW}{dy} \end{vmatrix} = 0.$$

is an invariant differential equation involving the constant c. This equation can be rewritten in the form

$$\begin{vmatrix} \frac{dU}{dx} & \frac{dV}{dx} \\ \frac{dU}{dy} & \frac{dV}{dy} \end{vmatrix} : \begin{vmatrix} \frac{dU}{dx} & \frac{dW}{dx} \\ \frac{dU}{dy} & \frac{dW}{dy} \end{vmatrix} = c.$$

Since c is an arbitrary constant, the left-hand side of this equation is certainly a differential invariant.

If we introduce U and W as independent variables instead of x, y, the resulting differential invariant can be expressed via the functional determinant of U and V with the independent variables U, W :

$$\begin{vmatrix} \frac{dU}{dU} & \frac{dV}{dU} \\ \frac{dU}{dW} & \frac{dV}{dW} \end{vmatrix} \equiv \frac{dV}{dW}\,.$$

85. Thus in our case the quantities (M) denoted by

$$u_\mu, u_\nu, v_\mu, v_\nu$$

are differential invariants. Moreover, we know *eight* differential invariants, namely

$$\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu \tag{N}$$

of order lower than *four.*

However, one can prove, that there exist no more than seven independent differential invariants of the fourth or lower order. In order to calculate all these differential invariants we add the expressions of increments

$$\delta\alpha = (-4\xi'\alpha - 6\xi''r - 4\xi''p - \xi^{(4)}z)\delta\tau,$$

$$\delta\beta = (-3\xi'\beta - 3\xi''s - \xi'''q)\delta\tau,$$

$$\delta\gamma = (-2\xi'\gamma - \xi''t)\delta\tau,$$

$$\delta\delta = -\xi'\delta \cdot \delta\tau$$

to the following formulae obtained above:

$$\delta x = \xi(x)\delta\tau, \quad \delta y = 0, \quad \delta z = -\xi' z\delta\tau,$$

$$\delta p = (-2\xi' p - \xi'' z)\delta\tau, \quad \delta q = -\xi' q\delta\tau,$$

$$\delta r = (-3\xi' r - 3\xi'' p - \xi''' z)\delta\tau,$$

$$\delta s = (-2\xi' s - \xi'' q)\delta\tau, \quad \delta t = -\xi' t\delta\tau.$$

The desired differential invariants are solutions of the complete system

$$\frac{\partial f}{\partial x} = 0, \quad z\frac{\partial f}{\partial a} = 0,$$

$$z\frac{\partial f}{\partial z} + 2p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} + 3r\frac{\partial f}{\partial r} + 2s\frac{\partial f}{\partial s} + t\frac{\partial f}{\partial t} + 4\alpha\frac{\partial f}{\partial \alpha} + 3\beta\frac{\partial f}{\partial \beta} + $$

$$+2\gamma\frac{\partial f}{\partial \gamma} + \delta\frac{\partial f}{\partial \delta} = 0,$$

$$z\frac{\partial f}{\partial p} + 3p\frac{\partial f}{\partial r} + q\frac{\partial f}{\partial s} + 6r\frac{\partial f}{\partial \alpha} + 3s\frac{\partial f}{\partial \beta} + t\frac{\partial f}{\partial \gamma} = 0,$$

$$z\frac{\partial f}{\partial r} + 4p\frac{\partial f}{\partial \alpha} + q\frac{\partial f}{\partial \beta} = 0$$

of five independent equations with *twelve* independent variables. The fifth-order determinant of the corresponding matrix has the value z^4. It follows that, on the one hand, every third-order invariant equation can be represented by a relation between differential invariants and, on the other hand, the number of independent third-order differential invariants is

$$12 - 5 = 7.$$

Whence we make a conclusion that the eight values (N) are connected with each other by *one and only one relation*

$$W(\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu) \equiv 0.$$

86. Now it is possible to develop a general integration theory for all second-order partial differential equations admitting the above group.

Indeed, the above considerations allow to write an invariant partial differential equation of the second order in the form

$$\Omega(\mu, \nu, u, v) = 0.$$

Let us solve the above equation, e.g. with respect to v :

$$v = V(\mu, \nu, u)$$

and calculate the derivatives v_μ and v_ν. Substitution of the resulting expressions for v, v_μ and v_ν in the equation $W = 0$ yields a relation

$$\Pi(\mu, \nu, u, u_\mu, u_\nu) = 0.$$

This gives a possibility to reduce the definition of u as a function of μ and ν to integration of a *first-order* partial differential equation, and hence to *ordinary* differential equations.

However, nothing special can be said about integration of $\Pi = 0$ since Π is an arbitrary function of its five arguments. Nevertheless, if the equation $\Pi = 0$ is already integrated then one can employ its solution

$$u = U(\mu, \nu)$$

together with $\Omega = 0$ to find v as a function of μ, ν, i.e. $v = V(\mu, \nu)$.

The resulting two equations

$$u - U(\mu, \nu) = 0, \quad v - V(\mu, \nu) = 0$$

provide, upon substitution

$$\mu = y, \quad \eta = \frac{q}{z}, \quad u = \frac{zs - pq}{z^3}, \quad v = \frac{t}{z}$$

a pair of second-order partial differential equations

$$\frac{z\,s - p\,q}{z^3} - U\left(y, \frac{q}{z}\right) = 0, \qquad \frac{t}{z} - V\left(y, \frac{q}{z}\right) = 0.$$

The latter equations have common integral surfaces and, moreover, constitute a second-order involutory system. Therefore, one can determine the common integral surfaces by integrating ordinary differential equations (cf. pp. 16–24).

Thus, if a second-order partial differential equation in variables x, y, z admits the infinite group

$$\xi(x)\frac{\partial f}{\partial x} - \xi' \cdot z\frac{\partial f}{\partial z},$$

its integration requires only successive integration of three systems of ordinary differential equations.

87. One can develop corresponding theories for arbitrary systems of differential equations in x, y, z, admitting our group. We shall restrict ourselves to the following simple observation.

Consider a third-order involutory system

$$F_1(x, y, z, p, q, r, s, t, \alpha, \beta, \gamma, \delta) = 0, \quad F_2 = 0$$

admitting our infinite group. According to the previous discussion it can be reduced to the form

$$\Phi_1(\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu) = 0, \quad \Phi_2 = 0.$$

Moreover, invoking the equation $W = 0$, we can express u and v as functions of two variables μ and ν by solving *three* partial differential equations of the first order:

$$\Phi_1 = 0, \quad \Phi_2 = 0, \quad W = 0.$$

Solution of the latter system is reduced to integration of ordinary differential equations (cf. p. 25 and further). Finally, we deal with the obtained equations

$$u = U(\mu, \nu), \quad v = V(\mu, \nu)$$

as in the previous example.

88. Given a *third-order* equation admitting our group, for instance the equation

$$F(x, y, z, \ldots, \gamma, \delta) = 0,$$

we bring it to the form

$$\Omega(\mu, \nu, u, v, u_\mu, u_\nu, v_\mu, v_\nu) = 0$$

and add the equation $W = 0$. Now we have to integrate the first-order involutory system

$$\Phi = 0, \quad W = 0$$

with the two unknown functions u and v. Upon differentiating it twice and eliminating v we substitute the first-order involutory system with two unknown functions by a *third-order* involutory system with one function u

$$\Theta_1(\mu, \nu, u, u_\mu, \ldots, u_{\nu\nu\nu}) = 0, \quad \Theta_2 = 0.$$

89. In order to generalize the obtained results I provide the following considerations.

An involutory system with variables x, y, z has several characteristic numbers and among them there is one characteristic number that I denote by ω, which is undoubtedly the most important. I call the number ω a *class* of the involutory system and define it as follows.

Given an involutory system of order m with variables x, y, z, we differentiate it q times. The difference between the number of derivatives of order $(m+q)$ in the resulting system and the number of independent differential equations of order $(m+q)$ is always the same no matter whether q is equal to or greater than zero. I denote this number by ω and call it the **class** *of the involutory system.*

Using this terminology and invoking observations given in Chapter II we arrive to the following theorem.

Theorem. *If the class of an involutory system with variables x, y, z is equal to zero or* 1, *the solution of the involutory system is reduced to ordinary differential equations.*

90. However, we can formulate the following more general statement.

Theorem. *Provided that an involutory system of class ω admits the infinite group*

$$\xi(x)\frac{\partial f}{\partial x} - \xi' z \frac{\partial f}{\partial z},$$

one can reduce it to an involutory system of class $(\omega - 1)$ *related to an involutory system of the first class.*

According to our earlier considerations the same statement holds for involutory systems admitting an infinite group

$$Z(z)\frac{\partial f}{\partial z}\cdot$$

Indeed, *the above theorem is only a particular case of a general theorem which can be extended to all involutory systems with an arbitrary number of variables as well as to all infinite groups.*

However, in this paper we restrict ourselves to particular cases of the general theorem, and later we will develop the general theorem in detail.

91. Let us consider partial differential equations admitting the group

$$\xi(x)\frac{\partial f}{\partial x} + \eta(y)\frac{\partial f}{\partial y}\cdot$$

Here the twice-extended infinitesimal transformation has the form

$$\xi\frac{\partial f}{\partial x}+\eta\frac{\partial f}{\partial y}-\xi' p\frac{\partial f}{\partial p}-\eta' q\frac{\partial f}{\partial q}-(2\xi' r+\xi'' p)\frac{\partial f}{\partial r}-$$
$$-(\xi'+\eta')s\frac{\partial f}{\partial s}-(2\eta' t+\eta'' q)\frac{\partial f}{\partial t}\cdot$$

Since the second-order differential invariants are the solutions of the complete system

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0,\quad \frac{\partial f}{\partial r}=0,\quad \frac{\partial f}{\partial t}=0,$$
$$p\frac{\partial f}{\partial p}+s\frac{\partial f}{\partial s}=0,\quad q\frac{\partial f}{\partial q}+s\frac{\partial f}{\partial s}=0,$$

they are functions of

$$z\quad\text{and}\quad \frac{s}{pq}\cdot$$

Hence, the second-order invariant differential equations can be represented by the following general formula:

$$\frac{s}{pq}=\varphi(z).$$

The methods elaborated on p. 50 and further provide two intermediate integrals:

$$e^{-\int\varphi dz}\frac{\partial z}{\partial x}=A'(x),\quad e^{-\int\varphi dz}\frac{\partial z}{\partial y}=B'(y),$$

that yield by means of quadrature the general integral

$$\int e^{-\int\varphi dz}dz=A(x)+B(y).$$

92. Our group has *four* third-order differential invariants:

$$u,\quad v,\quad \mu=z,\quad \nu=\frac{s}{pq}$$

and seven invariants of the fourth order. Thus there are precisely seven independent differential invariants of order ≤ 4. Since the following eight quantities:

$$\mu,\quad \nu,\quad u,\quad v,\quad u_\mu,\quad u_\nu,\quad v_\mu,\quad v_\nu$$

provide such invariants there exists an identical relation

$$W(\mu,\nu,u,v,u_\mu,u_\nu,v_\mu,v_\nu)\equiv 0.$$

Given any invariant differential equation of the third order, we bring it to the form

$$\Omega(\mu, \nu, u, v) = 0$$

and afterwards eliminate v from $W = 0$. Thus, we obtain a first-order partial differential equation determining u as a function of μ and ν. Upon integration of the resulting equation one arrives at third-order differential equations

$$u - U(\mu, \nu) = 0, \quad v - V(\mu, \nu) = 0$$

that constitute a third-order involutory system of the second class.

In order to make further reduction we use two infinite subgroups of our group, namely

$$\xi(x)\frac{\partial f}{\partial x} \quad \text{and} \quad \eta(y)\frac{\partial f}{\partial y}.$$

Moreover, both of them are *invariant subgroups.*

Therefore our involutory system of the second class can be reduced to a system of the first class, and hence to ordinary differential equations in two different ways. We content ourselves with this result and will not discuss the question whether the obtained ordinary differential equations can be simplified further.

93. Similar considerations lead us to the following theorem.

Theorem. *If an involutory system of class n with variables x, y, z admits the infinite group*

$$\xi(x)\frac{\partial f}{\partial x} + \eta(y)\frac{\partial f}{\partial y},$$

then the system can be reduced to an involutory system of class $n - 2$ and to ordinary differential equations.

The theorem holds also for involutory systems in x, y, z admitting the infinite group

$$\xi(x)\frac{\partial f}{\partial x} + \eta(y)\frac{\partial f}{\partial y} - z(\xi' + \eta')\frac{\partial f}{\partial z}\cdot$$

On the other hand, if an involutory system of the n-th class in variables x, y, z admits the infinite group

$$\xi(x)\frac{\partial f}{\partial x} + \eta(y)\frac{\partial f}{\partial y} + \zeta(z)\frac{\partial f}{\partial z}$$

with several known *invariant* subgroups then, our problem can be reduced to integration of an involutory system of the class $n - 3$, etc.

The extent of the reduction depends in every separate case on the number of arbitrary functions in the given infinite group or more specifically, on the class of the determining equations of the group.

94. Another question concerning an involutory system admitting an *unknown* group of contact transformations is how one can find this group most simply. In order to solve this question in general one has to begin with reduction of all groups to canonical forms. As I mentioned long ago, my general theories allow to solve all these problems rationally.

The developments presented in this paper are based, explicitly or implicitly, on my general transformation theory. In my next paper, the contents of which I already reported to the Scientific Society long ago, I investigate the question of how one can rationally apply the concept of groups to the theory of differential equations from a general viewpoint. Although the investigations may appear to give no definite answer to the question, I hope that my general results deserve attention of mathematicians.

95. During my lectures in the winter semester of 1893-1894 I illustrated theories of the last chapter by several further examples. Two of the participants of my lectures, namely Beudon and Williams, are likely to apply the theories I outlined. These theories also attracted attention of the Jablonowski Society*.

*In my recent work I gave a detailed presentation of my result, published long ago, containing determination of all surfaces admitting a continuous projective group. The reason for reproducing the details was that many mathematicians interested in this result could not reproduce the simple calculations I curtailed in earlier publication (cf. Archiv for Math., Bd. VII, Christiania, 1882[29]; see also Theorie der Transfgr., Bd. III, Leipzig, 1893).

Editor's Notes

(1) *Page 6.* Reprinted in [27], vol. 1, paper XI, XII; vol.2, paper I.

(2) *Page 6.* Reprinted in [27], vol. 3, paper I, V, VI.

(3) *Page 6.* Reprinted in [27], vol. 3, papers VII-XVI; vol.4, papers I and III.

(4) *Page 6.* Reprinted in [27], vol. 6, papers II and III.

(5) *Page 6.* This remark provided me with an incentive to begin in 1997 the systematic development of the infinitesimal approach to theory of invariants of families of differential equations. Subsequently this approach was applied to families of linear and nonlinear differential equations (see, e.g. N.H. Ibragimov [16], Chapter 10).

(6) *Page 10.* In other words, Lie expresses the quantities u and v as functions of τ by substituting in $u(x, y, z, p, q)$ and $v(x, y, z, p, q)$ the equations $x = X(\tau), y = Y(\tau), z = Z(\tau)$ and the corresponding relations $p = P(\tau), q = Q(\tau)$. Then he eliminates τ from $u = U(\tau), v = V(\tau)$ and obtains a relation $v - \varphi(u) = 0$.

(7) *Page 12.* Definition of unboundedly integrable systems is given on page 48.

(8) *Page 14.* Reprinted in [27], vol. 3, paper XIV, p. 205.

(9) *Page 16.* Reprinted in [27], vol 3, paper XXVII.

(10) *Page 18.* Recall the classical notation used by Lie: $r = z_{xx},\ s = z_{xy},\ t = z_{yy},\ \alpha = z_{xxx},\ \beta = z_{xxy},\ \gamma = z_{xyy},\ \delta = z_{yyy}.$

(11) *Page 19.* These equations are obtained from the system of four equations:

$$r + R = 0, \quad t + T = 0,$$

$$V_x + V_z p + V_p r + V_q s = 0, \quad V_y + V_s q + V_p s + V_q t = 0$$

on page 18. Specifically one has to substitute $r = -R = -(s\frac{V_q}{V_p} + m)$ and $t = -T = -(s\frac{V_p}{V_q} + n)$ in two last equations of the above system.

(12) *Page 21.* Reprinted in [27], vol. 3, paper IV, pp. 20–25.

(13) *Page 22.* Reprinted in [27], vol. 4, paper II, pp. 102–104.

(14) *Page 24.* Reprinted in [27], vol. III, paper I, p. 2.

(15) *Page 28.* Reprinted in [27], vol. 3, paper IV.

(16) *Page 31.* Reprinted in [27], vol. 3, paper XXVII.

(17) *Page 32.* Recall that $p = \partial z/\partial x, q = \partial z/\partial y$.

(18) *Page 32.* Lie denotes by

$$a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}$$

the infinitesimal translation $\bar{x} \approx x + a\varepsilon, \bar{y} \approx y + b\varepsilon, \bar{z} \approx z + c\varepsilon$ with the generator

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\cdot$$

(19) *Page 32.* In modern literature equations of the form

$$\xi(x,y,z)\frac{\partial z}{\partial x}+\eta(x,y,z)\frac{\partial z}{\partial y}-\zeta(x,y,z)=0$$

are termed quasi-linear instead of linear since their coefficients depend upon the dependent variable z. See N.H. Ibragimov [16], Chapter 4.

(20) *Page 33.* Here Lie develops the general theory of group invariant solutions. Lie was the pioneer in this field. His theory was rediscovered over and over again in the 1950s and has been widely used in group analysis ever since. Lie's role in the theory of invariant solutions has been mentioned in Section 9.4 of N.H. Ibragimov's book [16]

(21) *Page 33.* In the German original, this equation is labelled as (A). I changed the label since (A) had already been used on page 25.

(22) *Page 35.* This theorem means that the equation $Af = 0$ admitting a one-parameter group with an infinitesimal generator X has a one-parameter family of solutions invariant with respect to the group.

(23) *Page 37.* Reprinted in [27], vol. 3, paper I

(24) *Page 37.* Reprinted in [27], vol.4, paper III, p. 190, note 2.

(25) *Page 46.* Definition of union of elements is given on page 48.

(26) *Page 48.* The neighboring elements are written

$$(x_1,\ldots,x_n,z_1,\ldots,z_m,p_1^{(1)},\ldots,p_n^{(m)})$$

and

$$(x_1+dx_1,\ldots,x_n+dx_n,z_1+dz_1,\ldots,z_m+dz_m,p_1^{(1)}+dp_1^{(1)},\ldots,p_n^{(m)})+dp_n^{(m)}).$$

(27) *Page 48.* Lie's original term was *Elementverein.* We used the English translation *union of elements* suggested by J.E. Campbell and A. Cohen. See J.E. Campbell [3]; A. Cohen [5].

(28) *Page 53.* Reprinted in [27], vol. 5, paper XIII, p. 351.

(29) *Page 62.* Reprinted in [27], vol. 5, paper VII.

INTEGRATION OF A CLASS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS

By Sophus Lie

Translated from German by N.H.Ibragimov[(1)]

[Arch.for Math., Bd.VI, Heft 3, S.328-368, Kristiania 1881. Reprinted in [27], vol. 3, paper XXXV]

In this paper I define a wide class of linear second-order partial differential equations

$$R(x,y)r + S(x,y)s + T(x,y)t + P(x,y)p + Q(x,y)q + Z(x,y)z = 0, \quad (1)$$

for which a solution with two arbitrary functions can be obtained. Any of these arbitrary functions is contained in a definite integral, namely in a so-called particular integral. The corresponding differential equations, in general, cannot be solved by Laplace's method.

To give an idea about the theories developed in what follows, I shall first integrate two simple particular equations. Then I show how this integration method can be extended to a large class of differential equations.

Let's first assume the coefficients $R, S, ...Z$ of equation (1) to be independent of y. Then our equation admits a particular solution of the form $e^{cy}\Omega(x)$ with an arbitrary constant c. The quantity Ω depending on x and c, is defined by the linear ordinary differential equation

$$R\Omega'' + (cS + P)\Omega' + (c^2T + cQ + Z)\Omega = 0,$$

If R is not equal to zero, this equation has two independent solutions, viz.

$$\Omega_1(x,c) \quad \text{and} \quad \Omega_2(x,c).$$

Then

$$e^{cy}\Omega_1(x,c) \quad \text{and} \quad e^{cy}\Omega_2(x,c).$$

are two independent solutions of Equation 1, each containing an arbitrary constant. Hence the expression

$$z = \int\limits_{\alpha_1}^{\beta_1} f_1(c) e^{cy} \Omega_1(x, c) dc + \int\limits_{\alpha_2}^{\beta_2} f_2(c) e^{cy} \Omega_2(x, c) dc,$$

where $f_1(c)$ and $f_2(c)$ are arbitrary functions of c, and $\alpha_1, \beta_1, \alpha_2, \beta_2$ are arbitrary constants, is a solution of equation (1) with two arbitrary functions. Similarly, the formula

$$z = \sum_k f_1(c_k) e^{c_k y} \Omega_1(x, c_k) + \sum_k f_2(c_k) e^{c_k y} \Omega_2(x, c_k)$$

provides a solution of equation (1).

Now we assume the coefficients $R, S, ..., Z$ in (1) to be functions of $x+y$. Then there exist solutions of the form

$$e^{cy} W(x+y)$$

with an arbitrary constant c. The quantity W depending on $x+y$ and c, is defined by the linear ordinary differential equation

$$(R+S+T)W'' + (cS + 2cT + P + Q)W' + (c^2 T + cQ + Z)W = 0.$$

If $(R+S+T)$ is not equal to zero, this equation has two independent solutions, viz.

$$W_1(x+y, c) \quad \text{and} \quad W_2(x+y, c).$$

It follows, that there exist, in this case, two independent solutions of equation (1), namely:

$$e^{cy} W_1(x+y, c) \quad \text{and} \quad e^{cy} W_2(x+y, c),$$

each containing an arbitrary constant. Hence, an expression

$$z = \int\limits_{\alpha_1}^{\beta_1} e^{cy} W_1(x+y, c) f_1(c) dc + \int\limits_{\alpha_2}^{\beta_2} e^{cy} W_2(x+y, c) f_2(c) dc,$$

is a solution of equation (1) with two arbitrary functions.

I shall provide a background for both of the above integrations.

Given an arbitrary solution $z = f(x, y)$ for equation (1) with coefficients depending only on x, the function

$$z = f(x, y+k)$$

(it may be the constant k as well) is also a solution (in general, a new solution) of equation (1). On the other hand, if $R, S, ..., Z$ are functions of $x + y$, then, for the arbitrary solution $z = F(x, y)$, the expression

$$z = F(x + k, y - k)$$

with an arbitrary constant k is again a solution. In both of these particular cases there exists a transformation which enables one to deduce a solution with an arbitrary constant from any given solution. We shall prove later that both of the above integrations are based on this circumstance.

In the present paper, I begin by developing a complete transformation theory for linear equations (1). I demonstrate that any such equation admits infinitely many *infinitesimal* transformations of the form

$$\frac{\delta x}{\xi(x, y)} = \frac{\delta y}{\eta(x, y)} = \frac{\delta z}{z f(x, y) + \varphi(x, y)} \cdot \tag{2}$$

Here φ is always a solution, namely an arbitrary solution of equation (1). In general, the quantities ξ and η are equal to zero, and then f is constant. In this general case I gain no advantage from the given infinitesimal transformation.

However, if the coefficients $R, S, ..., Z$ are linked by certain relations, the quantities ξ and η may assume values different from zero. In this case f is defined by a quadrature that gives an arbitrary constant c. If

$$w(x, y) = \text{ const.}, \quad z\psi(x, y) = \text{ const.}$$

are two integrals of system (2) with $\varphi = 0$ then equation (1) always has a solution of the form

$$z = \frac{\Omega(w)}{\psi(x, y)} \cdot$$

Here Ω is determined by a linear ordinary differential equation of the second or the first order. If this auxiliary equation is of the second order, then one can find a solution of equation (1) with two arbitrary functions as was done previously for special cases. If the auxiliary equation is of the first order, one finds a solution with one arbitrary function only.

The method of integration outlined here always works out if the given equation can be reduced to the form

$$r + S(x)s + T(x)t + P(x)p + Q(x)q + Z(x)z = 0$$

by an arbitrary (known or unknown) contact transformation. Differential equations that belong to this vast category, can be integrated only occasionally by Laplace's method[(2)], i.e., with the help of the only general theory, known to me for integration of second order linear partial differential equations.

