Assignment 2

Quantum Computing

Ranjima Mohan P (DS & AI)

Department of Computer Science
Cochin University of Science And Technology, Kochi

January 9, 2023

Outline

Quantum Teleportation

- Quantum teleportation is a technique for transferring quantum information from a sender at one location to a receiver some distance away. While teleportation is commonly portrayed in science fiction as a means to transfer physical objects from one location to the next, quantum teleportation only transfers quantum information.
- The sender does not have to know the particular quantum state being transferred. Moreover, the location of the recipient can be unknown, but to complete the quantum teleportation, classical information needs to be sent from sender to receiver.
- Because classical information needs to be sent, quantum teleportation cannot occur faster than the speed of light.

- Alice wants to send quantum information to Bob. Specifically, suppose she wants to send the qubit state $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$. This entails passing on information about α and β to Bob.
- There exists a theorem in quantum mechanics which states that you cannot simply make an exact copy of an unknown quantum state. This is known as the **no-cloning theorem**.
- As a result of this we can see that Alice can't simply generate a copy of $|\Psi\rangle$ and give the copy to Bob. We can only copy classical states (not superpositions).
- However, by taking advantage of two classical bits and an entangled qubit pair, Alice can transfer her state $|\Psi\rangle$ to Bob. We call this **teleportation** because, at the end, Bob will have $|\Psi\rangle$ and Alice won't anymore.

Quantum Teleportation Protocol

- To transfer a quantum bit, Alice and Bob must use a third party (Telamon) to send them an entangled qubit pair.
- Alice then performs some operations on her qubit, sends the results to Bob over a classical communication channel, and Bob then performs some operations on his end to receive Alice's qubit.

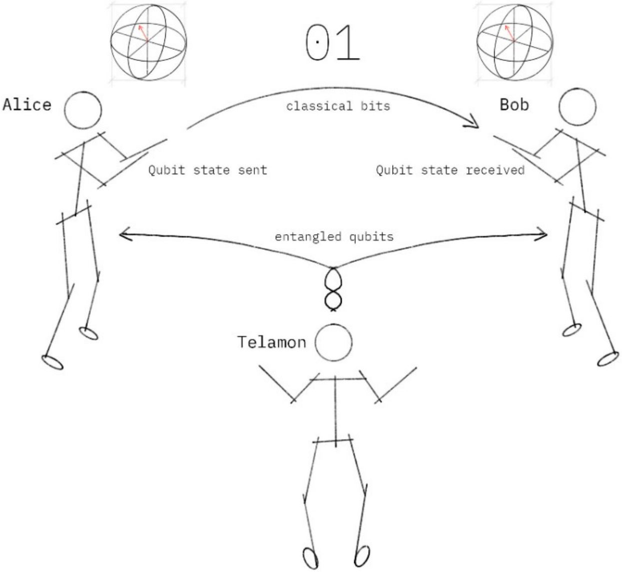
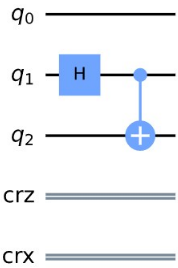


Figure: Quantum Teleportation Protocol

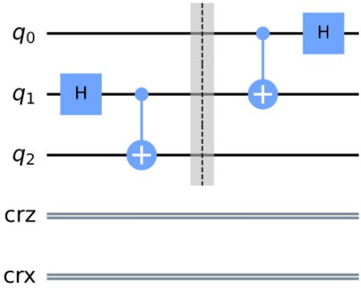
Step 1

- A third party, Telamon, creates an entangled pair of qubits and gives one to Bob and one to Alice. The pair Telamon creates is a special pair called a Bell pair.
- In quantum circuit language, the way to create a Bell pair between two qubits is to first transfer one of them to the X-basis ($|+\rangle$ and $|\text{-}\rangle$) using a Hadamard gate, and then to apply a CNOT gate onto the other qubit controlled by the one in the X-basis.



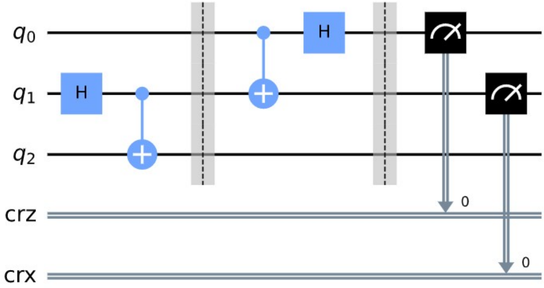
Step 2

- Alice applies a CNOT gate to q_1, controlled by $|\Psi\rangle$(the qubit she is trying to send Bob).
- Then Alice applies a Hadamard gate to $|\Psi\rangle$. In our quantum circuit, the qubit ($|\Psi\rangle$) Alice is trying to send is q_0.



