Why use fftshift(fft(fftshift(x))) instead of fft(x) in Matlab?

Wu, Kan

Oct. 1 2009

fft(x) has some intrinsic problem due to its realization. Here is an example.

Consider a Gaussian pulse in time domain

$$x(t) = e^{-\frac{t^2}{2A^2}}$$

Its Fourier transform is

$$\begin{aligned} X(f) &= \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2A^2} - j2\pi ft} dt \\ &= \int_{-\infty}^{+\infty} e^{-(\frac{t}{\sqrt{2}A} + j\sqrt{2}\pi Af)^2 - 2\pi^2 A^2 f^2} dt \\ &= \sqrt{2\pi} A e^{-2\pi^2 A^2 f^2} \end{aligned}$$

It is real and Gaussian.

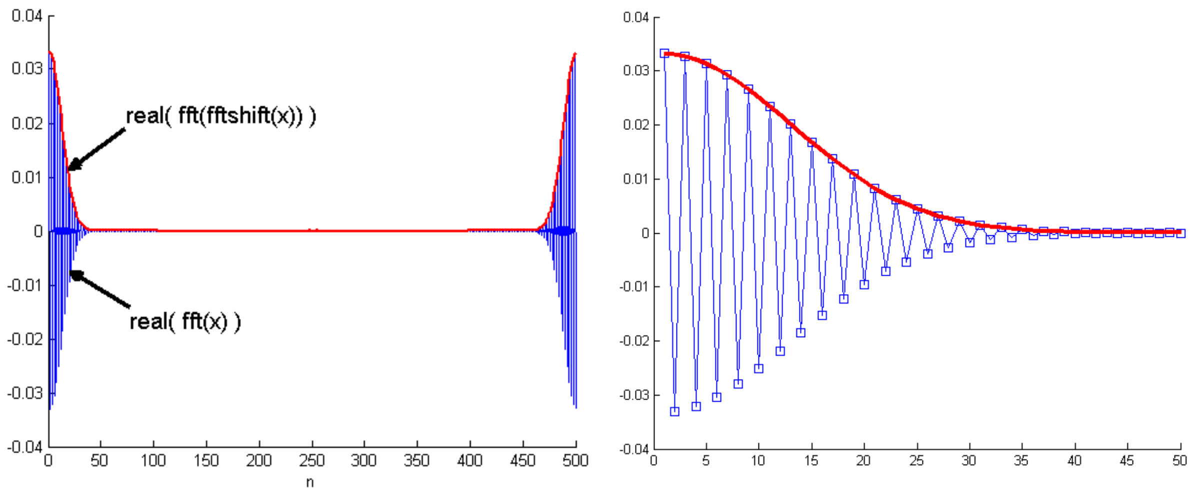
Codes in Matlab are:

```
%------------------------------------------------------------------
Bx = 10;
A = sqrt(log(2))/(2*pi*Bx);
fs = 500;           %sampling frequency
dt = 1/fs;          %time step
T=1;                %total time window
t = -T/2:dt:T/2-dt;     %time grids
df = 1/T;               %freq step
Fmax = 1/2/dt;          %freq window
f=-Fmax:df:Fmax-df;     %freq grids, not used in our examples, could be used by plot(f, X)
%------------------------------------------------------------------
% Numerical evaluations
%------------------------------------------------------------------
```

```
x = exp(-t.^2/2/A^2);

Xan = A*sqrt(2*pi)*exp(-2*pi^2*f.^2*A^2);      %X(f), analytical Fourier transform of x(t), real

Xfft = dt * fft(x);           %directly using fft()

Xfftshift = dt * fft(fftshift(x));    %using fftshift() before fft()

Xfinal = dt * fftshift(fft(fftshift(x)));       %identical with analytical X(f), also note dt

%------------------------------------------------------------------

hold on

plot(…)

plot(…)

%------------------------------------------------------------------
```

- fftshift() is a function to swap the left half and right half of a vector. e.g., x=[1 2 3 4 5 6] then fftshift(x)=[4 5 6 1 2 3]
- dt is to scale the amplitude of X (e.g. Xfft, Xfftshift, Xfinal) equal to its analytical Fourier transform X(f)

Real part of Xfft=dt * fft(x) is plotted in Fig.1(a). There exists obvious oscillation. But if we swap the left half and right half of x(t) before fft(), then oscillation disappears, as real(Xfftshift) in Fig.1(a). A zoomed region of left side of Fig.1(a) is shown in Fig.1(b). It is clear to shown that the real part of Xfft is periodically multiplied by 1,-1,1,-1…compared with Xfftshift. Also note that this oscillation could not be found if we merely observe its absolute value as shown in Fig.2. Compared with analytical Fourier transform X(f), a smooth curve real(fft(fftshift(x))) is correct.



