INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 7411–7422 doi:10.1088/0305-4470/39/23/016

Nonclassical properties of 'semi-coherent' quantum states

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Received 13 March 2006
Published 23 May 2006
Online at stacks.iop.org/JPhysA/39/7411

Abstract

We study the properties of 'semi-coherent' quantum states introduced by Mathews and Eswaran in 1973, establishing relations between them and other families of 'nonclassical' states, such as displaced number states and 'multiphoton' states. We also analyse the photon number distribution, photon statistics, the Wigner and marginal distribution functions, and show the ways for possible further generalizations.

PACS number: 42.50.Dv

1. Introduction

In 1973, Mathews and Eswaran [1] introduced the notion of 'semi-coherent states', defining them as those states of a harmonic oscillator which possess *time-independent* values of the quadrature variances σ_x and σ_p, different from the vacuum (or coherent state) values. They have shown that the necessary and sufficient condition for such states is

$$\langle \hat{a}^2 \rangle = \langle \hat{a} \rangle^2, \tag{1}$$

where $\hat{a} = (\hat{x} + \mathrm{i}\hat{p})/\sqrt{2}$ is the usual bosonic annihilation operator (we use the units $\hbar = m = \omega = 1$). Condition (1) is obviously satisfied for the usual coherent states $|\alpha\rangle$, for which $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Another trivial example is the Fock state $|n\rangle$, for which $\langle n|\hat{a}^2|n\rangle = \langle n|\hat{a}|n\rangle = 0$.

An immediate consequence of equation (1) is the equality of the quadrature variances, $\sigma_x = \sigma_p$, and zero value of the covariance, $\frac{1}{2}\langle \hat{x}\hat{p} + \hat{p}\hat{x}\rangle - \langle \hat{x}\rangle\langle \hat{p}\rangle = 0$. Thus 'semi-coherent' states cannot exhibit usual (second-order) squeezing. However, these states possess many other interesting properties, which are worth attention. One of the goals of our paper is to study these properties. In particular, we shall demonstrate a possibility of the *fourth-order* squeezing and sub-Poissonian statistics.

A nontrivial example of 'semi-coherent' states, only briefly discussed in [1], is a normalized superposition of two coherent states of the form

$$|\alpha_\perp\beta\rangle = \frac{|\alpha\rangle - |\beta\rangle\langle\beta|\alpha\rangle}{(1 - |\langle\beta|\alpha\rangle|^2)^{1/2}}. \tag{2}$$

We use the notation $|\alpha_\perp\beta\rangle$ in order to emphasize that the state (2) is orthogonal to the state $|\beta\rangle$:

$$\langle\beta|\alpha_\perp\beta\rangle = 0.$$

Therefore, the state $|\alpha_\perp\beta\rangle$ (we shall call it Mathews–Eswaran state, or simply MES) can be considered [1] as an orthogonal projection of the coherent state $|\alpha\rangle$ to another coherent state $|\beta\rangle$. In the state (2) one has [1]

$$\sigma_x = \sigma_p = \frac{1}{2} + \frac{|\langle\beta|\alpha\rangle|^2|\alpha - \beta|^2}{1 - |\langle\beta|\alpha\rangle|^2}, \tag{3}$$

so that, indeed, $\sigma_x = \sigma_p > 1/2$.

In the same 1973, a similar name 'semicoherent state' (without a dash in the first word) was used by Boiteux and Levelut [2] with respect to another family of quantum states, which are obtained by applying the displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$ to the Fock states $|n\rangle$:

$$|n; \alpha\rangle = \hat{D}(\alpha)|n\rangle. \tag{4}$$

Nowadays, states (4) are known under the name 'displaced number states'. As a matter of fact, the first examples of such states were given as far back as in 1954 by Plebański [3] and Senitzky [4]. The general construction,

$$|\psi_0; \alpha\rangle = \hat{D}(\alpha)|\psi_0\rangle, \tag{5}$$

where $|\psi_0\rangle$ is an arbitrary *fiducial state*, was considered by Klauder [5] (for other references see, e.g., the reviews in [6]). At first glance, the states defined according to equations (1) and (2) seem to be quite different from states (4) and (5). The second goal of our paper is to show that actually two families of states have much in common, being frequently (although not always) special cases of each other. In particular, we shall demonstrate that the limiting case of (2) when $\beta \to \alpha$ is nothing but the displaced first Fock state $\hat{D}(\alpha)|1\rangle$.

This paper is organized as follows. In section 2 we calculate average values of different products of the raising and lowering operators in the MES. The formulae obtained are used to show the existence of the fourth-order squeezing in these states. In section 3, we study the relations between MES and various other families of 'nonclassical' states. Section 4 is devoted to the Wigner representation of MES and the marginal probabilities. The photon statistics and dependence of the Mandel's Q-factor on the parameters of MES are studied in section 5. Section 6 contains a brief conclusion and perspectives for the further generalizations.

