Probability & Statistics for EECS: Homework #03

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Problem 1

1. **Step 1**

   Let's interpret the story into a mathematical problem, and denote that
   Event B : The level of trust the villagers have in the child, and let's assume that $P(B) = 0.8$.
   Event A : the child tell a lie, and let's assume that $P(A|B) = 0.3$, $P(A_i|B^c) = 0.6$.

   Step 2

   For the first time, if the child said that the wolf appeared, but the wolf didn't appear, which indicated that the child tell a lie, then the level of trust the villagers have in the child is

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A)} \\ &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \\ &= \frac{0.3 \times 0.8}{0.3 \times 0.8 + 0.6 \times 0.2} \\ &= \frac{2}{3} \end{aligned}$$

   And for the second time if the child said that the wolf appeared, but the wolf didn't appear, which indicated that the child tell a lie, we update $P(B)$ to $\frac{2}{3}$ and $P(B^c)$ to $\frac{1}{3}$, then the level of trust the villagers have in the child is

$$\begin{aligned} P(B|A) &= \frac{P(A|B)P(B)}{P(A)} \\ &= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)} \\ &= \frac{0.3 \times \frac{2}{3}}{0.3 \times \frac{2}{3} + 0.6 \times \frac{1}{3}} \\ &= \frac{1}{2} \end{aligned}$$

   So after the child told lies twice, the level of trust the villagers have in the child had faced a sharply decrease, which means that the villagers don't trust the child like before.

   (Note : though $\frac{1}{2}$ may not be so convincing, the result may show a more intuitive or obvious decline of the level of trust the villagers have in the child if I make a better assumption of the probability $P(A|B)$and$P(A_i|B^c)$.)

Problem 2

1. (a)

Let's approach this issue from a simpler perspective, which is, put our attention on the step before accessing exactly n, if we want the running total is n, then it means that our previous running total is exactly n-1 (we denoted this as p_{n-1}, and after a roll of the number 1 (whose probability is $\frac{1}{6}$, the running total will become n, and the probability of the total process is $\frac{1}{6}p_{n-1}$, similarly, if we want the running total is n, then it also means that our previous running total is exactly n-2 (we denoted this as p_{n-2}, and after a roll of the number 1 (whose probability is $\frac{1}{6}$, the running total will become n, and the probability of the total process is $\frac{1}{6}p_{n-2}$,...,ect. So the recursive equation can be denoted as

$$p_n = \frac{1}{6}(p_{n-1} + p_{n-2} + p_{n-3} + p_{n-4} + p_{n-5} + p_{n-6}$$

Since there is no possibility that the running total is actually a negative, so $p_k = 0$ for $k < 0$, and we always start with 0 running total so we get $p_0 = 1$.

(b)

$$\begin{aligned}
p_1 &= \frac{1}{6} \\
p_2 &= \frac{1}{6}(1+\frac{1}{6}) \\
p_3 &= \frac{1}{6}(1+\frac{1}{6})^2 \\
p_4 &= \frac{1}{6}(1+\frac{1}{6})^3 \\
p_5 &= \frac{1}{6}(1+\frac{1}{6})^4 \\
p_6 &= \frac{1}{6}(1+\frac{1}{6})^5 \\
p_7 &= \frac{1}{6}(p_1 + p_2 + p_3 + p_4 + p_5 + p_6) \\
&= \frac{1}{6} \times (\frac{1}{6} + \frac{1}{6}(1+\frac{1}{6}) + \frac{1}{6}(1+\frac{1}{6})^2 + ... + \frac{1}{6}(1+\frac{1}{6})^5) \\
&= \frac{1}{6} \times \frac{\frac{1}{6}[1-(1+\frac{1}{6})^6]}{1-(1+\frac{1}{6})} \\
&= \frac{1}{6}[(1+\frac{1}{6})^6 - 1] \\
&\approx 0.25
\end{aligned}$$

(c)

We can make a rough estimate of the average for one throw, which is $\frac{1+2+3+4+5+6}{6} = 3.5$, it means that a throw can cause the running total increased by approximately 3.5, so we can conclude that if the running total increased by 1, we need to generate $\frac{2}{7}$ throw so after 7 throwing, we can get a 2 running total increasing (which is a specific increase), and the probability that we want a specific running total is $\frac{2}{7}$.

