Digital Communication

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Preface

Chapter 2

Introduction

- In this chapter, we deal only with baseband communication:
  - Amplitude Spectrum of the signal is centered at 0 Hz
  - No carrier signal is used for transmission $\leftrightarrow$ no modulation
- Modulation Scheme will be addressed in later chapters
- Many of the important concept done in this chapter are generalized in modulation scheme also (this chapter isn't a waste of time)
  - Detection
  - Bits assignments
  - ...

Transmitting Binary Data

Transmitting Binary Data I

- One important thing to note in digital communication is that the message we transmit is a stream of bits: 0 and 1
- This has the advantage to let the receiver to decide between 2 values only when doing detection
  - In opposite to analog communication, where the received signal has infinite amount of values
  - So if we have noise, it is hard to do detection

Binary Communication System

- We begin first by binary communication system
  - We have only 1 bit to transmit: 0 or 1
  - We have 2 signals: $s_1(t)$ for bit 0 and $s_2(t)$ for bit 1
- These bits comes from the analog to digital process (digital signal processing classes)
  - In this class, we directly assume that these steps are done correctly and we have directly the stream of bits which we want to transmit

General Communication System and Mapping

General Communication System and Mapping I

- In general, we could also transmit more than 1 bit:
  - For example if we have 2 bits, we have 2^2 possible combination: 00, 01, 10 and 11
  - Thus we will have 4 signals $s_M(t)$ where $M = 2^n =$ number of symbols (signals) to be transmitted
- Usually in every digital communication system, we have to do a mapping of the stream bits to a set of signals
  - N bits
    $b_0 , b_1 , b_2 \cdots b_{N-1} \mapsto$ a set of M signals $s_i(t) : 0 \leq i \leq M - 1$
- In case we have more than 1 bit to transmit $\leftrightarrow$ more than 2 transmitted signal, the communication system is referred to as M-ary communication

General Communication System and Mapping II

- In this chapter we deal with the special case of an M-ary communication $\leftrightarrow$ binary communication system:
  - $M = 2 \rightarrow$ 2 signals $s_0(t)$ and $s_1(t)$, and we have 1 bit in each signal: 0 or 1

Digital Waveform

Digital Waveform

- The signals $s_i(t)$ are assumed to be time limited with a symbol duration denoted by T_{symbol}
- In binary communication system, in the period T_{symbol} we transmit only 0 or 1

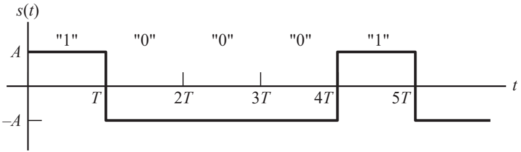


Figure 1: T here is T_{symbol}

- For transmission, the bits (0 and 1) are mapped to electrical signals using what we call line coding techniques

Synchronization

Synchronization and Receiver Knowledge I

- Throughout this chapter, we assume that the Rx knows where each pulse start and finish

Orthogonality Reminder

Orthogonality Reminder I

- 2 signals are orthogonal on $[0, t]$ if their inner product is 0

$$\int_0^t x(t)\, y(t)\, dt = 0 \tag{1}$$

  - This means that if we project $x(t)$ onto $y(t)$, $x(t)$ has no component on $y(t)$

- For a set of M signals as in the case of M-ary communication, we must verify the equality between all paired signals

$$\int_0^{T_{\text{symbol}}} s_i(t)\, s_j(t)\, dt = 0 \quad \forall\, i, j \text{ with } i \neq j \tag{2}$$

Introduction

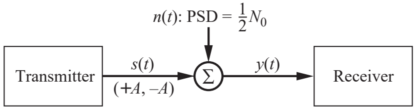
- Usually when explaining some concept, there are 2 ways to proceeds:
  1. Do directly a general derivation
  2. The opposite case: take a specific example, and then generalize this example
- In the following slides, we adopt approach 2 for receiver study for a base band binary communication system

A Proposed Model for the Receiver I

- The received signal is the sum of the transmitted signals plus the noise component which model the channel corruption

$$y_{\mathsf{R}_x}(t) = s_i(t) + n(t) \tag{3}$$

  - $s_i(t)$ is either $s_1(t)$ or $s_2(t)$ since we are in digital binary communication system



- Mathematical Assumption:
  - White Gaussian Noise with PSD $\dfrac{N_0}{2}$
  - The signals are: $\begin{cases} s_1(t) = -\mathsf{Amp} \times p_T(t) & \text{for bit 0} \\ s_2(t) = +\mathsf{Amp} \times p_T(t) & \text{for bit 1} \end{cases}$

A Proposed Model for the Receiver II

- We will consider a receiver model shown in figure 2.

