Chapter 8

The Free Dirac Field

8.1 Canonical Quantisation

Dirac Field Theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\overline{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \tag{8.1.1}$$

The Euler-Lagrange equation analagous to Eq.2.2.10

$$\frac{\partial}{\partial x^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})} - \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = 0 \tag{8.1.2}$$

leads to the Dirac equation.

The canonical momentum from Eq.2.2.5 is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\psi}(x)} = i\psi^\dagger(x) \tag{8.1.3}$$

The Hamiltonian density is analagous to Eq.2.2.7

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial\psi}{\partial t} \tag{8.1.4}$$

which is not positive definite. The general solution to the Dirac equation may be expanded in terms of plane waves analagously to Eq.2.3.19

$$\psi(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b_\alpha(\mathbf{k})u^\alpha(\mathbf{k})e^{-ik.x} + d^\dagger_\alpha(\mathbf{k})v^\alpha(\mathbf{k})e^{ik.x}] \tag{8.1.5}$$

$$\overline{\psi}(\mathbf{x},t) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{m}{k_0}\sum_{\alpha=1,2}[b^\dagger_\alpha(\mathbf{k})\overline{u}^\alpha(\mathbf{k})e^{ik.x} + d_\alpha(\mathbf{k})\overline{v}^\alpha(\mathbf{k})e^{-ik.x}] \tag{8.1.6}$$

The total Hamiltonian is

$$H = \int d^3x\mathcal{H} \tag{8.1.7}$$

After some algebra we find

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{k_0} k_0 \sum_{\alpha=1,2} [b^\dagger_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) - d_\alpha(\mathbf{k}) d^\dagger_\alpha(\mathbf{k})] \tag{8.1.8}$$

So far no commutation relations have been assumed, and H could quite easily be negative, unlike the Hamiltonian in the case of the charged scalars for example which was positive definite as seen in Eq.2.3.37. In order to give a positive definite Hamiltonian we require the creation and annihilation operators to satisfy *anticommutation* relations, first proposed by Wigner:

$$\left\{ b_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}') \right\} = (2\pi)^3 \frac{k_0}{m} \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\alpha\alpha'} \tag{8.1.9}$$

$$\left\{ d_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}') \right\} = (2\pi)^3 \frac{k_0}{m} \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\alpha\alpha'} \tag{8.1.10}$$

$$\{ b_\alpha(\mathbf{k}), b_{\alpha'}(\mathbf{k}') \} = 0 \tag{8.1.11}$$

$$\left\{ b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}') \right\} = 0 \tag{8.1.12}$$

$$\{ d_\alpha(\mathbf{k}), d_{\alpha'}(\mathbf{k}') \} = 0 \tag{8.1.13}$$

$$\left\{ d^\dagger_\alpha(\mathbf{k}), d^\dagger_{\alpha'}(\mathbf{k}') \right\} = 0 \tag{8.1.14}$$

The Hamiltonian is then defined as the normal ordered version of Eq.8.1.8 *but with a change of sign for each interchange of operator*

$$H = \int d^3x : \psi^\dagger i \frac{\partial \psi}{\partial t} : \tag{8.1.15}$$

which results in

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{k_0} k_0 \sum_{\alpha=1,2} [b^\dagger_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) + d_\alpha(\mathbf{k}) d^\dagger_\alpha(\mathbf{k})] \tag{8.1.16}$$

which is now postive definite.

Anticommutation implies Fermi statistics for example:

$$\left\{ b^\dagger_\alpha(\mathbf{k}), b^\dagger_{\alpha'}(\mathbf{k}') \right\} = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k}) b^\dagger_\alpha(\mathbf{k}) = 0$$

$$\Rightarrow b^\dagger_\alpha(\mathbf{k}) b^\dagger_\alpha(\mathbf{k}) |0>= 0$$

so that two quanta in the same state are not allowed (Pauli exclusion principle).

The charge operator is the analogue of Eq.2.3.36

$$Q = \int d^3\mathbf{x} : j_0(x) := \int d^3\mathbf{x} : \psi^\dagger i\partial\psi :$$

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{m}{k_0} k_0 \sum_{\alpha=1,2} [b^\dagger_\alpha(\mathbf{k}) b_\alpha(\mathbf{k}) - d^\dagger_\alpha(\mathbf{k}) d_\alpha(\mathbf{k})] \tag{8.1.17}$$

which shows that $b^\dagger$ creates fermions while $d^\dagger$ creates antifermions of opposite charge.