2. Higher order squeezing

A remarkable property of the state (2) is that it has the same average value of the *annihilation* operator $\hat{a}$ as the coherent state $|\alpha\rangle$, $\langle\alpha_\perp\beta|\hat{a}|\alpha_\perp\beta\rangle = \alpha$, *for any value* of β. Consequently, the average values of the quadrature components x and p in the state (2) are the same as in the coherent state $|\alpha\rangle$ [1]. Moreover, the average value of any power of the annihilation operator in the state (2) is the same as in the coherent state $|\alpha\rangle$:

$$\langle\hat{a}^n\rangle = \langle\hat{a}\rangle^n = \alpha^n. \tag{6}$$

However, the state is only 'semi-coherent', because average values of products of operators $\hat{a}$ and $\hat{a}^\dagger$ differ from average values of the same products in coherent states:

$$\langle \hat{a}^{\dagger m} \hat{a}^n \rangle = \alpha^{*m}\alpha^n + \frac{c}{1-c}(\beta^{*m} - \alpha^{*m})(\beta^n - \alpha^n), \tag{7}$$

where

$$c \equiv |\langle\beta|\alpha\rangle|^2 = \exp(-|\alpha-\beta|^2). \tag{8}$$

In particular, the average value of the number operator $\hat{n} \equiv \hat{a}^\dagger\hat{a}$ equals

$$\langle \hat{a}^\dagger \hat{a} \rangle = |\alpha|^2 + \frac{c}{1-c}|\beta-\alpha|^2. \tag{9}$$

The second term on the right-hand side of (9) depends only on the quantity $|\beta-\alpha|$; it tends to the unit value in the limiting case $\beta \to \alpha$, while it monotonically goes to zero when $|\beta-\alpha| \to \infty$. Taking into account such a behaviour and comparing equations (3) and (9), we conclude that the (equal) quadrature variances in the state (2) are confined within the interval $(1/2, 3/2)$, depending on the value of $|\beta-\alpha|$ (thus both the variances are greater than those in the vacuum states, but they are always smaller than those in the first Fock state). Consequently, as was mentioned already, there is no squeezing in MES in the usual sense. But what about higher order squeezing?

A concept of 'generalized squeezing' for arbitrary Hermitian operators $\hat{A}, \hat{B}$ was introduced by Walls and Zoller [7]. They said that fluctuations of the observable A are 'reduced' if

$$(\Delta A)^2 < \tfrac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|. \tag{10}$$

In particular, taking in (10) $\hat{A} = (\hat{a}^2 + \hat{a}^{\dagger 2})/2$ and $\hat{B} = (\hat{a}^2 - \hat{a}^{\dagger 2})/(2i)$, we arrive at the definition of *amplitude-squared squeezing* proposed by Hillery [8]. In this case inequality (10) can be written as

$$\langle(\hat{a}^2 + \hat{a}^{\dagger 2})^2\rangle - \langle \hat{a}^2 + \hat{a}^{\dagger 2}\rangle^2 < 4\langle \hat{a}^\dagger\hat{a}\rangle + 2.$$

An equivalent form of this inequality is

$$\langle \hat{a}^4 + \hat{a}^{\dagger 4} + 2\hat{a}^{\dagger 2}\hat{a}^2\rangle - \langle \hat{a}^2 + \hat{a}^{\dagger 2}\rangle^2 < 0. \tag{11}$$

Using relations (6) and (7), one can verify that the left-hand side of (11) is always positive, being equal to $2c|\beta^2 - \alpha^2|^2/(1-c)$. Consequently, MES do not exhibit amplitude-squared squeezing.

Another definition of higher-order squeezing was given by Hong and Mandel [9]. According to them, the state $|\psi\rangle$ is squeezed to the $2n$th order in the $\hat{x}$ quadrature component, if the mean value $\langle\psi|(\Delta\hat{x})^{2n}|\psi\rangle$ is less than the mean value of $(\Delta\hat{x})^{2n}$ in the coherent state, i.e., $\langle(\Delta\hat{x})^{2n}\rangle < 2^{-n}(2n-1)!!$. In particular, for $n = 2$ we have the requirement for the fourth-order squeezing $\langle(\Delta\hat{x})^4\rangle < 3/4$. A direct calculation of the fourth-order variance of coordinate in the simplest MES $|0_\perp\beta\rangle$ (when $\langle\hat{x}\rangle = \langle\hat{p}\rangle = 0$) yields the following dependence on parameter β:

$$\langle(\Delta\hat{x})^4\rangle = \frac{3}{4} + \frac{c|\beta|^2}{2(1-c)}(6 + 3|\beta|^2 + 2\beta^2 + 2\beta^{*2}), \qquad c = \exp(-|\beta|^2). \tag{12}$$