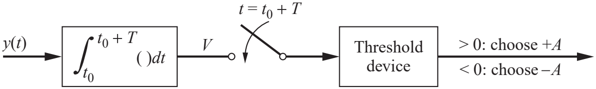


Figure 2: Integrate-and-dump Rx

- *Receiver Steps* : Several Steps of processing will be done to the received signal $y_{R_x}(t)$
  1. Integration over the symbol duration:
     - The output of the integrator will be some number
     - The integration done here is some kind of filtering
  2. Compare the output of the integration to some threshold

Integration Output

Probabilistic Integration

- The integration step involves a probabilistic quantity since the received signal $y_{R_x}(t)$ is random due to the noise component $n(t)$
- What we will do in this part is the computation of this integration to understand how the linear receiver of figure work
- Notation: We will denote the output of the integration by V
  - V is a decision statistics which depends on the transmitted signal $s_i(t)$
- To do the computation for V and derive its distribution, we do the following assumption that $s_2(t)$ is transmitted (positive pulse $+$Amp)
  - Hence V is know will be denoted by V_2 where the subscript denote the distribution of V given that $s_2(t)$ is transmitted $\leftrightarrow$ it is a conditional distribution
  - Same concept for $V_1 \leftrightarrow s_1(t)$ (negative pulse -Amp) is transmitted conditional distribution given that

Computation Steps for Probabilistic Integration of Positive Pulse I

1. General Formula at the output of the receiver:

$$V = \int_{0}^{T_{\text{symbol}}} s_i(t) + n(t) \tag{4}$$

2. Assume that $s_1(t)$ is transmitted hence $s_i(t) = s_2(t)$ and $Z \rightarrow V_2$

$$V_2 = \int_{0}^{T_{\text{symbol}}} s_2(t) + n(t) \tag{5}$$

Computation Steps for Probabilistic Integration of Positive Pulse II

3. Decompose the integral:

$$V_2 = \underbrace{\int_0^{T_{\text{symbol}}} s_1(t)}_{\text{Determnisitic}} + \underbrace{\int_0^{T_{\text{symbol}}} n(t)}_{\text{Random}} \quad (6)$$

- $s_2(t)$ is a rectangular pulse with a width of T_{symbol} and height equal to a certain amplitude $\rightarrow$ the integral is equal to Amp $\times$ T_{symbol}
- $n(t)$ is a Gaussian process, but the integral of a Gaussian is a Gaussian and if we sum a Gaussian random variable to a deterministic quantity, we obtain a Gaussian random variable also $\rightarrow$ Z_2 is a Gaussian random variable $\leftrightarrow$ we need its mean and variance to describe its conditional distribution

Computation Steps for Probabilistic Integration of Positive Pulse III

4. Hence we can write:

$$V_2 = \text{Amp} \times T_{\text{symbol}} + \int_0^{T_{\text{symbol}}} n(t)\,dt \tag{7}$$

Since Z_2 is Gaussian, we now need to compute its mean $E[Z_2]$ and its variance $\text{Var}(Z_2)$.

Some Rules regarding mean computation:

- Mean of a constant is a constant $\leftrightarrow$ $E[\text{constant}] = \text{constant}$
- Mean of a sum is the sum of the means $\leftrightarrow$ $E[X + Y] = E[X] + E[Y]$

Computation Steps for Probabilistic Integration of Positive Pulse IV

5. Mean Computation:

$$
\begin{aligned}
E[V_2] &= E\left[\mathsf{Amp} \times T_{\mathsf{symbol}} + \int_0^{T_{\mathsf{symbol}}} n(t)\,dt\right] \\
&= E\left[\,\mathsf{Amp} \times T_{\mathsf{symbol}}\,\right] + \underbrace{E\left[\int_0^{T_{\mathsf{symbol}}} n(t)\,dt\right]}_{\text{Exchange the integrale and the E operator}} \\
&= \mathsf{Amp} \times T_{\mathsf{symbol}} + \int_0^{T_{\mathsf{symbol}}} E\left[\,n(t)\,\right]\,dt
\end{aligned}
$$

