A guide to the Fast Fourier Transform
2nd Edition

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1 Preface to the 2nd edition

Three years ago, after the submission to The MathWorks site, I always had like a background process running in my mind that was looking for a demonstration or justification for the formula proposed for the use of the *FFT*.

This is until few days ago when, also thanks to discussions with a colleague of mine, Maurizio Crozzoli, I found the solution. This new edition tries to correct the inaccurate advices put in the first edition of this guide. Please accept my apologies and believe me: I have done all in good faith.

I have also to thank all the reviewers for theirs enthusiasm and gratitude and for suggestions to increase the quality of this guide: thank you very much!

2 Getting start with *FFT*

When a student faces for the first time the Fast Fourier Transform (*FFT*), to verify if obtains correct results, a common approach is to search tables of Fourier transform where are reported analytic expressions for signals and their transforms and try to obtain the same results with the *FFT*. These experiments can produce positive or negative results and sometime some signals produce correct results and other no. In the following I will try to clarify this strange situation.

The numerical algorithm of the *Discrete Fourier Transform* and his optimization, the *Fast Fourier Transform*, work correctly over an asymmetrical time domain from 0 to T.

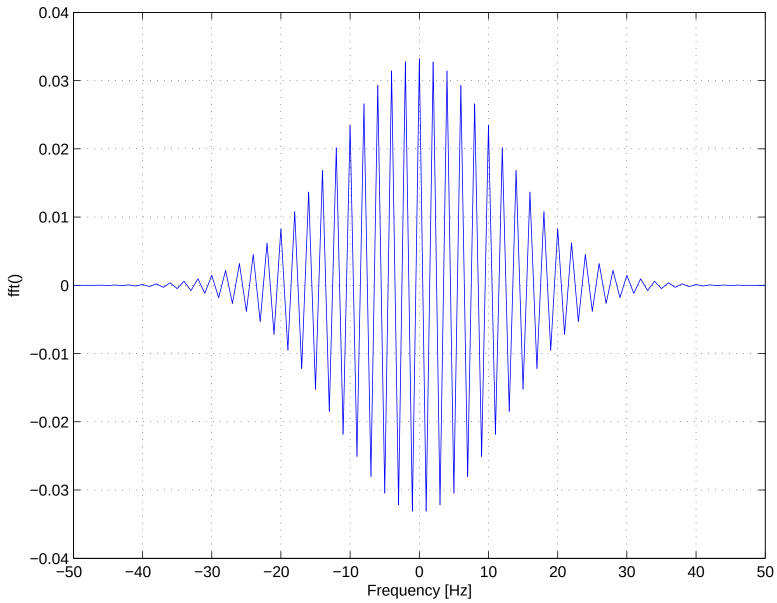

Figure 1: Strange output of the FFT

In the following $x(t)$ is the time domain signal to be transform; $X(f)$ his Fourier Transform (FT) in the frequency domain:

$$X(f) = \mathcal{F}\left\{x(t)\right\} \tag{1}$$

$$x(t) = \mathcal{F}^{-1}\left\{X(f)\right\} \tag{2}$$

and, if the time domain is chosen in the right way, it is possible to verify some properties of the FT like

$$x(t) \in \mathbb{R} \Rightarrow \mathrm{Re}\left\{X(f)\right\} = \mathrm{Re}\left\{X(-f)\right\} \tag{3}$$

$$x(t) \in \mathbb{R} \Rightarrow \mathrm{Im}\left\{X(f)\right\} = -\mathrm{Im}\left\{X(-f)\right\} \tag{4}$$

$$x(t) \in \mathbb{R} \Rightarrow |X(f)| = |X(-f)| \tag{5}$$

$$x(t) \in \mathbb{R} \Rightarrow \angle X(f) = -\angle X(-f) \tag{6}$$

and:

$$\left\{ \begin{array}{l} x(t) \in \mathbb{R} \\ x(t) = x(-t) \end{array} \right. \Rightarrow \left\{ \begin{array}{l} X(f) \in \mathbb{R} \\ X(f) = X(-f). \end{array} \right. \tag{7}$$

So if you try to transform signal like a rectangular pulse or a Gaussian, you obtain strange zigzag results not compliant to the theory (see Figure 1). A possible solution of this problem is the formula proposed in the first edition:

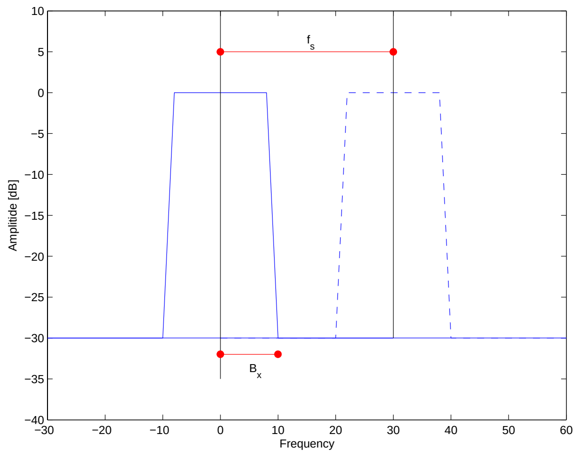

Figure 2: Right spectrum

- `fftshift(fft(fftshift(·)))*dt`
- `ifftshift(ifft(ifftshift(·)))*fs`[1]

Before to solve the question some generalities for FFT.