Finally the equal time commutation relations are (after some algebra):

$$\left\{\psi_i(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\} = \delta^3(\mathbf{x}-\mathbf{x}')\delta_{ij} \tag{8.1.18}$$

$$\{\psi_i(\mathbf{x},t),\psi_j(\mathbf{x}',t)\} = 0 \tag{8.1.19}$$

$$\left\{\psi_i^\dagger(\mathbf{x},t),\psi_j^\dagger(\mathbf{x}',t)\right\} = 0 \tag{8.1.20}$$

In fact at all times we have:

$$\{\psi(x),\psi(y)\} = 0 \tag{8.1.21}$$

8.2 Path Integral Quantisation

We have seen that Green's functions in quantum field theory may be generated from generating functionals, which are functional integrals over classical field functions. In the case of scalar field theory the classical field functions are commuting c-numbers. However in the case of spinor field theory the classical field functions cannot be regarded as commuting c-numbers. The problem is that the classical Hamiltonian is also given by Eq.8.1.8 and is similarly non-positive definite. Thus it would appear that classical spinor field theory does not make physical sense. Thus spinor fields are essentially non-classical, and this poses a problem for the path integral approach where functional integrals over classical fields will occur.

In chapter 2 we discussed the free Klein-Gordon field and showed how it could be regarded as a continuum of harmonic oscillators, one at each spacetime point, and each coupled to their nearest neighbours in a Lorentz invariant way. The path integral results in chapter 4 may be taken to apply to each of these little oscillators. In chapter 6 we generalised the path integral results from a single coordinate $q(t)$ to many coordinates $q_i(t)$, one for each oscillator. We then considered the continuum limit,

$$q_i(t) \to \phi(x,y,z,t) \tag{8.2.1}$$

just as we did to arrive at canonical field theory. Thus the Heisenberg operators $\hat{q}(t)$ in Eq.6.1.3 become Heisenberg fields $\hat{\phi}(x,y,z,t)$ whose eigenstates are given by

$$\hat{\phi}(x,y,z,t)|\phi(x,y,z,t) >= \phi(x,y,z,t)|\phi(x,y,z,t) > \tag{8.2.2}$$

and the generating functional involves the eigenvalue of the field $\phi(x,y,z,t)$ rather than the field operator.

Now we want to do the same thing for the spinor field theory, and the generating functional will involve the eigenvalue field on the right hand side of the equation:

$$\hat{\psi}(x,y,z,t)|\psi(x,y,z,t) >= \psi(x,y,z,t)|\psi(x,y,z,t) > \tag{8.2.3}$$

However if we apply the same field operator twice we encounter a problem:

$$\hat{\psi}^2(x,y,z,t)|\psi(x,y,z,t)>=\psi^2(x,y,z,t)|\psi(x,y,z,t)> \tag{8.2.4}$$

The problem is that the lhs is zero by the anticommutatation property assumed for the fermion field operator, but the rhs is apparently non-zero. In order to overcome this problem we must no longer regard the field eigenvalue $\psi(x,y,z,t)$ as being a commuting c-number, but instead regard it as an *anticommuting Grassmann variable* which obeys the anticommutation relations:

$$\{\psi(x),\psi(y)\}=0 \tag{8.2.5}$$

Then the product of two identical field eigenvalues is also zero and the previous problem is overcome.

What exactly are Grassmann variables? Well they are well defined mathematical objects which multiply operators and vectors just as if they were c-numbers, but have the property that they anticommute amongst themselves. They are discussed further in the Appendix. Since anticommuting field operators implies Grassmann field eigenvalues, the classical spinor field must be regarded as Grassmannian. Thus the classical Hamiltonian (i.e. the energy) becomes neither positive nor negative but rather as function of Grassmann variables, and this solves its positivity problem.

Clearly we must quantise the Dirac fields using functional methods based on spinor fields corresponding to Grassmann variables. By analogy with Eq.6.1.11 the generating functional for the free Dirac field is:

$$Z_0[\eta,\overline{\eta}]\propto\int\mathcal{D}\psi\mathcal{D}\overline{\psi}e^{i\left(\int d^4x(\mathcal{L}+\overline{\eta}\psi+\overline{\psi}\eta)\right)} \tag{8.2.6}$$

where we have introduced a source term $\overline{\eta}(x)$ for $\psi(x)$ and another one $\eta(x)$ for $\overline{\psi}(x)$, and both the sources and the fields are Grassmann variables. The Lagrangian is just the free Dirac Lagrangian introduced earlier.