It should be mentioned, by the way, that the integration theories developed in this paper can be extended to nth order linear partial differential equations with an arbitrary number of variables as well as to systems of such equations. This is provided that these equations admit infinitesimal transformations in addition to those demonstrating only the linearity of these equations. I hope to return back to this problem *if* I *shall not learn that my method was previously known.*

First Part: Transformation theory for linear second-order partial differential equations

It is well known that linear second-order partial differential equations with two independent variables can be reduced either to the form

$$s + Pp + Qq + Zz = 0,$$

or to the form

$$r + Pp + Qq + Zz = 0.$$

Therefore we can at first restrict the discussion to these two canonical forms. Then the obtained results can be easily extended to general equations.

I.

1. To simplify the development of the transformation theory for the equation

$$s' + P'p' + Q'q' + Z'z' = 0, \tag{1}$$

we first show that, without loss of generality, one can assume the coefficient P' to be equal to zero.

Suppose that Monge's equations for characteristics do not contain any integrable combinations except x' and y'. Then a contact transformation that converts equation (1) into an equation of the same form must have either the form

$$x' = X(x), \quad y' = Y(y), \quad z' = F(x, y, z, p, q),$$

or the form

$$x' = Y(y), \quad y' = X(y), \quad z' = F(x, y, z, p, q).$$

We can restrict our considerations to transformations of the first type. Then the known (derived by me) equations[3]

$$[x'z'] = 0, \quad [y'z'] = 0,$$

show that $F^{(4)}$ depends only on x, y, and z. Hence the transformation under consideration has the form

$$x' = X(x), \quad y' = Y(y), \quad z' = F(x, y, z).$$

It follows, by the way, that *any infinitesimal transformation that does not alter equation (1) nonintegrable by Monge's method*[5]*, has the form:*

$$\delta x = \xi(x)\delta t, \quad \delta y = \eta(y)\delta t, \quad \delta z = \zeta(x, y, z)\delta t.$$

This important comment will be needed soon.

We apply a transformation of the form

$$x' = x, \quad y' = y, \quad z' = F(x, y, z)$$

to our equation. The reckoning yields:

$$p' = \frac{dF}{dz}p + \frac{dF}{dx}, \quad q' = \frac{dF}{dz}q + \frac{dF}{dy},$$

$$s' = \frac{dF}{dz}s + \frac{d^2F}{dz^2}pq + \frac{d^2F}{dzdy}p + \frac{d^2F}{dzdx}q + \frac{d^2F}{dydx},$$

and

$$s' + P'p' + Q'q' + Z'z' = \frac{dF}{dz}s + \frac{d^2F}{dz^2}pq + p\left(\frac{d^2F}{dzdy} + P'\frac{dF}{dz}\right)$$

$$+q\left(\frac{d^2F}{dzdx} + Q'\frac{dF}{dz}\right) + \left(\frac{d^2F}{dxdy} + P'\frac{dF}{dx} + Q'\frac{dF}{dy} + Z'F\right).$$

Hence F must be linear in z :

$$F = zf(x, y) + \varphi(x, y),$$

provided that the transformed equation is linear along with the given one.

The coefficient of the quantity p, namely

$$\frac{df}{dy} + P'f$$

vanishes after a suitable choice of f. *Hence any equation (1) can be reduced to the form*

$$s + Qq + Zz = 0. \tag{2}$$

Thus, the problem of determining all the infinitesimal contact transformations leaving invariant an equation of the form (1), is reduced to determining all the infinitesimal point transformations

$$\delta x = \xi(x)\delta t, \quad \delta y = \eta(y)\delta t, \quad \delta z = (zf(x,y) + \varphi(x,y))\delta t, \tag{3}$$

that convert an equation of the form (2) into itself. It is assumed here only that Equation 2 is not integrable by Monge's method.

2. To solve this new problem, we calculate increments of the quantities p, q, s under the infinitesimal transformation (3). For this, we construct the equation

$$\delta(dz - pdx - qdy) = 0$$

or, by permuting the symbols d and δ :

$$d\delta z - pd\delta x - qd\delta y - \delta pdx - \delta qdy = 0,$$

or

$$d(zf + \varphi)\delta t - pd\xi - qd\eta - \delta pdx - \delta qdy = 0.$$

This equation splits into the following two equations:

$$\begin{aligned}
\frac{\delta p}{\delta t} &= \frac{d(zf+\varphi)}{dx} - p\frac{d\xi}{dx} = p\left(f - \frac{d\xi}{dx}\right) + z\frac{df}{dx} + \frac{d\varphi}{dx}, \\
\frac{\delta q}{\delta t} &= \frac{d(zf+\varphi)}{dy} - q\frac{d\eta}{dy} = q\left(f - \frac{d\eta}{dy}\right) + z\frac{df}{dy} + \frac{d\varphi}{dy}\cdot
\end{aligned}$$

Then we construct the equation

$$\delta(dp - rdx - sdy) = 0,$$

that yields the following determination for δs :

$$\begin{aligned}
\delta s &= \frac{d}{dy}\delta p - s\frac{d}{dy}\delta y, \\
\frac{\delta s}{\delta t} &= s\left(f - \frac{d\xi}{dx} - \frac{d\eta}{dy}\right) + p\frac{df}{dy} + q\frac{df}{dx} + z\frac{d^2 f}{dxdy} + \frac{d^2\varphi}{dxdy}\cdot
\end{aligned}$$

We substitute the values of increments $\delta x,\ \delta y,\ \delta z,\ \delta p,\ \delta q$, and δs into the equation

$$\delta(s+Qq+Zz)=0$$

$$=\delta s+Q\delta q+Z\delta z+\left(\frac{dQ}{dx}q+\frac{dZ}{dx}z\right)\delta x+\left(\frac{dQ}{dy}q+\frac{dZ}{dy}z\right)\delta y$$

and obtain the equation

$$s\left(f-\frac{d\xi}{dx}-\frac{d\eta}{dy}\right)+p\frac{df}{dy}+q\left\{\frac{df}{dx}+Q\left(f-\frac{d\eta}{dy}\right)+\frac{dQ}{dx}\xi+\frac{dQ}{dy}\eta\right\}$$

$$+z\left\{\frac{d^2f}{dxdy}+Q\frac{df}{dy}+Zf+\frac{dZ}{dx}\xi+\frac{dZ}{dy}\eta\right\}+\frac{d^2\varphi}{dxdy}+Q\frac{d\varphi}{dy}+Z\varphi=0,$$

which must become an identity upon substituting $s=-Qq-Zz$. In consequence, we find the following four equations:

$$\frac{df}{dy}=0, \tag{4}$$

$$\frac{d^2\varphi}{dxdy}+Q\frac{d\varphi}{dy}+Z\varphi=0, \tag{5}$$

$$Q\frac{d\xi}{dx}+\frac{df}{dx}+\frac{dQ}{dx}\xi+\frac{dQ}{dy}\eta=0, \tag{6}$$

$$Z\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)+\frac{d^2f}{dxdy}+Q\frac{df}{dy}+\frac{dZ}{dx}\xi+\frac{dZ}{dy}\eta=0. \tag{7}$$

They provide the necessary and sufficient conditions for Equation (2) to admit the infinitesimal transformation (3).

The quantity φ appears only in equation (5), which shows that φ must be a solution, namely an arbitrary solution of equation (2). Equation (4) shows that f depends only on x. It remains to satisfy, in general, the following two equations:

$$Q\frac{d\xi}{dx}+\frac{df}{dx}+\frac{dQ}{dx}\xi+\frac{dQ}{dy}\eta=0,$$
$$Z\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)+\frac{dZ}{dx}\xi+\frac{dZ}{dy}\eta=0. \tag{8}$$

Differentiation of the first equation with respect to y yields

$$\frac{dQ}{dy}\left(\frac{d\xi}{dx}+\frac{d\eta}{dy}\right)+\frac{d^2Q}{dxdy}\xi+\frac{d^2Q}{dy^2}\eta=0$$

or:

$$\frac{d}{dy}\left(\xi\frac{dQ}{dy}\right)+\frac{d}{dy}\left(\eta\frac{dQ}{dy}\right)=0.$$

Hence we can set

$$\xi\frac{dQ}{dy}=\frac{dU}{dy},\quad \eta\frac{dQ}{dy}=-\frac{dU}{dx}\cdot$$

The auxiliary function U satisfies the equation

$$\xi\frac{dU}{dx}+\eta\frac{dU}{dy}=0.$$

3. In the following discussion it is necessary to consider separately several subcases.

If both ξ and η are equal to zero, then the determining equations (8) reduce to

$$\frac{df}{dx}=0,\quad \text{or}\quad f=\text{const}.$$

This gives the following.

Theorem. Any partial differential equation

$$s+Qq+Zz=0$$

admits an arbitrarily many infinitesimal transformations of the form

$$\delta x=0,\quad \delta y=0,\quad \delta z=(cz+\varphi(x,y))\delta t,$$

where c is an arbitrary constant and φ is an arbitrary solution of the partial differential equation.

From the two cases: $\xi\gtrless 0,\eta=0$ and $\xi=0,\eta\gtrless 0$, which differ from each other non-essentially, we have to consider one case, e.g., $\xi\gtrless 0,\eta=0$. By introducing a suitable function of x as new x, we can set $\xi=1$. This implies:

$$\frac{dU}{dx}=0,\quad U=U(y),\quad Q=U(y)+\Omega(x),$$

$$\frac{df}{dx}=-\frac{d\Omega}{dx},\quad f=-\Omega(x)+\quad\text{const.},$$

$$\frac{dZ}{dx}=0, Z=Z(y).$$

The differential equation under consideration has then the form

$$s+(U(y)+\Omega(x))q+Z(y)z=0.$$

It admits the infinitesimal transformation

$$\delta x = \delta t, \quad \delta y = 0, \quad \delta z = \{(-\Omega(x) + c)z + \varphi(x, y)\}\delta t,$$

where again $\varphi(x, y)$ denotes an arbitrary solution of the partial differential equation. By introducing a suitable quantity of the form $zF(x)$ as new z, one can reduce $\Omega(x)$ to zero. Furthermore, one notices that the equation $s + U(y)q = 0$ corresponding to $Z = 0$, is integrable by Monge's method. Therefore, introducing a suitable function of y as new y, one can set Z to be equal to 1 without the loss of generality. This gives the folowing statement.

Theorem. If a linear second-order partial differential equation with two distinct families of characteristics admits an infinitesimal transformation under which one and only one family is transformed, then it can be reduced to the canonical form:

$$s + Y(y)q + z = 0, \tag{9}$$

and the infinitesimal transformation is defined by the equations:

$$\delta x = \delta t, \quad \delta y = 0, \quad \delta z = cz\delta t.$$

4. Let us now suppose that both ξ and η are different from zero. Then, without the loss of generality, we can set

$$\xi = 1, \quad \eta = 1.$$

The function U is defined by the equation

$$\frac{dU}{dx} + \frac{dU}{dy} = 0,$$

so that U has the form $U = U(x - y)$. Functions Q, Z and f are defined by the equations

$$\frac{dQ}{dy} = \frac{dU}{dy}, \quad Q = U(x - y) + \Omega(x),$$

$$\frac{dZ}{dx} + \frac{dZ}{dy} = 0, \quad \frac{df}{dx} = -\Omega',$$

whence

$$Q = U + \Omega(x), \quad Z = Z(x - y), f = -\Omega(x) + c.$$

It is clear that by introducing a suitable quantity of the form $zF(x)$ as new z, one can set $\Omega(x) = 0$.

Theorem. If a linear second order partial differential equation with two distinct families of characteristics admits an infinitesimal transformation under which both families are transformed, then it can be reduced to the canonical form

$$s + Q(x-y)q + Z(x-y)z = 0; \tag{10}$$

the infinitesimal transformation is defined by the equations:

$$\delta x = \delta t, \quad \delta y = \delta t, \quad \delta z = (cz + \varphi)\delta t.$$

II.

In the two classes of differential equations (9) and (10), there are certain particularly remarkable equations admitting several infinitesimal transformations. We shall identify these equations.

5. Every infinitesimal transformation converting an equation of the form

$$s + Y(y)q + z = 0$$

into itself, is determined by Equations (8):

$$Y\frac{d\xi}{dx} + \frac{df}{dx} + \frac{dY}{dy}\eta = 0, \quad \frac{d\xi}{dx} + \frac{d\eta}{dy} = 0.$$

We have from the second equation

$$\frac{d\xi}{dx} = -\frac{d\eta}{dy} = \alpha = \quad \text{const.},$$

$$\xi = \alpha x + \beta, \quad \eta = -\alpha y + \gamma.$$

Now the first equations yields

$$\alpha Y + \frac{df}{dx} + (-\alpha y + \gamma)\frac{dY}{dy} = 0,$$

whence

$$\frac{df}{dx} = \delta = \text{const.},$$

$$\alpha Y + \delta + (-\alpha y + \gamma)\frac{dY}{dy} = 0.$$

If $\alpha \gtrless 0$ we can set $\alpha = 1$ and obtain

$$Y = Ay + B.$$

If A is different from zero, one can set B equal to zero without loss of generality . The corresponding equation

$$s + Ayq + z = 0 \tag{11}$$

admits the infinitesimal transformation

$$\delta x = (x+\beta)\delta t, \quad \delta y = (-y+\gamma)\delta t, \quad \delta z = (-A\gamma x + c)z\delta t$$

with two arbitrary constants β and γ. Denoting the infinitesimal transformation

$$\delta x = \xi\delta t, \quad \delta y = \eta\delta t, \quad \delta z = \zeta\delta t$$

by the usual symbol

$$\xi p + \eta q + \zeta r,$$

we must see that equation (11) admits three and only three independent infinitesimal transformations, namely:

$$p, \quad q - Axzr, \quad xp - yq.$$

If, on the other hand, $A = 0$, and therefore $Y = B$, then one obtains the equation: $s + Bq + z = 0$ which, after introducing ze^{Bx} as new z, takes the form: $s + z = 0$. This is only a particular form of the previous equation $s + Ayq + z = 0$.

There is one assumption left: $\alpha = 0$. In this case we can set $\gamma = 1$. Hence Y again has the form

$$Y = Ay + B$$

discussed above.

6. Every infinitesimal transformation leaving invariant an equation of the form:

$$s + Q(x-y)\cdot q + Z(x-y)\cdot z = 0, \tag{12}$$

is determined by equations (8):

$$Q\frac{d\xi}{dx} + \frac{df}{dx} + Q'\cdot(\xi-\eta) = 0, \tag{13}$$

$$Z\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right) + Z'\cdot(\xi-\eta) = 0 \tag{14}$$

We supplement them with the equation obtained by differentiation of the first equation with respect to y:

$$Q'\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right) + Q''\cdot(\xi-\eta) = 0, \tag{15}$$

The last two equations yield:

$$(Q'Z' - ZQ'')\cdot(\xi-\eta) = 0.$$

Hence, if a given partial differential equation (12) admits infinitesimal transformations apart from the known one, i.e., from $p+q$, and therefore $\xi-\eta \gtrless 0$, then the quantities Z and Q' must satisfy a (not identical) relationship of the form:

$$\text{const.}Z + \text{const.}Q' = 0 \tag{16}$$

Further, we notice that: $Z = Q' = 0$ implies only the equation $s + Aq = 0$ integrable by Monge's method. Therefore, we differentiate either equation (14) or equation (15), by treating $x - y$ as a constant, and obtain the equation

$$(\xi - \eta)(\xi'' + \eta'') - \xi'^2 + \eta'^2 = 0 \tag{17}$$

for determining ξ and η.

We differentiate the last equation with respect to y :

$$-\eta'\xi'' + \eta'''\xi - \eta\eta''' + \eta'\eta'' = 0. \tag{18}$$

In the following discussion, it is convenient to handle separately with several subcases.

7. Let's first assume that $\eta = 0$. This assumption obviously leads only to the equations that we have treated before, namely, in section 5, and hence does not require a detailed discussion.

Further, let's assume that $\eta' = 0, \eta = B \gtrless 0$. In this case, if the corresponding equation admits an infinitesimal transformation of the form

$$\xi(x)p + Bq,$$

then it admits also the transformation $p + q$, and therefore the transformation

$$(\xi - B)p.$$

So, this assumption also can result only the equations that we have considered in section 5.

Now, we assume that $\eta' \gtrless 0,\ \eta'' = 0$. Then, in virtu of (18), $\xi'' = 0$, and hence (17) reduces to $\xi'^2 - \eta'^2 = 0$. Hence, we can set

$$\xi = \alpha x + \beta, \quad \eta = \pm\alpha y + \gamma.$$

Since the assumption $\alpha = 0$ leads to the equations that we have considered in Section 5, we can let $\alpha \gtrless 0$, or more specifically $\alpha = 1$. The substitution of the indicated values for ξ and η in (14) yields:

$$Z(1 \pm 1) + Z'(x \mp y + \beta - \gamma) = 0. \tag{19}$$

The upper sign gives:

$$Z = \frac{A}{(x - y + \beta - \gamma)^2}\,, \quad Q = \frac{B}{x - y + \beta - \gamma} + C,$$

where, without loss of generality, we can set $\beta = \gamma,\ C = 0$. The obtained significant equation

$$s + \frac{B}{x - y}\,q + \frac{A}{(x - y)^2}\,z = 0$$

will be soon investigated in detail.

If we take now the lower sign in (19), it follows:

$$Z' = 0, \quad Z = A, \quad Q = B(x - y) + C,$$

where C, without an essential restriction, can be taken equal to zero. The corresponding equation

$$s + B(x - y)q + Az = 0$$

by introducing $ze^{(1/2)Bx^2}$ as new z, turns into the equation

$$s - Byq + z = 0,$$

discussed in Section 5.

Let now $\eta'' \gtrless 0,\ \eta''' = 0$. Then Equation (18) yields

$$\xi'' = \eta'' = a = \text{const.},$$

where one can set $a = 2$ so that ξ and η assume the values

$$\xi = x^2 + bx + c, \qquad \eta = y^2 + \beta y + \gamma. \tag{20}$$

By substituting them into (17), one obtains the relation:

$$4(c - \gamma) + (\beta - b)(\beta + b) = 0. \tag{21}$$

Differentiation of (14), by treating $(x - y)$ as a constant, yields:

$$Z(\xi'' + \eta'') + Z' \cdot (\xi' - \eta') = 0,$$

or, substituting the values (20):

$$4Z + Z' \cdot (2(x - y) + b - \beta) = 0.$$

Therefore Z takes the form:

$$Z = \frac{A}{(x - y + \frac{1}{2}(b - \beta))^2}$$

where one can set $b - \beta = 0$ without loss of generality, and hence $c - \gamma = 0$ in virtu of (21). Formula (16) shows that

$$Q = \frac{B}{x - y} + C,$$

where C can be taken equal to zero by introducing ze^{Cx} as new z.

It remains to determine f; formula (13) yields

$$\frac{df}{dx} = -B, \quad f = -Bx.$$

Hence, the equation

$$s + \frac{B}{x - y} q + \frac{A}{(x - y)^2} z = 0$$

admits three independent infinitesimal transformations, namely:

$$p + q, \ xp + yq, \ x^2 p + y^2 q - Bxzr.$$

Since both x and y undergo threefold (driegliedrig) transformations, it is clear *a'priori* that there are no more infinitesimal transformations; however, this can be easily verified by means of equations (13), (14).

8. Finally, there remains the hypothesis that η''' differs from zero.

Equation (18) shows in this case that ξ satisfies an equation of the form

$$\xi'' = a\xi + b,$$

and therefore ξ has the form

$$\xi = A + B \sin m(x - \alpha).$$

We substitute this expression into the equation

$$\xi' \eta'' + (\xi - \eta)\xi''' - \xi' \xi'' = 0$$

to obtain

$$\eta'' = -m^2(\eta - A),$$

so that η has the form:

$$\eta = A + C \sin m(y - \beta).$$

It is clear that without a restriction we can set

$$A = 0, \quad m = 1, \quad \alpha = 0, \quad \beta = 0.$$

Then the substitution of the obtained values for ξ and η into (17) yields:

$$B^2 - C^2 = 0, \;\; B = \pm C,$$

where we can set $B = +C = 1$.

If we substitute the values for ξ and η in equation (14), it follows:

$$Z(\cos x + \cos y) + Z'(\sin x - \sin y) = 0$$

or:

$$\frac{Z'}{Z} = -\frac{\cos x + \cos y}{\sin x - \sin y} = -\mathrm{cotg}\frac{1}{2}(x - y),$$

whence:

$$Z = \frac{A}{\sin^2 \frac{1}{2}(x - y)}$$

and:

$$Q = B\,\mathrm{cotg}\frac{1}{2}(x - y) + C,$$

where C can be taken equal to zero by introducing ze^{Cx} as new z. By setting

$$\frac{1}{2}x = x', \;\; \frac{1}{2}y = y', \;\; z = z',$$

we reduce our partial differential equation to the form:

$$s' + B\,\mathrm{cotg}(x' - y') \cdot q' + \frac{A}{\sin^2(x' - y')}z' = 0.$$

We do not discuss this equation in details because it follows from the equation

$$s + \frac{B}{x - y}q + \frac{A}{(x - y)^2}z = 0,$$

obtained above, by the substitution

$$z = z'(\cos x')^B, \quad x = \mathrm{tg}x', \quad y = \mathrm{tg}y'.$$

9. We assemble the second-order equations with two distinct families of characteristics admitting infinitesimal transformations in the following table[(6)]:

$s+Y(y)\,q+z=0$ p	$s+Q(x-y)\,q+Z(x-y)z=0$ $p+q$
$s+Cyq+z=0$ $p,q-Cxzr,\ xp-yp$	$s+\dfrac{B}{x-y}q+\dfrac{A}{(x-y)^2}z=0$ $p+q,\ xp+yq,\ x^2p+y^2q-Bxzr$

III.

Now we turn to linear equations of second order with irreducible families of characteristics. It is known that they can assume the canonical form

$$r+Pp+Qq+Zz=0.$$

Furthermore, it is easy to prove that they can be reduced by quadratures to the following simple form:

$$r+\left(\frac{1}{2}\pm\frac{1}{2}\right)q+Z(x,y)z=0.$$

This reduction, that probably has been known long ago, should be done first in what follows.

10. Let an arbitrary equation of the form

$$r'+W(x',y',z',p',q')=0, \tag{22}$$

be given such that its characteristics admit only one integrable combination, corresponding to the integral y'.

If a contact transformation converts this equation into an equation of the similar form, then y' becomes a function of y. Therefore this transformation must have the form (this can be derived, e.g., from my previous investigations on contact transformations):

$$y'=\beta(y),\quad x'=f_1(x,y,z,p),\quad z'=f_2(x,y,z,p),$$

$$p'=f_3(x,y,z,p),\quad q'=qf_4(x,y,z,p)+f_5(x,y,z,p).$$

It follows:

$$dx'=\left(\frac{df_1}{dx}\right)dx+\left(\frac{df_1}{dy}\right)dy,\quad dy'=\beta' dy,$$

whence:

$$dy = \frac{1}{\beta'} dy', \quad dx = \frac{dx'}{(df_1/dx)} - \frac{(df_1/dy)dy'}{(df_1/dx)\beta'} \cdot$$

Thus, we have:

$$r' = \left(\frac{df_3}{dx'}\right) = \left(\frac{df_3}{dx}\right)\frac{dx}{dx'} + \left(\frac{df_3}{dy}\right)\frac{dy}{dx'}$$

or:

$$r' = \frac{(df_3/dx) + p\,(df_3/dz) + r\,(df_3/dp)}{(df_1/dx) + p\,(df_1/dz) + r\,(df_1/dp)} \cdot$$

We carry out such a transformation for any equation of the form

$$r' + P'p' + Q'q' + Z'z' = 0. \tag{23}$$

The transformed equation has the form:

$$r + \frac{C + Dq}{A + Q'\,q\,(df_1/dp)\,f_4} = 0.$$

Thus, in order that an equation of the form (22) *can assume the linear form* (23), *it must be linear in* q.

On the other hand, if a linear equation (23), that is not integrable by Monge's method, can be converted into a linear equation by a contact transformation, then one must have either

$$\frac{df_1}{dp} Q' f_4 = 0, \tag{24}$$

or

$$C : A = D : \frac{df_1}{dp} Q' f_4. \tag{25}$$

Equation (24) is valid only if $\frac{df_1}{dp} = 0$, because both f_4 and Q' must differ from zero, otherwise the equation (23) would be integrable by Monge's method. On the other hand if relation (25) is valid then, in virtu of

$$\begin{aligned}
C &= \frac{df_3}{dx} + p\frac{df_3}{dz} + \left(\frac{df_1}{dx} + p\frac{df_1}{dz}\right)(P'f_3 + Q'f_5 + Z'f_2),\\
D &= \left(\frac{df_1}{dx} + p\frac{df_1}{dz}\right) Q' f_4,\\
A &= \frac{df_3}{dp} + \frac{df_1}{dp}(P'f_3 + Q'f_5 + Z'f_2),
\end{aligned}$$

it follows the relation

$$\frac{df_1}{dp}\left(\frac{df_3}{dx}+p\frac{df_3}{dz}\right)-\frac{df_3}{dp}\left(\frac{df_1}{dx}+p\frac{df_1}{dz}\right)=0=[f_1f_3]=[x'p'],$$

that is impossible.