The Fast Fourier Transform suppose that the input $x(t)$ is a representation of a periodic signal in his fundamental period $(0, T)$ moreover, due to the nature of sampled signal, in the frequency domain you will see replica of the spectrum of $X(f)$ spaced of the sampled frequency f_s (see Figure 2).

For this reason the bandwidth of the signal (B_x) is fundamental to define in a right way the sampling frequency.[2] In fact, if f_s is less then $2B_x$ the spectrum is affected by the well know phenomenon of the aliasing how reported in the Figure 3

Now few rules to apply correctly a FFT.

- The time domain *must* be chosen between 0 and T;
- $f_s \geq 2B_x$;
- the sampling time $dt = 1/f_s$;
- the maximum time interval is $(N-1)dt = T$;

[1] Thanks to Renato Scotti that showed me the presence of ifftshift

[2] By definition, B_x is the frequency value where the *normalized amplitude* of $|X(f)|$ is below a predefined value. For this parameter, common choice is $\sqrt{0.5}$ that, expressed in logarithmic units, is equal to -3dB. In this case B_x is called the bandwidth at 3dB. B_x is in relation with the sampling frequency (f_s) according to Nyquist theorem: $f_s \geq 2B_x$

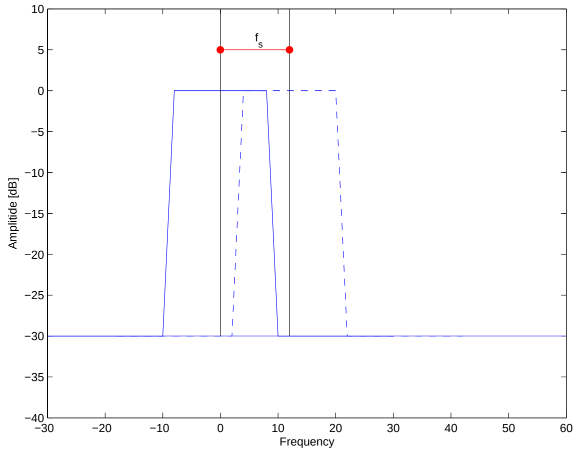

Figure 3: Aliasing

- the resolution frequency is $df = 1/T$;
- the extremes of the spectrum are $\pm F_{max} = 1/(2dt)$
- the freqeuncy domain is $f = -F_{max} : df : F_{max}$.

Arrived to this point, there are all the elements to solve the problems of the zigzag output. The mistake is the time domain symmetrical. For the FFT is like to apply a negative time translation of $T/2$. For the properties of the Fourier Trasform:

$$x(t+\alpha) \Rightarrow X(f)\exp(\mathrm{j}2\pi f\alpha) \tag{8}$$

Analyzing the exponential for $\alpha = T/2$:

$$\exp(\mathrm{j}\pi fT) = \exp\left[\mathrm{j}\pi\left(-F_{max} : df : F_{max}\right)T\right] \tag{9}$$

$$\exp(\mathrm{j}\pi fT) = \exp\left[\mathrm{j}\pi\left(-\frac{1}{2dt} : \frac{1}{T} : \frac{1}{2dt}\right)T\right] \tag{10}$$

$$\exp(\mathrm{j}\pi fT) = \exp\left[\mathrm{j}\pi\left(-\frac{N-1}{2} : 1 : \frac{N-1}{2}\right)\right] \tag{11}$$

This expoential, if N is even, is a sequence of $+$j, $-$j, $+$j etc. else if N is odd is a sequence of +1, -1, +1 etc. Looking now the Figure 1 it is possible to observe this alternance of a sequence of + and - that produce this zigzag behavior.

The use of `fftshift` for the input signal is equivalent to a shift of $T/2$ (remember the periodic sequence) that erases this sequence of + and - ones.