By analogy with Eq.6.1.13 the Green's functions may be defined by the VEV of the Heisenberg field operators

$$\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)\equiv<0|T(\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\overline{\hat{\psi}}(y_1)\cdots\overline{\hat{\psi}}(y_n))|0> \tag{8.2.7}$$

where the time ordering puts earlier times to the right as usual but in doing so we must introduce a minus sign every time two fermion fields are anticommuted. Again by analogy with Eq.6.1.14 the Green's functions are given from the generating functional as,

$$i^n\mathcal{G}(x_1,\cdots,x_n;y_1,\cdots,y_n)=$$

$$\left[\frac{\delta^{2n}Z_0[\eta,\overline{\eta}]}{\delta\overline{\eta}(x_1)\cdots\delta\overline{\eta}(x_n)\delta\eta(y_1)\cdots\delta\eta(y_n)}\right]_{\eta=0,\overline{\eta}=0} \tag{8.2.8}$$

8.3 The Feynman Propagator for the Dirac Field

We are familiar with the Feynman propagator of the scalar field as being the VEV of the time ordered product of two Heisenberg fields in the free field theory (or of two interaction picture fields in the interacting field theory which amounts to the same thing). In the language of Green's functions, the Feynman propagator is just the two point Green's function in the free field theory. Previously we calculated the Feynman propagator in two ways: by using canonical methods in section 3.5, and using functional methods in section 6.2. Here we shall use the functional approach, following the steps analagous to section 6.2.

We first consider the free Dirac field Lagrangian density

$$\mathcal{L} = \overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \tag{8.3.1}$$

The path integral expression for the generating functional is explictly:

$$Z_0[\eta, \overline{\eta}] \propto \int \mathcal{D}\psi \mathcal{D}\overline{\psi}$$

$$\exp\left(i \int d^4x \left[\overline{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) + \overline{\eta}\psi + \overline{\psi}\eta \right] \right)$$

As in the scalar case, in order to identify the Feynman propagator, we wish to perform the functional integral over the fields, and as in the previous case it is possible to do so for the free field theory. We first recall the Grassmannian version of the most important integral in the world from the Appendix:

$$Z'_0 = \int \prod_{i=1}^{n} da_i e^{-\sum_{i,j=1}^{n} a_i K_{ij} a_j + \sum_{k=1}^{n} \eta_k a_k} \tag{8.3.2}$$

$$Z'_0 = \sqrt{det(K)} e^{-\frac{1}{4}\sum_{i,j=1}^{n} \eta_i (K^{-1})_{ij} \eta_j} \tag{8.3.3}$$

The continuum version of this is:

$$Z_0[\eta] = \int \mathcal{D}\psi \exp(-\frac{1}{2} \int d^4x d^4x' \left[\psi(x') A(x', x) \psi(x)\right] + \int d^4x \eta(x) \psi(x))$$

$$Z_0[\eta] = \sqrt{det(A)} \exp(-\frac{1}{2} \int d^4x d^4x' \left[\eta(x') A^{-1}(x', x) \eta(x) \right]$$

This applies to real Grassmannian functions. For complex Grassmannian functions we need the following generalisation:

$$Z_0[\eta, \eta^*] = \int \mathcal{D}\psi \mathcal{D}\psi^*$$

$$\exp\left(i \int d^4x d^4x' \left[\psi^*(x') A(x', x) \psi(x)\right] + i \int d^4x \eta^*(x) \psi(x) + \psi^*(x) \eta(x) \right)$$

$$Z_0[\eta, \eta^*] = \sqrt{det(iA)} \exp(-i \int d^4x d^4x' \left[\eta^*(x') A^{-1}(x', x) \eta(x) \right] \tag{8.3.4}$$

In order to perform the functional integrations we first write the generating functional as:

$$Z_0[\eta,\overline{\eta}] \propto \int \mathcal{D}\psi\mathcal{D}\overline{\psi}$$
$$\exp(i\int d^4xd^4x'\{\overline{\psi}(x')\left[\delta^4(x-x')(i\gamma^\mu\partial_\mu^x - m)\right]\psi(x)$$
$$+i\int d^4x\overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x)\})$$