A similar expression for $\langle(\Delta\hat{p})^4\rangle$ is obtained by changing the signs before two last terms on the right-hand side of (12). We see that the difference $\langle(\Delta\hat{x})^4\rangle - 3/4$ is negative for pure imaginary $\beta = \mathrm{i}\gamma$ with $|\beta|^2 > 6$ (equivalently, $\langle(\Delta\hat{p})^4\rangle - 3/4$ is negative for real β with $|\beta|^2 > 6$). For such big values of $|\beta|^2$ we can neglect the term c in the denominator and find the value $|\beta|^2_{\min} = 4 + \sqrt{10} \approx 7$, which gives the minimum of the fourth-order variance

$\langle(\Delta\hat{x})^4\rangle_{\min} - 3/4 \approx -\frac{7}{2}\,\mathrm{e}^{-7}$ or more precisely $\langle(\Delta\hat{x})^4\rangle_{\min} - 3/4 \approx -0.003$. So, the maximal degree of fourth-order squeezing in MES is rather small. However, it is interesting to note in this connection that the states possessing higher-order squeezing in the absence of the second-order (usual) one were constructed in [9] in the form of a rather sophisticated *continuous* superposition of coherent states distributed along the line with the weight, which is a product of Gaussian and odd-power functions. At the same time, MES is a simple superposition of only *two* coherent states. It is also worth noting that a simple superposition of *four* coherent states with equal amplitudes and phases shifted by $\pi/2$ (the so-called òrthogonal-even' coherent state [10]) gives the values $\langle(\Delta\hat{x})^4\rangle = \langle(\Delta\hat{p})^4\rangle \approx 0.7$ for $|\alpha| \approx 0.67$. In this state, the fourth-order product $\Pi^{(4)} \equiv \langle(\Delta\hat{p})^4\rangle\langle(\Delta\hat{x})^4\rangle$ equals 0.49 instead of the coherent state value $9/16 = 0.5625$. The best minimal value $\Pi^{(4)}_{\min} = (0.6984)^2 = 0.4878$ was found in [11] by applying the variational approach to the states of the form $\sum_{n=0}^{K} c_n|4n\rangle$ with $K = 6$. Earlier, very close value $\Pi^{(4)} \approx 0.4901$ was obtained in [12] for the superposition of *two* Fock states $\sqrt{(1+\delta)/2}|0\rangle - \sqrt{(1-\delta)/2}|4\rangle$ with $\delta = \sqrt{150/151}$.

3. Relations between 'semi-coherent', 'displaced' and 'multiphoton' states

Taking into account the well-known properties of the unitary displacement operator

$$\hat{D}(\alpha) = \hat{D}^{-1}(-\alpha) = \hat{D}^{\dagger}(-\alpha), \qquad \hat{D}^{-1}(\alpha) f(\hat{a}) \hat{D}(\alpha) = f(\hat{a}+\alpha) \tag{13}$$

(where f is an arbitrary analytical function), we immediately conclude that if the property (6) (which includes as a special case (1)) holds for some state $|\psi_0\rangle$, then it holds automatically for the whole family of displaced states (5). In particular, *all displaced number states* are 'semi-coherent' states in the sense of (1), which means that 'semicoherent' states of Boiteux and Levelut [2] are, indeed, special cases of 'semi-coherent' states of Mathews and Eswaran. Obviously, not all displaced states are 'semi-coherent', but only those whose 'fiducial' state $|\psi_0\rangle$ satisfies the conditions

$$\langle\psi_0|\hat{a}|\psi_0\rangle = \langle\psi_0|\hat{a}^2|\psi_0\rangle = 0. \tag{14}$$

An infinite number of states satisfy these conditions. Writing $|\psi_0\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$, we obtain from (14) the restrictions

$$\sum_{n=0}^{\infty} c_n^* c_{n+1}\sqrt{n+1} = 0, \qquad \sum_{n=0}^{\infty} c_n^* c_{n+2}\sqrt{(n+1)(n+2)} = 0.$$

In particular, any superposition of Fock states, which has 'gaps' in the sequence $\{c_n\}$ whose widths always exceed 2, can serve as a required fiducial state. The problem is to choose 'the most interesting' families among such superpositions. One example corresponds to families of 'fractional photons' [13], whose simplest representative corresponds to the states such as $\sum_{n=0}^{\infty} z^n|kn\rangle/\sqrt{n!}$ with $k \geqslant 3$. Other families of 'multiphoton' states $|\alpha\rangle_k$, defined by the conditions

$$\hat{a}^k|\alpha\rangle_k = \alpha^k|\alpha\rangle_k, \qquad {}_k\langle\alpha|\hat{a}^j|\alpha\rangle_k = 0, \quad j = 1, 2, \ldots, k-1, \tag{15}$$

