MNIT Assignment1

RANJAN BARO

Latex Assignment

1. The basic for introduction, P_1 , is true, since $1^3 = 1^2$, now consider the induction step, assuming P_{n} is true and examining $P_{\mathrm{n+1}}$, by making use of result $(1+2+\cdots+n) = n(n+1)/2$ that is given as an example result in section 1, we see that

$$\begin{aligned}
1^3 + 2^3 + \cdots n^3 + (n+1)^3 &= (1^3 + 2^3 + \cdots n^3) + (n+1)^3 \\
&= (1^3 + 2^3 + \cdots n)^2 + (n+1)^3 \\
&= \left(\frac{n(n+1}{4}\right)^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{(n+1)^2}{4}(n^2+4n+4) \\
&= \frac{(n+1^2)(n+2)^2}{4} \\
&= (1+2+\cdots+n+(n+1))2
\end{aligned}$$

and thus $P_{\mathrm{n}}+1$ is satisfied. Hence, by the principle of mathematical induction, P_{n} is true for all $n \in \mathbb{N}$

2. (a) For this exercise, P_2 is treated as the basic for induction. We see that $22 = 4 > 2+1 = 3$, so P_2 is true. Now consider the induction step, assuming P_{n} is true and examining $P_{\mathrm{n+1}}$:

$$(n+1)^2 = n^2 + 2n + 1 > (n+1) + 2n + 1 > (n+1) + 1$$

Hence if P_{n} holds then $P_{\mathrm{n+1}}$ holds. By the principle of mathematical induction, P_{n} is true for all $n \in \mathbb{N}$ with $n \geq 2$.

(b) Here, P_4 is treated as the basic for induction. We see that $4! = 24 \geq 42 = 16$ so P_4 is true. Now consider the induction step, assuming P_{n} is true and examining $P_{\mathrm{n+1}}$:

$$(n+1)! = n!(n+1) > n^2(n+1).$$

Now, apply the result from (a), which is true for $n > 2$ and thus valid for all cases under consideration:

$$(n+1)^2 = n^2 + 2n + 1 > (n+1) + 2n + 1 > (n+1) + 1$$

Hence if P_{n} holds then $P_{\mathrm{n+1}}$ holds. By the principle of mathematical induction, P_{n} is true for all $n \in \mathbb{N}$ with $n \geq 4$.

3. (a) The golden ratio satisfies the relation

$$\varphi^2 = \frac{(1+\sqrt{5})^2}{4} = \frac{(6+2\sqrt{5})}{4} = 1 + \frac{(1+\sqrt{5})}{2} = 1 + \varphi$$

(b) first consider the induction hypothesis H_1:

$$f(0) = \frac{\varphi^0 - (1-\varphi)}{\sqrt{5}} = 0, \qquad f(1) = \frac{\varphi - (1-\varphi)}{\sqrt{5}} = \frac{2\varphi - 1}{\sqrt{5}} = 1$$

and hence H_1 is true. Now consider the induction step. Suppose H_{n} is true, and consider $H_{\mathrm{n+1}}$. Then $F_{\mathrm{n}} = f_{(\mathrm{n})}$ and

$$\begin{aligned} F_{n+1} &= F_n + F_{n+1} \\ &= \frac{\varphi^n - (1-\varphi)^n + \varphi^{n-1} - (1-\varphi)^{n-1}}{\sqrt{5}} \\ &= \frac{\varphi^{n-1}(1+\varphi) - (1-\varphi)^{n-1}(1+(1-\varphi))}{\sqrt{5}} \\ &= \frac{\varphi^{n-1}(\varphi^2) - (1-\varphi)^{n-1}(1-\varphi)}{\sqrt{5}} \end{aligned}$$

Since $(1-\varphi)^2 = 1 - 2\varphi + \varphi^2 = 1 - 2\varphi + (1+\varphi) = 2 - \varphi$, it follows that

$$\begin{aligned} F_{n+1} &= \frac{\varphi^{n-1}(\varphi^2) - (1-\varphi)^{n-1}(1-\varphi)^2}{\sqrt{5}} \\ &= \frac{\varphi^{n+1} - (1-\varphi)^{n-1}}{\sqrt{5}} \\ &= f_{n+1} \end{aligned}$$

So $H_n + 1$ is true. Hence by mathematical induction H_n is true for all n, so $F_n = f(n)$ for all $n \in \mathbb{N} \cup 0$.

4. To begin, search for a polynomial that is satisfied by $x = (5 - \sqrt{3})^{1/3}$:

$$\begin{aligned} x^3 &= 5 - \sqrt{3} \\ x^3 - 5 &= -\sqrt{3} \\ (x^3 - 5)^2 &= 3 \\ (x^6 - 10x^3 + 22) &= 0 \end{aligned}$$

Now suppose that is x is rational, so that it can be written as $x = p/q$ where $p, q \in Z$ and $q \neq 0$ with p and q co-prime. Then by the rational zeroes theorem, p divides 22 and q divides 1. Hence the only possible solutions are ± 1, ± 2, ± 11, and ± 22. The table below shows that none of these values satisfy the equation.

x	1	2	11	22
$x^6 - 10x^3 + 22$	13	6	1,758,273	113,273,446
$(-x)^6 - 10(-x)^3 + 22$	33	166	1,784,893	113,486,406