We identify the operator in square brackets as $A(x',x)$,

$$A(x',x) \equiv \left[\delta^4(x-x')(i\gamma^\mu\partial_\mu^x - m)\right] \quad (8.3.5)$$

and using Eq.8.3.4, replacing ψ^* by $\overline{\psi} = \psi^\dagger\gamma^0$, and similarly for η we find

$$Z_0[\eta,\overline{\eta}] \propto \sqrt{det(iA)}\exp(-i\int d^4xd^4x'\left[\overline{\eta}(x')S_F(x'-x)\eta(x)\right] \quad (8.3.6)$$

where

$$S_F(x'-x) = A^{-1}(x',x) \quad (8.3.7)$$

is the Feynman propagator for the Dirac field, as is readily verified by constructing the two point Green's function from Eq.8.3.6.

It only remains to explicitly find an expression for the Feynman propagator. From above we see that

$$S_F(x'-x) = \left[\delta^4(x-x')(i\gamma^\mu\partial_\mu^x - m)\right]^{-1} \quad (8.3.8)$$

Clearly we have

$$\int d^4y[S_F(x,y)]^{-1}S_F(y,z) = \delta(x-z)$$

Hence from its definition we have

$$\int d^4y\left[\delta^4(x-y)(i\gamma^\mu\partial_\mu^x - m)\right]S_F(y,z) = \delta(x-z)$$

Integrating over y, we find the equation which must be satisfied by $S_F(x,z)$,

$$(i\gamma^\mu\partial_\mu^x - m)S_F(x,z) = \delta(x-z) \quad (8.3.9)$$

The solution to this equation is formally

$$S_F(x,z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2}e^{-ik.(x-z)}$$

as may be verified by substitution. As in the scalar propagator case in section 6.2 the result is ambiguous due to the poles, and as in that case the resolution to the problem is to remember that the generating functional is strictly only defined off the real axis. The analysis is identical to that in section 6.2 and the result is:

$$S_F(x-z) = \int \frac{d^4k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2+i\epsilon}e^{-ik.(x-z)} \quad (8.3.10)$$

The Fourier transform of the Feynman propagator is thus

$$S_F(k) = \frac{\not{k}+m}{k^2-m^2+i\epsilon} \quad (8.3.11)$$

8.4 Appendix: Grassmann Variables

To begin with consider two Grassmann variables a, b which satisfy the anticommutation relations:

$$\{a, a\} = 0, \quad \{a, b\} = 0, \quad \{b, b\} = 0 \tag{8.4.1}$$

They are not operators; they do not act on any vectors in a vector space. However they do obey an algebra: the Grassmann algebra indicated above. We can think of them as "operators which do not act on anything" if we like (although please don't say this in the presence of a mathematician unless you want to see him or her turn blue.)

Without loss of generality any function of the two Grassmann variables may be written as:

$$f(a, b) = f_0 + f_1 a + \tilde{f}_1 b + f_2 ab \tag{8.4.2}$$

where the f's are non-Grassmannian c-numbers. The *left* derivatives of the function are:

$$\frac{\partial f}{\partial a} = f_1 + f_2 b \tag{8.4.3}$$

$$\frac{\partial f}{\partial b} = \tilde{f}_1 - f_2 a \tag{8.4.4}$$

so called because the infinitesimal denominator is really an inverse Grassmann variable which acts on the *left* in this case. Unless otherwise stated all our derivatives will be *left* derivatives. We find:

$$\frac{\partial^2 f}{\partial a \partial b} = -\frac{\partial^2 f}{\partial b \partial a} = -f_2 \tag{8.4.5}$$

We may also define integration with respect to Grassmann variables. The simplest Grassmann integral is zero:

$$\int da = 0 \tag{8.4.6}$$

(to prove this first show that $\left(\int da\right)^2 = -\left(\int da\right)^2$). The next simplest Grassmann integral is equal to unity:

$$\int daa = 1 \tag{8.4.7}$$

(in fact since $a^2 = 0$ the above integral serves as a sort of normalisation of the Grassmann variables.) Eqs.8.4.6 and 8.4.7 show that Grassmann integration has the same effect as differentiation.