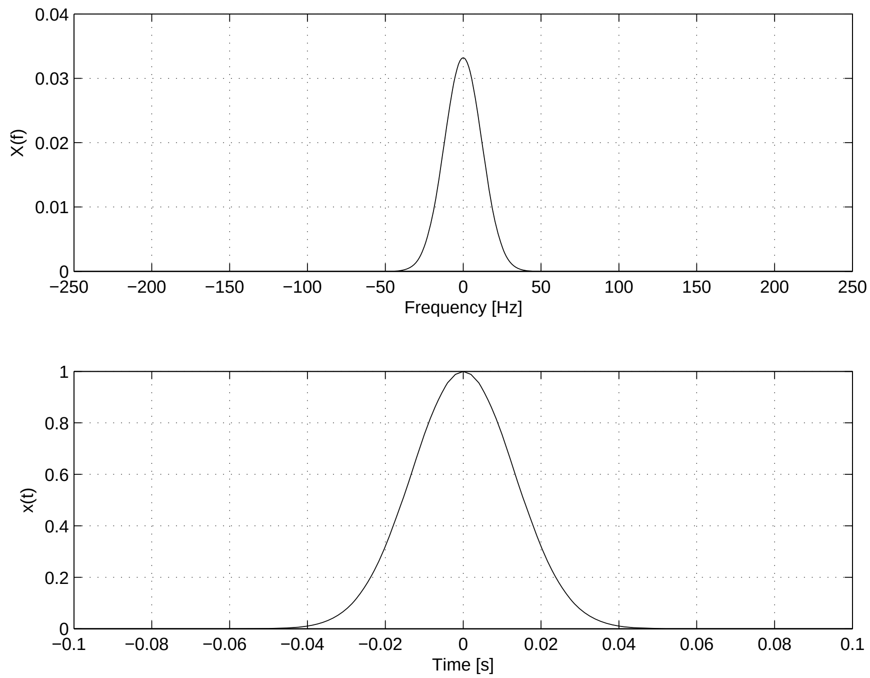

Figure 4: The functions test

Depending also from N (even or odd) the function instead of being real becomes imaginary, sometime violating properties before listed (eq.7).

In the following, three examples will be reported: the first one is a Gaussin with a time domain symmetric; the second one is a Raylaigh with a time domain asymmetric and the third one is cosine function, example of periodic function. For these cases it will be shown how to apply the FFT and, in the last one, the little difference to obtain theoretical results. This technique can be also used to delay a signal of any interval, without constrains due to the sampling time.

For each case, after the introduction of the test function and its analytical Fourier transform, the parameter A related to the bandwidth B_x of the spectrum ($|X(f)|$) will be evaluated.

To check the algorithm, plots with numerical and analytic approach are reported: $\mathcal{F}[x(t)]$ and $X(f)$; $\mathcal{F}^{-1}[X(f)]\}$ and $x(t)$. The analytic and numerical Fourier transform of a delayed version of $x(t)$ and the reconstruction of $x(t-\tau_n)$ (with τ_n equal to $25.5dt$) is also reported.