Therefore:

$$\frac{df_1}{dp}=0, \quad x'=f_1(x,y,z),$$

and in virtu of

$$[y'z']=0, \quad [x'z']=0$$

it follows also that

$$z'=f_2(x,y,z),$$

whence

$$p'=\frac{(df_2/dx)}{(df_1/dx)}, \quad q'=-\frac{1}{\beta'}\left(\frac{df_1}{dy}\right)\frac{(df_2/dx)}{(df_1/dx)}+\left(\frac{df_2}{dy}\right)\frac{1}{\beta'},$$

$$r'=\frac{(dp'/dx)}{(df_1/dx)}=\frac{(df_1/dx)(d^2f_2/dx^2)-(df_2/dx)(d^2f_1/dx^2)}{(df_1/dx)^3}.$$

If we carry out this transformation for the given linear equation (23), the transformed equation takes the form:

$$\frac{r\left[(df_1/dx)(df_2/dz)-(df_2/dx)(df_1/dz)\right]+\Omega(x,y,z,p)}{\left[(df_1/dx)+p(df_1/dz)\right]^3}$$

$$+\frac{Q'}{\beta'}q\left\{\frac{df_2}{dz}-\frac{df_1}{dz}\frac{(df_2/dx)+p(df_2/dz)}{(df_1/dx)+p(df_1/dz)}\right\}+W(x,y,z,p)=0.$$

Multiplying it by $(\frac{df_1}{dx}+p\frac{df_1}{dz})^3$ and then dividing by $\frac{df_1}{dx}\frac{df_2}{dz}-\frac{df_2}{dx}\frac{df_1}{dz}$, one obtains:

$$r+\frac{Q'}{\beta'}q\left(\frac{df_1}{dx}+p\frac{df_1}{dz}\right)^2+\Phi(x,y,z,p)=0.$$

Because this equation must be linear, it follows:

$$\frac{df_1}{dz}=0.$$

Then our transformation becomes:

$$y'=\beta(y),\ x'=\alpha(x,y),\ z'=F(x,y,z),$$

$$p'=\frac{F_x+pF_z}{\alpha_x},\ q'=\frac{1}{\beta'}\frac{-(F_x+pF_z)\alpha_y+(F_y+qF_z)\alpha_x}{\alpha_x},$$

$$r' = \frac{\alpha_x(F_{xx} + 2pF_{xz} + p^2 F_{zz} + rF_z) - (F_x + pF_z)\alpha_{xx}}{\alpha_x^3},$$

so that

$$F_{zz} = 0, \ F = zf(x,y) + \varphi(x,y).$$

This proves the following.

Theorem. If a linear equation

$$r' + P'p' + Qq' + Zz' = 0,$$

non-integrable by Monge's method, is converted to a linear equation of the same form by a contact transformation, then this transformation must have the form:

$$y' = \beta(y), \ x' = \alpha(x,y), \ z' = zf(x,y) + \varphi(x,y),$$

$$p' = \frac{fp + zf_x + \varphi_x}{\alpha_x}, \quad q' = \frac{\alpha_x(fq + zf_y + \varphi_y) - \alpha_y(fp + zf_x + \varphi_x)}{\beta'\alpha_x},$$

$$r' = \frac{\alpha_x(fr + 2f_x p + zf_{xx} + \varphi_{xx}) - \alpha_{xx}(fp + zf_x + \varphi_x)}{\alpha_x^3}.$$

We require that the transformed equation has the form:

$$r + q + Z_1(x,y)z = 0.$$

Then, as it will be shown we can set $\varphi = 0$. This yields the equation

$$\frac{f}{\alpha_x^2} r + \frac{2f_x\alpha_x - \alpha_{xx} f + Pf\alpha_x^2 - Q\alpha_y\alpha_x^2(1/\beta')f}{\alpha_x^3} p + \frac{Qf}{\beta'} q$$

$$+z\left\{\frac{f_{xx}}{\alpha_x^2} - \frac{\alpha_{xx} f_x}{\alpha_x^3} + P\frac{f_x}{\alpha_x} + Q\frac{f_y}{\beta'} - Q\frac{\alpha_y f_x}{\beta'\alpha_x} + Zf\right\}$$

$$= \frac{f}{\alpha_x^2}\{r + q + Z_1 z\},$$

that splits into the following:

$$2f_x\alpha_x - \left(\alpha_{xx} - P\alpha_x^2 + Q\alpha_y\alpha_x^2\frac{1}{\beta'}\right) f = 0, \quad \frac{Q}{\beta'} = \frac{1}{\alpha_x^2}.$$

Here, one can take an arbitrary $\beta(y)$, then the last equation gives α by a quadrature, and finally the first equation gives f by a quadrature.

Thus, every linear equation of the form

$$r + Pp + Qq + Zz = 0, \quad \text{where} \quad Q \gtrless 0,$$

can be reduced with the help of two quadratures to the simple form:

$$r + q + Z(x,y)z = 0.$$

11. Now we determine all the infinitesimal transformations, that leave invariant an equation of the form

$$r + q + Z(x,y)z = 0. \tag{26}$$

According to the previous results, such a transformation must be necessarily of the form

$$\delta x = \xi(x,y)\delta t, \quad \delta y = \eta(y)\delta t, \quad \delta z = (zf(x,y) + \varphi(x,y))\delta t.$$

Therefore:

$$\frac{\delta p}{\delta t} = \left(\frac{d\zeta}{dx}\right) - p\left(\frac{d\xi}{dx}\right) - q\left(\frac{d\eta}{dx}\right) = (f - \xi_x)p + f_x z + \varphi_x,$$

$$\frac{\delta q}{\delta t} = \left(\frac{d\zeta}{dy}\right) - p\left(\frac{d\xi}{dy}\right) - q\left(\frac{d\eta}{dy}\right) = -\xi_y p + (f - \eta_y)q + f_y z + \varphi_y,$$

$$\frac{\delta r}{\delta t} = \left(\frac{d\delta p}{dx}\right) - r\left(\frac{d\xi}{dx}\right) - s\left(\frac{d\eta}{dx}\right) = (f - 2\xi_x)r + (2f_x - \xi_{xx})p + f_{xx}z + \varphi_{xx}.$$

It follows, in view of (26):

$$(f - 2\xi_x)(-q - Zz) + (2f_x - \xi_{xx})p + f_{xx}z + \varphi_{xx}$$

$$-\xi_y p + (f - \eta_y)q + f_y z + \varphi_y + Z(zf + \varphi) + z(Z_x\xi + Z_y\eta) = 0,$$

whence:

$$\begin{aligned} 2\xi_x - \eta_y &= 0, \\ 2f_x - \xi_{xx} - \xi_y &= 0, \\ 2Z\xi_x + f_{xx} + f_y + Z_x\xi + Z_y\eta &= 0, \\ \varphi_{xx} + \varphi_y + Z\varphi &= 0, \end{aligned} \tag{27}$$

While discussing these equations, two different cases can occur, i.e., when η is equal to zero or is different from zero.

If $\eta = 0$ then:

$$\xi_x = 0, \quad \xi = \xi(y), \quad 2f_x = \xi_y, \quad Z_x = 0,$$

so that our equation has the form

$$r + q + Z(y)z = 0.$$

In this case, we always can annul Z by introducing an appropriate quantity of the form $zF(y)$ as new y.

We shall determine the most general infinitesimal contact transformation which transforms the equation found here, i.e., $r + q = 0$, into itself. We obtain the relations

$$2\xi_x - \eta_y = 0, \quad 2f_x - \xi_y = 0, \quad f_{xx} + f_y = 0.$$

Hence,

$$\xi = \frac{1}{2}\, x\, \frac{d\eta}{dy} + Y(y), \quad \frac{df}{dx} = \frac{1}{4}\, x\, \frac{d^2\eta}{dy^2} + \frac{1}{2}Y',$$

$$f = \frac{1}{8}x^2\frac{d^2\eta}{dy^2} + \frac{1}{2}xY' + Y_1(y),$$

$$\frac{1}{4}\frac{d^2\eta}{dy^2} + \frac{1}{8}x^2\frac{d^3\eta}{dy^3} + \frac{1}{2}xY'' + Y_1' = 0,$$

and

$$\eta = \alpha y^2 + \beta y + \gamma, \quad Y = my + n, \quad Y' = -\frac{1}{2}\alpha y + \delta.$$

Thus, the equation $r + q = 0$ will be transformed into itself by the following infinitesimal transformations:

$$p, \quad q, \quad 2yp + xzr, \quad xp + 2yq, \quad 2xyp + 2y^2q + \left(\frac{1}{2}x^2 - y\right) zr.$$

12. We now turn to the case $\eta \gtrless 0$. Let's recall, that, when reducing the equation (23) to the normal form $r + q + Zz = 0$, we could introduce any function of y as new variable y; so, we find out that without loss of generality, we can let $\eta = 1$. It follows that the first three equations (27) become

$$\xi_x = 0, \quad 2f_x - \xi_y = 0, \quad f_{xx} + f_y + Z_x\xi + Z_y = 0,$$

so that ξ is independent of x :

$$\xi = \xi(y).$$

If we introduce now an appropriate quantity $x + \Psi(y)$ as new x [and $ze^{(1/2)x\Psi'(y)}$ as new z], that changes the form of the equation $r+q+Z(x,y)z = 0$ non essentially, then we can always let

$$\xi = 0.$$

This yields:

$$f_x = 0, \quad f_y + Z_y = 0,$$
$$f = f(y), \quad Z = -f(y) + \Omega(x),$$

so that our second-order equation has the form

$$r + q + (-f(y) + \Omega(x))z = 0.$$

To obtain a more simple form, we set

$$z = z'Y(y), \quad x = x', \quad y = y',$$

whence:

$$r = r'Y, \quad q = q'Y + z'Y',$$

so that the transformed equation

$$r' + q' + \left(-f(y) + \Omega(x) + \frac{Y'}{Y}\right) z' = 0,$$

after a suitable choice of Y, takes the simple form:

$$r + q + Z(x)z = 0.$$

If a linear second order equation with an irreducible family of characteristics admits an infinitesimal transformation that alters characteristics, then it can take the form:

$$r + q + Z(x)z = 0. \tag{28}$$

13. Now we look for all equations of the form determined above that admit several infinitesimal transformations.

The most general infinitesimal transformation for such equation is determined by relations (27):

$$2\xi_x = \eta_y, \quad 2f_x = \xi_y, \quad 2Z\xi_x + f_{xx} + f_y + Z_x\xi = 0 \tag{29}$$

whence:

$$\xi = \frac{1}{2}x\eta_y + Y(y), \quad f_x = \frac{1}{4}x\eta'' + \frac{1}{2}Y',$$
$$f = \frac{1}{8}x^2\eta'' + \frac{1}{2}xY' + Y_1,$$

and:

$$Z\eta' + \frac{1}{4}\eta'' + \frac{1}{8}x^2\eta''' + \frac{1}{2}xY'' + Y_1' + Z_x\left(\frac{1}{2}x\eta' + Y\right) = 0.$$

Considering this functional equation, it is not difficult to find the possible values of the unknown functions.

The easiest way to the desired goal is as follows. First of all, we notice that there is no infinitesimal transformation with vanishing η, because this equation according to proceeding statements can be reduced to the form: $r + q = 0$ discussed above. Taking into account my old investigations on transformation groups of a one-dimensional manifold (y) we find out that our equation admits at most three infinitesimal transformations. Moreover, we can assume that the corresponding expression for η has, in general, the form:

$$A + By + Cy^2$$

Indeed, let

$$H_1 = \xi_1 p + \eta_1 q + \zeta_1 r, \quad H_2 = \xi_2 p + \eta_2 q + \zeta_2 r$$

be two infinitesimal transformations of our partial differential equation that transform y and obey the condition

$$(H_1 H_2) = H_1. \tag{30}$$

Then, as before, we always can choose variables x and y such that equation (28) takes the form:

$$r + q + Z(x)z = 0,$$

and H_1 becomes q. Therefore, in virtu of (30):

$$\frac{d\xi_2}{dy} = 0, \quad \frac{d\eta_2}{dy} = 1, \quad \frac{d\zeta_2}{dy} = 0,$$

so that H_2 has the form:

$$H_2 = \xi(x)p + yq + f(x)zr.$$

Equations (29) yield:

$$2\xi_x = \eta_y = 1, \quad \xi = \frac{1}{2}x + a, \quad 2f_x = 0, \quad f = f(y),$$

$$f_y + Z + Z_x\left(\frac{1}{2}x + a\right) = 0,$$

where the constant a, without loss of generality, can be taken equal to zero, whereas f_y must be equal to a constant B. Then the last equation shows that we can set

$$Z = \frac{A}{x^2} - B.$$

Moreover, one can reduce B to zero by introducing an expression of the form $zF(y)$ as new z. Then the infinitesimal transformation H_2 accepts the form:

$$H_2 = xp + 2yq + Czr.$$

The question arises of whether the partial differential equation that was found:

$$r + q + \frac{A}{x^2} z = 0,$$

admits a third infinitesimal transformation of the form:

$$H_3 = \xi p + y^2 q + f(x, y) zr.$$

To find ξ and f we construct the equations

$$\xi_x = y, \quad \xi = xy + Y, \quad 2f_x = x + Y',$$

$$2Zy + f_{xx} + f_y + Z_x \xi = 0. \tag{31}$$

But now there is the relation:

$$\left(\frac{1}{2} xp + yq,\ \xi p + y^2 q\right) = \xi p + y^2 q,$$

whence:

$$\frac{1}{2} x\xi_x + y\xi_y - \frac{1}{2}\xi = \xi, \quad \frac{1}{2} xy + y(x + Y') = \frac{3}{2}(xy + Y),$$

$$yY' = \frac{3}{2} Y, \quad Y = Ky^{3/2}.$$

On the other hand:

$$(q, \xi p + y^2 q) = 2(\frac{1}{2} xp + yq),$$

whence:

$$\xi_y = x, \quad Y' = 0, \quad K = 0.$$

It remains to determine f from equation (31). It follows:

$$2f_x = x, \quad f = \frac{1}{4} x^2 + Y_1,$$

$$2\frac{A}{x^2} y + \frac{1}{2} + Y_1' - 2\frac{A}{x^3} xy = 0,$$

whence:

$$Y_1 = -\frac{1}{2} y + b, \quad f = \frac{1}{4} x^2 - \frac{1}{2} y + b.$$

Therefore, the second-order partial differential equation

$$r + q + \frac{A}{x^2} z = 0$$

admits three infinitesimal transformations

$$q, \quad xp + 2yq, \quad xyp + y^2 q + \left(\frac{1}{4} x^2 - \frac{1}{2} y \right) zr.$$

Thus, we know canonical forms of all the linear second-order partial differential equations with an irreducible family of characteristics that admit infinitesimal transformations. These canonical forms are as follows:

$$\boxed{\begin{array}{c} r + q + Z(x)z = 0 \\ q \end{array}}$$

$$\boxed{\begin{array}{c} r + q = 0 \\ p,\ q,\ 2yp + xzr,\ xp + 2yq,\ xyp + y^2 q + \left(\frac{1}{4} x^2 - \frac{1}{2} y \right) zr \end{array}}$$

$$\boxed{\begin{array}{c} r + q + \frac{A}{x^2} z = 0 \\ q,\ xp + 2yq,\ xyp + y^2 q + \left(\frac{1}{4} x^2 - \frac{1}{2} y \right) zr \end{array}}$$

Second Part: Integration of linear partial differential equations admitting infinitesimal transformations

IV.

After we have defined, in the previous part, canonical forms of all the linear second-order partial differential equations admitting infinitesimal transformations, we turn now to the integration of these canonical forms. Then we show, in the next sections, that a preliminary reduction to the appropriate canonical form is not necessary.

We will consider successively four canonical forms presented on Page 81 and three canonical forms given on Page 90.

14. Every equation of the form

$$s + Y(y)q + z = 0 \tag{32}$$

has particular integrals of the form:

$$z = e^{cx}\Omega(y),$$

where Ω is defined by the equation:

$$(Y + c)\Omega' + \Omega = 0.$$

It follows:

$$\log \Omega = -\int \frac{dy}{Y + c} = -Y_1, \quad \Omega = e^{-Y_1}, \quad z = e^{cx - Y_1}.$$

If an arbitrary function of c is denoted by $f(c)$ then the definite integral:

$$z = \int_{\alpha}^{\beta} e^{cx - Y_1} f(c) dc$$

between the constant limits α and β, is a solution of (32) with an arbitrary function.

Let's consider now the equation

$$s + Cyq + z = 0 \tag{33}$$

with the infinitesimal transformations

$$p, \quad q - Cxzr, \quad xp - yq.$$

The infinitesimal transformation:

$$\frac{\delta x}{1} = \frac{\delta y}{0} = \frac{\delta z}{cz}$$

shows as before that:

$$z = \int_\alpha^\beta e^{cx}(Cy + c)^{-\frac{1}{C}} f(c)dc = z_1$$

is a solution with an arbitrary function. On the other hand, the infinitesimal transformation $q - Cxzr$, or in general form:

$$\frac{\delta x}{0} = \frac{\delta y}{1} = \frac{\delta z}{(-Cx + k)z} \quad (k = \text{const}),$$

reveals that (33) has a solution of the form:

$$z = e^{ky} e^{-Cxy} \Omega(x).$$

Here Ω is defined by the linear differential equation

$$(k - Cx)\Omega' = (C - 1)\Omega,$$

and hence equation (33) has a solution of the form:

$$z = \int_{\alpha_1}^{\beta_1} e^{(k - Cx)y}(Cx - k)^{-(1-C)/C} \varphi(c)dc = z_2.$$

Thus: $z = z_1 + z_2$ is a solution of equation (33) with *two* arbitrary functions.

The third canonical form

$$s + Q(x - y)q + Z(x - y)z = 0 \tag{34}$$

admits the infinitesimal transformation $p + q$ and simultaneously the more general transformation $p + q + czr$ or equivalently:

$$\frac{\delta x}{1} = \frac{\delta y}{1} = \frac{\delta z}{cz}$$

Hence, equation (34) has solutions of the form

$$z = e^{cx}\Omega(x-y),$$

where Ω is defined by the linear ordinary differential equation:

$$\Omega'' + (c+Q)\Omega' - Z\Omega = 0.$$

If Ω_1 and Ω_2 are two independent solutions of this equation then

$$z = \int_{\alpha_1}^{\beta_1} \Omega_1(x-y,c)e^{cx}f_1(c)dc + \int_{\alpha_2}^{\beta_2} \Omega_2(x-y,c)e^{cx}f_2(c)dc$$

is a solution of (34) with two arbitrary functions.

Specifically, if one deals with the integration of the important equation

$$s + \frac{B}{x-y}q + \frac{A}{(x-y)^2}z = 0 \tag{35}$$

with three infinitesimal transformations:

$$p+q, \quad xp+yq, \quad x^2p+y^2q-Bxzr, \tag{36}$$

then, according to our method, one should first integrate the linear equation

$$\Omega'' + \left(c + \frac{B}{x-y}\right)\Omega' - \frac{A}{(x-y)^2}\Omega = 0.$$

Because I don't know whether this auxiliary equation can be solved, in general, by the known methods, *I* develop the other noteworthy method in which all the three infinitesimal transformations are utilized.

If λ denotes an arbitrary constant, then equation (35) admits the infinitesimal transformation:

$$p+q+2\lambda(xp+yq)+\lambda^2(x^2p+y^2q-Bxzr)$$

and simultaneously the transformation

$$\frac{\delta x}{(1+\lambda x)^2} = \frac{\delta y}{(1+\lambda y)^2} = \frac{\delta z}{-B\lambda z(1+\lambda x)}.$$

It follows that equation (35) has solutions of the form:

$$z = (1+\lambda x)^{-B} W\left(\frac{\lambda(x-y)}{(1+\lambda x)(1+\lambda y)}\right).$$

Here W is defined by the equation

$$-\frac{\lambda^2}{(1+\lambda x)^2(1+\lambda y)^2}W'' - B\lambda\frac{W'}{(x-y)(1+\lambda x)(1+\lambda y)}$$

$$+A\frac{W}{(x-y)^2} = 0,$$

which, after setting

$$\frac{\lambda(x-y)}{(1+\lambda x)(1+\lambda y)} = \omega$$

accepts the integrable form

$$-W'' - \frac{B}{\omega}W' + \frac{A}{\omega^2}W = 0.$$

This auxiliary equation is satisfied by the assumption $W = \omega^m$, when the constant m is defined by the equation of second degree:

$$m(m-1) + Bm - A = 0 = m^2 + (B-1)m - A.$$

Hence, it is always possible to obtain a solution of equation (35) with two arbitrary functions. If the roots of the last equation are distinct, the corresponding solution has the form

$$z = \int_{\alpha_1}^{\beta_1}(1+\lambda x)^{-B}\omega^{m_1}f_1(\lambda)d\lambda + \int_{\alpha_2}^{\beta_2}(1+\lambda x)^{-B}\omega^{m_2}f_2(\lambda)d\lambda.$$

15. Now turn to the integration of the three canonical forms presented on pages 356-357 [boxed formulas in Section 13. N.H.I.]. Since the equation

$$r + q = 0$$

admits five independent infinitesimal transformations:

$$p,\ q,\ 2yp + xzr,\ xp + 2yq,\ 2xyp + 2y^2q + \left(\frac{1}{2}x^2 - y\right)zr,$$

we can obtain its solution with two arbitrary functions using several ways.

The infinitesimal transformation q, or more generally

$$q + kzr$$

shows that there exist solutions of the form:

$$e^{ky}X(x).$$

Here

$$X'' + kX = 0,$$

$$X = A \sin x\sqrt{k} + B \cos x\sqrt{k},$$

so that

$$z = \int_{\alpha_1}^{\beta_1} \sin x\sqrt{k} \cdot e^{ky} f_1(k) dk + \int_{\alpha_2}^{\beta_2} \cos x\sqrt{k} \cdot e^{ky} f_2(k) dk$$

represents a solution with two arbitrary functions. A different (by the form) solution with two arbitrary functions is obtained as follows. One constructs first a solution with one arbitrary function using the infinitesimal transformation $p + czr$, then likewise one constructs a solution with one arbitrary function using the infinitesimal transformation $2yp + xzr$. After that one adds together these solutions.

A third solution with two arbitrary functions is obtained with the help of the infinitesimal transformation

$$p + \varepsilon q + \rho zr.$$

In the particular solution obtained here, one treats, for example ρ, as an arbitrary function of ε and then integrates with respect to ε, etc.

In order to find a solution with two arbitrary functions for an arbitrary equation of the form

$$r + q + Z(x)z = 0, \tag{37}$$

one makes use of the infinitesimal transformation $q + czr$, which shows that there exist particular solutions of the form

$$z = e^{cy} X(x).$$

Here X is defined by the equation

$$X'' + (c + Z)X = 0.$$

Let $X_1(x, c)$ and $X_2(x, c)$ be its two solutions. Then

$$z = \int_{\alpha_1}^{\beta_1} e^{cy} X_1(x, c) f_1(c) dc + \int_{\alpha_2}^{\beta_2} e^{cy} X_2(x, c) f_2(c) dc$$

is a solution with two arbitrary functions.