Computation Steps for Probabilistic Integration of Positive Pulse V

Since we are dealing with zero mean Gaussian process

$$\rightarrow E\,[\,n(t)\,] = 0 \rightarrow \int_{0}^{T_{\mathsf{symbol}}} E\,[\,n(t)\,]\,dt = \int_{0}^{T_{\mathsf{symbol}}} 0\,dt = 0$$

and hence we have:

$$E[V_2] = \mathsf{Amp} \times T_{\mathsf{symbol}} \tag{8}$$

Computation Steps for Probabilistic Integration of Positive Pulse VI

6. Variance Computation:
   - Recall the variance rule: $\mathsf{Var}(X) = E\left[(X - E[X])^2 \right]$

$$\mathsf{Var}(V_2) = E\left[(V_2 - E[V_2])^2 \right]$$

where

- $V_2 = \mathsf{Amp} \times T_{\mathsf{symbol}} + \int_0^{T_{\mathsf{symbol}}} n(t)\, dt$
- $E[V_2] = \mathsf{Amp} \times T_{\mathsf{symbol}}$

Plugin those equations into equation (6) we get:

Computation Steps for Probabilistic Integration of Positive Pulse VII

$$
\begin{aligned}
\mathsf{Var}(V_2) &= E\left[\left(\mathsf{Amp} \times T_{\mathsf{sym}} + \int_0^{T_{\mathsf{sym}}} n(t)\,dt - \mathsf{Amp} \times T_{\mathsf{sym}}\right)^2\right] \\
&= E\left[\left(\int_0^{T_{\mathsf{sym}}} n(t)\,dt\right)^2\right] \quad \text{decompose the integral} \\
&= E\left[\int_0^{T_{\mathsf{sym}}} n(t)\,dt \times \int_0^{T_{\mathsf{sym}}} n(s)\,ds\right]
\end{aligned}
$$

Exchange the E[.] and $\int$

Computation Steps for Probabilistic Integration of Positive Pulse VIII

$$\mathrm{Var}(V_2) = \int_0^{T_{\mathsf{sym}}} \int_0^{T_{\mathsf{sym}}} E\left[\, n(t) \times n(s) \,\right] dt\, ds$$

- $E\left[\, n(t) \times n(s) \,\right]$ is the autorcorroleation at (t, s) and its equal to $\frac{N_0}{2}\,\delta(\,t - s\,)$.

$$\mathrm{Var}(V_2) = \frac{N_0}{2} \times \int_0^{T_{\mathsf{sym}}} \int_0^{T_{\mathsf{sym}}} \delta(\,t - s\,)\, dt\, ds$$

Computation Steps for Probabilistic Integration of Positive Pulse IX

- Let's think about the inner integral $\int_0^{T_{\text{sym}}} \delta(t - s)\, dt$: here the $\delta(.)$ is a function of t, and it is located at $t = s$, and its area value (hence its integral) equal to 1. Hence we have:

$$\begin{aligned}
\text{Var}(V_2) &= \frac{N_0}{2} \times \int_0^{T_{\text{sym}}} 1\, dt \\
&= \frac{N_0\, T_{\text{sym}}}{2}
\end{aligned}$$

Computation Steps for Probabilistic Integration of Positive Pulse X

Note: need to review the integral trick we did in the double integration when we talk about the inner integral: why did we drop down ds and let dt. This is done from random processes class when we deal with double integration for a $\delta(.)$ function.

Summary of Computation

- $E[V_2]$ = Amp $\times$ T_{symbol} $\leftrightarrow$ location of the distribution Z_2
  - So when I have symbol $s_1(t)$ transmitted, on average, I land on Amp $\times$ T_{symbol}
- Var(V_2) = $\sigma^2 = \dfrac{N_0 T_{\text{sym}}}{2}$ $\leftrightarrow$ the spread of the distribution Z_2
- Same computation can be made for the conditional distribution Z_1: case where have as a given signal $s_1(t)$ (negative pulse -Amp) is transmitted

Distribution Plot

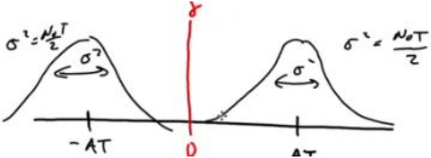


Figure 3: Conditional Distribution Output of the Linear Receiver

- Regarding the threshold k, we have chosen $k = 0$
- The reason to choose this and its effect will be discussed later
  - How we can choose optimum threshold value

Some Observation I

- The variance for both signals $s_1(t)$ and $s_2(t)$ is the same
  - This indicates that the channel corrupt the signal in the same way, independently of the signals we transmit
  - Notice also that the variance σ^2 doesn't depends on the signals $\rightarrow$ there is no index $i \leftrightarrow$ it depends only on the noise (N_0) and the duration T_{symbol}
- Regarding the mean which control the position of the curves:
  - The amplitude controls the energy of the signals
  - The higher the energy in our signals and if the noise still the same $\rightarrow$ the curves should be pushed further $\rightarrow$ the probability of error will decrease allot

And this is very logical: the more energy we have $\rightarrow$ the more error decrease

Some Observation II

- Always pay attention that our probability distribution should make sense
  - We can't obtain for example the opposite result: the more energy we have, the more error we get $\leftrightarrow$ area of error will get bigger since the curves will get closer. That doesn't make sense.