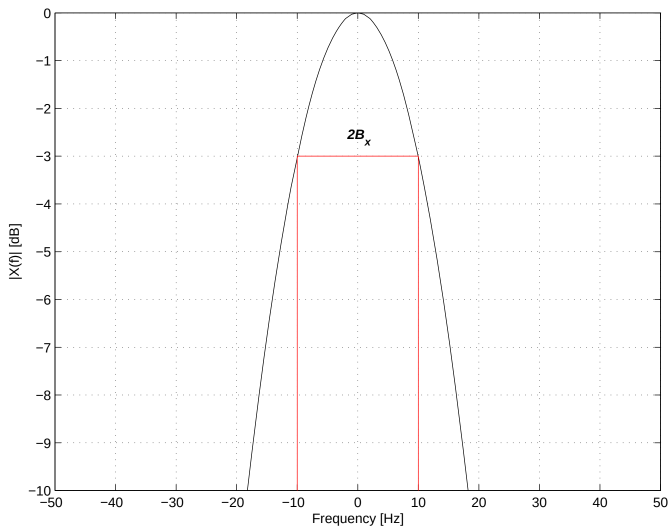

Figure 5: 3dB Bandwidth

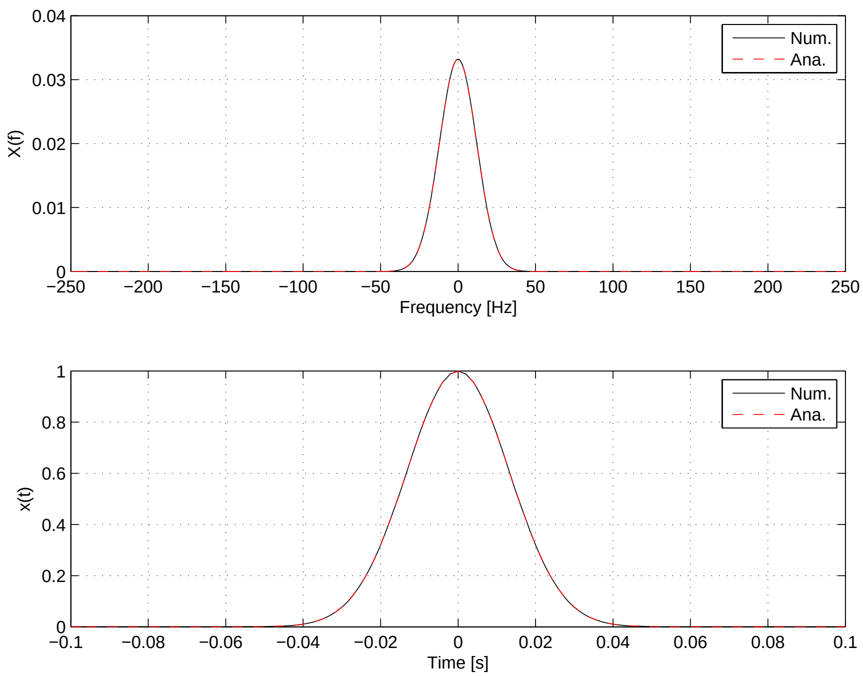

Figure 6: Numerical vs Analytical results

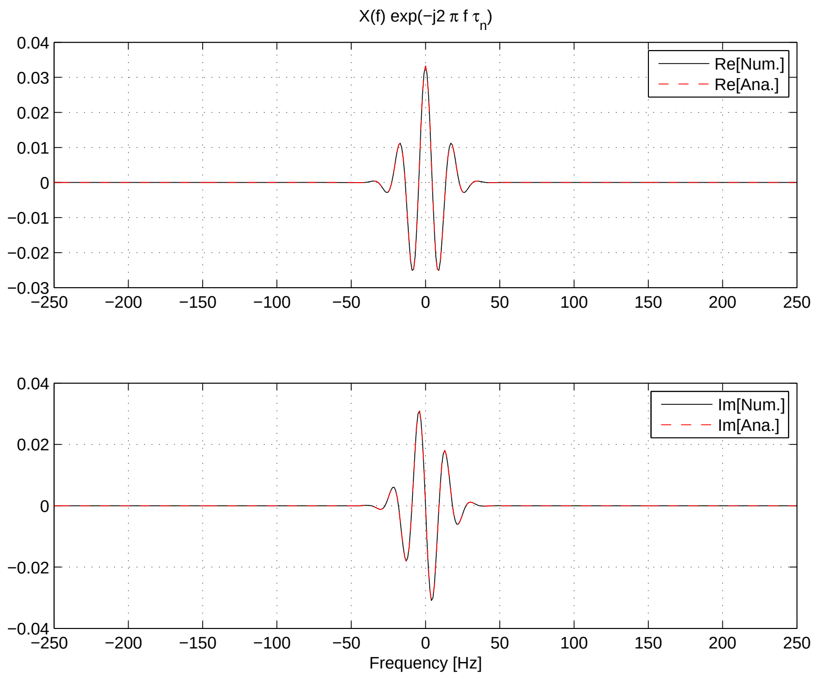

Figure 7: Fourier Transform of $x(t - \tau_n)$

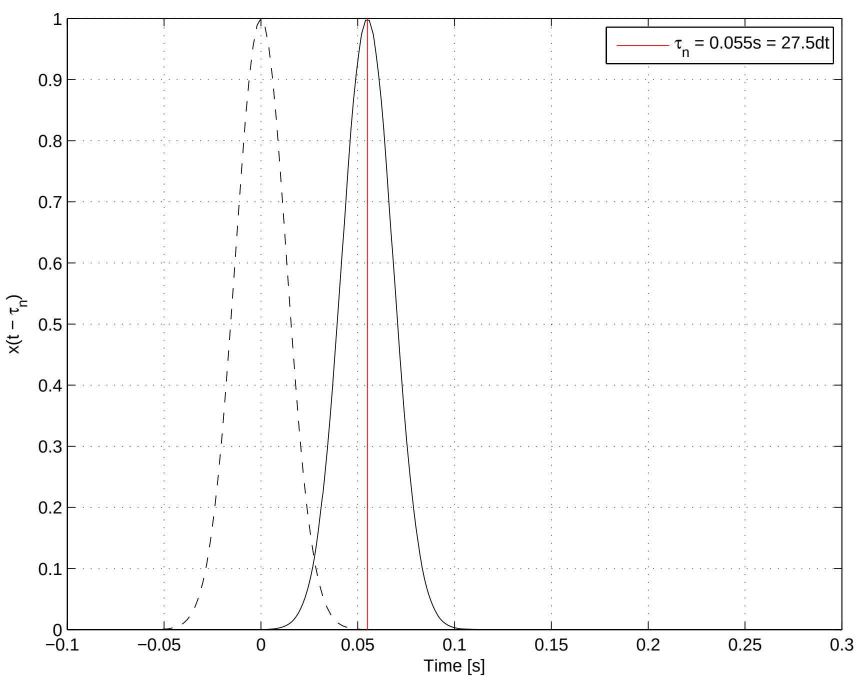

Figure 8: $x(t - \tau_n)$

3 Signal $x(t)$ with symmetric time domain

The test function is:

$$x(t) = \exp\left(-\frac{t^2}{2A^2}\right) \tag{12}$$

and the analytic FT is (fig.4)

$$X(f) = A\sqrt{2\pi}\exp\left(-2\pi^2 A^2 f^2\right) \tag{13}$$

Neglecting the term $A\sqrt{2\pi}$ to obtain the normalized expression:

$$X_N(f) = \exp\left(-2\pi^2 A^2 f^2\right) \tag{14}$$

$$\exp\left(-2\pi^2 A^2 {B_x}^2\right) = \frac{1}{\sqrt{2}} \tag{15}$$

$$\log\left[\exp\left(-2\pi^2 A^2 {B_x}^2\right)\right] = \log\left(\frac{1}{\sqrt{2}}\right) \tag{16}$$

$$-2\pi^2 A^2 {B_x}^2 = \frac{1}{2}\log 2 \tag{17}$$

$$B_x = \frac{\sqrt{\log 2}}{2\pi A}. \tag{18}$$

The (18) is useful to set the variable A (fig.5). In the example, the parameters are:

$$\begin{aligned} &B_x = 10\,\mathrm{Hz} \\ &f_s = 500\,\mathrm{Hz} \\ &T = 1\,\mathrm{s} \\ &dt = f_s^{-1} = 2\,\mathrm{ms} \\ &df = T^{-1} = 1\,\mathrm{Hz} \\ &F_{max} = f_s/2 = 250\,\mathrm{Hz} \end{aligned} \tag{19}$$

where T is the temporal lenght of the signal $x(t)$, df the frequency resolution for the Fourier Transform $X(f)$ represented between $-F_{max}$ and F_{max}. The sampling frequency is much higher respect to the Nyquist frequency only for representation problems. The signal under test is real an even and for these reasons $X(f)$ must be real and even.

The delayed case is reported in figures 7 and 8.