Using Eq.8.4.2 we find that

$$\int dadbf(a, b) = \frac{\partial^2 f}{\partial a \partial b} \tag{8.4.8}$$

An interesting result is:

$$\int dadbe^{-ba} = \int dadb(1 - ba) = 1 \tag{8.4.9}$$

Finally recall the most important integral in the world from Eq.5.6.10:

$$Z_0 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dq_i e^{-\sum_{i,j=1}^{n} q_i K_{ij} q_j + \sum_{k=1}^{n} J_k q_k} \tag{8.4.10}$$

This is an n dimensional integral and K_{ij} is an $n \times n$ symmetric matrix, while J_k is an n component column vector. Amazingly the above integral can be done for all invertible K_{ij}:

$$Z_0 = \frac{\pi^{n/2}}{\sqrt{det(K)}} e^{+\frac{1}{4}\sum_{i,j=1}^{n} J_i (K^{-1})_{ij} J_j} \tag{8.4.11}$$

Now we consider the analagous integral but involving Grassmann variables $a_1, \cdots a_n$ and Grassmann variables $\eta_1, \cdots \eta_n$,

$$Z_0 = \int \prod_{i=1}^{n} da_i e^{-\sum_{i,j=1}^{n} a_i K_{ij} a_j + \sum_{k=1}^{n} \eta_k a_k} \tag{8.4.12}$$

and we assume that K_{ij} is now *antisymmetric* rather than symmetric otherwise we would get a zero result. For a similar reason we must assume that n is even. In this case the result is:

$$Z_0 = \sqrt{det(K)} e^{-\frac{1}{4}\sum_{i,j=1}^{n} \eta_i (K^{-1})_{ij} \eta_j} \tag{8.4.13}$$

The proof of this is found in Bailin and Love 8.1.

Chapter 9

The Free Electromagnetic Field

9.1 The Classical Electromagnetic Field

The four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

It is straightforward to show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

In covariant form,

$$\partial_\mu j^\mu = 0$$

where $j^\mu = (c\rho, \mathbf{j})$.

It is convenient (and even essential) to introduce scalar and vector potentials ϕ and $\mathbf{A}$ by defining

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla \phi - \partial \mathbf{A}/\partial t.$$

whence two of the Maxwell equations become automatic.

Recall the gauge invariance of electrodynamics which says that $\mathbf{E}$ and $\mathbf{B}$ are unchanged when

$$\mathbf{A} \to \mathbf{A} + \nabla \Lambda \quad \text{and} \quad \phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

for any scalar function Λ. Gauge invariance corresponds to a lack of uniqueness of the scalar and vector potentials. This lack of uniqueness can be reduced by imposing a further condition on the scalar and vector potentials, for example

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial \phi}{\partial t}$$

Assuming that ϕ and $\mathbf{A}$ can be combined into a four vector

$$A^\mu = (\phi/c, \mathbf{A})$$

this can be written as

$$\partial_\mu A^\mu = 0 \tag{9.1.1}$$

which is known as the *Lorentz gauge* condition. Gauge invariance in four-vector notation is just:

$$A^\mu \to A^\mu + \partial_\mu \Lambda \tag{9.1.2}$$

Note that even the imposition of the Lorentz gauge condition does not completely fix the vector potential; it merely restricts the function Λ to satisfy

$$\partial^2 \Lambda = 0 \tag{9.1.3}$$

With the Lorentz gauge condition Maxwell's equations are equivalent to

$$\partial^2 A^\mu = \mu_0 j^\mu$$

The tensor $F_{\mu\nu}$ is defined by

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ clearly has six independent components, and can be written:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

It is straightforward to show that,

$$F_{\mu\nu} F^{\mu\nu} = -2 \left(\frac{\mathbf{E}^2}{c^2} - \mathbf{B}^2 \right)$$

$$\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = -\frac{8}{c} \mathbf{E} \cdot \mathbf{B}$$

where

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of 0123} \\ -1 & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

This gives the relativistic invariants which can be constructed from $\mathbf{E}$ and $\mathbf{B}$.

It is easy to see that in any gauge the Maxwell equations can be written,

$$\partial_\mu F^{\mu\nu} = j^\nu$$

The Maxwell equations, in this compact form, can be reproduced by the following Lagrangian density,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{9.1.4}$$

via the Euler-Lagrange equations for each of the four A_μ fields separately.