Performance Measure and Probability of Error

Performance Measure and Probability of Error

- How we can know if the Rx is good or not for such a task?
- A performance measure can be take is the probability of error. In this section we will compute it
- In binary digital communication, the error can occur in 2 ways:
    1. $s_1(t)$ is transmitted but the Rx detect $s_2(t) \rightarrow P\left(\text{Error} \,|\, s_1(t) \text{ is sent}\right)$
    2. $s_2(t)$ is transmitted but the Rx detect $s_1(t) \rightarrow P\left(\text{Error} \,|\, s_2(t) \text{ is sent}\right)$
- Since the error can occur in 2 ways, so what we need to compute is the total probability of error
    - This will be done using total probability rule

Total Probability Rule Reminder I

- The probability of errors can be computed using total probability rule, which compute complicated events (in this case the error) in disjoint scenarios.
- Suppose we have 2 disjoint event A_1 and A_2, and we want to compute probability of event B under A_1 and A_2, this can be done as:

$$\begin{aligned} P(B) &= P(A_1 \cap B) + P(A_2 \cap B) \\ &= P(A_1) \times P(B \mid A_1) + P(A_2) \times P(B \mid A_2) \end{aligned} \tag{9}$$

Total Probability Rule Reminder II

- Application to Binary Baseband Communication System

$$
\begin{aligned}
P(\text{Error}) &= P\left(\text{Error} \cap s_1(t)\right) + P\left(\text{Error} \cap s_2(t)\right) \\
&= P\left[s_1(t)\right] \times P\left(\text{Error} \,|\, s_1(t) \text{ is sent}\right) \\
&\quad + P\left[s_2(t)\right] \times P\left(\text{Error} \,|\, s_2(t) \text{ is sent}\right)
\end{aligned} \qquad (10)
$$

- $P[s_1(t)]$ and $P[s_2(t)]$ are the prior beliefs, and will be take as uniform distribution $\rightarrow$ $P[s_1(t)] = P[s_2(t)] = 0.5$ and $P[s_1(t)] + P[s_2(t)] = 1$
- The next step is to compute the conditional probabilities: $P\,(\text{Error} \,|\, s_2(t) \text{ is sent})$ and $P\,(\text{Error} \,|\, s_1(t) \text{ is sent})$

Conditional Probabilities of Errors

- $P\,(\text{Error}\,|\,s_1(t)$ is sent$)$ and $P\,(\text{Error}\,|\,s_2(t)$ is sent$)$ are nothing but the distribution we have computed in the sections before, and they are illustrated in figure 3
- $P\,(\text{Error}\,|\,s_1(t)$ is sent$)$ can be computed when the redarea under the curve is below the threshold k as shown in figure 4.

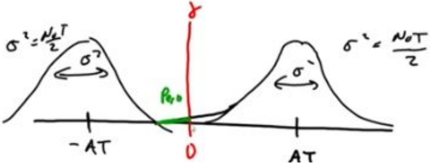


Figure 4: Error Probability as Area under the curve: the tails of the Gaussian distribution

Computing $P\,(\text{Error}\,|\,s_2(t)\text{ is sent})$ I

- Computing:

$$\begin{aligned} P\,(\text{Error}\,|\,s_2(t)\text{ is sent}) &= P(V \le \underbrace{k}_{=0}\,|\,s_2(t)\text{ is sent}) \\ &= P(V_2 \le 0\,) \end{aligned} \tag{11}$$

- $P(V_2 \le 0\,)$ can be computed using the cumulative distribution of the distribution

Computing $P\left(\text{Error} \mid s_2(t) \text{ is sent}\right)$ II

- Cumulative distribution reminder for a Gaussian:

$$\begin{aligned} F(X = x) &= P(X \leq x) \\ &= \Phi\left(\frac{x - E[X]}{\sqrt{\text{Var}(X)}}\right) \end{aligned} \tag{12}$$

  - The random variable X is the the conditional distribution V_2
  - $x = k$ the threshold value
  - $E[X] = E[V_2] = \text{Amp} \times T_{\text{sym}}$
  - $\text{Var}(X) = \text{Var}(V_2) = \dfrac{N_0\, T_{\text{sym}}}{2}$