Specifically, if one deals with the integration of equation

$$r + q + \frac{A}{x^2} z = 0 \tag{38}$$

with the infinitesimal transformations

$$q, \quad xp + 2yq, \quad xyp + y^2q + \left(\frac{1}{4}x^2 - \frac{1}{2}y\right) zr,$$

then, following our method, one should first integrate the auxiliary equation

$$X''(x) + \left(c + \frac{A}{x^2}\right) X(x) = 0.$$

However, it is also possible to proceed the following way:

Equation (38) admits the infinitesimal transformation

$$q + \lambda(xp + 2yq - \frac{1}{2}zr) + \lambda^2(xyp + y^2q + \left(\frac{1}{4}x^2 - \frac{1}{2}y\right) zr).$$

We construct the simultaneous system

$$\frac{dx}{\lambda x(1+\lambda y)} = \frac{dy}{(1+\lambda y)^2} = \frac{dz}{\lambda^2 z\left(\frac{1}{4}x^2 - \frac{1}{2}y\right) - \frac{1}{2}\lambda z}$$

with two solutions:

$$\frac{x}{1+\lambda y} = w, \quad \log z - \frac{\lambda^2 yx^2}{4(1+\lambda y)^2} + \frac{1}{2}\log(1+\lambda y).$$

Therefore equation (38) has solutions of the form

$$z = (1+\lambda y)^{-\frac{1}{2}} e^{\frac{\lambda^2 yx^2}{4(1+\lambda y)^2}} F\left(\frac{x}{1+\lambda y}\right).$$

One can find that $F(w)$ is determined, as a function of w, from the linear ordinary differential equation

$$F''(w) - \lambda w F' + \left(\frac{1}{4}\lambda^2 w^2 - \frac{1}{2}\lambda + \frac{A}{w^2}\right) F = 0.$$

The substitution

$$F = e^{\frac{1}{4}\lambda w^2}\Phi(w)$$

reduces this equation to the integrable form

$$\Phi'' + \frac{A}{w^2}\Phi = 0.$$

The latter equation is satisfied by the assumption

$$\Phi = w^m,$$

where

$$m(m-1)+A=0.$$

If this equation of the second degree is valid for two different values m_1 and m_2 of a constant m, then

$$z=\int_{\alpha_1}^{\beta_1}\psi(\lambda)e^{\frac{1}{4}\lambda w^2}w^{m_1}f_1(\lambda)d\lambda+\int_{\alpha_2}^{\beta_2}\psi(\lambda)e^{\frac{1}{4}\lambda w^2}w^{m_2}f_2(\lambda)d\lambda$$

is a solution of (38) with two arbitrary functions; here, for the sake of brevity, the following notation is used:

$$(1+\lambda y)^{-\frac{1}{2}}e^{\frac{\lambda^2 yx^2}{4(1+\lambda y)^2}}=\psi(\lambda).$$

16. Thus, we have found solutions with two arbitrary functions for six canonical forms:

$$\begin{aligned} s+Cyq+z&=0,\\ s+Q(x-y)\cdot q+Z(x-y)\cdot z&=0,\\ s+\frac{A}{x-y}q+\frac{B}{(x-y)^2}z&=0,\\ r+q&=0,\\ r+q+Z(x)z&=0,\\ r+q+\frac{A}{x^2}z&=0. \end{aligned}$$

On the contrary, we have found for the equation

$$s+Y(y)q+z=0$$

a solution only with one arbitrary function.

Some of these equations have been integrated long ago. It should be noted, however, that the method that I have used can be applied actually to all equations, that can be reduced to any of these canonical forms by a suitable contact transformation. This is to be shown in the next sections.

V.

17. Let an arbitrary linear second order partial differential equation

$$Rr+Ss+Tt+Pp+Qq+Zz=0, \tag{39}$$

nonintegrable by Monge's method, be given. It follows from the preceding that any infinitesimal contact transformation that converts this equation into itself, must have the form

$$\delta x=\xi(x,y)\delta t,\quad \delta y=\eta(x,y)\delta t,\quad \delta z=\{zf(x,y)+\varphi(x,y)\}\delta t,$$

where φ is an arbitrary solution of equation (39). If we set $\varphi = 0$ and seek the most general infinitesimal transformation:

$$\delta x = \xi \delta t, \quad \delta y = \eta \delta t, \quad \delta z = z f \delta t, \tag{40}$$

that transforms equation (39) into itself, then we find five equations for determining three quantities ξ, η and f. Then one obtains, by differentiation, still further equations.

Here, various cases may take place: either there is no system of values ξ, η, f that satisfies the relations under consideration (this is a general case) , or there are several such systems of values, namely $1, 3$ or 5. The corresponding systems of values are determined by ordinary differential equations, that can be integrated by a rational theory.

18. If an infinitesimal transformation (40) is found in this way, then the partial differential equation[(7)]

$$\xi p + \eta q - z(f + c) = 0$$

with an arbitrary constant c has ∞^2 solutions, common with the given second order equation. These common solutions are obtained by integrating an ordinary differential equation of the second order. Then, as before, one obtains a solution with two arbitrary functions.

The given method always leads to the aim, when all the characteristics of equation (39) are transformed by the infinitesimal transformation. The correctness of the theory developed in the present section follows without difficulties from the theories, explained in the previous sections.

VI.

An interesting application of the previously explained theory is the following:

19. Let two families of curves, c and c_1, be given on a sphere, that form an orthogonal system. Then it's well known that all surfaces having c and c_1 as the spherical image of their lines of curvature, are defined by a known linear second-order partial differential equation. If we assume that both families of curves c and c_1 are transformed into themselves under the infinitesimal rotation (or infinitesimal conformal transformation) of the spherical image, then the corresponding partial differential equation admits a known infinitesimal transformation of the form considered previously. If we exclude the simple case when c and c_1 are families of meridian and parallel circles, then our partial differential equation has the form:

$$s + Q(x - y) \cdot q + Z(x - y) \cdot z = 0. \tag{41}$$

Its solution with two arbitrary functions is obtained by integrating the linear second-order differential equation

$$W'' + (c + Q)W' - ZW = 0.$$

In general, the integration of this auxiliary equation cannot be done.

20. Suppose that the auxiliary equation for $c = 0$ (or for any special value of c) can be integrated. Then one obtains a special solution

$$z = W(x - y)$$

of the partial differential equation (41). Further, by means of successive quadratures, one finds an arbitrary number (∞^∞) of solutions for equation (41).

Then equation (41) admits the infinitesimal transformation

$$\frac{\delta x}{1} = \frac{\delta y}{1} = \frac{\delta z}{W(x - y)}$$

and therefore has a solution of the form

$$z = xW(x - y) + W_1(x - y),$$

where W_1 is defined by the equation

$$W_1'' + QW_1' - ZW_1 + W' = 0,$$

which is always integrable. Furthermore, equation (41) admits the infinitesimal transformation

$$\frac{\delta x}{1} = \frac{\delta y}{1} = \frac{\delta z}{xW + W_1}$$

and therefore has the solution of the form:

$$z = \frac{1}{2}x^2 W(x - y) + xW_1(x - y) + W_2,$$

where W_2 is again defined by an integrable equation. This procedure can be repeated infinitely; thus one finds, by successive quadratures, an arbitrary number (∞^∞) of solutions.

21. Similarly, one obtains, for example, by successive quadratures an infinite (∞^∞) surfaces, the lines of curvature of which have the same spherical image as those for any given screw surface.

On the other hand, if one takes any surface which transforms into itself by any infinitesimal transformation then first one finds lines of curvature for this surface by a quadrature. Then by successive quadratures one finds an arbitrary number (∞^∞) of surfaces, curvature lines of which have the same spherical image. By the way, it should be noticed, that the method can be applied to any linear second-order equation admitting an infinitesimal transformation.

Review by the author

[F.d.M., Bd.XIII, Jahrg. 1881, S.298-300. Berlin 1883. Reprinted in [27], vol.3, paper XXXVa.]

This paper begins with a complete transformation theory for all linear homogeneous second order partial differential equations with two independent variables and one dependent variable.

Each of these equations admits an infinite set of invariance transformations. Namely, if $z = f(x, y)$ is an arbitrary solution then the equation admits *eo ipso* the transformation

$$x' = x, \quad y' = y, \quad z' = cz + f,$$

where c denotes an arbitrary constant. This evident transformation leaves x and y invariant.

There exist several classes of equations, that admit, in addition, invariance transformations changing x and y as well. All such equations can be reduced, by a suitable choice of coordinates, to one of the following forms:

$$s + Y(y)q + z = 0, \tag{1}$$

$$s + Q(x - y) \cdot q + Z(x - y) \cdot z = 0, \tag{2}$$

$$s + \text{const. } yq + z = 0, \tag{3}$$

$$s + \frac{A}{x - y}q + \frac{B}{(x - y)^2}z = 0, \tag{4}$$

$$r + q + Z(x)z = 0, \tag{5}$$

$$r + q = 0, \tag{6}$$

$$r + q + \frac{A}{x^2}z = 0. \tag{7}$$

Equations of canonical forms (1), (2) or (5) admit only one infinitesimal transformation that changes x and y. Each of the equations (3), (4) and (7) admits three independent infinitesimal transformations that change x and y. Finally, Equation (6) admits five infinitesimal transformations of that kind.

It should be noticed, that the differential equation of minimal surfaces is a special case of the type (4). This type of equations involves, however, a set of easily definable equations, the solutions of which belong to the Ampére's first class.

The above-mentioned equations can not be solved, in general, by Laplace's method. However, it is possible to get a particular solution with two arbitrary constants. These constants appear in a nonlinear way, whence one immediately obtains a solution with two arbitrary functions that appear in particular integrals.

To accomplish this integration procedure it is not necessary to reduce the given equation to its canonical form. Thus, given a linear second order partial differential equation between z, x, y admitting an infinitesimal transformation that changes x and y, it is always possible to get a solution with arbitrary functions by integrating ordinary differential equations.

Although the integration accomplished here is not complete because arbitrary functions occur in partial integrals, the present theory seems to be a substantial advance.

It is in the nature of the method that these investigations can be extended to linear equations of any order with an arbitrary number of variables.

Editor's Notes

(1) *Page 6.* First published in [15]. Reproduced here by courtesy of CRC Press.

(2) *Page 67.* Laplace [22]. See also Darboux [7] Chap. 3, and Goursat [13].

(3) *Page 70.* See e.g. Lie [26].

(4) *Page 70.* In the German original, F is misprinted as Z.

(5) *Page 70.* Equation (1) is said to be integrable by Monge's method if it possesses an intermediate integral with one arbitrary function, or, equivalently, if differential equations for one of the systems of characteristics admit two distinct integrable combinations. See, e.g., E. Goursat [13].

(6) *Page 80.* In the German original, the last but one equation in the table is misprinted with A and B permuted.

(7) *Page 98.* This equation defines invariant solutions.

SURFACE TRANSFORMATIONS

By A.V. Bäcklund

Translated from German by
N.H.Ibragimov and R.M. Yakushina

[Ueber Flächentransformationen, A.V. Bäcklund, Mathematische Annalen, IX, 1876, S. 297–320.]

I.

Recently, I became concerned with the question if there are surface transformations of a three-dimensional space that leave invariant the second-order tangency (osculation) rather than the first-order tangency. I discussed this question in volume 10 of Annual Reports of Lund University (Sept. 1874) and came to the conclusion that the transformations that leave invariant the first-order tangency, i.e. Lie's contact transformations, are the only ones for which higher-order tangency conditions are invariant(1). Simultaneously, Lie's paper* appeared in volume 8 of Mathematischen Annalen where the similar question on osculating transformations was raised. Therefore I would like to undertake a detailed presentation of the previous investigation and to dwell upon certain points that were only hinted before.

I begin with the proof of nonexistence of a proper osculating transformation of plane curves, giving first pure geometric reasoning and then an analytic proof. Only then I proceed to an exhaustive survey of the problem.

§ 1. Geometric proof of nonexistence of proper osculating transformations of plain curves

1. The osculating transformation transforms any curve of a plane into one or several, but not infinitely many, curves of the same plane. Moreover, it converts any two curves in osculation into two likewise osculating curves. Furthermore, application of an osculating transformation to a figure, consisting of a curve C and two curves C' and C'' that are infinitely close to

*S. Lie, 'Begründung einer Invariantentheorie der Berührungstransformationen', *Math. Annalen,* **8**, 1874, 215-288.

each other and osculate with C in two neighboring points, results in a new figure that consists of a curve Γ, the image of C, and two infinitely close to each other curves Γ' and Γ'', images of C' and C'', respectively, that osculate with the curve Γ in two neighboring points. Since C' and C'' are in osculation with one and the same curve in two neighboring points, they should contact each other. Likewise, the curves Γ', Γ'' should contact each other. We conclude that *any osculating transformation should posses a property to map any two infinitely close curves contacting each other into two curves of the same kind.*

But this property characterizes, *as I will show bellow*, Lie's contact transformations. Hence, a proper osculating transformation does not exist.

2. Consider on the plane (x, y) a transformation of the above kind. It maps any curve of the plane to a curve and converts two contacting each other curves into two other likewise contacting curves. Furthermore, let $\lambda_1\,\lambda_2\,\lambda_3$ be parameters of a three-fold system[(2)] of curves $\psi(x\,y\,\lambda_1\,\lambda_2\,\lambda_3) = 0$ and let

$$\varphi(\lambda_1\,\lambda_2\,\lambda_3\,d\lambda_1\,d\lambda_2\,d\lambda_3) = 0 \tag{1}$$

be the contact condition for two neighboring curves $(\lambda)(\lambda + d\lambda)$. Let the curves obtained from the curves (λ) by means of the above transformation be represented by the equation

$$f(x\,y\,\lambda_1\,\lambda_2\,\lambda_3) = 0. \tag{2}$$

Then eliminating $x\ y\ p$ from (2) and from the following three equations [(3)]:

$$\begin{cases} f'(x) + pf'(y) = 0, \\ \sum \dfrac{df}{d\lambda} d\lambda = 0, \\ \sum \dfrac{df'(x)}{d\lambda} d\lambda + p \sum \dfrac{df'(y)}{d\lambda} d\lambda = 0, \end{cases} \tag{3}$$

one arrives again at equation (1). Hence, the last equation represents the contact condition for two consecutive curves (λ) as well as for two consecutive curves (2). Thus, when two neighboring curves (λ) satisfy the contact condition, the corresponding curves (2) must also obey the contact condition.

Conversely, when two three-fold systems of curves provide a definite differential equation as the contact condition, they furnish a transformation of the above character.

Given a three-parameter system of curves regarded as the system (λ), one can obtain a significant correlation between this system and a system (2) in the following way.

Let us consider the parameters λ as coordinates of points in a three-dimensional space R_3 and the variables $x\,y$ as arbitrary constants. Then equation (2) represents a system of ∞^2 surfaces in R_3. Furthermore, equation (1) associates with every point of R_3 an elementary complex-cone (Complexkegel), and since (1) results from (2) via the equations (3), any two infinitely approaching surfaces (2) must intersect each other along a curve. All the linear elements $(\lambda\, d\lambda)$ of this curve satisfy equation (1) and hence belong to the elementary complex-cones (1). In other words the curves of intersection should belong to complex (1). – Note that ∞^1 surfaces (2) pass through any point of R_3. The envelope of their tangent plains at any point is a cone that coincides near the point with the cone, formed by the linear elements, through the given point, of the curves of intersection of every two neighboring ∞^1 surfaces. However, this cone corresponds to elementary complex-cone (1) with the vertex at the above point. Consequently, any of the ∞^2 surfaces (2) will contact cone (1) in all the points of the corresponding surface. Thus: *the surfaces (2) build up a solution with two arbitrary constants $x\,y$ of the first-order differential equation the elementary complex-cones of which are represented by equation (1).*

Hence, two neighboring integrals of the first-order partial differential equation (I denote it briefly $\Phi = 0$) should intersect along a characteristic of this equation. In consequence, the curve represented by the equations

$$f = 0, \quad f'(x) + p f'(y) = 0$$

should be a characteristic of $\Phi = 0$, and a certain characteristic passes through every point* $x\,y\,p$. It means that infinitely many one-fold curves (2) having one common element $x\,y\,p$ correspond to ∞^1 points (λ) that constitute a characteristic of $\Phi = 0$. Let us represent *the system of curves* (λ) via its equation in point coordinates $x\,y$

$$\varphi(x\,y\,\lambda_1\,\lambda_2\,\lambda_3) = 0. \tag{4}$$

*The result of the above speculation is the following: any ordinary nonlinear differential equation $\varphi(\lambda\,d\,\lambda) = 0$ can in infinitely many ways be replaced by ∞^3 curves $\varphi(x\,y\,p\,\lambda_1\,\lambda_2\,\lambda_3) = 0, \quad \chi(x\,y\,p\,\lambda_1\,\lambda_2\,\lambda_3) = 0$. In other words, according to Lie, $\varphi(\lambda\,d\,\lambda) = 0$ forms the basis of a certain curve-complex. There exists only one system of curves of this complex that can be represented by a system of equations

$$\psi(x\,y\,\lambda_1\,\lambda_2\,\lambda_3) = 0, \quad \psi'(x) + p\,\psi'(y) = 0.$$

These are the characteristics of the first-order partial equation connected with $\varphi(\lambda\,d\,\lambda) = 0$.

There is one more implication of the above: any ordinary nonlinear differential equation $\varphi(\lambda\,d\,\lambda) = 0$ can be considered as contact condition of two neighboring curves taken from a three-parameter system of curves.

If $x\,y\,\lambda_1\,\lambda_2\,\lambda_3$ are interpreted in the similar way, then this system provides a solution of $\Phi = 0$ where $x\,y$ are two arbitrary constants. Besides, every family of curves (λ) (i.e. the curves (4)) that contact each other in one point must correspond to a characteristic of $\Phi = 0$. But since there are no more than ∞^3 characteristics of $\Phi = 0$, one characteristic does not, in general, envelope a family of ∞^1 characteristics. Therefore, conversely, to the points (λ) of any characteristic of $\Phi = 0$ there correspond infinitely many one-fold curves (λ), curves (4), that contact each other in one and the same point. As one and the same characteristic corresponds to both families of curves, ∞^1 curves (λ) osculating in one point correspond to ∞^1 curves (2) that osculate in one point.

Thus, the following statement is proved. Supposed that *between two three-parameter systems of curves there is given a correspondence such that two neighboring tangent curves of one system correspond to two curves of the same kind from the other one. Then all the osculating in one point curves of one of the systems correspond to similar curves of the other system.* Consequently, the transformation that maps one system of curves into the other is a transformation of linear elements $(x\,y\,p)$. Furthermore, it must map any two connected elements into two similar elements, because two connected linear elements always belong to a curve (real or imaginary) of one of the systems and the corresponding elements will join the corresponding curve. *Hence, every transformation of the above kind is also Lie's contact transformation. q.e.d.*

§ 2. Analytical proof of the same theorem

3. Since any curve of the plane determines definite values of $xypp'\ldots$* which conversely determine the curve, any curve transformation between two domains (xy) and (XY) (these domains are supposed to be overlapping) of the plane is a transformation between $xypp'\ldots$ and $XYPP'\ldots$. In particular, an osculating transformation converts the quantities $(xypp')$ into $(XYPP')$; and naturally all the values $(xypp')$ belonging to a curve in the plane (xy) are mapped into the corresponding values of a curve in the plane (XY). Thus, every osculating transformation is defined by equations of the form:

$$\begin{aligned} x &= F(XYPP'), \\ y &= F_1(\qquad\quad), \\ p &= \Phi_1(\qquad\quad), \\ p' &= \Phi_2(\qquad\quad), \end{aligned}$$

---

*$p = dy/dx, \quad p' = dp/dx, \ldots$

where $F \dots \Phi_2$ are to be determined in such a way that the system of equations

$$dy - p\,dx = 0, \qquad dp - p'dx = 0 \tag{a}$$

is transformed into the similar system:

$$dY - PdX = 0, \qquad dP - P'dX = 0. \tag{b}$$

The latter statement represents the analytical proof to the assertion: if two neighboring elements $(x \,.\,.\, p'), (x + dx \,.\,.\, p' + dp')$ belong to a curve then the corresponding elements $(X \,.\,.\, P'), (X + dX \,.\,.\, P' + dP')$ also belong to a certain curve.

Let us consider the following series of ∞^1 consecutive elements $(xypp')$:

$$\begin{aligned}
&x_0\, y_0\, p_0\, p', \\
&x_0\, y_0\, p_0\, p' + dp', \\
&x_0\, y_0\, p_0\, p' + 2dp', \\
&\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Since any two neighboring elements of this series satisfy the equations (a), (because here $dx = dy = dp = 0$), and since any two neighboring elements of the corresponding ∞^1 consecutive elements $(XYPP')$ must also satisfy the equations (b), we conclude that the latter ∞^1 elements should belong to some curve. Upon eliminating P, P' from the equations

$$\begin{aligned}
x_0 &= F(XYPP'), \\
y_0 &= F_1(\qquad\quad), \\
p_0 &= \Phi_1(\qquad\quad),
\end{aligned}$$

one obtains the equation of the curve that *corresponds to the linear element* $(x_0 y_0 p_0)$. Likewise, one obtains the equation of a curve in (xy) that *corresponds to the linear element* (XYP). – In other words after the elimination of the quantity P' from the equations of transformation, every osculating transformation must result in two equations:

$$f(x\,y\,p\,X\,Y\,P) = 0, \qquad \varphi(x\,y\,p\,X\,Y\,P) = 0, \tag{c}$$

which have one common integral in both variables xyp and XYP. Conversely, two equations (c) possessing the above properties determine an osculating transformation provided that the equations*

$$f=0,\quad \varphi=0,\quad \frac{df}{dx}+p\frac{df}{dy}+p'\frac{df}{dp}=0,\quad \frac{df}{dX}+P\frac{df}{dY}+P'\frac{df}{dP}=0$$

associate to an arbitrary system of quantities $(XYPP')$, resp. $(xypp')$, a system of quantities $(xypp')$, resp. $(XYPP')$, or several systems of this kind.

I will show, however, that systems of equations possessing the properties (c) can not provide all the quantities $(xypp')$ *of the plane.* Indeed, these equations associate with three-fold infinite (XYP) only a two-fold infinite set of curves. Hence, the above construction can provide only the ∞^3 elements $(xypp')$ of these curves. Thus, it is proved that a proper osculating transformation does not exist.

4. Elimination of P, resp. p, from the equations (c) leads to two equations:

$$p=f(xyXY),\qquad P=\varphi(xyXY). \tag{d}$$

They replace the equations (c) and should have a common integral both in the (xy) and (XY) spaces. Then the relations between the equations (d) are expressed algebraically as follows:

$$\frac{d\varphi}{dx}+f\frac{d\varphi}{dy}=0,\qquad \frac{df}{dX}+\varphi\frac{df}{dY}=0.$$

Elimination of f results in an equation for determining φ:

$$\frac{d}{dX}\left(\frac{d\varphi}{dx}:\frac{d\varphi}{dy}\right)+\varphi\frac{d}{dY}\left(\frac{d\varphi}{dx}:\frac{d\varphi}{dy}\right)=0.$$

which can be rewritten in the form:

$$\frac{d}{dx}\left(\frac{d\varphi}{dX}+\varphi\frac{d\varphi}{dY}\right)-\left(\frac{d\varphi}{dx}:\frac{d\varphi}{dy}\right)\cdot\frac{d}{dy}\left(\frac{d\varphi}{dX}+\varphi\frac{d\varphi}{dY}\right)=0.$$

If $\frac{d\varphi}{dx}+p\frac{d\varphi}{dy}=0$ then the following equation is also valid:

$$\left(\frac{d}{dx}+p\frac{d}{dy}\right)\left(\frac{d\varphi}{dX}+\varphi\frac{d\varphi}{dY}\right)=0.$$

*Combined when necessary with the equations

$$\frac{d\varphi}{dx}+p\frac{d\varphi}{dy}+p'\frac{d\varphi}{dp}=0,\quad \frac{d\varphi}{dX}+P\frac{d\varphi}{dY}+P'\frac{d\varphi}{dP}=0,$$

which are compatible with the above ones due to the relations between f and φ.

Thus the differentials[(4)] of φ vanish and therefore

$$\frac{d\varphi}{dX} + \varphi\frac{d\varphi}{dY} = \psi(X, Y, \varphi).$$

The solution of this equation is given by

$$\Phi(\Psi_1(X, Y, \varphi), \Psi_2(X, Y, \varphi)\, x, y) = 0,$$

where Φ is an arbitrary function of four variables. If one replaces here φ by P, one obtains the second equation from (d). This equation is only two-fold infinite with respect to XYP. Therefore, the ∞^3 elements (XYP) belong only to ∞^2 curves in the xy plane. Thus, there are no proper osculating transformations of curves on a plane given by one-to-one mappings. Q.e.d.

II.

The following question arises. How far can one extend the previous results to multi-dimensional spaces? I will consider this question and will deal with the general problem of determining all those transformations of an $(n+1)$-dimensional space converting the n - dimensional varieties, i.e. the surfaces in this space, into one another. These transformations are obviously divided into two distinctly different classes. The first class comprises transformations that convert any surface of the space $(zx_1x_2 \ldots x_n)$ into only one surface (or several surfaces) of the space $(ZX_1X_2 \ldots X_n)$. The second class consists of transformations that map any surface into infinitely many surfaces.