(a) (b)

Fig. 1 Real part of fft(x) and fft(fftshift(x))

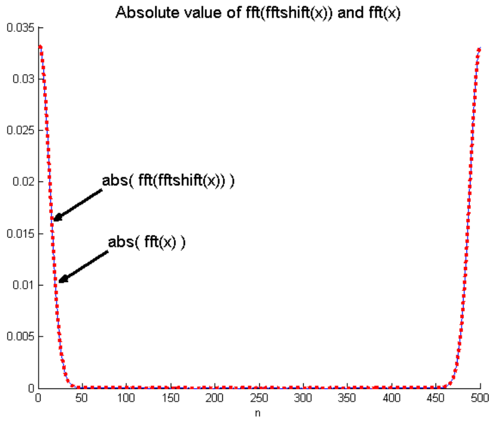
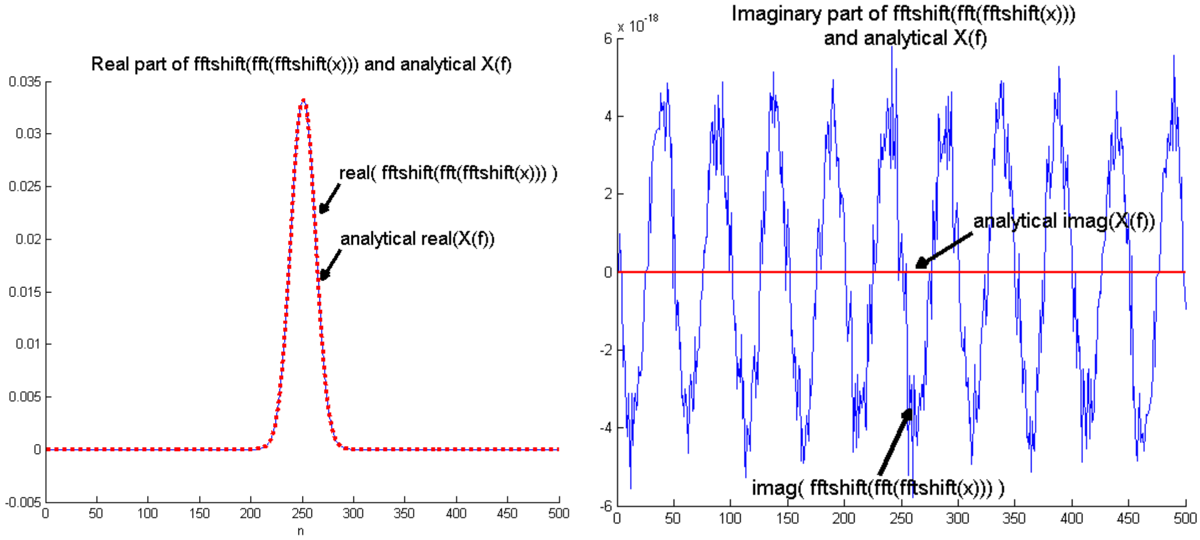


Fig. 2 Absolute value of fft(x) and fft(fftshift(x))

Then we consider second step: since fft() is intrinsically single sided, the f=0 point is at left side of horizontal axis and the right half is just the mirror of left half. For double side case, we need to move the f=0 point to the center of horizontal axis by swapping the right half and left half of the data. This leads to:

Xfinal = fftshift(Xfftshift) = dt * fftshift(fft(fftshift(x))); %don't forget to multiply dt!

As plotted in Fig.3, both real and imaginary part of Xfinal well fit the analytical Fourier transform X(f). The error shown in imaginary part is due to the numerical error and very small.



(a) (b)

Fig. 3 Real and imaginary part of fftshift(fft(fftshift(x))) and analytical X(f)

Finally, we have the formula for FFT and IFFT in Matlab:

FFT: fftshift(fft(fftshift(x))) * dt

IFFT: fftshift(ifft(fftshift(X))) * fs %fs = 1/dt

For other language, we need to multiply IFFT by 1/N based on the theory of discrete Fourier transform, which is done by Matlab automatically. E.g. type “help fft” in Matlab command window, we would find that:

```
                N
      X(k) = sum  x(n)*exp(-j*2*pi*(k-1)*(n-1)/N), 1 <= k <= N
               n=1
                      N
      x(n) = (1/N) sum  X(k)*exp( j*2*pi*(k-1)*(n-1)/N), 1 <= n <= N
                     k=1
```

Acknowledge

This little note is based on Daniele Disco ‘s work in *A guide to the Fast Fourier Transform, 2nd Edition*. More details of fft() and Matlab codes could be found in the following link: http://www.mathworks.com/matlabcentral/fileexchange/5654