Computing $P\left(\text{Error} \,|\, s_2(t) \text{ is sent}\right)$ III

- Applying:

$$
\begin{aligned}
P(V_2 \le 0) &= \Phi\left(\frac{\gamma - E[V_2]}{\sqrt{\mathsf{Var}(V_2)}}\right) \\
&= \Phi\left(\frac{0 - \mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0 T_{\mathsf{sym}}}}\right) \\
&= \Phi\left(\frac{-\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0 T_{\mathsf{sym}}}}\right) \\
&= 1 - \Phi\left(\frac{+\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0 T_{\mathsf{sym}}}}\right)
\end{aligned}
\tag{13}
$$

Computing $P\,(\text{Error}\,|\,s_2(t)$ is sent) IV

$$
\begin{aligned}
P(V_2 \le 0) &= 1 - \Phi\left(\frac{+\,\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0\,T_{\mathsf{sym}}}}\right) \\
&= Q\left(\frac{+\,\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0\,T_{\mathsf{sym}}}}\right)
\end{aligned}
\tag{14}
$$

Computing $P\,(\text{Error}\,|\,s_1(t)$ is sent$)$ I

- Now we compute the 2^{nd} type of error
- Be aware that in the computation of 2^{nd} type error, we compute the probability above the threshold k

$$
\begin{aligned}
P\,(\text{Error}\,|\,s_1(t)\text{ is sent}) &= P(V > \underbrace{k}_{=0} \,|\, s_1(t)\text{ is sent}) \\
&= P(V_1 > 0) \\
&= 1 - \underbrace{P(V_1 \leq 0)}_{\text{Using CDF}} \\
&= 1 - \Phi\left(\frac{0 - (-\text{Amp} \times T_{\text{sym}})}{\sqrt{0.5 \times N_0\, T_{\text{sym}}}}\right)
\end{aligned}
$$

Computing $P\left(\text{Error} \,|\, s_1(t) \text{ is sent}\right)$ II

$$\begin{aligned} P\left(\text{Error} \,|\, s_1(t) \text{ is sent}\right) &= 1 - \Phi\left(\frac{+\,\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0\, T_{\mathsf{sym}}}}\right) \\ &= Q\left(\frac{+\,\mathsf{Amp} \times T_{\mathsf{sym}}}{\sqrt{0.5 \times N_0\, T_{\mathsf{sym}}}}\right) \end{aligned}$$

Summary for Conditional Probabilities of Errors

- $P\,(\text{Error}\,|\,s_1(t)\text{ is sent}) = Q\left(\dfrac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0\,T_{\text{sym}}}}\right)$
- $P\,(\text{Error}\,|\,s_2(t)\text{ is sent}) = Q\left(\dfrac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0\,T_{\text{sym}}}}\right)$
- What we have is that the conditional probabilities of both error types are the same:
  - $P\,(\text{Error}\,|\,s_1(t)\text{ is sent}) = P\,(\text{Error}\,|\,s_2(t)\text{ is sent}) = Q\left(\dfrac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0\,T_{\text{sym}}}}\right)$

Total Probability of Error I

- We haven't done yet: don't forget that we are interested by the total probability of the error

$$
\begin{aligned}
P(\text{Error}) &= P\left(\text{Error} \cap s_1(t)\right) + P\left(\text{Error} \cap s_2(t)\right) \\
&= P\left[s_1(t)\right] \times P\left(\text{Error} \,|\, s_1(t) \text{ is sent}\right) \\
&\quad + P\left[s_2(t)\right] \times P\left(\text{Error} \,|\, s_2(t) \text{ is sent}\right) \\
&= \frac{1}{2} Q\left(\frac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0 T_{\text{sym}}}}\right) + \frac{1}{2} Q\left(\frac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0 T_{\text{sym}}}}\right) \\
&= Q\left(\frac{+\text{Amp} \times T_{\text{sym}}}{\sqrt{0.5 \times N_0 T_{\text{sym}}}}\right) = Q\left(\frac{+\sqrt{\text{Amp}^2 \times T_{\text{sym}}^2}}{\sqrt{0.5 \times N_0 T_{\text{sym}}}}\right)
\end{aligned}
$$

Total Probability of Error II

$$P(\text{Error}) = Q\left(\sqrt{\frac{+2\,\text{Amp}^2 \times T_{\text{sym}}}{N_0}}\right)$$