with $k = 3, 4, \ldots$, result in the Fock decompositions of the form (in the simplest case) $\sum_{n=0}^{\infty} \alpha^{kn}|kn\rangle/\sqrt{(kn)!}$. Such states (they include, in particular, the 'orthogonal-even' coherent states mentioned in the preceding section) have been extensively studied, e.g., in papers [14–19] (for other references see [6]). Contrary to the displaced number states with quantized values of quadrature variances $\sigma_x = \sigma_p = n + 1/2$, and the Mathews–Eswaran states (2), whose variances are confined in the interval $(1/2, 3/2)$, the quadrature variances in the k-photon states (15) can assume any value (greater than 1/2), because these states can be represented as superpositions of k coherent states, whose labels are uniformly distributed along the circle $|\alpha| = \text{const}$: $|\alpha\rangle_k = \sum_{m=1}^{k} c_m |\alpha \exp(\mathrm{i}\phi_m)\rangle$, $\phi_m = 2\pi m/k$, thus the value of the variance is determined mainly by the distance between coherent components, which grows unlimitedly with increase of $|\alpha|$.

The expansion of the state (2) over the Fock basis has the form

$$|\alpha_{\perp}\beta\rangle = \frac{\exp(-|\alpha|^2/2)}{\sqrt{1-c}} \sum_{n=0}^{\infty} [\alpha^n - \beta^n \, \mathrm{e}^{\beta^*(\alpha-\beta)}] \frac{|n\rangle}{\sqrt{n!}}. \qquad (16)$$

Formally, the right-hand sides of equation (2) or (16) are not defined for $\beta = \alpha$. However, taking $\beta = \alpha + \rho \, \mathrm{e}^{\mathrm{i}\chi}$, one can easily verify that in the limit $\rho \to 0$ the state (2) goes to the state $\mathrm{e}^{\mathrm{i}\chi} |\alpha_{\perp}\alpha\rangle$, where

$$|\alpha_{\perp}\alpha\rangle = \exp(-|\alpha|^2/2) \left[\alpha^* |0\rangle + \sum_{m=1}^{\infty} \frac{\alpha^{m-1}}{\sqrt{m!}} (|\alpha|^2 - m)|m\rangle \right]. \qquad (17)$$

A distinguished feature of the state (17) is its orthogonality to the coherent state $|\alpha\rangle$. However, this state is not new. Comparing (17) with the explicit decomposition of the displaced number states [20, 21][3]

$$\hat{D}(\alpha)|n\rangle = \exp(-|\alpha|^2/2) \sum_{m=0}^{\infty} \left(\frac{n!}{m!} \right)^{1/2} \alpha^{m-n} L_n^{(m-n)}(|\alpha|^2)|m\rangle \qquad (18)$$

(where $L_n^{(k)}(x)$ is the associated Laguerre polynomial defined as in [24]), one can see that (17) coincides with the displaced first Fock state $\hat{D}(\alpha)|1\rangle$ (with an opposite sign). The properties of this state (photon statistics, quasiprobability distributions, etc) have been studied in detail in [25].

Various finite superpositions of coherent states have been considered in the literature for decades, beginning with the studies [14, 26, 27] (for reviews see, e.g., [6, 28]). In particular, different superpositions of *two* coherent states $c_1|\alpha\rangle + c_2|\beta\rangle$ were studied in [29]. However, as a rule, the coefficients c_1 and c_2 of these superpositions were independent of the parameters of coherent components α and β. The distinguished feature of MES (2) is that the coefficient of the superposition is determined by the scalar product between its components (a specific superposition of the vacuum and coherent state $|\alpha\rangle - \exp(-|\alpha|^2/2)|0\rangle$, which is nothing but the state $|\alpha_{\perp}0\rangle$, was considered in [30] as an example of the 'most nonclassical' *truncated* states).

4. Wigner function and marginal probabilities

Frequently, squeezing properties of quantum states are illustrated with the aid of some quasidistributions in the phase space, such as the Wigner or Husimi function. Since the contour plots of these functions for coherent or Fock states are nothing but circles, the absence of squeezing in these states becomes quite evident (as well as different levels of quadrature fluctuations in the squeezed states, whose contour plots are ellipses). In this connection, it seems interesting to consider the Wigner function corresponding to the Mathews–Eswaran state (2). Since the form of the quasiprobability distribution is not changed by the displacement operator (and quadrature variances are left intact by the operator $\hat{D}(\alpha)$), it is sufficient to consider the Wigner function

$$W(x, p) = \int \psi\,(x + y/2)\, \psi^*\,(x - y/2) \exp(-\mathrm{i}py)\,\mathrm{d}y$$

[3] Actually, matrix elements $\langle m|\hat{D}(\alpha)|n\rangle$ have been known long before [22, 23].