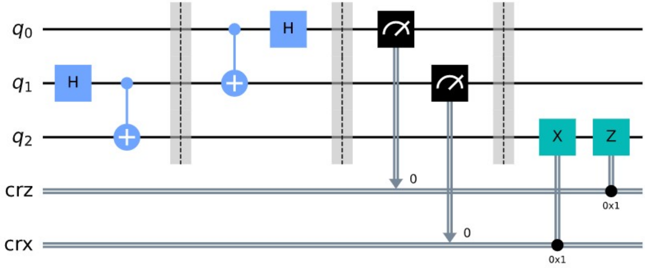
Step 3

- Next, Alice applies a measurement to both qubits that she owns, q_1 and $|\Psi\rangle$, and stores this result in two classical bits.
- Alice then sends these two bits to Bob.



Step 4

- Bob, who already has the qubit q_2, then applies the following gates depending on the state of the classical bits:
  $00 \rightarrow$ Do nothing
  $01 \rightarrow$ Apply X gate
  $10 \rightarrow$ Apply Z gate
  $11 \rightarrow$ Apply ZX gate



- At the end of this protocol, Alice's qubit has now teleported to Bob.

Outline

Super Dense Coding

- Quantum teleportation and superdense coding are closely related, to avoid confusion we need to clarify the difference.
- Superdense coding is a procedure that allows someone to send two classical bits to another party using just a single qubit of communication.
- The teleportation protocol can be thought of as a flipped version of the superdense coding protocol, in the sense that Alice and Bob merely "swap their equipment."

Teleportation	Superdense Coding
Transmit one qubit using two classical bits	Transmit two classical bits using one qubit

Figure:

Procedure

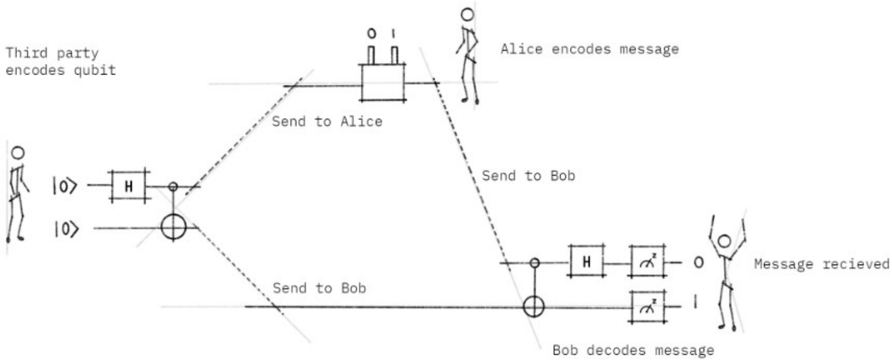


Figure: Super Dense Coding Process

Step 1

- The process starts with a third party, who we'll call Charlie.
- Two qubits are prepared by Charlie in an entangled state. He initially starts the 2 qubits in the basis state $|0\rangle$.
- He applies Hadamard gate (H) to the first qubit to create superposition. He then applies CNOT gate (CX) using the first qubit as a control and the second as the target.
- This is the entangled state (Bell pair).

Step 2

- Charlie sends the first qubit to Alice and the second qubit to Bob.
- The goal of the protocol is for Alice to send 2 classical bits of information to Bob using her qubit.
- But before she does, she needs to apply a set of quantum gates to her qubit depending on the 2 bits of information she wants to send:

Encoding Rules for Superdense Coding (Alice protocol):

Intended Message	Applied Gate	Resulting State $(\cdot\frac{1}{\sqrt{2}})$
00	I	$\|00\rangle + \|11\rangle$
01	X	$\|10\rangle + \|01\rangle$
10	Z	$\|00\rangle - \|11\rangle$
11	ZX	$-\|10\rangle + \|01\rangle$

- Thus if she wants to send a 00, she does nothing to her qubit (apply the identity (I) gate).
- If she wants to send a 01, then she applies the X gate.
- Depending on what she wants to send, she applies the appropriate gate, then sends her qubit to Bob for the final step in the process.