Consider again two-dimensional spaces, i.e. planes. Note that any curve of the plane is completely determined by the quantities $(xypp' \ldots)$, and that two neighboring quantities $(xypp' \ldots)$ and $(x + dx\, y + dy\, p + dp\, p' + dp' \ldots)$ belong to one and the same curve if and only if the following equations hold:

$$dy - p\, dx = 0, \qquad dp - p' dx = 0 \ldots. \tag{A}$$

Therefore any curve transformation between (xy) and (XY) planes is described by the requirement that equations (A) are mapped into the similar equations:

$$dY - PdX = 0, \qquad dP - P' dX = 0, \ldots. \tag{B}$$

Thus, in order to determine a curve transformation, it is sufficient to specify two arbitrary equations

$$\begin{cases} X = F(xypp' \cdots p^k), \\ Y = F_1(xypp' \cdots p^l) \end{cases} \tag{C}$$

and deduce, employing (A), (B), the following equations:

$$\left\{\begin{array}{l} P = \Phi(xypp' \cdots), \\ P' = \Phi_1(\qquad), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right. \tag{D}$$

In general the equations (C), (D) cannot be solved for $xypp' \cdots$. Then the transformation belongs to the second class mentioned above, i.e. it is a *multivalued* transformation. Though any curve in the (xy) plane is mapped only into one curve in the (XY) plane, a curve of the latter plane corresponds to infinitely many curves in the (xy) plane, namely, all solutions of a certain differential equation*. However, if the equations (C) together with the first k equations from (D) form a system that can be solved with respect to $xyp \ldots p^{k-1}$ so that these equations can be represented in the form

$$\begin{array}{l} x = f(XYP\ldots), \\ y = f_1(\qquad), \\ p = \varphi(\qquad), \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}$$

then the transformation will belong to the first class, i.e. it will be a single-valued (or finite-valued) transformation. In this case this transformation will be, first of all, a transformation of curve-pieces $(xyp \ldots p^{k-1})$ and $(XYP \ldots P^{k-1})$ with the same length. Consequently, to such pieces $(xyp \ldots p^{k-1})$ of a curve there correspond pieces of the same length belonging to another curve.

Regarding the transformations of this type there is a theorem that they are exclusively contact transformations according to the definition of Sophus Lie. Hence, all single-valued transformations are transformations of (xyp) in (XYP). It is already proved that there exist no other curve transformations of $(xypp')$ on $(XYPP')$. In the following sections it will be shown that proper contact transformations of higher orders do not exist as well.

Let us now consider spaces of an arbitrary dimension $n+1$. In order to define the most general transformation mapping all surfaces (M_n) into surfaces one can take any $n+1$ equations

$$\left\{\begin{array}{l} Z = F(z\, x_1 \,.\,.\, x_n\, p_1 \,.\,.\, p_n\, p_{11}\, p_{12} \,.\,.\, p_{nn} \,.\,.\, p_{klm} \cdots), \\ X_1 = F_1(\qquad\qquad\qquad\qquad\qquad), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ X_n = F_n(\qquad\qquad\qquad\qquad\qquad) \end{array}\right. \tag{C'}$$

*Or, possibly, of a system of several differential equations.

and derive from them by differentiation and elimination the equations

$$\begin{cases} P_k = \Phi_k(z..x_k..p_k..p_{kl}..p_{klm}...), \\ P_{kl} = \Phi_{kl}(\qquad\qquad\qquad\qquad), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \qquad (k, l, m, .. = 12..n), \end{cases} \tag{D'}$$

so that the following system of equations[(5)], Section 6.2.1. remains invariant:

$$dz - \sum p_k\, dx_k = 0, \quad dp_k - \sum p_{kl}\, dx_l = 0, \dots \text{ to inf.} \tag{A'}$$

In general, the system (C') defines a multivalued transformation. The surface transformation is single-valued if and only if (C') is Lie's contact transformation (see §4).

I have already discussed the assumption that probably there are no single-valued surface transformations other than Lie's contact transformations. The proof of this statement for two and three dimensions was given, as mentioned in the preamble to this paper, in one of the issues of Annuals Reports of Lund University[(6)]. Simultaneously the same question was raised by Lie in one of his papers in Mathematischen Annalen where he also questioned if partial differential equations of higher orders admit transformations which are not contact transformations[(7)]. In my paper mentioned above I proved non-existence of higher-order contact transformations. The proof of non-existence which concerns *all surfaces* manifested inter alia, as Lie also mentioned to me, that the statement cannot be directly applied to transformations valid only on integral surfaces of a higher-order partial differential equation; in the present work I represent it as a corollary of my earlier theorem.

In §5 I single out from transformations discussed in the previous sections those mapping a first-order partial equation in an $n + 1$-dimensional space onto an n-dimensional space. This allows one to make a conclusion about the transformation of first-order equations.

Such a mapping is based on a contact transformation. It was employed by Lie in his synthetic investigations as I concluded from his remarks in the paper "General theory of first-order partial differential equations" (Abh. der Gesellschaft der Wissenschaften zu Christiania, 1874, p.218).

Finally, I provide several remarks on a class of multivalued transformations of three-dimensional spaces.

I would like to mention, as I already did in my previous paper, that last summer in Munich I discussed osculating transformations on the plane with Felix Klein. At that time the problem was not solved yet and he offered new view points to the problem that considerably promoted its handling.

§ 3. Single-valued transformations of plane curves

5. I am going to resume the speculations presented in no° 2 in a more detailed form. Instead of a three-fold system of curves I will deal with a system involving $n+1$ arbitrary parameters λ:

$$f(xy\lambda_1\lambda_2\dots\lambda_{k+1})=0 \tag{5}$$

and use the procedure that was applied previously to system (4). Here $\lambda_1\lambda_2\dots\lambda_{k+1}$ will be considered as coordinates of a point in a $k{+}1$-dimensional space R_{k+1}, whereas $x\,y$ will be regarded as arbitrary constants. Elimination of $x\,y\,p$ from the equations

$$\begin{cases} f=0, \quad f'(x)+pf'(y)=0, \\ \sum \dfrac{df}{d\lambda}d\lambda=0, \quad \sum \dfrac{df'(x)}{d\lambda}d\lambda+p\sum\dfrac{df'(y)}{d\lambda}d\lambda=0 \end{cases} \tag{6}$$

leads to an equation

$$\psi(\lambda\,d\lambda)=0 \tag{7}$$

that provides the contact condition for two neighboring curves (5).

Regarding this equation one should note first of all the following. One can infer from the equations (6), letting $xyp\lambda$ be constant (then the values of λ are determined by two first equations (6)), that to every such system of values there correspond ∞^{k-2} values of $\frac{d\lambda_i}{d\lambda_{k+1}}$. Specifically, these values have the form

$$d\lambda_i=\alpha_1 d\lambda_i^{(1)}+\alpha_2 d\lambda_i^{(2)}+\ \dots.\ \alpha_{k-1}d\lambda_i^{(k-i)},$$
$$(i=1\,2\dots k+1),$$

where α are arbitrary. The rays of the cone $\psi=0$ belonging to some point (λ) provide an infinite number of plane bundles of dimensions $k-2$. Hence, the cone itself must be represented in plane coordinated by $k-1$ equations. Let these $k-1$ equations,

$$\begin{cases} \psi_1(\lambda_1\dots\lambda_{k+1}\,\pi_1\dots\pi_{k+1})=0, \\ \psi_2(\qquad\qquad\qquad\qquad)=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \psi_{k-1}(\qquad\qquad\qquad\qquad)=0, \end{cases} \tag{8}$$

be homogeneous with respect to π. Then contact condition (7) can be replaced by this system of partial differential equations of the first order in

R_{k+1}. However, this is not a complete characteristic of equation (7). Namely, the surface elements $(\lambda\,\pi)$ of manifold (5), M_k in the space R_{k+1}, satisfy equations (8), and, hence, manifold-system (5) is a common solution for the equations (8) with two arbitrary constants xy.

Conversely, every system of $k-1$ *partial equations of the first order in* R_{k+1} *possessing a common solution with two arbitrary constants leads to an equation* $\psi(\lambda\, d\,\lambda) = 0$, *which can be interpreted as a contact condition for two neighboring curves of a* $k+1$*-fold system:*

$$f(xy\lambda_{\,1}\ldots\lambda_{\,k+1}) = 0.$$

Furthermore, the equation of every system of curves of this kind, where $x\,y$ *are treated as arbitrary constants, represents a common complete solution for the system of partial differential equations.*

The above mapping of the system of equations (8) to the plane associates with every linear element $(x\,y\,p\,)$ of the plane a characteristic M_{k-1} (it is a $k-1$ dimensional manifold - the intersection of two neighboring integrals M_k), with every element $x\,y\,p\,p'$ a characteristic M_{k-2} (the intersection of three consecutive integrals M_k), etc. The points of a characteristic M_{k-1} correspond to the curves (5) that contact each other in one point, the points of a characteristic M_{k-2} correspond the curves (5) that osculate each other in one point, etc.

An additional point to emphasize is that if between two $k+1$-fold systems of curves

$$\begin{aligned} &f\,(x\,y\,\lambda_{\,1}...\,\lambda_{\,k+1}) = 0,\\ &\varphi\,(\qquad\qquad\qquad) = 0 \end{aligned}$$

there exists a correspondence such that to two neighboring and contacting each other curves of one system correspond two likewise contacting each other curves of the other system, then all contacting in one point curves $f = 0$ correspond to curves $\varphi = 0$ possessing exactly the same property. Therefore, both systems of curves provide the same system of partial differential equations (8), and one and the same characteristic M_{k-1} corresponds to both contacting at one point families $f = 0$, resp. $\varphi = 0$. It follows that for all transformations of the plane that map two neighboring contacting each other curves into similar curves, the first-order tangency condition must leave invariant. Thus, all these transformations are Lie's contact transformations. In fact it was already proved in no° 2.

6. As it was already shown, a curve transformation leaving the second-order tangency unaltered is in fact a usual contact transformation. A transformation leaving invariant the third-order tangency, maps every two neighboring curves, that have the second-order tangency, into two similar curves;

or even more, I claim that this transformation must belong to the class of transformations that map two neighboring curves having the first-order tangency into similar curves. Indeed, if curves $C' C''$ are first-order tangent to each other, then there exists a curve C osculating with $C' C''$ in two points, neighboring the tangency point of these curves. Every transformation of the above kind will convert $C' C'' C$ into $\Gamma' \Gamma'' \Gamma$, where Γ will osculate both Γ' and Γ'' in two neighboring points. Since Γ' and Γ'' osculate one and the same curve Γ in two neighboring points they are first-order tangent. Thus, two first-order tangent curves $C' C''$ are converted into two similar curves $\Gamma' \Gamma''$ - as was stated. Thus, the above transformation is Lie's contact transformation. Likewise there are no transformations for which the third-order tangency is an invariant condition other than Lie's contact transformations.

The same reasoning shows that there are no proper tangent transformations of the 4th, 5th ... order. But, as it has already been proved, every single-valued curve transformation is to be a transformation of curve-pieces of the same length, $(x\,y\,p \ldots . p^k)$, $(X\,Y\,P \ldots P^k)$, and hence, it is to be a contact transformation of either the first or the second, third, fourth ... order. It follows that *every single-valued transformation of curves on the plane is Lie's contact transformation.*

§ 4. Transformations of n-dimensional manifolds M_n in an $n+1$-dimensional space

7. I note first of all that if a surface M_n has the tangency of order r with two infinitely neighboring each other surfaces in two neighboring points $p\,p'$, respectively, then the two latter surfaces should possess a tangency of order $r-1$ at the point p'. And vice versa, if two infinitely neighboring surfaces have a tangency of order $r-1$, then there exist an infinite number of ways to construct surfaces having a tangency of order r with the above surfaces in the vicinity of contact points.

Every surface transformation that converts surfaces having second-order tangency into similar ones, will consequently map two infinitely close surfaces which are first-order tangent into two surfaces of the same kind. Likewise, a transformation leaving invariant third-order tangency must convert any two osculating surfaces into two second-order tangent surfaces. One can follow now the procedure suggested in no° 6. Namely, given two surfaces $C' C''$ having first-order tangency at a point, we construct a surface C that has second-order tangency with them at two points neighboring the tangency point of C' and C''. It is self evident from this construction that our transformation converts any two first-order tangent surfaces into two similar surfaces, etc. Finally, every surface transformation leaving invariant tangency of a certain order, should convert any two first-order tangent surfaces into two similar surfaces. Now every single-valued surface transformation should be a transformation of surface elements of the same length $(z\, x_k\, p_k\, p_{kl} ...)$ $(Z\, X_k\, P_k\, P_{kl} ...)$, and for every such transformation the tangency of a certain order is an invariant condition. Consequently, *every single-valued surface transformation must be a transformation leaving invariant the first-order tangency of two infinitely neighboring surfaces.*

8. Let us consider an $n+2$-fold system of surfaces

$$f(z\, x_1 ... x_n\ \lambda_1 ... \lambda_{n+2}) = 0. \tag{9}$$

The condition that two surfaces corresponding to the parameters $\lambda \quad \lambda + d\lambda$ contact each other is obtained by elimination of $z\, x\, p$ from the following $2n+2$ equations:

$$\left\{ \begin{array}{c} f = 0, \quad f'(x_k) + p_k f'(z) = 0, \\ \sum \dfrac{df}{d\lambda} d\lambda = 0, \quad \sum \dfrac{df'(x_k)}{d\lambda} d\lambda + p_k \sum \dfrac{df'(z)}{d\lambda} d\lambda = 0, \\ (k = 1\, 2 ... n)\,, \end{array} \right. \tag{10}$$

Then the desired condition is given by an ordinary differential equation

$$\varphi\,(\lambda\, d\,\lambda) = 0 \tag{11}$$

If $z\, x$ are interpreted as arbitrary constants and $\lambda_1\, \lambda_2 ... \lambda_{n+2}$ as coordinates of a point in a space R_{n+2}, then equation (11) represents in this space a system of elementary cones, while equation (9) defines in the same space an $n+1$-fold system of manifolds M_{n+1}. These manifolds intersect in every point (in consequence of the equations (10)) with ∞^{n-1} neighboring M_{n+1} along a one-dimensional manifold linear elements of which constitute rays of elementary cones (11). In consequence, the ∞^n manifolds (9) passing through one and the same point (λ), define a cone (11) by means of their surface elements in this point. Let $\Phi = 0$ be the partial equation of the first order, the characteristic cone (or the elementary complex-cone) of which is represented by equation (11). *Then every system of surfaces (9), for which $\varphi = 0$ is a tangency condition, provides a complete solution of the partial equation $\Phi = 0$ with $n+1$ arbitrary constants $z\, x_1 ... x_n$.*

When two $n+2$-fold systems of surfaces

$$f(z\, x_1 ... x_n\ \lambda_1 ... \lambda_{n+2}) = 0, \quad \varphi(z\, x_1 ... x_n\ \lambda_1 ... \lambda_{n+2}) = 0 \tag{12}$$

are related to each other in such a way, that to two contacting each other surfaces $f(\lambda^{(1)}) = 0$, $f(\lambda^{(1)} + d\lambda) = 0$ correspond two likewise contacting surfaces $\psi(\lambda^{(1)}) = 0$, $\psi(\lambda^{(1)} + d\lambda) = 0$, then each of the above two equations is a complete solution of one and the same partial differential equation $\Phi = 0$, provided that z, x are regarded as constants and the λs as variables. The parameters λ of ∞^1 surfaces (corresponding to one of the solutions) contacting in one point, and hence having a common system of values $(z\,x\,p)$, represent in the space R_{n+2} coordinates of points of a characteristic for the equation $\Phi = 0$, and vice versa. We conclude that the systems of surfaces (12) are related in such a way that if ∞^1 surfaces of one system contact each other in one point, the corresponding surfaces of the other system also contact in one point. Therefore, one of the systems of surfaces can be derived from the other via a Lie contact transformation.

Thus, *every single-valued surface transformation is Lie's contact transformation.*

9. Among infinite number of $n+k$-fold surfaces

$$f(z\,x_1 \dots x_n\; \lambda_1 \dots \lambda_{n+k}) = 0, \tag{13}$$

there are ∞^{k-1} surfaces with a given element $(z\,x\,p)$. If $\lambda_1 \dots \lambda_{n+k}$ are regarded as coordinates of points in a space R_{n+k}, then the tangency condition for two surfaces (13) is given by a differential equation

$$\psi(\lambda\, d\,\lambda) = 0$$

which is actually a system of $k-1$ first-order equations (homogeneous with respect to π) in R_{n+k} :

$$\begin{cases} \Psi_i(\lambda_1 \dots \lambda_{n+k}\, \pi_1 \dots \pi_{n+k}) = 0, \\ \qquad (i = 1\,2 \dots k-1). \end{cases} \tag{14}$$

The above system has a common solution containing $n+1$ arbitrary constants. *This common solution is given by the equation $f = 0$, where $z\,x$ are arbitrary constants, as well as by every system of ∞^{n+k} manifolds M_n in the space R_{n+1}, provided that the tangency condition for the system is defined by $\varphi = 0$.*

The above consideration furnishes a relation between the space R_{n+1} and the space composed by the elements $(\lambda\,\pi)$ of the equations (14). I will return to this matter later[(8)].

§ 6. A class of multivalued transformations of three-dimensional spaces

16. As it was already shown in the introduction any three equations

$$\begin{cases} X = F\,(z\,x\,y\,p\;q), \\ Y = F_1(\qquad\qquad), \\ Z = F_2(\qquad\qquad) \end{cases} \tag{15}$$

completely determine a surface transformation provided that one requires the invariance of the system of equations

$$dz = p\,dx + q\,dy, \quad dp = r dx + s\,dy, \quad dq = s dx + t dy, \ldots \text{to inf.}$$

i.e., it is transformed into the similar system:

$$dZ = PdX + QdY,\ dP = RdX + SdY,\ dQ = SdX + TdY, \ldots \text{to inf.}$$

This surface transformation will be single valued if the quantities P, Q obtained from the above equations contain only $z\,x\,y\,p\;q$, but not the derivatives of higher orders.

Otherwise, if these quantities have the form

$$P = \Phi_1(z\,x\,y\,p\;q\,r\;s\,t),$$
$$Q = \Phi_2(\qquad\qquad\qquad),$$

the equations (15) define a multivalued transformation. This transformation assigns to every point $(X\,Y\,Z)$ a set of ∞^2 elements $(z\,x\,y\,p\;q)$, and to every surface element $(Z\,X\,Y\,P\,Q)$ on each of the elements $(z\,x\,y\,p\;q)$ associated with the point $(X\,Y\,Z)$ - a set of ∞^1 values of $(r\,s\,t)$. Furthermore, every surface in the space $(x\,y\,z)$ is mapped onto a surface of the space $(X\,Y\,Z)$, whereas every surface of the latter space is transformed into all the integrals of a first-order partial differential equation $f\,(F\,F_1\,F_2) = 0$.

The transformation (15) maps* a first-order partial differential equation $\varphi\,(Z\,X\,Y\,P\,Q) = 0$ into a second-order partial differential equation that has a first integral with two arbitrary constants $\lambda\;\mu$:

$$f\,(F\,F_1\,F_2\lambda\,\mu) = 0$$

*Transformation (15) is also referred to by P. du Bois-Reymond in the work "Beiträge zur Interpretation der partiellen Differentialgleichungen mit drei Variablen", Leipzig, 1864, p. 173.

Linear partial differential equations of the first order in the space $(X\,Y\,Z)$ are mapped to second-order partial differential equations in the space $(x\,y\,z)$. The latter second-order equations are linear in $r\,s\,t\;rt-s^2$ and have the first integral of the form:

$$f\,(F\,F_1\,F_2) \text{ equals to an arbitrary function of } \varphi\,(\mathrm{F}\,\mathrm{F}_1\,\mathrm{F}_2).$$

17. I will pay a special attention to the following transformation:

$$\begin{cases} X = x, \\ Y = y, \\ Z = q \end{cases} \tag{16}$$

due to its application to a known class of second-order partial equations.

The equations (16) yield

$$\begin{cases} P = s, \\ Q = t, \\ R = v, \quad \left(v = \dfrac{d^3z}{dx^2\,dy}, \quad w = \dfrac{d^3z}{dx\,dy^2}, \quad \widetilde{\omega} = \dfrac{d^3z}{dy^3}\right) \\ S = w, \\ T = \widetilde{\omega}, \\ \text{etc.} \end{cases} \tag{16$'$}$$

Consider in the space $(x\,y\,z)$ a second-order equation that does not contain $z\;p$ and hence, has the form:

$$F(x\,y\,q\,r\,s\,t) = 0,$$

or upon solving for r:

$$r = f(x\,y\,q\,s\,t)^* \tag{17}$$

and find its image in the space $(X\,Y\,Z)$.

Now differentiate equation (17) with respect to y:

$$v = \frac{df}{dy} + t\frac{df}{dq} + w\frac{df}{ds} + \widetilde{\omega}\frac{df}{dt},$$

*In the same way one can handle the following equation:

$$r = f(x\,y\,q\,s\,t) + z\,\varphi(x) + p\,\psi(x)$$

and use the equations (16), (16′) to obtain:

$$R - S\frac{df}{dP} - T\frac{df}{dQ} = \frac{df}{dY} + Q\frac{df}{dZ}. \tag{18}$$

This provides the image of all equations of the form

$$r = f\,(x\,y\,q\,s\,t) + F(x),$$

where $F(x)$ is an arbitrary function.

To every surface of the space $(X\,Y\,Z)$ there correspond the integrals of an equation

$$q = F(x\,y),$$

whose solution has the form:

$$z = \varphi\,(x\,y) + \Psi(x), \tag{19}$$

where Ψ is an arbitrary function.

Since to each integral of equation (18) there must correspond, inter alia, integrals of equation (17), the arbitrary function Ψ in (19) can be determined by the requirement that equation (19) becomes an integral of (17) provided that $Z = F(X\,Y)$ is an integral surface for equation (18). In consequence, the function Ψ is determined (by two quadratures) as a function of x, namely, it is obtained from $F(x)$ by multiplying it by $cx + c'$ where c, c' are arbitrary constants.

Thus, the problem of integration of the second-order equation (17) reduces to the problem of integration of the linear second-order equation (18).

The above transformation associates any two integrals of equation (17) having the n^{th}-order tangency in one point with two integrals of equation (18) having the $n-1^{th}$-order tangency. Accordingly, the characteristics of equation (17) correspond to the characteristics of equation (18).

The above considerations generalize the well-known Legendre's theory [*] of the equations[†]

$$F(r\,s\,t) = 0.$$

In order to find the linear equation (18) corresponding to Legendre's form one has to use the transformation

$$X' = s, \quad Y' = t, \quad Z' = sx + ty - q$$

obtained from (16) via the reciprocal transformation

$$X' = P, \quad Y' = Q, \quad Z' = PX + QY - Z.$$

[*] Cf. Boole: Differential equations, Cambridge 1859, p. 369.

[†] I learned recently from Lie about Legendre's theory.

The required equation has the form:

$$R\frac{df}{dY} - S\frac{df}{dX} - T = 0.$$

The above considerations hold true if the equations (16), (17) undergo any contact transformations.

Helsingborg, 18. July 1875.

Editor's Notes

(1) *Page 103.* See A.V. Bäcklund [1]. The reader is referred to Lars Gårding [11], *Chapter 5: Bäcklund,* for a brief account on the life of Albert Viktor Bäcklund (1845-1922) and interesting comments on the influence of Lie's 1872 work on Bäcklund's interest to the theory of tangent transformations.

(2) *Page 104.* The author's German term is *ein dreifaches Curvensystem.*

(3) *Page 104.* Here, $f'(x)$ and $f'(y)$ stand for partial derivatives of $f(x, y, \lambda_1, \lambda_2, \lambda_3)$ with respect to x and y, respectively.

(4) *Page 109.* Read *total differentials.* Specifically, the author's expression *the differential of φ vanishes* means that its total derivative vanishes, i.e. $D_x(\varphi) = 0$ where $D_x = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y}$ is the symbol of the total derivation.

(5) *Page 111.* The equations (A') are rudiments to infinite-order tangent transformations which we [19] called later Lie-Bäcklund transformations. See also N.H. Ibragimov [15], Section 6.2.1 for more detailed historical remarks on Lie-Bäcklund transformation groups.

(6) *Page 111.* See Editor's Note (1).

(7) *Page 111.* Bäcklund refers to the footnote on page 223 in: S. Lie [24]. See also R.L. Anderson and N.H. Ibragimov [19], §1 and §4.

(8) *Page 116.* We did not include in the translation the material of no° 10, 11 and §5 since they comprise a mere reiteration of some conclusions of the above considerations.

GROUP PROPERTIES OF THE CHAPLYGIN EQUATION

By L.V. Ovsyannikov

Translated from Russian by N.H.Ibragimov

[Gruppovye svoistva uravnenia S.A. Chaplygina, L.V. Ovsyannikov, Prikladnaia Mekhanika i Tekhnicheskaia Fizika (Applied Mechanics and Technical Physics), Novosibirsk, 1960, No. 3, p. 126–145.]