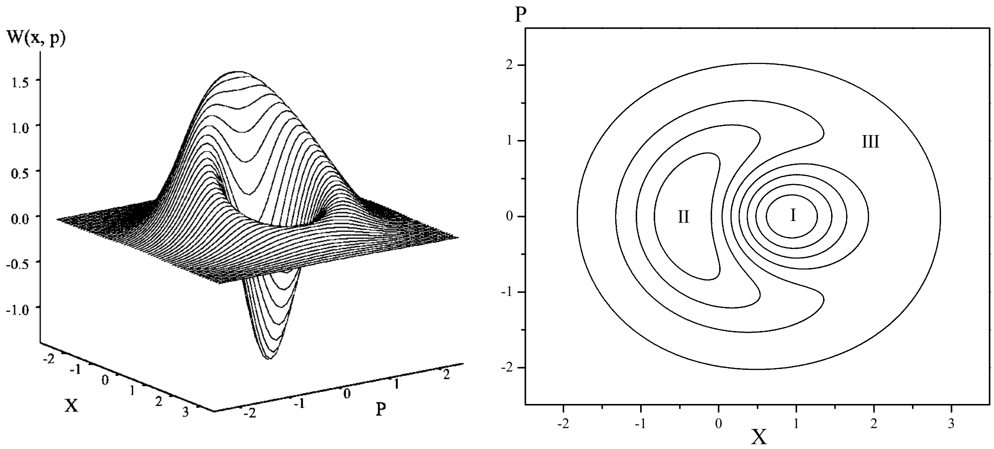

Figure 1. The surface and contour plots of the Wigner function (19) for $\beta = 1$. Region I in the right plot corresponds to negative values of the Wigner function.

of the family of 'fiducial' states $|0_\perp\beta\rangle$ centred at the origin. In this case, parameter β can be chosen real, because changing its phase is equivalent to rotating coordinate axes. The explicit expression is as follows:

$$W_{0_\perp\beta}(x,p) = \frac{2\exp(-p^2)}{1-\exp(-\beta^2)}\left\{\exp(-x^2) + \exp[-(x-\sqrt{2}\beta)^2 - \beta^2] - 2\cos(\sqrt{2}\beta p)\exp\left[-\frac{1}{2}(\sqrt{2}x-\beta)^2 - \frac{\beta^2}{2}\right]\right\}. \tag{19}$$

The surface and contour plots of this quasidistribution, given in figure 1, show the existence of the domain with negative values of W, indicating the 'nonclassical' nature of the MES. Another striking feature is the strong asymmetry between x and p variables. Nonetheless, the quadrature variances σ_x and σ_p are the same, and they are preserved with the course of time, when the plots rotate around the origin.

There is also strong asymmetry with respect to positive and negative values of the x coordinate, which is well pronounced in the probability density $\rho(x) = |\psi(x)|^2$. For a complex parameter $\beta = |\beta|\exp(\mathrm{i}\varphi)$ we obtain

$$\rho_{0_\perp\beta}(x; |\beta|, \varphi) = \left\{\exp(-x^2) + \exp[-(x-\sqrt{2}|\beta|\cos\varphi)^2 - |\beta|^2] - 2\exp\left[-\frac{1}{2}(\sqrt{2}x - |\beta|\cos\varphi)^2 - \frac{|\beta|^2}{2}(1+\cos^2\varphi)\right] \times \cos[|\beta|\sin\varphi(\sqrt{2}x - |\beta|\cos\varphi)]\right\}(\sqrt{\pi}[1-\exp(-|\beta|^2)])^{-1}. \tag{20}$$

The plots of function (20) for different values of parameters $|\beta|$ and φ are shown in figure 2. Varying the phase φ with the fixed value of $|\beta|$ we obtain quite different distributions, but all of them have the same mean values $\langle x\rangle = 0$ and $\langle x^2\rangle = 1/2 + |\beta|^2/[\exp(|\beta|^2) - 1]$. Note that the shift $\varphi \to \varphi + \pi/2$ is equivalent to the transformation from the coordinate representation to the momentum one.

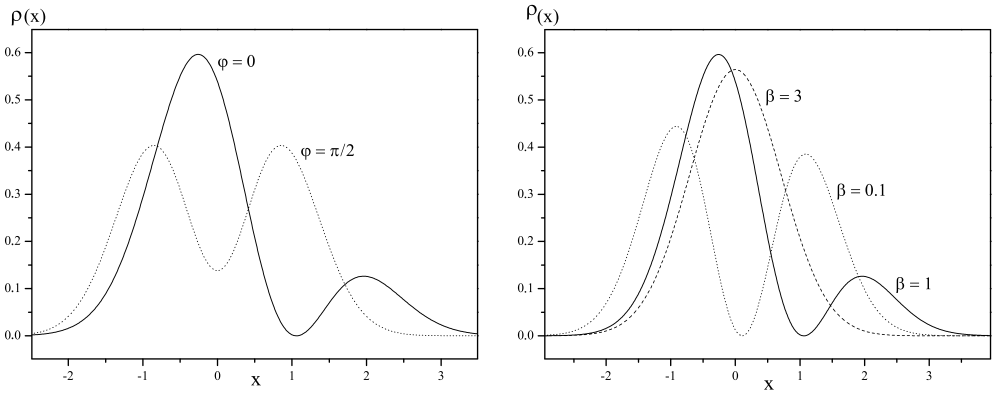

Figure 2. The coordinate probability density (20) for different values of parameters $|\beta|$ and φ. The left plot corresponds to $|\beta| = 1$ (as in figure 1). In the right plot $\varphi = 0$.