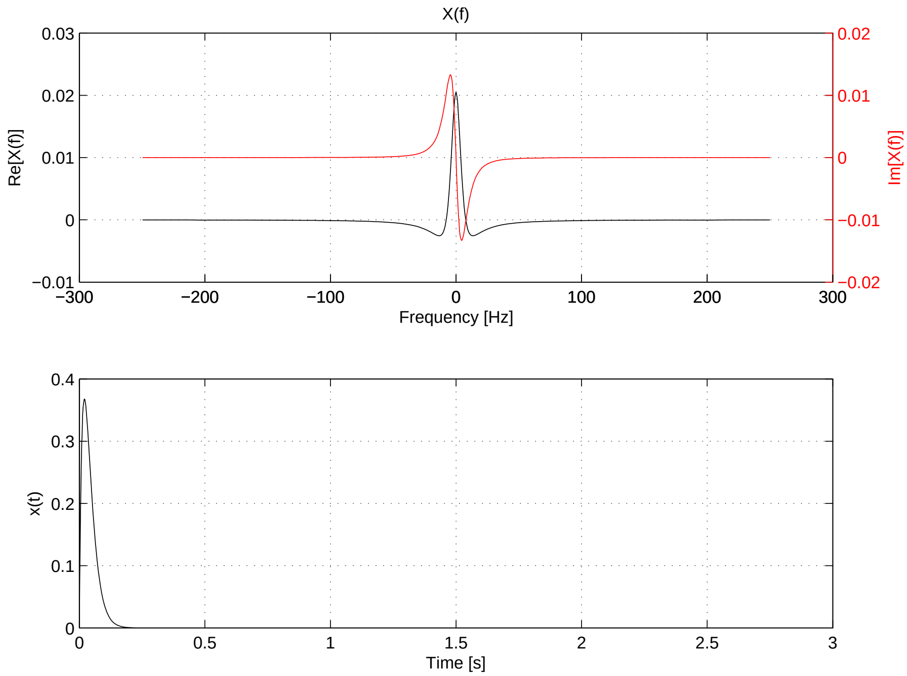

Figure 9: The functions test

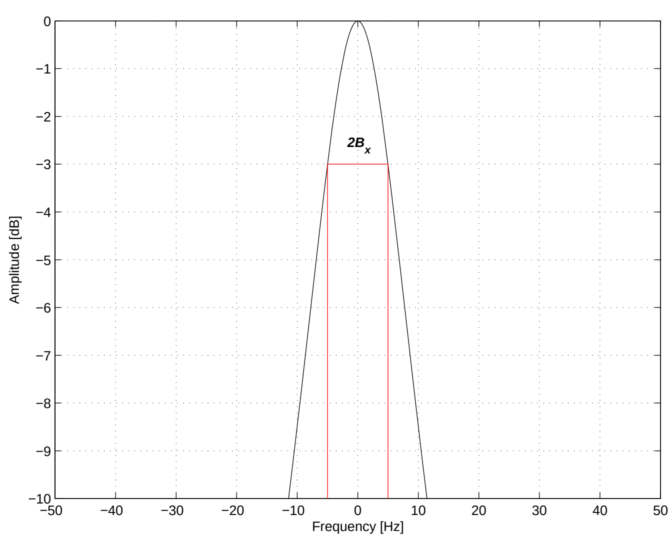

Figure 10: 3dB Bandwidth

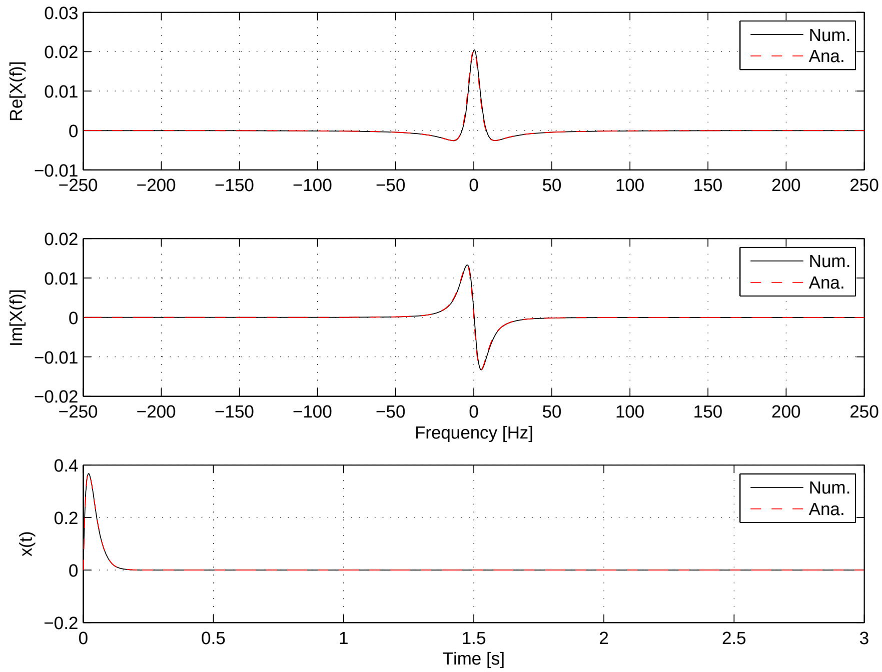

Figure 11: Numerical vs Analytical results

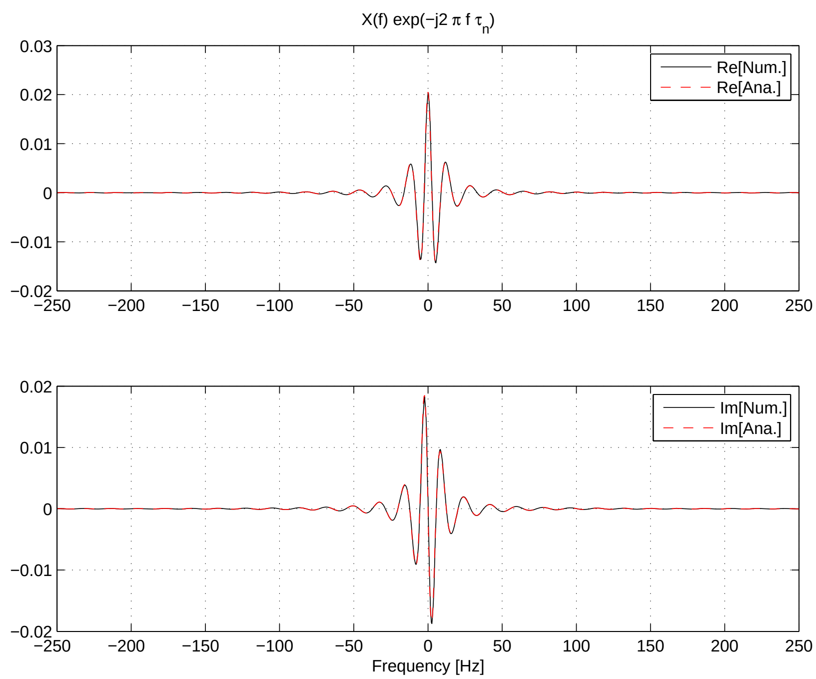

Figure 12: Fourier Transform of $x(t - \tau_n)$

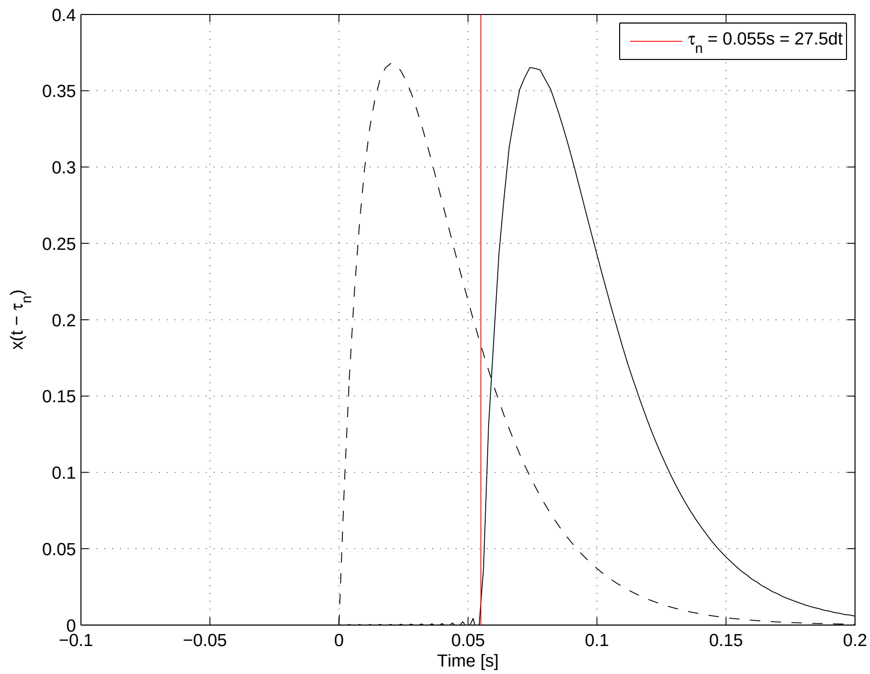

Figure 13: $x(t-\tau_n)$

4 Signal $x(t)$ with asymmetric time domain

The test function is

$$x(t) = Au(t)t\exp(-At) \tag{20}$$

with

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{21}$$

and the Fourier transform (fig.9)

$$X(f) = \frac{A}{(A + 2\mathrm{j}\pi f)^2}. \tag{22}$$

For the bandwidth B_x, starting form the normalized expression of the module of $X(f)$:

$$X_N(f) = \frac{A^2}{\sqrt{(A^2 - 4\pi^2 f^2)^2 + (4\pi A f)^2}} = \frac{1}{\sqrt{2}} \tag{23}$$

$$\frac{A^4}{\sqrt{(A^2 - 4\pi^2 f^2)^2 + 16\pi^2 A^2 f^2}} = \frac{1}{2} \tag{24}$$

and simplifying:

$$16\pi^4 f^4 + 8\pi^2 A^2 f^2 - A^2 = 0 \tag{25}$$

that has a *physical* solution for (fig.10):

$$B_x^2 = \frac{A^2\left(\sqrt{2}-1\right)}{4\pi^2} \tag{26}$$