- Another observation we have: total probability of error is the same as the conditional probability of errors
  - $P(\text{Error}) = P\left(\text{Error}\,|\,s_1(t)\text{ is sent}\right) = P\left(\text{Error}\,|\,s_2(t)\text{ is sent}\right) = Q\left(\sqrt{\frac{+2\,\text{Amp}^2 \times T_{\text{sym}}}{N_0}}\right)$

Probability of Error and SNR

Probability of Error and SNR I

- It is very common to use to express $P(\text{Error})$ as function of SNR
- Computing the energy of the signals $s_i(t)$:

$$\begin{aligned}
\text{Energy} &= \int_0^{T_{\text{sym}}} [\, s_i(t) \,]^2 \, dt \\
&= \int_0^{T_{\text{sym}}} [\, \text{Amp} \,]^2 \, dt \\
&= \text{Amp} \times T_{\text{sym}}
\end{aligned}$$

Probability of Error and SNR II

- Linking Formula of $P(\text{Error})$ and SNR using Energy expression:

$$
\begin{aligned}
P(\text{Error}) &= Q\left(\sqrt{\frac{+2\,\text{Amp}^2 \times T_{\text{sym}}}{N_0}}\right) \\
&= Q\left(\sqrt{\frac{+2\,\text{Energy}}{N_0}}\right)
\end{aligned}
$$

Probability of Error and SNR III

- The quantity $\frac{\text{Energy}}{N_0}$ is what we call signal to noise ration (SNR) and will commonly used in our analysis
- SNR is often a big quantity, so we in practice we express it in dB

$$\left(\frac{\text{Energy}}{N_0}\right)_{\text{dB}} = 10\log_{10}\left(\frac{\text{Energy}}{N_0}\right)_{\text{Linear}} \tag{15}$$

- Figure shows the $P(\text{Error})$ as function of SNR in dB

Probability of Error and SNR IV

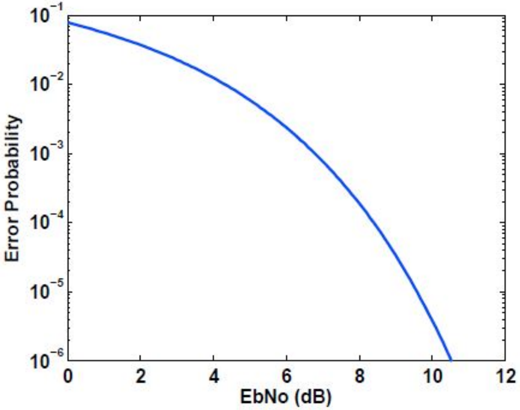


Figure 5: $P(\text{Error})$ as function of SNR in dB scale

Common Mistake

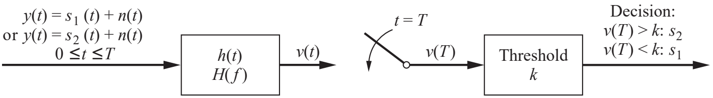
- When using the formula $P(\text{Error}) = Q\left(\sqrt{\dfrac{+2\,\text{Energy}}{N_0}}\right)$ to know for example for a given energy of a signal, what is the probability of error, remember to convert the SNR to linear before plug it int the equation of $P(\text{Error})$

General Receiver

Generalization of Receiver Computation

- Now we have understand the basic computation for receiver, we will generalize the concepts done before
  - However, we will still in the case of binary digital communication system
- The receiver job was to do some kind of processing to give us a decision statistics which help us distinguish between the signals transmitted
- This decision statistics was compared to some threshold k
- What we will generalize:
  - Integration $\rightarrow$ filtering:
    - Before the filtering operation was an integration operation, but is is the best one ?
  - $P(\text{error})$
  - Signals Shape:
    - The transmitted signals $s_i(t)$ had a rectangular shape, what happen if we change the shape of these signals $\leftrightarrow$ try other forms then rectangular pulse ?

Generalization of Systems I



- We can see now that instead of $\int_0^{T_{\text{sym}}}$ we have a general linear system $h(t)$
  - The output of $h(t)$ is a random process $n(t)$
  - It will be assumed to be Zero-Mean Wide-Sense Stationary Gaussian Random Process
- We sample a value from this random process at a time T_0
  - So instead of V we will have $V(T_0)$
  - Before, the time instant was $T_0 = T_{\text{sym}}$
  - We determine later the optimum sampling time

Generalization of Systems II

- As a receiver decision, we still have to compare $V(T_0)$ to some threshold k
- Decision Rule:
  1. $s_2(t)$ is sent if $V(T_0) > k$
  2. $s_1(t)$ is sent if $V(T_0) < k$
- We could also reverse the above convention if we consider signal $s_2(t)$ is the one need to be in the left side and $s_1(t)$ in the right side
  1. $s_1(t)$ is sent if $V(T_0) > k$
  2. $s_2(t)$ is sent if $V(T_0) < k$