5. Photon statistics

The photon distribution function $p_n \equiv |\langle n|\psi\rangle|^2$ is not invariant with respect to the unitary transformation performed by means of the displacement operator. For the generic Mathews–Eswaran state (2) the expansion (16) yields

$$p_n(\alpha;\beta) = \frac{1}{n!(1-c)}\{|\alpha|^{2n}\,\mathrm{e}^{-|\alpha|^2} + c[|\beta|^{2n}\,\mathrm{e}^{-|\beta|^2} - 2\,\mathrm{Re}((\alpha\beta^*)^n\,\mathrm{e}^{-\alpha\beta^*})]\}. \tag{21}$$

The special cases of formula (21) are as follows:

$$p_n(0;\beta) = \frac{|\beta|^{2n}}{n!}\frac{\exp(-2|\beta|^2)}{1-\exp(-|\beta|^2)}, \qquad n>0, \qquad p_0(0;\beta) = 1-\exp(-|\beta|^2), \tag{22}$$

$$p_n(\alpha;0) = \frac{|\alpha|^{2n}}{n!}\frac{\exp(-|\alpha|^2)}{1-\exp(-|\alpha|^2)}, \qquad n>0, \qquad p_0(\alpha;0) = 0, \tag{23}$$

$$p_n(\alpha;-\alpha) = \frac{|\alpha|^{2n}}{n!}\exp(-|\alpha|^2)\times\begin{cases}\tanh(|\alpha|^2), & n \text{ even}\\ \coth(|\alpha|^2), & n \text{ odd}.\end{cases} \tag{24}$$

Distribution (23) was considered in [30] as an example of *truncated coherent state*. Formula (24) is especially interesting because the photon distribution functions (PDF) for even and odd values of n show the same behaviour as in the case of *even and odd coherent states* [27]

$$|\alpha\rangle_\pm = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2[1\pm\exp(-2|\alpha|^2)]}}, \tag{25}$$

provided the PDFs corresponding to the states $|\alpha\rangle_\pm$ are rescaled (due to normalization) by the factors $\frac{1}{2}[1\mp\exp(-2|\alpha|^2)]$, respectively.

The photon distribution generating function

$$G(z) \equiv \sum_{n=0}^{\infty} z^n p_n \tag{26}$$

is a useful tool for calculating the factorial moments

$$\overline{n^{(k)}} \equiv \sum_{n=0}^{\infty} n(n-1)\cdots(n-k+1)p_n \tag{27}$$

due to the formula

$$\overline{n^{(k)}} = (d^{(k)}G/\mathrm{d}z^k)|_{z=1}. \tag{28}$$

In the case of the photon distribution (21) we have

$$G(z) = (1-c)^{-1}\{\exp[(z-1)|\alpha|^2] + c[\exp[(z-1)|\beta|^2] - 2\,\mathrm{Re}(\exp[(z-1)\alpha\beta^*])]\} \tag{29}$$

and

$$\overline{n^{(k)}} = (1-c)^{-1}\{|\alpha|^{2k} + c[|\beta|^{2k} - 2\,\mathrm{Re}[(\alpha\beta^*)^k]]\}. \tag{30}$$

One of frequently used characteristics of the photon statistics is Mandel's Q-factor

$$Q = \frac{\overline{n^2} - \bar{n}^2 - \bar{n}}{\bar{n}} \equiv \frac{\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2}{\langle\hat{a}^\dagger\hat{a}\rangle}. \tag{31}$$

Using formula (7) or (30) one can obtain the following expression:

$$Q(\alpha;\beta) = \frac{c|\beta-\alpha|^2[|\beta+\alpha|^2 - 2|\alpha|^2 - 2|\beta|^2 c]}{(1-c)[|\alpha|^2 + c(|\beta-\alpha|^2 - |\alpha|^2)]}. \tag{32}$$

Obviously, the photon statistics is super-Poissonian ($Q > 0$) for $|\beta| \gg |\alpha|$. If $|\alpha|$ is not very small ($|\alpha| > 1$, for example), then Q is only slightly different from zero in this case, due to the factor c, which rapidly goes to zero when $|\beta - \alpha| \gg 1$. On the other hand, for $\alpha = 0$ we have

$$Q(0;\beta) = |\beta|^2 \frac{1 - 2\exp(-|\beta|^2)}{1 - \exp(-|\beta|^2)}, \tag{33}$$

and this expression monotonically increases from -1 at $\beta \to 0$ to arbitrarily large values when $|\beta| \to \infty$, changing the sign at $|\beta|^2 = \ln 2$. However, if $|\alpha| \ll 1$, but $\alpha \neq 0$, then the right-hand side of (33) should be multiplied by the factor

$$\frac{|\beta|^2\exp(-|\beta|^2)}{|\alpha|^2 + |\beta|^2\exp(-|\beta|^2)},$$

which results in the exponential decay of Q as

$$\frac{|\beta|^4\exp(-|\beta|^2)}{|\alpha|^2 + |\beta|^2\exp(-|\beta|^2)}$$

for $|\beta| \to \infty$, after Q attains the maximal value $Q_{\max} \approx \ln(1/|\alpha|^2)$ at $|\beta|^2 \approx \ln(1/|\alpha|^2)$.