Step 3

- Bob receives Alice's qubit (leftmost qubit) and uses his qubit to decode Alice's message. Notice that he does not need to have knowledge of the state in order to decode it – he simply uses the restoration operation.
- Bob applies a CNOT gate using the leftmost qubit as control and the rightmost as target. Then he applies a Hadamard gate and finally performs a measurement on both qubits to extract Alice's message.

Bob Receives ($\cdot \frac{1}{\sqrt{2}}$)	After CNOT-gate ($\cdot \frac{1}{\sqrt{2}}$)	After H-gate
$\|00\rangle + \|11\rangle$	$\|00\rangle + \|10\rangle$	$\|00\rangle$
$\|10\rangle + \|01\rangle$	$\|11\rangle + \|01\rangle$	$\|01\rangle$
$\|00\rangle - \|11\rangle$	$\|00\rangle - \|10\rangle$	$\|10\rangle$
$-\|10\rangle + \|01\rangle$	$-\|11\rangle + \|01\rangle$	$\|11\rangle$

Figure:

Outline

Schmidt Decomposition

- The **Schmidt decomposition** refers to a particular way of expressing a vector in the tensor product of two inner product spaces.
- It has numerous applications in quantum information theory, for example in entanglement characterization and in state purification, and plasticity.
- We have seen that for any given orthonormal basis $\{\eta_\alpha\}$ of H_2, any vector Ψ in $H_1 \otimes H_2$ can be written in the form

  $$\Psi = \Sigma_\alpha |\zeta_\alpha > |\epsilon_\alpha >,$$

  and we shall now show that by being more flexible about the choice of basis for H_2 it is possible to get a more symmetric expansion.

The Schmidt Decomposition Theorem

Let $|\Psi>$ be a pure state of a composite quantum system; then there exists a set of orthonormal states ξ_j of H_1 and a set of orthonormal states η_j of H_2 (j = 1, 2, . . .) such that the following condition hold

$$|\Psi> = \sum_j c_j |\xi_j> |\eta_j>,$$

where c_j are non-negative real numbers such that $\Sigma_i c_j$=1. The numbers c_j are called the **Schmidt coefficients**.
There are density operators 1 on H1 and 2 on H2 such that

$$¡\Psi|(A \otimes 1)\Psi¿ = \mathrm{tr}[A\rho_1], ¡\Psi|(1 \otimes B)\Psi¿ = \mathrm{tr}[B\rho_2],$$

for all observables A and B on H_1 and H_2, respectively, and the ξ_j may be chosen to be the eigenvectors of ρ_1 corresponding to non-zero eigenvalues p_j, the vectors η_j, the corresponding eigenvectors for ρ_2, and the positive scalars $c_j = \sqrt{p_j}$.

Proof

The Schmidt decomposition is essentially a restatement of the singular value decomposition in a different context. Fix orthonormal bases $\{e_1, \ldots, e_n\} \subset H_1$ and $\{f_1, \ldots, f_m\} \subset H_2$.

We can identify an elementary tensor $e_i \otimes f_j$ with the matrix $e_i f_j^{\mathsf{T}}$, where f_j^{T} is the transpose of f_j.

A general element of the tensor product

$$w = \sum_{1 \leq i \leq n, 1 \leq j \leq m} \beta_{ij} e_i \otimes f_j$$

can then be viewed as the n × m matrix

$$M_w = (\beta_{ij})$$

By the singular value decomposition, there exist an n × n unitary U, m × m unitary V, and a positive semidefinite diagonal m × m matrix σ such that

$$M_w = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^*.$$

Write $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$ where U_1 is n × m and we have $M_w = U_1 \Sigma V^*$. Let $\{u_1, \ldots, u_m\}$ be the m column vectors of U_1, $\{v_1, \ldots, v_m\}$ the column vectors of $\overline{V}$, and $\alpha_1, \ldots, \alpha_m$ the diagonal elements of σ. The previous expression is then

$$M_w = \sum_{k=1}^{m} \alpha_k u_k v_k^{\mathsf{T}},$$

Then

$$w = \sum_{k=1}^{m} \alpha_k u_k \otimes v_k,$$

which proves the claim.

References

1. https://qiskit.org/textbook/ch-algorithms/teleportation.html
2. https://qiskit.org/textbook/ch-algorithms/superdense-coding.html
3. https://en.wikipedia.org/wiki/Schmidt_decomposition
4. KC Hannabuss: Notes on Quantum Computing, Oxford, HT 2007