Generalization of Signals

- General transmitted signals $s_i(t)$ will be written as:

$$\begin{aligned} s_1(t) &= \mathsf{Amp} \times \psi_1(t) \\ s_2(t) &= \mathsf{Amp} \times \psi_2(t) \end{aligned} \tag{16}$$

where $\psi_1(t)$ and $\psi_2(t)$ are arbitrary shape signal (rectangular, triangular, cos, $\cdots$) with finite energy signals on $[\,0\,,\,T_{\mathsf{sym}}\,]$

Filter Output I

- The received signal $y_{R_x}(t)$:

$$y_{R_x}(t) = s_i(t) + n(t) \tag{17}$$

- $y_{R_x}(t)$ will be inputted to linear system $h(t)$, in which will give some output $V(t)$
- We will apply superposition property to compute $V(t)$:

$$s_i(t) \rightarrow h(t) \rightarrow s_{i_{\text{filtered}}}(t)$$

$$n(t) \rightarrow h(t) \rightarrow n_{\text{filtered}}(t)$$

$$y_{R_x}(t) \rightarrow h(t) \rightarrow V(t)$$

Filter Output II

$$V(t) = \underbrace{s_{i_{\text{filtered}}}(t)}_{\text{deterministic}} + \underbrace{n_{\text{filtered}}(t)}_{\text{probabilistic}} \tag{18}$$

- The deterministic part of the filter output will give us the mean of $V(t)$ $\leftrightarrow$ will tell us where to land on the real axis
  - $s_{i_{\text{filtered}}}(t)$ can computed using convolution

$$\begin{aligned} s_{i_{\text{filtered}}}(t) &= (s \star h)(t) \\ &= \int_{-\infty}^{+\infty} s_i(t-\tau)\, h(\tau)\, d\tau \end{aligned} \tag{19}$$

  - $s_i(t-\tau)$: we do a time reverse then shifting

Filter Output III

- The challenging part as always is the probabilistic part $n_{\text{filtered}}(t)$
- However, we are assuming that $n(t)$ is a Gaussian random process with 0 mean
  - If we filter a Gaussian process, we obtain a Gaussian process
  - The mean of the output is 0 since we are assuming the input Gaussian process has 0 mean
  - Need to compute its variance
- From random process theory, we can characterize the variance of the output of a random process from 2 perspective:
  1. Time using autorcorrolation
  2. Frequency using power spectral density

Filter Output: Variance Computation using Time Domain I

- The autocorrolation $R_{n_{\text{filtered}}}(\tau)$:

$$R_{n_{\text{filtered}}}(\tau) = \int_{-\infty}^{+\infty} f(\tau - \lambda)\, R_{n_{\text{input}}}(\lambda)\, d\lambda \tag{20}$$

where

$$f(t) = \int_{-\infty}^{+\infty} h(u - t)\, h(u)\, du \tag{21}$$

- Equation (20) are general equation for any input random process $n_{\text{input}}(t)$
  - However, in our case $n_{\text{input}}(t)$ is a Gaussain white noise so its autocorrolation is $N_0\, \delta(\lambda)$

Filter Output: Variance Computation using Time Domain II

- Developing equation (20) we have:

$$\begin{aligned} R_{n_{\text{filtered}}}(\tau) &= \int_{-\infty}^{+\infty} f(\tau - \lambda)\, R_{n_{\text{input}}}(\lambda)\, d\lambda \\ &= \int_{-\infty}^{+\infty} f(\tau - \lambda)\, 0.5\, N_0\, \delta(\lambda)\, d\lambda \end{aligned} \tag{22}$$

Filter Output: Variance Computation using Time Domain III

$$
\begin{aligned}
R_{n_{\text{filtered}}}(\tau) &= 0.5\, N_0 \int_{-\infty}^{+\infty} f(\tau - \lambda)\, \delta(\lambda)\, d\lambda \\
&= 0.5\, N_0\, f(\tau) \int_{-\infty}^{+\infty} \delta(\lambda)\, d\lambda \\
&= 0.5\, N_0\, f(\tau) \times 1 \\
&= 0.5\, N_0\, f(\tau)
\end{aligned} \tag{23}
$$

And $\text{Var}(n_{\text{filtered}}) = R_{n_{\text{filtered}}}(\tau = 0)$

Filter Output: Variance Computation using Frequency Domain

- Using power spectral density we have:

$$S_{n_{\text{filtered}}}(\omega_c) \;=\; |H(\omega_c)| \;\times\; S_{n_{\text{input}}}(\omega_c) \tag{24}$$