Moreover, for any value of α one can find the (finite) regions in the complex plane β, which result in the sub-Poissonian statistics ($Q < 0$). For example, this happens if one extracts from the coherent state its projection to the vacuum state [30]:

$$Q(\alpha;0) = -\frac{|\alpha|^2\exp(-|\alpha|^2)}{1 - \exp(-|\alpha|^2)} < 0, \tag{34}$$

moreover, $Q(\alpha;0) \to -1$ when $\alpha \to 0$. In figure 3 we show the dependence of Q-factor on parameters α and β.

For $|\beta| = |\alpha|$, using the parametrization $\beta = \alpha\exp(\mathrm{i}\varphi)$, we find the expression

$$Q(|\alpha|;\varphi) = \frac{8c|\alpha|^2\sin^2(\varphi/2)(\cos\varphi - c)}{(1-c)[1 + c(1 - 2\cos\varphi)]}, \qquad c = \exp[-4|\alpha|^2\sin^2(\varphi/2)]. \tag{35}$$

In this case, the Q-factor is obviously negative for $\varphi \geqslant \pi/2$. For example, if $\varphi = \pi$, then

$$Q(\alpha;-\alpha) = -\frac{8|\alpha|^2 c(1+c)}{(1+3c)(1-c)} < 0, \qquad c = \exp(-4|\alpha|^2), \tag{36}$$

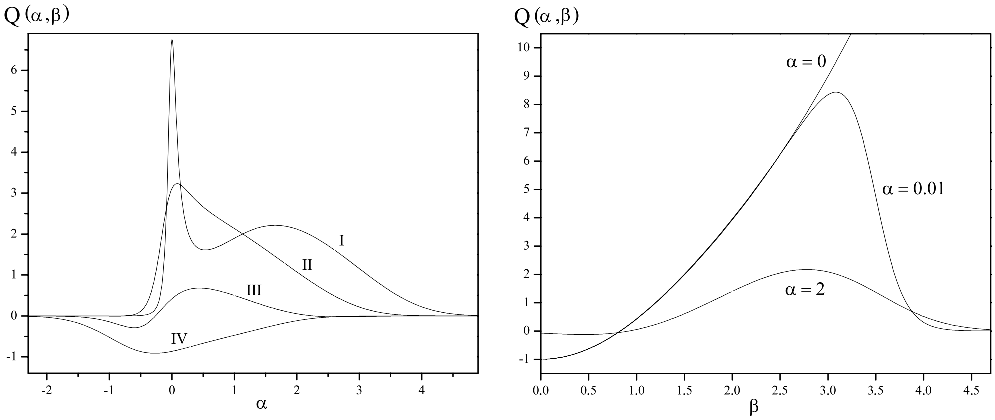

Figure 3. Mandel's Q-factor of the ME state as a function of α with fixed values of parameter β (at the left) and as a function of β with fixed values of α (at the right). The values of β in the left plot are as follows: I – $\beta = 2.6$, II – $\beta = 1.8$, III – $\beta = 1$, IV – $\beta = 0.3$.

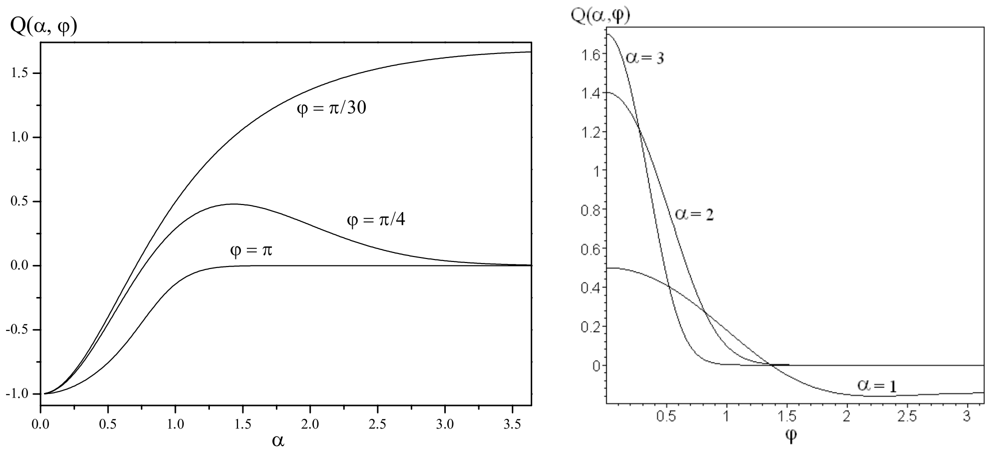

Figure 4. Mandel's Q-factor of the ME state with $\beta = \alpha \exp(\mathrm{i}\varphi)$ as a function of α for different fixed values of the phase φ (at the left) and as a function of φ for different fixed values of α (at the right).