and an expression for A is

$$A = \frac{2\pi B_x}{\sqrt{\left(\sqrt{2}-1\right)}} \tag{27}$$

The parameters for this case are:

$$\begin{aligned} &B_x = 5\,\mathrm{Hz} \\ &f_s = 500\,\mathrm{Hz} \\ &T = 3\,\mathrm{s} \\ &dt = f_s^{-1} = 2\,\mathrm{ms} \\ &df = T^{-1} = 1\,\mathrm{Hz} \\ &F_{max} = f_s/2 = 250\,\mathrm{Hz}. \end{aligned} \tag{28}$$

Respect to the previous case, the signal input for the *FFT*, must be represented in a symmetric domain: in other words, $x(t)$ will be:

$$x(t) = \begin{cases} 0 & -\dfrac{T}{2} \le t < 0 \\ At\exp(-At) & 0 \le t < \dfrac{T}{2} \end{cases} \tag{29}$$

The figures 11,12 and 13 demonstrate the validity of the approach.

5 Signal $x(t)$ periodic

The formula in this case become:

- `fftshift(fft(fftshift(·)))`$/N_p$
- `ifftshift(ifft(ifftshift(·)))*fs`

A possible interpretation of this modification is related to how to build the signal in the fondamental period. The start point is the periodic signal that is cutted in his fondamental period by a rect function of width T. This function in frequency is a $sinc(x) = sin(x)/x$ that is sampled in all his nulls