In Lorentz gauge the Lagrangian density has the more general form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.1.5}$$

where ξ is a free parameter. The EL equations then imply

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{9.1.6}$$

which reduce to Maxwell's equations in Lorentz gauge. The extra term in the Lagrangian density $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$ thus has no effect on physics in Lorentz gauge. In fact it is possible to turn the argument around and use this term to fix the gauge to be Lorentz gauge by imposing current conservation instead of obtaining it as a consequence of Maxwell's equations. If one adds the extra term to the Lagrangian and imposes current conservation then Eq.9.1.6 implies immediately the Lorentz gauge condition by the antisymmetry of $F^{\mu\nu}$. For this reason the extra term is referred to as a *gauge fixing term* and ξ is a Lagrange multiplier. The choice $\xi = 1$ is known as Feynman gauge although it is within the framework of the Lorentz gauge.

As usual we can expand the field $A_\mu(x)$ in its Fourier components

$$A_\mu(x)(\mathbf{x}, t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega}[a_\mu(k)e^{-ik.x} + a^*_\mu(k)e^{ik.x}] \tag{9.1.7}$$

where $\omega = k_0 = |\mathbf{k}|$. The Lorentz gauge condition implies

$$k.a(k) = 0 \tag{9.1.8}$$

This implies that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k)$$

where $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$. Thus the time component of a_μ equals the longitudinal component $\mathbf{a}.\mathbf{k}$. Of course only the transverse components are physical (since the $\mathbf{E}$ and $\mathbf{B}$ fields are always orthogonal to the three momentum) and it can be shown that the contribution to the Hamiltonian from the time component and longitudinal component cancel against each other. In fact it is possible to completely specify the gauge by requiring that

$$a_0(k) = \hat{\mathbf{k}}.\mathbf{a}(k) = 0$$

which is called Coulomb gauge. In Coulomb gauge we can write

$$a_\mu(k) = \sum_{\lambda=1,2} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where $\epsilon^\lambda_\mu(k)$ are two orthonormal spacelike vectors in the plane transverse to $\mathbf{k}$.

In a general Lorentz gauge we can write:

$$a_\mu(k) = \sum_{\lambda=0,1,2,3} a^\lambda(k)\epsilon^\lambda_\mu(k)$$

where now $\epsilon^\lambda_\mu(k)$ are arbitrary unit four-vectors. Suppose that $\mathbf{k}$ is along the third axis, $k = (\omega, 0, 0, \omega)$ then we can define the basis vectors as:

$$\epsilon^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.1.9}$$

so that we call $\lambda = 1, 2$ the physical transverse polarisations, $\lambda = 0$ the unphysical timelike polarisation and $\lambda = 3$ the unphysical longitudinal polarisation. Clearly,

$$k.\epsilon^{1,2} = 0$$

and

$$\epsilon^\lambda.\epsilon^{\lambda'} = g^{\lambda\lambda'}$$

which is in fact a basis independent result, although we shall always work in this basis.

9.2 Canonical Quantisation

We shall now quantise the free e.m. theory ($j^\mu = 0$). To quantise the theory canonically we introduce the canonical momenta

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} \tag{9.2.1}$$

and impose the equal time covariant canonical commutation relations

$$[A_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = i g_{\mu\nu}\delta^3(\mathbf{x} - \mathbf{x}') \tag{9.2.2}$$

$$[A_\mu(\mathbf{x}, t), A_\nu(\mathbf{x}', t)] = 0 \tag{9.2.3}$$

$$[\pi_\mu(\mathbf{x}, t), \pi_\nu(\mathbf{x}', t)] = 0 \tag{9.2.4}$$

Now if the Lagrangian were simply

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.2.5}$$

then we would find that

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

which would imply that π^0 always commutes with A^0, which loses us both covariance and quantum mechanics at a stroke!

We clearly need a π^0 that does not vanish. In order to do this we need to change the Lagrangian without changing the physics. But we have learned how to do this in Lorentz gauge which corresponds to the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.2.6}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.2.7}$$

Henceforth for simplicity we shall take $\xi = 1$ which is called Feynman gauge (a sub-class of Lorentz gauge).

At first sight this doesn't help us because we find

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu$$

which apparently vanishes in Lorentz gauge. However we shall only assume that matrix elements of $\partial_\mu A^\mu$ vanish rather than imposing the operator condition that it vanish.