so the state $|\alpha_{\perp} - \alpha\rangle$ always shows the sub-Poissonian statistics (it is worth noting that the photon statistics of even/odd coherent states is super/sub-Poissonian). However, Q can be positive for small values of φ, depending on the value of $|\alpha|$. For example, if $\varphi = 0$, then we obtain

$$Q(\alpha; \alpha) = \frac{2|\alpha|^2 - 1}{|\alpha|^2 + 1}, \tag{37}$$

and this expression goes to -1 for $|\alpha| \to 0$, but it is positive for $|\alpha|^2 > 1/2$, going to the asymptotical value 2 when $|\alpha| \to \infty$. If φ is close to zero (but different from zero), then Q approaches some maximal value (smaller than 2) but finally goes to zero as $|\alpha| \to \infty$. In figure 4 we show the dependence of Q-factor on parameters α and φ.

To analyse the evolution of the domain of sub-Poissonian statistics in the complex β-plane in dependence of α, we can always assume that α is real and nonnegative. For $\alpha = 0$ this domain is simply the internal part of the circle $|\beta|^2 = \ln 2$. With the increase of α, this circle exhibits some deformations, expanding more to the left semiplane than to the right one. When $\alpha \gg 1$, then the boundary again becomes close to a circle $|\beta + \alpha| = \alpha\sqrt{2}$, whose centre is shifted to the point $-\alpha$ and radius equals $\alpha\sqrt{2}$. Note, however, that the absolute value of Q-factor is very small in this case, and even at the centre of the circle we have $Q(\alpha; -\alpha) \approx -8\alpha^2 \exp(-4\alpha^2)$.

The following generalization of the Mandel parameter was proposed in [31]:

$$S_k = \overline{n^{(k)}}/\langle n\rangle^k - 1 \tag{38}$$

(for $k = 2$ one has $S_2 = Q/\langle n\rangle$). Using formula (30) one can calculate the parameter $S_k(\alpha, \beta)$ for any values of k, α and β. We confine ourselves here to the cases which can be analysed in the most simple way.

$$S_k(\alpha, 0) = [1 - \exp(-|\alpha|^2)]^{k-1} - 1. \tag{39}$$

This quantity is negative for any $k \geqslant 2$, meaning the sub-Poissonian statistics for any order.

$$S_k(0, \beta) = [\exp(|\beta|^2) - 1]^{k-1} - 1. \tag{40}$$

This quantity is negative if $|\beta|^2 < \ln 2$, i.e., the boundary of the sub-Poissonian behaviour does not depend on k.

6. Conclusion

We have demonstrated that the Mathews–Eswaran states defined by equation (2) possess interesting 'nonclassical' properties, which include the higher-order squeezing (in the Hong–Mandel sense) and sub-Poissonian statistics. These properties are the consequences (to certain extent) of relation (1) or (6), which are fulfilled for the eigenstates of operator $\hat{a}$. It is worth noting in this connection that deriving equations (6)–(9) we did not use explicit form of operator $\hat{a}$ or the commutation relation between $\hat{a}$ and $\hat{a}^\dagger$ (except for the second equality in (8), which holds for the scalar product of Klauder–Glauber's coherent states). This gives a possibility of introducing a large family of 'generalized Mathews–Eswaran states', which satisfy property (6) for an *arbitrary* operator $\hat{A}$. All such states have the same structure as the superposition (2), where the components $|\alpha\rangle$ and $|\beta\rangle$ should be understood as eigenstates of the operator $\hat{A}$. For example, taking $\hat{A} = \hat{a}^k$ we thus obtain 'semi-coherent' generalizations of the 'Barut–Girardello coherent states' [32] or even/odd coherent states [27] for $k = 2$, and of 'multiphoton' states for $k \geqslant 3$. Choosing $\hat{A} = f(\hat{a}^\dagger\hat{a})\hat{a}^k$ (with $k = 1, 2, \ldots$), we arrive at 'semi-coherent' generalizations of 'nonlinear' coherent states [33–37]. Some specific choices of the function $f(\hat{n})$ could be especially interesting because they result in the same commutation relation $[\hat{A}, \hat{A}^\dagger] = 1$ as for the operators $\hat{a}$ and $\hat{a}^\dagger$, although $\hat{A} \neq \hat{a}$ [38]. However, all such generalizations deserve separate studies.

Acknowledgments

We thank Brazilian agencies CNPq and FAPESP for support. We also thank the referee for careful reading the manuscript and valuable suggestions.

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