- For a white noise $S_{n_{\text{input}}}(\omega_c) \;=\; 0.5\, N_0$

Decision Statistics V

- Once we analyze the filter output, the next step is to grad a sample at some time instant T_0
- The decision statistic $V(T_0)$ which is the input to the threshold device is written as:

$$V(T_0) = s_{i_{\text{filtered}}}(t = T_0) + n_{\text{filtered}}(t = T_0) \tag{25}$$

- Assuming signal $s_2(t)$ is transmitted, we can write (same as we did in before) conditioned decision statistic $V_2(T_0)$

$$\begin{aligned} V_2(T_0) &= V(T_0 \,|\, s_2(t) \text{ is transmitted}) \\ &= s_{2_{\text{filtered}}}(t = T_0) + n_{\text{filtered}}(t = T_0) \end{aligned} \tag{26}$$

- We will compute its mean $E[\, V_2(T_0)\,]$ and its variance $\text{Var}(\, V_2(T_0)\,)$ as function of the grabbing time T_0 so we can optimize upon T_0 later

Expectation of Decision Statistics V_2 I

$$
\begin{aligned}
E[V_2(T_0)] &= E\left[s_{2_{\mathsf{filtered}}}(t = T_0) + n_{\mathsf{filtered}}(t = T_0)\right] \\
&= E\left[\underbrace{s_{2_{\mathsf{filtered}}}(t = T_0)}_{\text{deterministic}}\right] + E\left[n_{\mathsf{filtered}}(t = T_0)\right] \\
&= s_{2_{\mathsf{filtered}}}(t = T_0) + 0 \\
&= \int_{-\infty}^{+\infty} s_2(T_0 - \tau)\, h(\tau)\, d\tau
\end{aligned}
$$

Expectation of Decision Statistics V_2 II

- As summary we have:

$$E[\,V_2(T_0)\,] = \int_{-\infty}^{+\infty} s_2(T_0 - \tau)\, h(\tau)\, d\tau \tag{27}$$

Equation (27) is a generic equation:

  - We have a general system h
  - A general sampling time T_0 to choose from
  - A general shape of the transmitted signal $s_2(t)$

Variance of Decision Statistics V_2 I

$$\begin{aligned}
\mathrm{Var}[\,V_2(T_0)\,] &= E\left[\,(\,V_2(T_0)\,-\,E[\,V_2(T_0)\,]\,)^2\,\right] \\
&= E[\,(\,s_{2_{\text{filtered}}}(t\,=\,T_0)\,+\,n_{\text{filtered}}(t\,=\,T_0)\,- \\
&\quad s_{2_{\text{filtered}}}(t\,=\,T_0)\,]\,)^2\,] \\
&= E\left[\,(\,n_{\text{filtered}}(t\,=\,T_0)\,)^2\,\right]
\end{aligned}$$

- As always, the variance is a function of the noise $n_{\text{filtered}}(t = T_0)$, and has nothing to do with the transmitted signals $s_2(t)$
- Now the expression $E\left[\,(\,n_{\text{filtered}}(t = T_0)\,)^2\,\right]$ is of the form $E[X^2]$ for a random variable X

Variance of Decision Statistics V_2 II

- We can do also the same steps when we did the computation for $\text{Var}(V_2)$ in the integration-and-dump rx and decompose the integral to an multiplication of integrals $\leftrightarrow \displaystyle\int \times \int$
  - But we will skip the computation for now
  - To be done later
- As summary we have as final answer:

$$\text{Var}[\, V_2(T_0)\,] \;=\; R_{n_{\text{filtered}}}(\tau \;=\; 0) \tag{28}$$

Conditional Distribution Plots

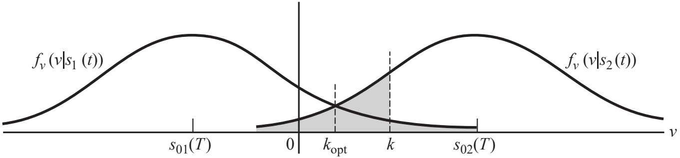


Figure 6: Conditional probability density functions of the filter output at time $t\,T_0$

- Since we have 2 signals to choose from, 1 threshold line is enough to decide which signal I have

Probability of Error

- Once I have decided my threshold line where, the next step is to compute the $P(\text{Error})$
  - For now, we skip the problem of finding the optimum value for the threshold k
  - We do it in next sections

References I