In Feynman gauge we have the field equations:

$$\partial^2 A_\mu = 0 \tag{9.2.8}$$

and we can once again expand the A_μ field in plane wave solutions similar to the previous section:

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega} \sum_{\lambda=0}^{3} [\epsilon_\mu^\lambda(k) a^\lambda(k) e^{-ik.x} + \epsilon_\mu^{\lambda *}(k) a^\lambda(k)^\dagger e^{ik.x}] \tag{9.2.9}$$

Here $\epsilon_\mu^\lambda(k)$ are the set of four linearly independent vectors defined in Eq.9.1.9, but now we regard a and its hermitian conjugate as operators whose commutation relations readily follow from Eq.9.2.2

$$[a^\lambda(k), a^{\lambda'}(k')^\dagger] = -g^{\lambda\lambda'} 2k_0 (2\pi)^3 \delta^3(\mathbf{k} - \mathbf{k}') \tag{9.2.10}$$

For longitudinal and transverse photons quantisation proceeds in the usual way. But for timelike photons with $\lambda = \lambda' = 0$ we have a negative quantity on the rhs which gives problems. This leads to timelike photons with negative norm. However it is possible to overcome these problems using the Gupta-Bleuler formalism. However at this point we prefer to abandon the canonical approach and move on to the path integral approach which has its own problems.

9.3 Path Integral Quantisation

We have seen that the freedom to make gauge transformations means that the A^μ fields are not uniquely specified, and this causes problems with the theory in the canonical formalism. It should be no surprise that these problems persist in the path integral approach.

The generating functional in this case is

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x(\mathcal{L}+J^\mu A_\mu\right)} \tag{9.3.1}$$

where $\mathcal{L}$ is the Lagrangian for the free photon field which we might naively take to be

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{9.3.2}$$

(since we have already found problems with this form in the canonical formalism it really is naive to expect it to work here). The field equations in this case are as in Eq.9.1.6

$$\partial_\mu F^{\mu\nu} = 0 \tag{9.3.3}$$

which can be written as

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)A^\mu = 0 \tag{9.3.4}$$

After partial integration and discarding surface terms we can write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x\frac{1}{2}A^\mu[g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu]A^\nu+J^\mu A_\mu\right)} \tag{9.3.5}$$

By now we know that the photon propagator $D_{\mu\nu}$ is going to be the inverse of the operator in square brackets, and it will satisfy the equation:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)D^{\nu\lambda}(x-y) = \delta_\mu^\lambda\delta^4(x-y) \tag{9.3.6}$$

If we multiply this equation by ∂^μ we get zero multiplying $D^{\nu\lambda}(x-y)$ on the lhs and something non-zero on thhe rhs, which would seem to imply that $D^{\nu\lambda}(x-y)$ is infinite. In fact the problem is that the operator in square brackets does not have an inverse! To show this all we need to do is show that it has a zero eigenvalue, and this can easily be done:

$$(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\partial^\mu\Omega = 0$$

for any function Ω.

From the point of view of the path integral the problem is that the functional integral is taken over all A_μ including those related by a gauge transformation, leading to an infinite overcounting in the calculation of the generating functional, and hence an infinite overcounting for the Green's functions which are obtained from it by functional differentiation. To cure this problem we need to fix a particular gauge, and we do this by imposing the Lorentz gauge condition:

$$\partial_\mu A^\mu = 0 \tag{9.3.7}$$

Recall the Lagrangian with gauge fixing term,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{9.3.8}$$

and the field equations:

$$\partial^2 A_\mu - (1 - \frac{1}{\xi})\partial_\mu(\partial_\nu A^\nu) = 0 \tag{9.3.9}$$

After partial integration and discarding surface terms we can now write the generating functional as

$$Z_0[J] \propto \int \mathcal{D}A_\mu e^{i\left(\int d^4x \frac{1}{2}A^\mu[g_{\mu\nu}\partial^2 + (\frac{1}{\xi}-1)\partial_\mu\partial_\nu]A^\nu + J^\mu A_\mu\right)} \tag{9.3.10}$$

and the operator in square brackets now has an inverse given by

$$D_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} - \frac{\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} e^{-ik.(x-y)} \tag{9.3.11}$$

The Fourier transform of the Feynman propagator is thus

$$D_{\mu\nu}(k) = -\frac{\left[g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right]}{k^2 + i\epsilon} \tag{9.3.12}$$

Amongst this class of gauge choices two common choices are Feynman gauge ($\xi = 1$) and Landau gauge ($\xi = 0$).