5. To begin, we find a polynomial that is satisfied by $x = \sqrt{2} + \sqrt{3}$:

$$\begin{aligned} x^2 &= (\sqrt{2} + \sqrt{3})^2 \\ x^2 &= 5 + 2\sqrt{2}\sqrt{3} \\ (x^2 - 5) &= 2\sqrt{2}\sqrt{3} \\ (x^2 - 5)^2 &= 24 \\ (x^4 - 10x^2 + 25) &= 24 \\ (x^4 - 10x^2 + 1) &= 0 \end{aligned}$$

By the application of the rational zeroes theorem, if $x = p/q$ with p and q coprime, then p divides 1 and q divides 1. Hence the only possible rational solutions to this polynomial are ± 1. However

$$\begin{aligned} 1^4 - 10(1)^2 + 1 &= -8 \neq 0 \\ (1^4 - 10(-1)^2 + 1) &= -8 \neq 0 \end{aligned}$$

and since neither possibility satisfies the equation, $\sqrt{2} + \sqrt{3}$ is irrational.

6. We start by considering the statement $0 < 1$. Suppose that this was untrue, so that $1 \geq 0$. Then by applying axiom O4, we know that

$$\begin{aligned} 1 + (-1) &\leq 0 + (-1) \\ 0 &\leq -1 \end{aligned}$$

Now, apply axiom O5, using $a = 0$, $b = -1$ and $c = -1$ to show that $0 \leq (-1) \cdot (-1)$. Then by making use of Theorem 3.1(iv), we know that

$$0 \leq 1 \cdot 1 \;=\; 1$$

But now $0 \neq 1$ and $0 \geq 1$, so by axiom O2, $0 = 1$. Since zero and one are distinct elements, this is a contradiction, and thus $0 < 1$.

Why is $0 \leq 1$? In the textbook's treatment, the rational numbers $\mathbb{Q}$ are assumed to exist and then the axioms A1–A4, M1–M4, and DL are subsequently introduced to create the field structure. If the numbers 0 and 1 are constructed apriori as elements of $\mathbb{Q}$ then it is safe to assume that they are distinct objects. In other presentations of the field axioms, which describe them in the abstract and not in relation to an already given set, it is assumed that the additive and multiplicative inverses are unique, which would immediately disallow $0 = 1$. Regardless, it is clear that $0 = 1$ leads to catastrophic problems: using Theorem 3.1(i), we would see that for any element a,

$$a = a \cdot 1 = a \cdot 0 = 0$$

implying that all elements are zero.
We now wish to show that for all non-zero a, $b \in \mathbb{R}$, $0 < a < b$ implies $0 < b^{-1} < a^{-1}$. By the transitivity axiom O3, we know that $0 \leq a$ and $a \leq b$ implies that $0 \leq b$, and since b is non-zero, $0 < b$. Thus by Theorem 3.2(vi), $0 < b^{-1}$. Now suppose that $b^{-1} \geq a^{-1}$. Since $b > 0$, axiom O5 can be applied to show that

$$\begin{aligned} b \cdot b^{-1} &\geq b \cdot a^{-1} \\ 1 &\geq b \cdot a^{-1} \end{aligned}$$

and since $a > 0$, axiom O5 can be applied to show that

$$\begin{aligned} a \cdot 1 &\geq a \cdot b \cdot a^{-1} \\ a &\geq b. \end{aligned}$$

This is a contradiction, and hence $b^{-1} < a^{-1}$. Thus $0 < b^{-1} < a^{-1}$.

7. (a) First suppose that $-a \leq b \leq a$. Then either $|b| = b$ in which case $|b| \leq a$, or $|b| = -b$ in which case $-|b| \geq -a$ and thus $|b| \leq a$. Thus, $|b| \leq a$ in all cases. Now suppose that $|b| \leq a$. Since $|b| \geq 0$ for all b, it follows that $a \geq 0$ by transitivity. If $b \geq 0$, then $|b| = b$ and hence $b \leq a$. In addition $b \geq 0 \geq -a$, and hence $-a \leq b \leq a$. If $b \leq 0$, then $|b| = -b$, and hence $-b \leq a$, so $-a \leq b$. In addition, $b \leq 0 \leq a$, and thus $b \leq a$ also. Thus $-a \leq b \leq a$ in all cases.

   (b) The triangle inequality states that for all c, $d \in \mathbb{R}$,

$$|c| + |d| \leq |c + d|.$$

For any two numbers a, $b \in \mathbb{R}$, substituting $c = a - b$ and $d = b$ into the triangle inequality gives

$$\begin{aligned} |a - b| + |b| &\geq |a - b + b| \\ |a - b| &\geq |a| - |b|. \end{aligned}$$

Similarly, a substituting of $c = a$ and $d = b - a$ gives

$$\begin{aligned} |a| + |b - a| &\geq |a + b - a| \\ |b - a| &\geq |b| - |a|. \end{aligned}$$

Applying Theorem 3.2(i) to this expression gives $-|b - a| \leq |a| - |b|$. Also, $|b - a| = |a - b|$, and hence

$$-|a - b| \leq |a| - |b| \leq |a - b|. \tag{1}$$

By using the result from part (a), it follows that

$$||a| - |b|| \leq |a - b|.$$

8. How to include figure.

Figure 1: hawa mahal palace