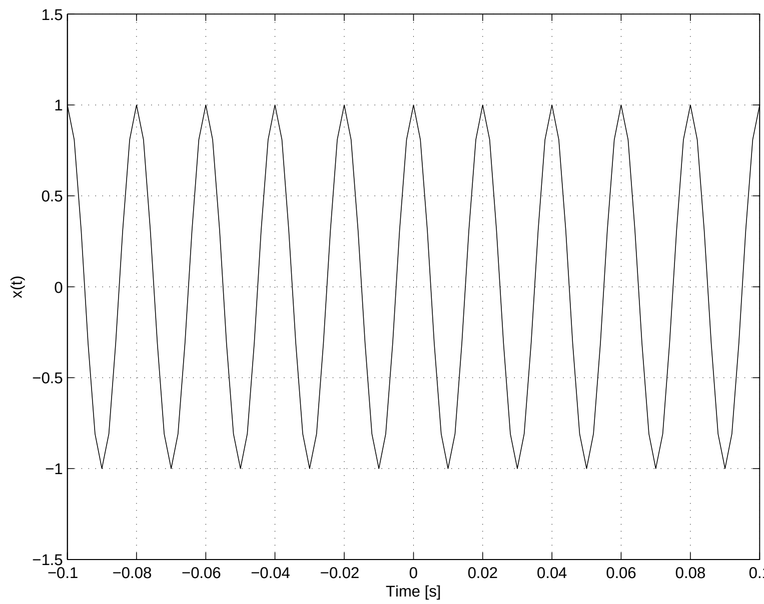

Figure 14: $x(t)$

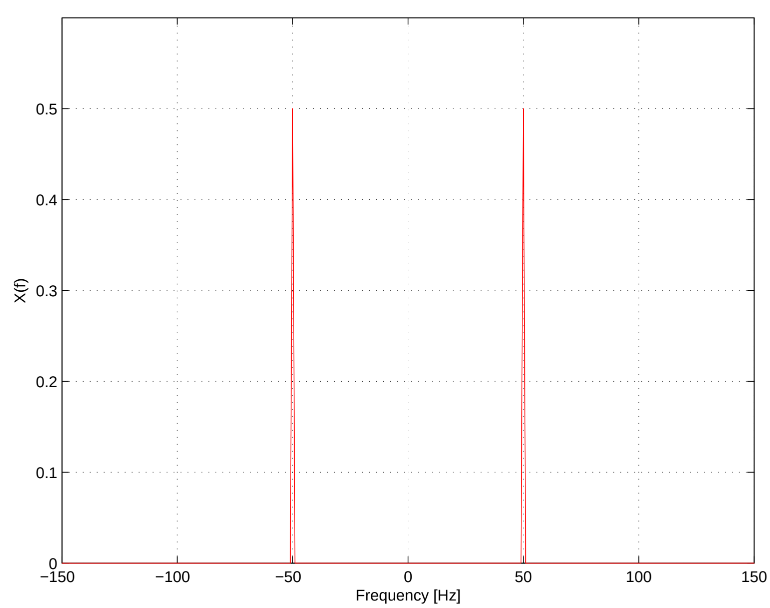

Figure 15: $X(f)$

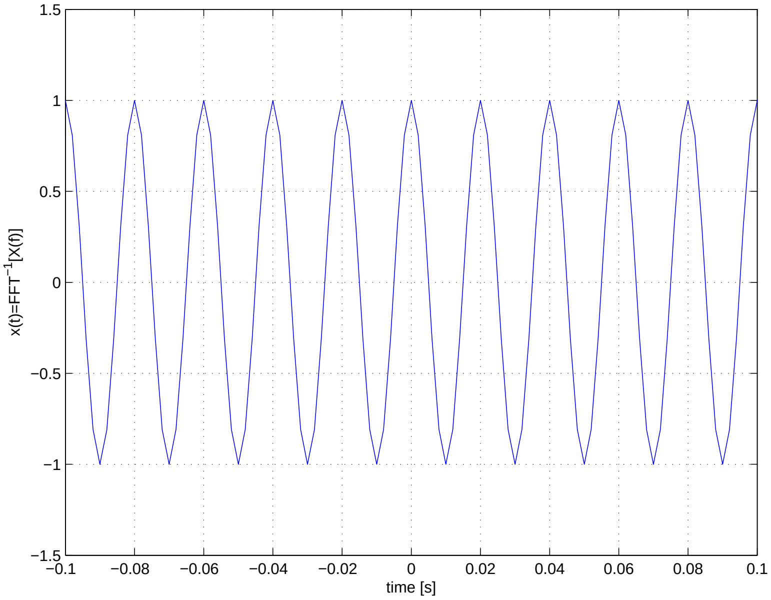

Figure 16: $x(t) = fft^{-1}[x(f)]$

and in the central point where ths *sinc* is equal to T. This last point his also where you can find the peaks of the periodic function. To restore the amplitude is necessary to divide by T and the original coefficient dt become $1/N$.

The example:

$$x(t) = \cos(2\pi f_0 t) \tag{30}$$

with analytic transform equal to:

$$X(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right] \tag{31}$$

Also in this case, the figure 14, 15 and 16 are a result of this method.

6 Conclusions

In this script a practical use of the fast Fourier transform has been reported. With this paper, there is also a script Matlab® file that realizes the numerical evaluations and the pictures reported in this document.

I hope someone can find this useful and in case of problems please, send me a feedback: arkkimede@gmail.com.

Have fun!