Methods based on transition to the plane of the hodograph of velocities are of great importance in studying plane steady-state irrotational problems of gas dynamics. This transition originates a linear partial differential equation of the second-order known as the *Chaplygin equation.* However, boundary value problems remain, in general, nonlinear. Furthermore, in accordance with the nature of the phenomenon, the Chaplygin equation has a mixed elliptic-hyperbolic type. To overcome the resulting difficulties many researchers tried to find such approximations to the Chaplygin equation for which the general solution had a relatively simple form. Then one can regard an approximate equation as an equation describing the motion of some fictitious gas in which the dependence of pressure on density (or density on velocity) approximates, in a certain way, the real dependence.

It is well known that the first approximation of this kind was suggested by Chaplygin [4] in 1902. The next consideration to the problem was given in forty years' time. S.A. Khristianovich [20] developed Chaplygin's method in order to apply it to the problem of subsonic flow around a body. F.I. Frankle [10] and S.V. Falkovich [9] suggested that the Chaplygin equation be approximated by the Tricomi equation. Various approximations were considered by S.A. Khristianovich [21], L.I. Sedov [29], Tomotika and Tamada [30], Germain and Liger [12], M.A. Lavrentyev and A.V. Bitsadze [23], G.A. Dombrovskii [8], S.V. Vallander [31], I.M. Yur'ev [32], A.A. Grib and A.G. Ryabinin [14], etc. A more comprehensive bibliography on the problem can be found in L. Bers' book [2].

The majority of the suggested approximations were obtained by using one and the same approach. Namely, an approximating equation was required to possess particular solutions of a specified structure or to be reduced to such an equation by simple substitutions. However, it is clear that this approach is not regular enough to embrace all "good" approximations from a unified point of view.

In the present paper the quality of an approximation is estimated by the dimension of the transformation group admitted by the approximating equation. The principle of estimation here is as follows: the broader the group, the "better" the equation. This method requires a preliminary investigation of group properties of second-order differential equations. Group classification of such equations (with two independent variables) was given by Lie [25]. However, Lie's classification lacked an invariant formulation that caused difficulties in applying it to particular problems. Once this deficiency is made up all Chaplygin equations admitting a three-parameter group as well as all "good" approximations in the problem of gas dynamics can be enumerated. The obtained result meets the expectations: the majority of the known approximations turn out to obey the condition that the approximate equation admits as large a group as possible. The only exception is the approximation of Tomotika and Tamada [30] which cannot be referred to as "good" approximations from the group viewpoint. However, employing the approximation, the authors found only solitary examples of interesting flows.

Thus, the present paper resumes a new approach to approximate methods in investigating the plane problem of gasdynamics. Moreover, the presented results have independent significance as well. They specify general peculiarities possessed by second-order linear equations in two independent variables.

The structure of the paper is as follows. Firstly, a brief derivation of the Chaplygin equation is given (§1) and general equivalence properties related to second-order equations and their Laplace invariants (§2, Lemmas 1-4) are recalled. Secondly, Lie's problem on group classification of such equations is considered (§3). The solution of the problem is given in terms of the Laplace invariants of the initial equation (Theorem 1 and its Corollary). Then the group classification of the Chaplygin equations is presented (§4, Theorem 3) in terms of the group properties of an auxiliary system of ordinary differential equations (Theorem 2). Further, canonical forms of the "admissible" Chaplygin equations are indicated and the rules reducing them to these forms are clarified (§5). Finally, we single out among the Chaplygin equations, admitting a three-parameter group, the Tricomi type equations, i.e. the simplest equations of the mixed type (§6, Theorems 4 and 5), as well as the equations that asymptotically tend to the Laplace equation and hence furnish a good approximation of the Chaplygin equation in the case of low velocities (§7, Theorem 6).

All "arbitrary" functions encountered in the paper are supposed to be analytical thus making it possible to ignore the difference in types of second-order equations obtained at intermediate stages of reasoning. The final results are independent of this assumption.

§ 1. The Chaplygin equation and the Chaplygin function

Let u and v be the projections of velocity to the x and y axes, respectively, where x and y are referred to the rectangular Cartesian coordinates system, and let ρ be density. Then the equations of the plane-parallel steady irrotational gas motion are written in the form:

$$u_y - v_x = 0, \quad (\rho u)_x + (\rho v)_y = 0, \quad \rho = \rho_0 R(w) \tag{1.1}$$

where $w = \sqrt{u^2 + v^2}$ is the velocity related to the critical velocity, $R(w)$ is a given function, $R(0) = 1$, and ρ_0 is a constant. For instance, for the polytropic gas with the polytropic index γ we have:

$$R = \left(1 - \frac{\gamma - 1}{\gamma + 1} w^2\right)^{1/\gamma - 1} \qquad (1 < \gamma < \infty). \tag{1.2}$$

By virtue of Equations (1.1) there exist a velocity potential φ and a flow function ψ determined by the equations

$$d\varphi = udx + vdy, \quad d\psi = Rudy - Rvdx.$$

Inversion of these equations with the following change to the polar coordinates in the hodograph plane $u = w\cos\theta, \quad v = w\sin\theta$ yields:

$$dx = \frac{\cos\theta}{w} d\varphi - \frac{\sin\theta}{Rw} d\psi, \quad dy = \frac{\sin\theta}{w} d\varphi + \frac{\cos\theta}{Rw} d\psi \tag{1.3}$$

Hence, for φ and ψ considered as functions of w, Θ it follows:

$$\varphi_w = w \frac{d}{dw}\left(\frac{1}{Rw}\right)\psi_\theta, \quad \varphi_\theta = \frac{w}{R}\psi_w \tag{1.4}$$

Elimination of φ leads to a single equation for the flow function:

$$w\frac{d}{dw}\left(\frac{1}{Rw}\right)\psi_{\theta\theta} = \frac{\partial}{\partial w}\left(\frac{w}{R}\psi_w\right) \tag{1.5}$$

If following Chaplygin one introduces a new independent variable $\sigma = \sigma(w)$ defined by the equations

$$\frac{d\sigma}{dw} = -\frac{R(w)}{w}, \quad \sigma(1) = 0, \tag{1.6}$$

one reduces Equation (1.5) to the form

$$K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0, \tag{1.7}$$

where

$$K(\sigma) = \frac{wR' + R}{R^3}\cdot$$

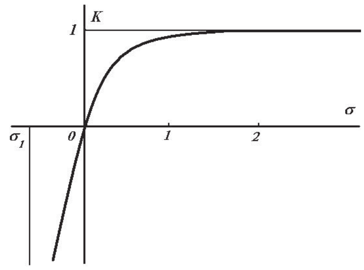

Figure 1: Function $K(\sigma)$.

Hereafter, an equation of the form (1.7) is called the Chaplygin equation, and the function $K(\sigma)$ is called the Chaplygin function.

A qualitative graph of the Chaplygin function in the case (1.2) is plotted in Fig. 1. A refinement of the behaviour of $K(\sigma)$ in this case is given by the following asymptotic formulae:

$$K(0) = 0, \quad K'(o) = 2\left(\frac{\gamma+1}{2}\right)^{\frac{2}{\gamma-1}}, \tag{1.8}$$

$$\lim_{\sigma\to+\infty} \mathrm{e}^{4\sigma}[K(\sigma)-1] = -\frac{1}{\gamma+1}\exp\left[2\int_0^1\left[\left(1-\frac{\gamma-1}{\gamma+1}x\right)^{\frac{1}{\gamma-1}}-1\right]\frac{dx}{x}\right], \tag{1.9}$$

$$\lim_{\sigma\to\sigma_1}(\sigma-\sigma_1)^{\frac{\gamma+1}{\gamma}}K(\sigma) = -\frac{2}{\gamma}\left(\frac{\gamma-1}{\gamma}\right)^{\frac{1}{\gamma}}, \tag{1.10}$$

where

$$\sigma_1 = -\int_1^W\left(1-\frac{\gamma-1}{\gamma+1}w^2\right)^{\frac{1}{\gamma-1}}\frac{dw}{w}, \quad W = \sqrt{\frac{\gamma+1}{\gamma-1}}, \tag{1.11}$$

and by the inequalities:

$$K'(\sigma) > 0, \quad K''(\sigma) < 0 \qquad (\sigma_1 < \sigma < +\infty). \tag{1.12}$$

The transformation of the Chaplygin equation to the characteristic variables

$$\lambda = L(\sigma) + \theta, \quad \mu = L(\sigma) - \theta, \quad \frac{dL}{d\sigma} = \sqrt{-K} \tag{1.13}$$

yields:

$$\psi_{\lambda\mu} + N(\lambda+\mu)(\psi_\lambda + \psi_\mu) = 0 \quad \left(N = -\frac{1}{4}\frac{d}{d\sigma}\frac{1}{\sqrt{-K}}\right) \tag{1.14}$$

Equation (1.14) will be referred to as the Chaplygin equation as well. Let us introduce the quantity $t = 2L(\sigma) = \lambda + \mu$. Then

$$N(t) = -\frac{1}{2}\frac{d}{dt}\log\sigma' \quad \left(\sigma' = \frac{d\sigma}{dt}\right). \tag{1.15}$$

If the function $\sigma(t)$ defining the dependence $t = t(\sigma)$ is known the Chaplygin function is determined from (1.13) and has the form:

$$K(\sigma) = -\frac{1}{4}\left(\frac{dt}{d\sigma}\right)^2. \tag{1.16}$$

§ 2. The Laplace invariants

Consider an equation with an unknown function $z = z(x, y)$

$$z_{xy} + Az_x + Bz_y + C_z = 0 \tag{2.1}$$

where A, B, C are given functions of x, y.

Two equations of the form (2.1) are said to be equivalent if they are connected by transformation:

$$x_1 = \alpha(x), \quad y_1 = \beta(y), \quad z = \omega(x_1, y_1)z_1, \tag{2.2}$$

where $z_1 = z_1(x_1, y_1)$ is a new unknown function. Furthermore, equations will be called *equivalent by function* if they are transformed into each other by (2.2) with $\alpha(x) \equiv x, \beta(y) \equiv y$.

Equivalence properties is conveniently formulated in terms of the Laplace invariants of equation (2.1):

$$h = A_x + AB - C, \quad k = B_y + AB - C. \tag{2.3}$$

The following statements hold. They were presented by Darboux [6] in a slightly different form.

Lemma 1. For an equation of the form (2.1) with the invariants h, k to be equivalent by function to an equation with the invariants h_1, k_1 it is necessary and sufficient that $h_1 = h, \quad k_1 = k$.

Proof. The proof of the necessity is obtained by substitution $z = \omega(x, y)z_1$ and calculation of the invariants of the resulting equation for z_1. The proof of the sufficiency results from the observation that if the coefficients of the second equation are A_1, B_1, C_1, then the equations $h_1 = h, \quad k_1 = k$ yield

$$(A_1 - A)_x = (B_1 - B)_y.$$

Therefore there exists a function $\omega = \omega(x,y)$ such that

$$A_1 = A + \frac{\omega_y}{\omega}, \quad B_1 = B + \frac{\omega_x}{\omega}.$$

Moreover, it follows from $h_1 = h$ that

$$C_1 = C + A\frac{\omega_x}{\omega} + B\frac{\omega_y}{\omega} + \frac{\omega_{xy}}{\omega}$$

The direct substitution into Equation (2.1) shows that the change $z = \omega z_1$ maps (2.1) into the equation with the coefficients A_1, B_1, C_1.

A simple consequence of the Lemma states that those and only those equations (2.1) are equivalent by function to the equation $z_{xy} = 0$ for which $h \equiv k \equiv 0$.

Lemma 2. An equation (2.1) with the invariants $h(x,y), k(x,y)$ is equivalent to an equation of the same type with the invariants $h_1(x,y), k_1(x,y)$ if and only if there exist two functions, $\alpha(x)$ and $\beta(y)$ such that

$$\frac{h(x,y)}{h_1[\alpha(x),\beta(y)]} = \frac{k(x,y)}{k_1[\alpha(x),\beta(y)]} = \alpha'(x)\beta'(y). \tag{2.4}$$

Proof. Here again proof of the necessity results from direct substitution which in this case can be considerably simplified due to Lemma 1. Namely, one can consider only those transformations (2.2) where $\omega \equiv 1$. Now let equations (2.4) be satisfied for some $\alpha(x), \beta(y)$. We apply to the equation with the invariants h, k the change of variables (2.2) with the above $\alpha(x), \beta(y)$ and $\omega = 1$. Then it will go over into an equation with the invariants h_2, k_2 such that the equations (2.4) hold with h_1, k_1 replaced by h_2, k_2 :

$$\frac{h(x,y)}{h_2[\alpha(x),\beta(y)]} = \frac{k(x,y)}{k_2[\alpha(x),\beta(y)]}\,.$$

The above equation together with (2.4) yield that

$$h_2(x_1,y_1) = h_1(x_1,y_1), \quad k_2(x_1,y_1) = k_1(x_1,y_1).$$

Hence, according to Lemma 1, the equation with the invariants h_2, k_2 is equivalent by function to the equation with the invariants h_1, k_1.

In addition to transformations (2.2) there exists another type of transformations preserving the structure of equations (2.1). In order to obtain such transformations we note that Equation (2.1) can be derived by eliminating the auxiliary function z^* from the system

$$z_y + A_z = z^*, \quad z^*_x + Bz^* = hz \tag{2.5}$$

or by eliminating z^{**} from the system

$$z_x + Bz = z^{**}, \quad z_y^{**} + Az^{**} = kz. \tag{2.6}$$

Conversely, provided that $h \neq 0$, elimination of z from the system (2.5) gives an equation of the form (2.1) for z^{*}.. This new equation will be called Laplace's x-transformation of Equation (2.1). Let h^*, k^* denote its Laplace invariants. Then simple calculations yield:

$$h^* = 2h - k - \frac{\partial^2 \log h}{\partial x \partial y}, \quad k^* = h. \tag{2.7}$$

Likewise, if $k \neq 0$, elimination of z from the system (2.6) yields an equation for z^{**} called Laplace's y-transformation of Equation (2.1). Its invariants are:

$$h^{**} = k, \quad k^{**} = 2k - h - \frac{\partial^2 \log k}{\partial x \partial y} \cdot \tag{2.8}$$

Now let us agree to denote equations of the form (2.1) with the invariants h, k by the symbol (h, k). Then, Laplace's x-transformation of the equation (h, k) results in the equation (h^*, k^*) while the y-transformation results in the equation (h^{**}, k^{**}).

Lemma 3. Laplace's y-transformation of the equation (h^*, k^*) is equivalent by function to the equation (h, k). Likewise, Laplace's x-transformation of the equation (h^{**}, k^{**}) is equivalent by function to the equation (h, k).

Proof. The direct calculation of the invariants shows that

$$(h^*)^{**} = h, \quad (k^*)^{**} = k$$

and

$$(h^{**})^* = h, \quad (k^{**})^* = k.$$

Now Lemma 1 completes the proof.

It follows from Lemma 3 that the set of equations derived from the original equation by Laplace's transformations is "one-dimensional" in the following sense. We set in the original equation $h = h^0, k = h^{-1}$ and hence, denote our equation by (h^0, h^{-1}). Then we define h^n for any n by the recurrent formula:

$$h^{n+1} + h^{n-1} = 2h^n - \frac{\partial^2 \log h^n}{\partial x \partial y} \quad (n = 0, \pm 1 \pm 2, \ldots). \tag{2.9}$$

Due to (2.7), the formula (2.9) yields the invariant $h^* = h^1$ of the equation (h^*, k^*) when $n = 0$, and the invariant $k^{**} = h^{-2}$ of the equation (h^{**}, k^{**})

when $n = -1$. It is easy to verify now that Laplace's x-transformation, in general, maps (h^n, h^{n-1}) into h^{n+1}, h^n, whereas the y-transformation maps (h^{n+1}, h^n) into (h^n, h^{n-1}), respectively. Thus the Laplace series is obtained:

$$\ldots; (h^{-2}, h^{-3}); (h^{-1}, h^{-2}); (h^0, h^{-1}); (h^1, h^0); (h^2, h^1); \ldots \tag{2.10}$$

The Laplace series can be continued until some invariant h^n becomes identical to zero. A remarkable property of the series (2.10) is that if $h^n \equiv 0$ for a certain n, the general solution with two arbitrary functions involving quadratures can be found for the original equation. Moreover, if the series (2.10) is "cut off" at both ends, then the general solution with two arbitrary functions can be obtained without quadratures [6]. For further considerations, the case when the invariants h^n have a constant ratio will be significant.

Lemma 4. If the invariants h, k of the original equation have the constant ratios$^{(1)}$

$$\frac{k}{h} = p, \quad \frac{1}{h}\frac{\partial^2 \log h}{\partial x \partial y} = q, \tag{2.11}$$

then all the invariants of the Laplace series (2.10) have constant ratios.

Proof. Let us rewrite (2.11) in the form:

$$h^{-1} = ph^0, \quad \frac{\partial^2 logh^0}{\partial x \partial y} = qh^0.$$

The formula (2.9) with $n = 0$ proves the constancy of the ratio h^1/h^0 :

$$\frac{h^1}{h^0} = 2 - p - q.$$

Now the constancy of the ratio h^n/h^0 follows from (2.9) by induction. To evaluate this ratio for any n, we consider (2.9) as a finite-difference equation:

$$h^{n+1} - 2h^n + h^{n-1} = -qh^0$$

with the initial conditions $h^{-1} = ph^0,\ h^0 = h^0$. The solution of this problem (which is obviously unique) is given by:

$$\frac{h^n}{h^0} = 1 + (1-p)n - \frac{1}{2}qn(n+1). \tag{2.12}$$

In the particular case when the invariants of the original equation (h^0, h^{-1}) are equal i.e. when $p = 1$, the formula (2.12) has the form:

$$\frac{h^n}{h^0} = 1 - \frac{1}{2}qn(n+1) \tag{2.13}$$

§ 3. Calculation of the group for the second-order equation

Let us proceed to determination of transformations preserving Equation (2.1), or as it is practice to say, transformations admitted by Equation (2.1). Specifically, the term a *group admitted by Equation* (2.1) will stand for the quotient group of the group of all admitted transformations by its trivial normal deviser. The latter is comprised of the transformations

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{z} = az + z_0(x, y)$$

where a is a parameter and $z_0(x, y)$ is any fixed solution of the equation. The infinitesimal generator of the group is written in the form:

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \pi z \frac{\partial}{\partial z} \cdot \tag{3.1}$$

It is known [28] that the functions ξ, η, π depend at most on x, y. Moreover, the function π is defined up to a constant addend. Applying the notation of the paper [28] to Equation (2.1), we find that the only component K_{12} of the tensor K_{ij} and the invariant H have the form:

$$K_{12} = 2J = 2(B_y - A_x), \quad H = A_x + B_y + 2(AB - C). \tag{3.2}$$

The system of the determining equations given in [28] is reduced now to the following:

$$\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 0, \tag{3.3}$$

$$\frac{\partial}{\partial x}(\pi + B\xi + A\eta) = -J\eta, \quad \frac{\partial}{\partial y}(\pi + B\xi + A\eta) = J\xi, \tag{3.4}$$

$$\frac{\partial}{\partial x}(J\xi) + \frac{\partial}{\partial y}(J\eta) = 0, \quad \frac{\partial}{\partial x}(H\xi) + \frac{\partial}{\partial y}(H\eta) = 0. \tag{3.5}$$

Equations (3.3) show that $\xi = \xi(x)$, and $\eta = \eta(y)$. Equations (3.4) determine $\pi(x, y)$ provided that ξ and η are found. The first equation in (3.5) is the compatibility condition for the system (3.4), while the second equation in (3.5) is an additional condition imposed on the functions ξ, and η.

Thus, the size of the group admitted by Equation (2.1) is determined by the general solution of system (3.5). Since (3.2) yields that $J = k - h,\ H = k + h$, Equations (3.5) can be rewritten in terms of the Laplace invariants of Equation (2.1) as follows:

$$\frac{\partial}{\partial x}(k\xi) + \frac{\partial}{\partial y}(k\eta) = 0, \quad \frac{\partial}{\partial x}(h\xi) + \frac{\partial}{\partial y}(h\eta) = 0. \tag{3.6}$$

We restrict ourselves to the case when at least one of the invariants h, k is distinct from zero since otherwise equation (2.1) is equivalent to the equation $z_{xy} = 0$. Let $h \neq 0$. Using the notation (2.11) we obtain from (3.6) the following equation

$$\xi \frac{\partial p}{\partial x} + \eta \frac{\partial p}{\partial y} = 0. \tag{3.7}$$

This equation shows that p is either an invariant of the group with the operator X or $p = \text{const}$.

It is easily seen that if $p \neq \text{const.}$, Equation (2.1) admits at most a one-parameter group. Indeed, if along with (3.7) we have

$$\xi_1 \frac{\partial p}{\partial x} + \eta_1 \frac{\partial p}{\partial y} = 0,$$

then, invoking that $p \neq \text{const}$, we will have $\xi\eta_1 - \xi_1\eta = 0$ or $\xi_1 = c\xi,\ \eta_1 = c\eta$. Here c is a constant since it is a function only of x and only of y, simultaneously. Then (3.4) yields that $\pi_1 = c\pi$ as well. It means that the operator

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} + \pi_1 z \frac{\partial}{\partial z}$$

is linearly dependent on the operator X.

Let us consider now the case $p = \text{const}$. From the two equations (3.6) here remains only one, namely, the second one. Rewriting it in the form

$$\xi \frac{\partial \log h}{\partial x} + \eta \frac{\partial \log h}{\partial y} + \xi'(x) + \eta'(y) = 0, \tag{3.8}$$

and applying the operator $\frac{\partial^2}{\partial x \partial y}$ we obtain:

$$\xi \frac{\partial}{\partial x} \log \left(\frac{\partial^2 \log h}{\partial x \partial y} \right) + \eta \frac{\partial}{\partial y} \log \left(\frac{\partial^2 \log h}{\partial x \partial y} \right) + \xi'(x) + \eta'(y) = 0.$$

Whence, subtracting the equation (3.8) and using the notation (2.11) we have:

$$\xi \frac{\partial q}{\partial x} + \eta \frac{\partial q}{\partial y} = 0. \tag{3.9}$$

This equation shows again that either q is an invariant of the group admitted by Equation (2.1) and then the admitted group depends at most on one parameter, or $q = \text{const}$.

Thus, Equation (2.1) can admit more than a one-parameter group only when both p and q are constants. Let us show that if q is constant then the invariant h must have a very peculiar form. According to Lie [25], Equation (3.10) considered below had been integrated by Liouville.

Lemma 5. The general solution of the equation

$$\frac{1}{h}\frac{\partial^2 \log h}{\partial x \partial y} = q, \qquad q = \text{const.}, \tag{3.10}$$

has the form:

$$h = \frac{2}{q}\frac{\alpha'(x)\beta'(y)}{[\alpha(x)+\beta(y)]^2} \quad (q \neq 0), \qquad h = \alpha'(x)\beta'(y) \quad (q = 0), \tag{3.11}$$

where $\alpha(x)$ and $\beta(y)$ are arbitrary functions.

Proof. The case $q = 0$ is trivial. If $q \neq 0$ the formula (3.11) provides a solution of equation (3.10) for arbitrary $\alpha(x)$ and $\beta(y)$. This can be checked by straightforward substitution of the expression for h into (3.10). Conversely, let h be a solution of Equation (3.10). We set $h = \chi^{-2}$ and rewrite (3.10) in the form

$$\chi\chi_{xy} - \chi_x\chi_y = -\frac{q}{2}\,\cdot \tag{3.12}$$

Whence, differentiating with respect to x, we obtain an equation equivalent to

$$\frac{\partial}{\partial y}\log\frac{\chi_{xx}}{\chi} = 0, \quad \text{or} \quad \chi_{xx} = r(x)\chi\,.$$

If χ_1 and χ_2 are two linearly independent solutions of the above ordinary differential equation, then its general solution is

$$\chi = c_1(y)\chi_1(x) + c_2(y)\chi_2(x).$$

We write it in the form

$$\chi = \chi_1(x)c_2(y)\left[\frac{\chi_2(x)}{\chi_1(x)} + \frac{c_1(y)}{c_2(y)}\right]$$

or

$$\chi = m(x)n(y)[\alpha(x)+\beta(y)],$$

substitute the latter expression in Equation (3.12) and arrive at the following equation:

$$(mn)^2 = \frac{q}{2\alpha'\beta'}\,\cdot$$

Consequently, $h = \chi^{-2}$ has the form (3.11). The main result on Equation (2.1) can be formulated as follows.

