Theorem (Cauchy Test or Cauchy condensation test) : If $a_n \geq 0$ and $a_{n+1} \leq a_n \forall$ n, then

$\Sigma_{n=1}^{\inf} a_n$converges if and only if $\Sigma_{k=0}^{\inf} 2^k a_{2k}$ converges.

## Proofs :-

Let $S_n = a_1 + a_2 + \cdots\cdots + a_n$

and $T_k = a_1 + 2a_2 + \cdots\cdots + 2^k a_{2^k}$

Suppose (T$_k$)converges.For a fixed n, choose k such that $2^k \geq n$.Then

$$S_n = a_1 + a_2 + \cdots\cdots + a_n$$
$$\leq a_1 + (a_2 + a_3) + \cdots\cdots + (a_{2^k} + \cdots\cdots + a_{2^{k+1}-1})$$
$$\leq a_1 + 2a_2 + \cdots\cdots + 2^k a_{2^k}$$
$$= T_k$$

This shows that (S$_n$) is bounded above; hence(S$_n$) converges.

Suppose (S$_n$) converges. For a fixed k, choose n such that $n \geq 2^k$ Then

$$S_n = a_1 + a_2 + \cdots\cdots + a_n$$

$$\geq a_1 + a_2 + (a_3 + a_4) + \cdots\cdots (a_{2^{k-1}+1} + \cdots + a_{2^k})$$
$$\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots\cdots + 2^{k-1}a_{2^k}$$
$$= T_k$$

This shows that ($T_{k)}$) is bounded above; hence ($T_k$) converges.