NCERT-6.5.24

EE24BTECH11039 - MANDALA RANJITH

Proof Using Gradient Descent

Question: Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Solution:

Objective Function and Constraint

The **Curved Surface Area (CSA)** of the cone is:

$$\text{CSA} = \pi r \sqrt{r^2 + h^2}. \tag{1}$$

The volume constraint is:

$$V = \frac{1}{3}\pi r^2 h, \tag{2}$$

which gives:

$$h = \frac{3V}{\pi r^2}. \tag{3}$$

Substituting h into the CSA:

$$\text{CSA}(r) = \pi r \sqrt{r^2 + \left(\frac{3V}{\pi r^2}\right)^2}. \tag{4}$$

Gradient of CSA

To minimize CSA, we compute its gradient:

$$\frac{d}{dr}[\text{CSA}(r)] = \pi \left(\sqrt{f(r)} + \frac{r}{2\sqrt{f(r)}} \cdot f'(r) \right), \tag{5}$$

where:

$$f(r) = r^2 + \left(\frac{3V}{\pi r^2}\right)^2, \tag{6}$$

$$f'(r) = 2r - 2\left(\frac{3V}{\pi r^2}\right) \cdot \frac{6V}{\pi r^3}. \tag{7}$$

Gradient Descent Algorithm

We minimize CSA using gradient descent:

$$r_{\text{new}} = r_{\text{old}} - \eta \frac{d}{dr}[\text{CSA}(r)], \tag{8}$$

where η is the learning rate.

Numerical Results

After running gradient descent with:

- Learning rate (η) = 0.01,
- Tolerance = 10^{-6},
- Maximum iterations = 10,000,

we obtained:

$$r \approx 0.877308077654739, \tag{9}$$

$$h \approx 1.2407009817987995, \tag{10}$$

$$\frac{h}{r} \approx \sqrt{2}. \tag{11}$$

Computational Solution: We use the method of gradient descent to find the minimum/maximum of the given function, since the objective function is convex. Since the coefficient of $r^2 > 0$, we expect to find a local minimum.

$$r_{n+1} = r_n - \alpha f'(r_n) \tag{12}$$

$$r_{n+1} = r_n - \alpha\left(2r_n - 2\frac{3V}{\pi r_n^2} \cdot \frac{6V}{\pi r_n^3}\right) \tag{13}$$

$$r_{n+1} = r_n - 2\alpha r_n - 2\alpha\frac{3V}{\pi r_n^2} \cdot \frac{6V}{\pi r_n^3}. \tag{14}$$

Conclusion

The gradient descent results confirm that the cone of least curved surface area for a given volume satisfies:

$$h = \sqrt{2}r. \tag{15}$$

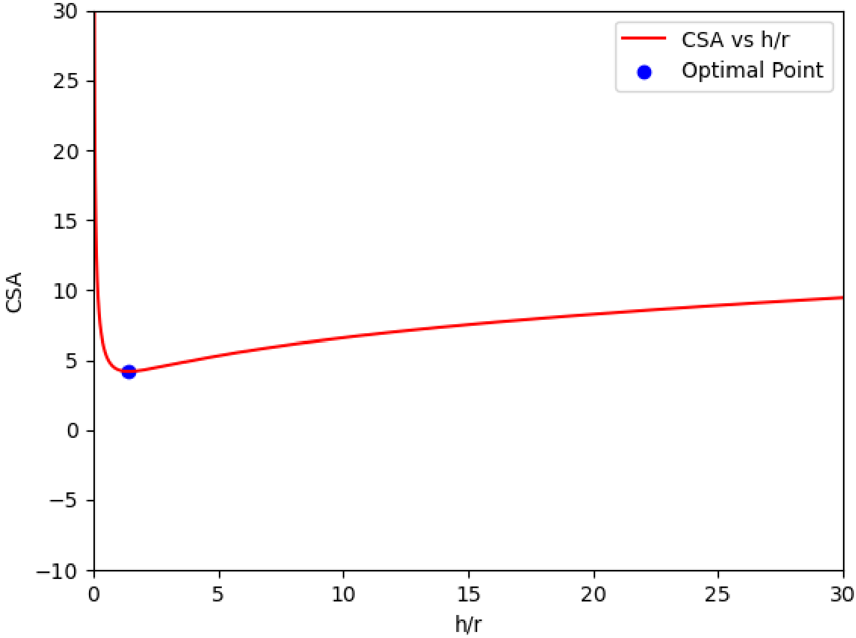

Fig. 0.1: $\frac{h}{r}$ vs CSA graph and minimum point