Theorem 1. Equation (2.1) admits more than a one-parameter group if and only if the quantities

$$\frac{k}{h} = p, \quad \frac{1}{h}\frac{\partial^2 \log h}{\partial x \partial y} = q$$

have constant values. Provided that this condition is satisfied, Equation (2.1) admits a three-parameter group and is equivalent either to the Euler-Poisson equation (if $q \neq 0$)

$$z_{xy} - \frac{2/q}{x+y} z_x - \frac{2p/q}{x+y} z_y + \frac{4p/q^2}{(x+y)^2} z = 0 \tag{3.13}$$

or to the equation (if $q = 0$)

$$z_{xy} + x z_x + p y z_y + p x y z = 0. \tag{3.14}$$

Proof. The necessity for p and q to be constant has been already established. Now let p and q be constant. Then by virtue of Lemma 5 the invariants h, k have the form (provided that $q \neq 0$):

$$h = \frac{2}{q} \frac{\alpha'(x)\beta'(y)}{[\alpha(x) + \beta(y)]^2}, \quad k = \frac{2p}{q} \frac{\alpha'(x)\beta'(y)}{[\alpha(x) + \beta(y)]^2}\,. \tag{3.15}$$

Let us compare the above equation with an equation having the invariants

$$h_1 = \frac{2}{q} \frac{1}{(x+y)^2}, \quad k_1 = \frac{2p}{q} \frac{1}{(x+y)^2} \tag{3.16}$$

and employ Lemma 2. It is evident that the functions $\alpha(x)$ and $\beta(y)$ involved in the formulae (3.15) obey the condition (2.4) of Lemma 2. Therefore Equation (2.1) is equivalent to the equation with the invariants (3.16). Since Equation (3.13) has precisely the invariants (3.16), Lemma 1 yields that Equation (2.1) is equivalent to Equation (3.13).

In the case $q = 0$ the invariants h, k have the form (see Lemma 5):

$$h = \alpha'(x)\beta'(y), \quad k = p\alpha'(x)\beta'(y).$$

Hence, by virtue of Lemma 2, Equation (2.1) is equivalent to an equation with the invariants $h_1 = 1$, $k_1 = p$, the latter equation being equivalent by function to Equation (3.14).

Since equivalent equations admit similar groups, the proof of Theorem 1 will be finished if we show that each of the equations (3.13) and (3.14) admits a three-parameter group. Recall that the only equation that ξ, η should satisfy is Equation (3.8).

If $q \neq 0$ Equation (3.8) takes the form

$$x\xi' - 2\xi + y\eta' - 2\eta + y\xi' + x\eta' = 0. \tag{3.17}$$

Applying the operation $\partial^2/\partial x\partial y$ we obtain $\xi'' + \eta'' = 0$, whence

$$\xi'' = -\eta'' = 2a_0 = \text{const}.$$

Consequently,

$$\xi = a_0x^2 + a_1x + a_2, \quad \eta = -a_0y^2 + b_1y + b_2.$$

Equation (3.17) yields that

$$b_1 = a_1, b_2 = -a_2.$$

Thus, the general form of the coordinates ξ, η satisfying (3.8) in the case of invariants (3.16) is the following:

$$\xi = a_0x^2 + a_1x + a_2, \quad \eta = -a_0y^2 + a_1y - a_2.$$

Furthermore, π is determined by Equation (3.4) and has the form:

$$\pi = a_0\,\frac{2}{q}\,(px - y).$$

Ultimately, we arrive at the conclusion that Equation (3.13) admits a three-parameter group of transformations generated by the following operators:

$$X_1 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\,, \quad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\,,$$

$$X_3 = x^2\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} + \frac{2}{q}\,(px - y)\,z\frac{\partial}{\partial z}\,\cdot \tag{3.18}$$

If $q = 0$ equation (3.8) has the form

$$\xi' + \eta' = 0$$

and yields:

$$\xi = a_0x + a_1, \quad \eta = -a_0y + b_1.$$

Furthermore, π is determined by Equation (3.4) and has the form:

$$\pi = -a_1y - b_1p\,x.$$

Hence, Equation (3.14) admits a three-parameter group with the generators

$$X_1 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\,, \quad X_2 = \frac{\partial}{\partial x} - yz\frac{\partial}{\partial z}\,, \quad X_3 = \frac{\partial}{\partial y} - pxz\frac{\partial}{\partial z}\,\cdot \tag{3.19}$$

Corollary 1. Two equations of the form (2.1) admitting a three-parameter group are equivalent if and only if they have equal values of the parameters p and q.

Indeed, if two equations have equal p and q then by Theorem 1 each of them is equivalent either to Equation (3.13) or to Equation (3.14). Conversely, if two equations are equivalent then Equation (2.4) and Lemma 2 show that they have equal values of the parameter p. The equality of the values of the parameter q follows from the equations

$$\frac{1}{h}\frac{\partial^2 \log h}{\partial x \partial y} = \frac{1}{\alpha'\beta' h'}\frac{\partial^2 \log(\alpha'\beta' h_1)}{\partial x \partial y} = \frac{1}{h_1}\frac{\partial^2 \log h_1}{\partial \alpha \partial \beta}$$

obtained by using (2.4).

§ 4. Determination of the admissible Chaplygin functions

In this section we will find all Chaplygin functions $K(\sigma)$ for which Equation (1.7) admits a three-parameter group. Any Chaplygin function of this kind and the corresponding equation will be called an *admissible* Chaplygin function and an *admissible* equation, respectively. In order to find all admissible Chaplygin functions, it suffices to consider Equation (1.14) and utilize the results of § 3 since the equations (1.7) and (1.14) admit similar groups.

The Laplace invariants of equation (1.14) have the form

$$h = k = N' + N^2,$$

where $N = N(t) = N(\lambda + \mu)$ and the prime denotes the differentiation with respect to t.

Let us first consider the case when $h = 0$. Then the abo ve equations yields that either $N = 0$ or $N = (t + t_0)^{-1}$, where t_0 is a constant. The case $N = 0$ corresponds evidently to the function $K(\sigma) = \text{const}$. In the second case Equation (1.15) has the form

$$\frac{\partial}{\partial t}\log \sigma'(t) = -\frac{2}{t + t_0}; \quad \text{or} \quad \sigma - \sigma_0 = -\frac{\mathrm{c}_1}{\mathrm{t} + \mathrm{t}_0}$$

and hence, according to (1.16), we have

$$K(\sigma) = -\left(\frac{c}{\sigma - \sigma_0}\right)^4, \qquad c, \sigma_0 = \text{const}. \tag{4.1}$$

According to Lemma 1, these are the only cases when the Chaplygin equation is equivalent to the wave equation for vibrating strings (or to the Laplace equation).

Let $h \neq 0$. The equation $h = k$ and Theorem 1 show that Equation (1.14) is admissible if and only if the quantity

$$\frac{1}{h}\frac{\partial^2 \log h}{\partial\lambda\partial\mu} = \frac{1}{h}\frac{\partial^2 \log h}{dt^2} = q \tag{4.2}$$

is constant. This condition provides a differential equation for $h = h(t)$. Given any solution of Equation (4.2), one can obtain the function $N(t)$ by solving the Riccati equation

$$N' + N^2 = h. \tag{4.3}$$

Let us introduce a new independent variable s and an auxiliary function $\zeta = \zeta(s)$ via the equations

$$\frac{ds}{dt} = h, \tag{4.4}$$

$$N = h\,\frac{\zeta'(s)}{\zeta(s)}\,\cdot \tag{4.5}$$

Considering the invariant h as a function of s, we obtain that Equation (4.2) takes the form

$$\frac{d^2 h}{ds^2} = q \tag{4.6}$$

and Equation (4.3) reduces to the following linear equation for ζ :

$$\frac{d}{ds}\left(h\,\frac{d\zeta}{ds}\right) - \zeta = 0. \tag{4.7}$$

Invoking the formulae (1.15) (1.16) and using Equation (4.5), one can readily show that any solution $\zeta = \zeta(s)$ of Equation (4.7) provides the following parametric representation of the admissible Chaplygin functions:

$$K(\sigma) = -\frac{1}{4}\zeta^4(s), \quad \sigma = \int \frac{ds}{h\zeta^2}\,\cdot \tag{4.8}$$

In the above formulae, the arbitrary constant that appears while determining σ' from (1.15) can be taken, without loss of generality, to be equal to one. Indeed, this condition can be achieved by multiplying the solution ζ of Equation (4.4) by a constant factor.

In order to work out the quadrature in the second equation (4.8), we consider a solution $\zeta_0 = \zeta_0(s)$ of Equation (4.7) that is linearly independent of $\zeta(s)$. The Wronskian

$$W[\zeta_0, \zeta] = \zeta_0' \, \zeta - \zeta_0 \, \zeta'$$

of these two solutions is equal to

$$W[\zeta_0, \zeta] = \frac{c}{h}, \quad c = \text{const.}$$

Therefore

$$\frac{d}{ds}\left(\frac{\zeta_0}{\zeta}\right) = \frac{W[\zeta_0, \zeta]}{\zeta^2} = \frac{c}{h\zeta^2}\,.$$

Consequently, one can take σ in the following form:

$$\sigma = \frac{1}{c}\,\frac{\zeta_0(s)}{\zeta(s)} \tag{4.9}$$

since the additive constant of integration can be arbitrarily changed due to the choice of a solution ζ_0 when ζ is fixed.

Note that, given any fixed q, we obtain a set of admissible functions $K(\sigma)$ depending on the arbitrary constants that appear while solving the equations (4.6), (4.7) and evaluating the quadrature (4.8). In order to give a precise description of the above set, we will use group properties of the system of ordinary differential equations

$$h'' = q, \quad (h\,\zeta')' = \zeta, \quad h\,\zeta^2\,\sigma' = 1, \quad 4K = -\zeta^4. \tag{4.10}$$

determining $K(\sigma)$ when q is given. Here, the primes denote the derivatives with respect to s.

It is easy to verify that the system (4.10) admits a five-parameter group with the following generators:

$$\begin{gathered} Y_1 = \frac{\partial}{\partial s}, \quad Y_2 = s\frac{\partial}{\partial s} - \sigma\frac{\partial}{\partial \sigma} + 2h\frac{\partial}{\partial h}, \quad Y_3 = \frac{\partial}{\partial \sigma}\,, \\ Y_4 = 2\sigma\frac{\partial}{\partial \sigma} - \zeta\frac{\partial}{\partial \zeta} - 4K\frac{\partial}{\partial K}, \quad Y_5 = \sigma^2\frac{\partial}{\partial \sigma} - \sigma\zeta\frac{\partial}{\partial \zeta} - 4\sigma K\frac{\partial}{\partial K}\,\cdot \end{gathered} \tag{4.11}$$

The operators (4.10) generate the following one-parameter groups of transformations with the respective parameters $a_i (i = 1, 2, 3, 4, 5)$ (the unwritten quantities are invariant under the corresponding operator):

$$
\begin{aligned}
&(Y_1) \quad \bar{s} = s + a_1,\\
&(Y_2) \quad \bar{s} = a_2 s, \quad \bar{\sigma} = \frac{1}{a_2}\sigma, \quad \bar{h} = a_2^2 h,\\
&(Y_3) \quad \bar{\sigma} = \sigma + a_3,\\
&(Y_4) \quad \bar{\sigma} = a_4^2\sigma, \quad \zeta = \frac{1}{a_4}\zeta, \quad \overline{K} = \frac{1}{a_4^4}K,\\
&(Y_5) \quad \bar{\sigma} = \frac{\sigma}{1 - a_5\sigma}, \quad \bar{\zeta} = (1 - a_5\sigma)\zeta, \quad \overline{K} = (1 - a_5\sigma)^4 K.
\end{aligned}
\tag{4.12}
$$

We will denote by D the group composed of the transformations (4.12). The system (4.10) admits the group D, and every transformation from D maps any solution of the system (4.10) into a solution of this system. This yields the following result.

Theorem 2. If $K_0(\sigma)$ is an admissible Chaplygin function corresponding to a certain value of q then

$$
K(\sigma) = \frac{M}{(c\sigma + d)^4} K_0\left(\frac{a\sigma + b}{c\sigma + d}\right) \tag{4.13}
$$

is also an admissible Chaplygin function corresponding to the same value of q. Here M, a, b, c, d are arbitrary constants (in general, complex) such that $M \neq 0$ and $ad - bc \neq 0$.

Proof. Let us take any constants M', a', b', c', d' such that $M' \neq 0$, $a'd' - b'c' \neq 0$ and, using the superposition of the transformations (4.12), consider the transformation

$$
\bar{\sigma} = \frac{a'\sigma + b'}{c'\sigma + d'}, \quad \overline{K} = M'(c'\sigma + d')^4 K.
$$

The group property yields that if $K(\sigma)$ is an admissible function then $\overline{K}(\bar{\sigma})$ is also an admissible function. Furthermore, we have:

$$
\sigma = \frac{d'\bar{\sigma} - b'}{-c'\bar{\sigma} + a'}, \quad c'\sigma + d' = \frac{a'd' - b'c'}{-c'\bar{\sigma} + a'}
$$

and

$$
\overline{K}(\bar{\sigma}) = \frac{M'(a'd' - b'c')^4}{(-c'\bar{\sigma} + a')^4} K\left(\frac{d'\bar{\sigma} - b'}{-c'\bar{\sigma} + a'}\right).
$$

Since the last formula differs from (4.12) only in notation, the theorem is proved.

The formula (4.13) defines the most general transformation of the functions $K(\sigma)$ induced by the transformations of the group D. Consequently, regarding two functions to be equivalent if they are connected via (4.13), we divide the whole family of functions $K(\sigma)$ into the classes of *equivalent functions* with respect to D. The problem is to find these classes via their simplest representatives.

We first note that the group D transforms, along with the functions $K(\sigma)$, the solutions $h(s)$ of the equation (4.6) as well. Since the group D acts on the variables s and h by means of the transformations (Y_1) and (Y_2) from (4.12), the most general transformation of the function $h(s)$ under the group D can be written in the form

$$h(s) = \frac{1}{l^2}\, h_0(ls + m), \tag{4.14}$$

where l, m are arbitrary constants, $l \neq 0$.

The functions $K(\sigma)$ obtained as solutions of system (4.10) with a fixed function $h(s)$ belong to one class. Indeed, letting $h(s)$ be given, we see from the equations (4.8) and (4.9) that the variation of $K(\sigma)$ is possible only due to arbitrariness in the choice of two linearly independent solutions $\xi_0(s)$ and $\xi(s)$ of a given equation (4.7). It is manifest, however, that this arbitrariness is settled by the transformation (4.13).

Let us introduce the classes of equivalent functions $h(s)$ by putting two functions in one class if they are connected by a transformation of the form (4.14). Then to every class of equivalent $h(s)$ there will correspond one class of equivalent $K(\sigma)$. Indeed, let $K_0(\sigma)$ and $K(\sigma)$ be obtained via $h_0(s)$ and $h(s)$, respectively, where $h(s)$ is given by (4.14). Then $h(s)$ is mapped to $h_0(s)$ by a proper transformation from D while $K(\sigma)$ will be mapped into $K_1(\sigma)$. According to the previous remark, $K_1(\sigma)$ is equivalent to $K_0(\sigma)$, and hence $K(\sigma)$ is equivalent to $K_0(\sigma)$. Conversely, the functions $h(s)$ determined via a given function $K(\sigma)$ belong to one class. This is readily seen, e.g. from the equations

$$h = -\frac{1}{\zeta^3}\frac{d^2}{d\sigma^2}\left(\frac{1}{\zeta}\right), \quad s = -\int \frac{d^2}{d\sigma^2}\left(\frac{1}{\zeta}\right)\frac{d\sigma}{\zeta}$$

resulting from (4.7) and (4.8).

Thus, there is a one-to-one correspondence between the classes of equivalent functions $K(\sigma)$ and $h(s)$. This means that we will find all classes of the admissible $K(\sigma)$ if we proceed as follows. At first, we fill find the classes of functions $h(s)$ and then we will construct, for every representative $h_0(s)$, any single function $K_0(\sigma)$.

Equation (4.6) determining $h(s)$ has the general solution

$$h(s) = 1/2qs^2 + C_1 s + C_2, \tag{4.15}$$

where C_1 and C_2 are arbitrary constants. It is apparent now that if $q \neq 0$ all solutions $h(s)$ are divided into two classes according to the character of roots of the function (4.15). Namely, class I comprises the solutions having repeated roots and class II comprises the solutions with simple roots. Any two solutions of the same class are connected by the transformation (4.14) while solutions of different classes cannot be transformed into one another via this transformation. If $q = 0$ we will assign the solutions (4.15) with $C_1 = 0$ to class I while the solutions (4.15) with $C_1 \neq 0$ will be referred to class II (if $q = 0$ one or both roots can be formally assumed to be at ∞; then classes I and II will differ according to the multiplicity of the root $s = \infty$). The corresponding classes of the functions $K(\sigma)$ will be denoted by (I, q) and (II, q).

It remains now to take two linearly independent solutions $\zeta_0(s)$ and $\zeta(s)$ of equation (4.7) for each of the elementary representatives $h_0(s)$ of the defined classes and to determine the representatives $K_0(\sigma)$ of the above classes via the parametric representation (4.8), (4.9). Due to Theorem 2, the latter can be written in a more simple form:

$$K = \zeta^4(s), \quad \sigma = \frac{\zeta_0(s)}{\zeta(s)} \cdot \tag{4.16}$$

We proceed now to calculations.

Class $(I, 0)$. One can take here $h_0(s) = 1$. Then solutions to Equation (4.7) will be, e.g. $\zeta_0 = e^{\sigma}$ and $\zeta = e^{-\sigma}$. Elimination of s from (4.16) yields:

$$K_0(\sigma) = \frac{1}{\sigma^2} \cdot \tag{4.17}$$

Class $(II, 0)$. In this case, it is convenient to take $h_0(s) = -2s$. Then Equation (4.7) has the form:

$$2(s\zeta')' + \zeta = 0.$$

The change of variables $x = \sqrt{2s}, \quad \zeta(s) = y(x)$ reduces the above equation to the Bessel equation with the index zero:

$$y'' + \frac{1}{x} y' + y = 0$$

Taking the Bessel functions $y = J_0(x)$ and $y_0 = Y_0(x)$ of the first and second kind, respectively, as two linearly independent solutions of this equation, we obtain:

$$K_0(\sigma) = J_0^4(\sqrt{2s}), \quad \sigma = \frac{Y_0(\sqrt{2s})}{J_0(\sqrt{2s})} \cdot \tag{4.18}$$

In the case when $q \neq 0$, it is convenient to introduce a new parameter ν instead of q by setting

$$\frac{2}{q} = \nu(\nu+1). \tag{4.19}$$

Class (I, q) with $q \neq 0$. One can take here $h_0(s) = \frac{1}{2}qs^2$. Invoking the notation (4.9), we write Equation (4.7) in the form

$$(s^2\zeta')' - \nu(\nu+1)\zeta = 0,$$

i.e. as the Euler equation with the indices ν and $-\nu - 1$. If $\nu \neq 1/2$, two linearly independent solutions of the above equation are

$$\zeta_0 = s^{-\nu-1}, \quad \zeta = s^\nu.$$

Elimination of s from Equations (4.16) yields:

$$K_0(\sigma) = \sigma^{-\frac{4\nu}{2\nu+1}}. \tag{4.20}$$

The case when $\nu = -1/2$ corresponds to $q = -8$. Hence, the *class* $(I, -8)$ should be singled out. Two solutions of Equation (4.7) can be taken in the form:

$$\zeta_0 = -2\,\frac{\log s}{\sqrt{s}}\,, \quad \zeta = \frac{1}{\sqrt{s}}\,\cdot$$

Then $K_0 = s^{-2}$ and $\sigma = -2\log s$ or $s = e^{-\sigma/2}$. Consequently, the representative of this class is

$$K_0(\sigma) = e^\sigma. \tag{4.21}$$

Class (II, q) with $q \neq 0$. Let us assume that the roots of $h_0(s)$ are at the points ± 1, and hence

$$h_0(s) = \frac{1}{2}\,q(s^2 - 1).$$

Then equation (4.7) becomes an equation for the Legendre functions:

$$[(1 - s^2)\zeta']' + \nu(\nu+1)\zeta = 0. \tag{4.22}$$

Two linearly independent solutions are furnished by the Legendre functions of the νth degree $\zeta = P_\nu(s)$ and $\zeta_0 = Q_\nu(s)$ of the first and second kind, respectively. We obtain:

$$K_0(\sigma) = P_\nu^4(s), \qquad \sigma = \frac{Q_\nu(s)}{P_\nu(s)}\,\cdot \tag{4.23}$$

The final result of these calculations and the preceding discussions can be formulated as follows.

Theorem 3. Any admissible Chaplygin function either coincides with one of the functions (4.17), (4.18), (4.20), (4.21), (4.23) or is obtained from one of them via group transformation (4.13).

Let us note that the case $h = 0$ fits the above classification as well if, by convention, we set $q = \infty$ and take $K_0(\sigma) = 1$ as a representative of the class (II, ∞). This is due to the fact that the transformation (4.13) maps $K_0(\sigma) = 1$ to the function (4.1).

§ 5. Canonical forms of the admissible Chaplygin equation

It is manifest that $p = 1$ for every Chaplygin equation. Hence, due to the corollary of Theorem 1, two admissible Chaplygin equations are equivalent if and only if they have identical values of the parameter q. The Chaplygin equation with the invariant h is equivalent by function to the equation

$$\psi^\circ_{\lambda\mu} - h\,\psi^\circ = 0 \tag{5.1}$$

since the latter has the Laplace invariants equal to h as well. The reckoning shows that Equation (1.14) is mapped into (5.1) by the following substitution:

$$\psi = \frac{1}{\zeta}\,\psi^\circ, \tag{5.2}$$

where ζ is determined by Equation (4.5).

Let us consider the canonical forms of admissible Equations (5.1) and the corresponding transformations to the canonical forms. Applying to (5.1) the reasoning used in the proof of Theorem 1, one obtains the following results.

If $q = 0$ then there exist functions $\alpha(\lambda)$ and $\beta(\mu)$ such that

$$h = \alpha'(\lambda)\beta'(\mu). \tag{5.3}$$

Furthermore, the change of variables

$$\alpha = \alpha(\lambda), \quad \beta = \beta(\mu), \quad \psi^\circ(\lambda, \mu) = \Psi(\alpha, \beta) \tag{5.4}$$

reduces equation (5.1) to the form

$$\Psi_{\alpha\beta} = \Psi. \tag{5.5}$$

If $q \neq 0$ then there exist functions $\alpha(\lambda)$ and $\beta(\mu)$ such that

$$h = \frac{2}{q}\,\frac{\alpha(\lambda)\beta'(\mu)}{[\alpha(\lambda) + \beta(\mu)]^2}\,. \tag{5.6}$$

Now the same change of variables (5.4) reduces (5.1) to the equation

$$\Psi_{\alpha\beta} = \frac{\nu(\nu+1)}{(\alpha+\beta)^2}\Psi, \tag{5.7}$$

where the parameter q is replaced by ν defined via Equation (4.19).

Thus, any admissible Chaplygin equation is equivalent either to Equation (5.5) or to Equation (5.7).

Let us find the possible forms of the functions α and β. For this purpose the invariant h should be calculated as a function of $t = \lambda + \mu$ by integrating Equation (4.4). Note that it is sufficient to consider only representatives of classes of equivalent functions $h(s)$.

Class $(I, 0)$. Here $h = 1$, and (4.4) yields $s = t$. Thus

$$\alpha = \lambda, \quad \beta = \mu.$$

Class $(II, 0)$. Here $h = -2s$, and (4.4) yields $2s = -\mathrm{e}^{-2t}$. Thus $h = \mathrm{e}^{-2t} = \mathrm{e}^{-2\lambda}\mathrm{e}^{-2\mu}$ and consequently

$$\alpha = \frac{1}{2}\,\mathrm{e}^{-2\lambda}, \quad \beta = \frac{1}{2}\,\mathrm{e}^{-2\mu}.$$

Class (I, q) with $q \neq 0$. Here $h = \frac{1}{2}\,qs^2$, and (4.4) yields $s = -\nu(\nu+1)/t$. Thus $h = \nu(\nu+1)/t^2$ and consequently

$$\alpha = \lambda, \quad \beta = \mu.$$

Class $(I, -8)$ has no differences from all the other classes (I, q).

Class (II, q) with $q \neq 0$. Here $h = \frac{1}{2}\,q(s^2 - 1)$. Integrating (4.4) we obtain

$$s = -\mathrm{cth}\frac{qt}{2}, \quad h = \frac{q}{2}\,\mathrm{sh}^2\frac{qt}{2}\,\cdot$$

Since

$$\mathrm{sh}^2\frac{qt}{2} = \frac{1}{4}\mathrm{e}^{-q\lambda}\mathrm{e}^{q\mu}(\mathrm{e}^{q\lambda} - \mathrm{e}^{-q\mu})^2,$$

we have

$$\alpha = \mathrm{e}^{q\lambda}, \quad \beta = -\mathrm{e}^{-q\mu}.$$

Let us also specify the group transformation that maps any solution of an admissible Chaplygin equation again into a solution of the same equation. It is sufficient to do this for each of the canonical forms (5.5) and (5.7) since the change of variables reducing any admissible equation to one of them is already known. Using the results of §3 we obtain that a three-parameter transformation of the family of solutions for Equation (5.5) into itself has the form:

$$\overline{\Psi}(\alpha, \beta) = \Psi\left(a\alpha + b, \frac{1}{a}\beta + c\right), \quad a \neq 0. \tag{5.8}$$

The similar transformation for equation (5.7) has the form:

$$\bar{\Psi}(\alpha, \beta) = \Psi\left(\frac{a\alpha + b}{c\alpha + d}, \quad \frac{a\beta + b}{c\beta + d}\right), \quad ad - bc \neq 0. \tag{5.9}$$

Equations (5.8) and (5.9) involve arbitrary constants a, b, c, d.

In conclusion of this section, let us consider the Laplace series for every admissible Chaplygin equation. Due to the equivalence properties it is sufficient to investigate the series for the canonical form of such an equation. In the case of the canonical form (5.5) all terms of the Laplace series (2.10) coincide with the original equation. However, application of Laplace's method to Equation (5.7) leads to new equations that differ from the original one by the values of the parameter ν. Equation (2.13) shows that the Laplace series cuts off at a certain n if and only if the following equation holds:

$$\frac{2}{q} = n(n+1). \tag{5.10}$$

Comparing (5.10) with Equation (4.19) defining ν we see that Equation (5.10) is satisfied when $\nu = n$. Moreover, since $n(n+1) = (-n-1)(-n)$, the Laplace series cuts off at both ends.

Consequently, if ν is an integer ($\nu \neq 0$ and $\nu \neq -1$), one can find the general solution for the admissible Chaplygin equation in an explicit form containing two arbitrary functions and not involving quadratures. For the canonical equation (5.7) the solution has the form [6]:

$$\Psi(\alpha, \beta) = (\alpha + \beta)^{n+1} \frac{\partial^{2n}}{\partial\alpha^n \partial\beta^n} \frac{F(\alpha) + G(\beta)}{\alpha + \beta}\,, \tag{5.11}$$

where $F(\alpha)$ and $G(\beta)$ are arbitrary functions.

If ν is not an integer then the Laplace series is infinite. But since the admissible Chaplygin function is equivalent to Equation (5.7), its Riemann function is expressed by a hypergeometric function [6]. Hence, a formula for the general solution containing two arbitrary functions can be written for the general admissible Chaplygin equation as well. However, the arbitrary functions will appear now under a sign of some quadratures.

Note that in the problem of gas dynamics, the approximations due to S.A. Khristianovich [21] and G.A. Dombrovsky [8] correspond to the value $\nu = 1$ which leads to the approximating equations of classes $(I, 1)$ and $(II, 1)$, respectively.

§ 6. The Tricomi type equations

Let us investigate admissible Chaplygin functions that can be used for approximation in the vicinity of the point of transition over the acoustic speed $\sigma = 0$. A necessary condition for reliability of such an approximation is validity of (1.8) type equations.

We call an admissible Chaplygin equation (1.7) a *Tricomi type equation* (in short: T-type) if $K(\sigma)$ is an analytic function of σ in a vicinity of the point $\sigma = 0$ and if $K(0) = 0$ whereas $K'(0) \neq 0$. The corresponding function $K(\sigma)$ will also be termed a T-type function. The problem is to find all T-type equations.

The *Tricomi equation*

$$\sigma\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \tag{6.1}$$

itself is a T-type equation since $K(\sigma) = \sigma$ is one of the admissible functions. It is obtained from (4.20) when

$$\nu = -\frac{1}{6}, \tag{6.2}$$

i.e it is a representative of the class $(I, -72/5)$.

Theorem 4. Any T-type equation is equivalent to the Tricomi equation.

Proof. The proof can be obtained by inspecting the corresponding $K(\sigma)$ in different classes. Let us consider the formulae (4.16) of parametric representation of the admissible Chaplygin functions. A point where $K = 0$ and σ is finite will be called a *transition point.* Let $s = s_0$ at a transition point. Then Equations (4.16) yield that $\zeta(s_0) = 0$ and $\zeta_0(s_0) = 0$. It follows that a transition point should be a critical point of Equation (4.7).

Note that if a critical point of equation (4.7) is a transition point for some $h_0(s)$ then after transformation (4.14) the transformed point will be a transition point for the new equation with the transformed function $h(s)$. Therefore, in order to obtain transition points it is sufficient to consider the standard equations (4.7) corresponding to the simplest representatives of classes of equivalent functions $h(s)$.

If $q = 0$, the only transition points could be $s = 0$ and $s = \infty$. However, neither when $h = 1$ nor when $h = -2s$ the above points are transition points. The case $h = 1$ is a trivial one. If $h = -2s$ this conclusion results from the behavior of $J_0(x)$ and $Y_0(x)$ at $x \to 0$ and $x \to \infty$. Consequently, there are no T-type equations in the classes $(I, 0)$ and $(II, 0)$.

Turning to the class (I, q), the only transition point there can be $s = \infty$. Indeed, the form of two linearly independent solutions $\zeta_0 = s^{-\nu-1}$ and $\zeta = s^{\nu}$

of Equation (4.7) shows that the requirement for $s = 0$ to be a transition point leads to the contradictory inequalities $\nu > 0$ and $-\nu - 1 > 0$. Furthermore, $s = \infty$ will be a transition point for the function (4.20) if $-4\nu/(2\nu+1) = 1$, i.e. $\nu = -1/6$. This leads to the Tricomi equation (6.1).

Finally, let us consider the classes (II, q) with $q \neq 0$. Here $h = \frac{1}{2}q(s^2 - 1)$ and transition points can be provided only by the critical points $s = \pm 1$ and $s = \infty$. However, the indices of equation (4.22) at the critical points $s = \pm 1$ equal to zero. Therefore one of the solutions at any of these points will be regular while the other will possess the logarithmic singularity. It follows that $s = \pm 1$ are not transition points. Hence, every admissible function in the classes (II, q) can have at most one transition point corresponding to $s = \infty$. Equation (4.22) has at $s = \infty$ the indices $-\nu$ and $\nu + 1$. If this point is a transition one, then the indices should be different, otherwise (i.e. if $\nu = -1/2$) one of the solutions would have the logarithmic singularity at $s = \infty$. Consequently, there exist two linearly independent solutions that can be presented when $|s| > 1$ in the form:

$$\zeta = s^{\nu} f_1(s^{-1}), \quad \zeta_0 = s^{-\nu-1} f_2(s^{-1}), \tag{6.3}$$

where $f_1(s^{-1})$ and $f_2(s^{-1})$ are power series in s^{-1} converging in the circle $|s^{-1}| < 1$. Moreover, $f_1(0) = f_2(0) = 1$. Therefore, we obtain by using (4.16):

$$K(\sigma) = \sigma^{-\frac{4\nu}{2\nu+1}} g(\sigma), \quad g(0) = 1.$$

Thus, the equation

$$-\frac{4\nu}{2\nu+1} = 1$$

is a necessary condition for $K(\sigma)$ to be a T-type function. The above equation leads again to the value $\nu = -1/6$.

This proves Theorem 4 since we obtained that the admissible T-type functions $K(\sigma)$ can be contained only in the classes $(I, -72/5)$ and $(II, -72/5)$. It remains only to employ the corollary of Theorem 1.

We will show now that the class (II, -72/5) indeed contains T-type function $K(\sigma)$. For this purpose we will construct explicitly the power series in the formulae (6.3) and, in addition, will prove the analyticity of the obtained function $K(\sigma)$.

With this aim in view, we will introduce in Equation (4.22) the new independent variable τ defined by thje equations

$$s^2 = \tau, \quad \zeta(s) = \delta(\tau). \tag{6.4}$$

Then equation (4.22) takes the form:

$$\tau(\tau - 1)\,\delta'' + \left(-\frac{1}{2} + \frac{3}{2}\,\tau\right)\delta' - \frac{1}{4}\,\nu(\nu+1)\,\delta = 0. \tag{6.5}$$

The hypergeometric equation (6.5) has at $\tau = \infty$ the indices $-\frac{1}{2}\nu$ and $\frac{1}{2}(\nu+1)$. Consequently, its two linearly independent solutions are written via hypergeometric series as follows:

$$\begin{aligned} \delta &= \tau^{\frac{\nu}{2}} F\left(-\frac{\nu}{2},\ -\frac{\nu}{2}+\frac{1}{2};\ -\nu+\frac{1}{2};\ \frac{1}{\tau}\right), \\ \delta_0 &= \tau^{-\frac{\nu+1}{2}} F\left(\frac{\nu+1}{2},\ \frac{\nu}{2}+1;\ \nu+\frac{3}{2};\ \frac{1}{\tau}\right). \end{aligned} \tag{6.6}$$

Returning to the variable s and taking the value $\nu = -\frac{1}{6}$, the one we are interested in, we obtain two linearly independent solutions to Equation (4.22) that have the required form (6.3):

$$\zeta = s^{-\frac{1}{6}} F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; s^{-2}\right), \quad \zeta_0 = s^{-\frac{5}{6}} F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; s^{-2}\right). \tag{6.7}$$

The representation (4.16) of the corresponding admissible Chaplygin function has the form:

$$K = s^{-\frac{2}{3}} \left[F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; s^{-2}\right)\right]^4, \quad \sigma = s^{-\frac{2}{3}} \frac{F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; s^{-2}\right)}{F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; s^{-2}\right)}. \tag{6.8}$$

The second equation in (6.8) can be written in the form

$$\sigma^3 = s^{-2} f(s^{-2}),$$

where $f(s^{-2})$ is analytic near the point $s^{-2} = 0$ and besides $f(0) = 1$. Hence, s^{-2} is an analytic function of σ^3. Consequently, the ratio

$$\frac{K}{\sigma} = \frac{\left[F\left(\frac{1}{12}, \frac{7}{12}; \frac{2}{3}; s^{-2}\right)\right]^5}{F\left(\frac{5}{12}, \frac{11}{12}; \frac{4}{3}; s^{-2}\right)}$$

as well as the admissible function $K(\sigma)$ determined by (6.8) will be analytic functions of σ. Consequently, $K(\sigma)$ is a T-type function.

Finally we note that if $K_0(\sigma)$ is an admissible T-type function then, due to group property (4.13), the function

$$K(\sigma) = \frac{M}{(c\sigma + d)^4} K_0\left(\frac{\sigma}{c\sigma + d}\right) \tag{6.9}$$

will be also an admissible T-type function for all values of the constants M, c, d $(M \neq 0, d \neq 0)$. It turns out that the above constructions furnish all admissible T-type functions. Thus the following assertion holds.

Theorem 5. Any T-type admissible Chaplygin function can be obtained by means of the group transformation (6.9) either from the function $K_0(\sigma) = \sigma$ or from the function $K_0(\sigma)$ determined by Equations (6.8).

Proof. Recall that the T-type functions $K(\sigma)$ equivalent to one of such functions $K_0(\sigma)$ belong to family (4.13). The requirement $K(0) = 0$ leads to the equation $K_0(b/d) = 0$ meaning that $\sigma = b/d$ must be a transition point. This is possible only if $b = 0$ since otherwise the function $K_0(\sigma)$ would have two transition points. However, the proof of Theorem 4 revealed that none of the admissible functions $K(\sigma)$ can have more than one transition point. If $b = 0$ then $K(\sigma) = 0$ is satisfied due to $K_0(0) = 0$. Moreover,

$$K'(0) = Mad^{-5}K_0'(0) \tag{6.10}$$

and therefore $K'(0) \neq 0$ if $K_0'(0) \neq 0$. It is manifest that if $b = 0$ one can assume $a = 1$. This proves that every subclass of equivalent T-type functions is described by the formula (6.9). Since the obtained T-type functions, namely, $K = \sigma$ and the function (6.8) are representatives of classes $(I, -72/5)$ and $(II, -72/5)$ of equivalent functions and since no other classes contain T-type functions, the theorem is proved.

§ 7. The Laplace type equations

Here we will obtain all admissible Chaplygin appropriate for approximating the gasdynamical function $K(\sigma)$ in the limiting case of low speeds, i.e. when $\sigma \to +\infty$. The accuracy of such an approximation is guaranteed by Equation (1.9) showing that the Chaplygin equation turns into the Laplace equation asymptotically when $\sigma \to +\infty$.

We call an admissible equation (1.7) a *Laplace type equation* (briefly L-type) if $K(\sigma)$ meets the condition of the form (1.9) when $\sigma \to +\infty$. The corresponding function $K(\sigma)$ will also be called an L-type function. It is apparent that the Laplace equation itself is an L-type equation.

Theorem 6. The L-type Chaplygin functions can be only functions of the classes (II, q). Moreover, the class (II, q) with any q contains an L-type function. If $K_0(\sigma)$ is a function of this type then the general form of L-type functions of the same class is given by

$$K(\sigma) = K_0(a\sigma + b), \tag{7.1}$$

where a and b are arbitrary constants, $a > 0$.

Proof. We will call a point a *critical point* and denote it by s_∞ if, at this point, $\sigma = \infty$ and $K \neq 0, K \neq \infty$. The representation (4.16) shows that at the critical point one should have $\zeta \neq 0$, $\zeta \neq \infty$ and $\zeta_0 = \infty$. Hence, only a singular point of Equation (4.7) can be a critical point.

It is easily seen that there are no critical points in the classes (I, q). For the class $(I, 0)$, the statement results from the observation that the only possible critical point is $s_\infty = \infty$ and none of the solutions of equation (4.7) has a finite value different from zero at this point. In the classes (I, q) with $q \neq 0$ there can be two critical points, $s_\infty = 0$ and $s_\infty = \infty$. However, again there are no solutions having a finite non-vanishing value at the point s_∞.

Let us show that there is an L-type function in any class (II, q). Indeed, in this case Equation (4.7) has a regular singular point where both its indices equal to zero. One of the solutions is finite at the singular point and can be taken as the function ζ in the representation (4.16) while the other has necessarily a logarithmic singularity and can be taken as the function ζ_0.

In the case of the class $(II, 0)$, it is the function (4.18) that represents an L-type function, whereas in the classes (II, q) with $q \neq 0$ it is the function (4.23). In the first case the critical point is $s_\infty = 0$, and in the second case it is $s_\infty = 1$.

Finally, let $K_0(\sigma)$ be an L-type function determined by Equations (4.16) via linearly independent solutions $\zeta(s)$ and $\zeta_0(s)$ of Equation (4.7). The general form of the admissible functions of the class represented by $K_0(\sigma)$ is defined by (4.13). To finish the proof of the theorem, it remains to demonstrate that if the formula (4.13) furnishes again an L-type function then it reduces to (7.1), i.e. $c = 0$.

Let us assume that $c \neq 0$. Then to the point $\sigma_* = a/c$ there corresponds a point s_* such that

$$\zeta(s_*) = \infty, \quad \frac{\zeta_\theta(s_*)}{\zeta(s_*)} \neq \infty. \tag{7.2}$$

These equations show that s_* is a singular point for Equation (4.7).

Let us show that $s_* \neq \infty$. It is manifestly true for the class $(II, 0)$ since both solutions $J_0(\sqrt{2s})$ and $Y_0(\sqrt{2s})$ tend to zero when $s \to +\infty$.

Consider now a class (II, q) with $q \neq 0$. The indices of two linearly independent solutions of Equation (4.7) at $s = \infty$ equal to $-\nu$ and $\nu + 1$. To satisfy (7.2) at $s_* = \infty$, the parameter ν should be outside the interval $[-1, 0]$. Indeed, if $-1 < \nu < 0$ both solutions vanish at $s = \infty$, and the values $\nu = -1$ and $\nu = 0$ do not correspond to any finite value q. In what follows, one can assume without loss of generality, that $\nu > 0$ since otherwise we would replace ν by $-\nu - 1$ without changing the value of q.

The analytical continuation of the solutions $\zeta(s)$ and $\zeta_0(s)$ into the region

$|s| > 1$ has the form:

$$\zeta = As^{\nu} f_1(s^{-1}) + Bs^{-\nu-1} f_2(s^{-1}),$$
$$\zeta_0 = A_0 s^{\nu} f_1(s^{-1}) + B_0 s^{-\nu-1} f_2(s^{-1}), \tag{7.3}$$

where A, B, A_0, B_0 are constants such that $AB_0 - A_0 B \neq 0$ while $f_1(s^{-1})$ and $f_2(s^{-1})$ are power series in integral powers of s^{-1} such that $f_1(0) = f_2(0) = 1$. Since $\nu > 0$, the condition that $\zeta \to \infty$ when $s \to \infty$ implies $A \neq 0$. Hence, we obtain the following expression for σ :

$$\sigma = \frac{\zeta_0}{\zeta} = \frac{A_0 + B_0 s^{-2\nu-1} f_3(s^{-1})}{A + Bs^{-2\nu-1} f_3(s^{-1})} \cdot \tag{7.4}$$

Here $f_3(s^{-1})$ is a function that has the same properties as $f_2(s^{-1})$. Due to (7.2) we have $A_0 = A\sigma_*$. Therefore, the formula (7.4) yields that

$$(\sigma - \sigma_*) s^{2\nu+1} \to \frac{AB_0 - A_0 B}{A^2} \neq 0$$

when $s \to \infty$. Consequently, we have in the vicinity of $\sigma = \sigma_*$:

$$s \sim (\sigma - \sigma_*)^{-\frac{1}{2\nu+1}}.$$

The expression (7.3) for ζ and the representation (4.16) in the vicinity of $\sigma = \sigma_*$ yield:

$$K_0(\sigma) \sim (\sigma - \sigma_*)^{-\frac{4\nu}{2\nu+1}}.$$

Now, invoking (4.13), we have at $\sigma \to \infty$:

$$K(\sigma) = \frac{M}{(c\sigma + d)^4} K_0 \left(\sigma_* + \frac{bc - ad}{c(c\sigma + d)} \right) \sim (c\sigma + d)^{\frac{4\nu}{2\nu+1} - 4}.$$

It follows that $K(\sigma)$ can be an L-type function only if the exponent in the latter expression vanishes, i.e. if $\nu = -1$. This cannot correspond, however, to any final value of q. It follows that $s_* \neq \infty$ in the formula (7.2).

The class $(II, 0)$ contains only two singular points, $s = 0$ and $s = \infty$. The first of them was already used in constructing the L-type function (4.18). Therefore, the condition $c \neq 0$ leads here to contradiction.

The class (II, q) with $q \neq 0$ contains three singular points, $s = \pm 1$ and $s = \infty$. The assumption $c \neq 0$ eliminates the point $s_* = \infty$. Let us assume that $K_0(\sigma)$ is a function for which $s_\infty = 1$ is a critical point. It is obvious that this function is determined up to the transformation (7.1). If the transformation (4.13) maps $K_0(\sigma)$ into an L-type function $K(\sigma)$, the latter can have as a critical point only $s_\infty = -1$. However, the equation used in constructing $K(\sigma)$ remains unaltered when s is replaced by $-s$. Therefore the variety of L-type functions with the critical point $s_\infty = -1$ coincides with the set of similar functions for $s_\infty = 1$. Therefore the critical point $s_\infty = -1$ can be disregarded.

We obtain that the transformations mapping an L-type function $K_0(\sigma)$ again into an L-type function can be identified with the transformations preserving the critical point associated with $K_0(\sigma)$. Obviously, these are only the transformations with $c = 0$, i.e. the transformations of the form (7.1). This completes the proof of Theorem 6.

Let us show that there exists an admissible function $K(\sigma)$ which is T-type and L-type simultaneously. According to Theorem 5 and Theorem 6, the desired function can belong only to the class $(II, -72/5)$. Here the transition point is $s_0 = \infty$ while $s_\infty = 1$ can be taken as a critical point.

Starting from the T-type function $K_0(\sigma)$ defined by (6.8), one can choose an appropriate transformation (6.9) so that to obtain an L-type function. To this end, it suffices to replace the solution ζ defined according to (6.7) by a linear combination ζ_1 of solutions ζ and ζ_0 such that the analytical continuation of ζ_1 will be regular at the point $s = 1$. One can obviously use, instead of s, the variable τ defined by (6.4). Then we have for Equation (6.5) the unique solution (up to a constant factor) which is regular at $\tau = 1$, namely (when $\nu = -1/6$):

$$\delta_1 = F\left(\frac{1}{12},\ \frac{5}{12};\ 1;\ 1-\tau\right).$$

By extending this hypergeometric series into the domain $|\tau| > 1$, we obtain:

$$\delta_1 = c_1\delta + d_1\delta_0,$$

where δ and δ_0 are determined by (6.6) and the coefficients c_1 and d_1 are:

$$c_1 = \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{12}\right)\Gamma\left(\frac{11}{12}\right)}, \qquad d_1 = \frac{\Gamma\left(-\frac{2}{3}\right)}{\Gamma\left(\frac{1}{12}\right)\Gamma\left(\frac{7}{12}\right)}. \tag{7.5}$$

Thus, a function $K_*(\sigma)$ which is simultaneously of the T-type and the L-type, is determined by the equations

$$\begin{gathered} K_* = \left[F\left(\frac{1}{12},\ \frac{5}{12};\ 1;\ 1-s^2\right)\right]^4, \\ \sigma = \frac{F\left(\frac{5}{12},\ \frac{11}{12};\ \frac{4}{3};\ s^{-2}\right)}{s^{2/3}F\left(\frac{1}{12},\frac{7}{12};\frac{2}{3};s^{-2}\right) - QF\left(\frac{5}{12},\ \frac{11}{12};\ \frac{4}{3};\ s^{-2}\right)}, \end{gathered} \tag{7.6}$$

where

$$Q = -\frac{d_1}{c_1} = \frac{3\Gamma(\frac{5}{12})\Gamma(\frac{5}{12})}{2\Gamma(\frac{1}{12})\Gamma(\frac{7}{12})} \cdot \tag{7.7}$$

It can be readily verified that the function $K_*(\sigma)$ is obtained from the "standard" T-type function (6.8), denoted by $K_0(\sigma)$, by means of the group transformation

$$K_*(\sigma) = \frac{c_1^4}{(1+Q\sigma)^4} K_0\left(\frac{\sigma}{1+Q\sigma}\right). \tag{7.8}$$

The function $K_*(\sigma)$ provides a good approximation of the gasdynamical Chaplygin function in the whole subsonic as well in the transonic domain. This function was obtained in different way and used in applications by Germain and Liger [12].

In conclusion we note that one can find likewise all admissible equations meeting the condition (1.10). This condition requires that the function $K(\sigma)$ have a "limiting" point, i.e. a point s_1 where $\sigma = \sigma_1$ and $K = \infty$. At the limiting point, one should have $\zeta = \infty$ and $\zeta_0 = \infty$ while the ration ζ_0/ζ should be finite and different from zero. Hence, the limiting point must also be a singular point of Equation (4.7), namely, $s_1 = \infty$. It is easily seen that the corresponding functions $K(\sigma)$ are contained in the classes (I, q) and (II, q_1), where

$$\frac{2}{q_1} = \nu_1(\nu_1 + 1), \qquad \nu_1 = \frac{\gamma+1}{2(\gamma-1)} \cdot \tag{7.9}$$

Furthermore, there are no admissible T-type functions that meet the condition (1.10). Hence, it is impossible to obtain an approximation of the gasdynamical Chaplygin equation that will be good in the supersonic and transonic domains simultaneously.

Editor's Notes

(1) *Page 130.* The quantities p and q are invariants, Ovsyannikov's invariants, of the hyperbolic equations (2.1) with respect to the general group of equivalence transformations (2.2). The complete set of all invariants was found recently by N.H. Ibragimov [18], see also [17].

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Lie Group Analysis: Classical Heritage

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Sophus Lie (1842-1899)
Created a new branch in mathematics - Lie groups, Lie algebras and group analysis of differential equations.

"If the historical exposition given here is correct, I can claim that I have been the first to use the concept of groups in the integration theory of differential equations." (S. Lie, Paper I in this volume, Section 12).

Albert Victor Bäcklund (1845-1922)
Nowadays his name is associated with Bäcklund transformations and Lie-Bäcklund transformation groups. However, he contributed one more result of fundamental mathematical importance, namely the theory of characteristics for second-order partial differential equations with an arbitrary number of independent variables (for two variables, characteristics were known since Monge and Ampére).

Lev Vasilyevich Ovsyannikov (born 1919)
The spearhead in the restoration of group analysis of differential equations in the 1960s and establishing modern group analysis. He made a fundamental contribution to theory of invariant and partially invariant solutions, group classification of differential equations and applications in fluid mechanics.

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