Problem 3

1. (a)

We can see that A_2 denotes the event that the first and second trial are both failed (the number of successful trials is 0) or the first and second trial are both successful (the number of successful trials is 2).So

$$\begin{aligned}P(A_2) &= q_1q_2 + (1-q_1)(1-q_2)\\ &= 2q_1q_2 - (q_1+q_2) + 1\\ &= 2(b_1+\frac{1}{2})(b_2+\frac{1}{2}) - (b_1+\frac{1}{2}+b_2+\frac{1}{2}) + 1\\ &= 2b_1b_2 + b_1 + b_2 + \frac{1}{2} - b_1 - b_2 - 1 + 1\\ &= \frac{1}{2} + 2b_1b_2\end{aligned}$$

(b)

We can find that $P(A_1) = q_1 = \frac{1}{2} + b_1$ for **n=1**.
From (a), we have $P(A_2) = \frac{1}{2} + 2b_1b_2$, and let's assume that
if **n=k**, then we have $P(A_k) = \frac{1}{2} + 2^{k-1}b_1b_2...b_k$.
if **n=k+1**,

$$\begin{aligned}P(A_{k+1}) &= q_{k+1}P(A_k) + (1-q_{k+1})(1-P(A_k)\\ &= (\frac{1}{2}+b_{k+1})(\frac{1}{2}+2^{k-1}b_1b_2...b_k) + (\frac{1}{2}-b_{k+1})(\frac{1}{2}-2^{k-1}b_1b_2...b_k)\\ &= \frac{1}{4}+\frac{1}{2}\times 2^{k-1}b_1b_2...b_k + \frac{1}{2}b_{k+1} + 2^{k-1}b_1b_2...b_{k+1} + \frac{1}{4} - \frac{1}{2}\times 2^{k-1}b_1b_2...b_k - \frac{1}{2}b_{k+1} + 2^{k-1}b_1b_2...b_{k+1}\\ &= \frac{1}{2} + 2^{(k+1)-1}b_1b_2...b_{k+1}\end{aligned}$$

which proves that our assumption is true by the method of induction.

(c)

If $p_i = \frac{1}{2}$ for some i,
on one hand, we can just assume that the probability of the last trial ends in "success" is $\frac{1}{2}$, and we can find that the number of successful trials is even or odd is actually depends on the last trial, if the last trial is successful, the number of successful trials is even or odd share the same probability of $\frac{1}{2}$, and the same case with the failed last trial, so $P(A_n) = \frac{1}{2}$. On the other hand, if $p_i = \frac{1}{2}$ for some i, then there exist $b_i = 0$ for $i = \frac{1}{2}$ and $P(A_n) = \frac{1}{2} + 2^{n-1}\times 0 = \frac{1}{2}$, which proves that our result of (b) is true.

If $p_i = 0$ for all i, it means that for all trials, we are bound to face faliure and there is no chance that we got successful trials thus the number of successful trials is 0 (a even number) , so $b_i = \frac{1}{2}$ for all i. And $P(A_n) = \frac{1}{2} + 2^{n-1}\times\frac{1}{2}\times\frac{1}{2}...\times\frac{1}{2}$**(n items)** $= \frac{1}{2}+\frac{1}{2} = 1$.

If $p_i = 1$ for all i, it means that for all trials, we are bound to face successful and whether the the number of successful trials is even or odd depends on n. And $b_i = -\frac{1}{2}$. If n is odd, $P(A_n) = \frac{1}{2} + 2^{n-1}\times(-\frac{1}{2})\times(-\frac{1}{2})...\times(-\frac{1}{2}) = \frac{1}{2}-\frac{1}{2} = 0$. If n is even, $P(A_n) = \frac{1}{2}+2^{n-1}\times(-\frac{1}{2})\times(-\frac{1}{2})...\times(-\frac{1}{2}) = \frac{1}{2}+\frac{1}{2} = 1$.

Problem 4

1. (a)

If the message do have errors but go undetected, it means that there are 2 digits in the message get corruption, or there are 4 digits in the message get corruption, and it can be denoted as below:

$$P = \binom{5}{2} \times 0.1^2 \times 0.9^3 + \binom{5}{4} \times 0.1^4 \times 0.9^1 = 0.07335$$

(b)

$$\begin{aligned} P &= \binom{n}{2} \times p^2 \times (1-p)^{n-2} + \binom{n}{4} \times p^4 \times (1-p)^{n-4} + ... \\ &+ \binom{n}{n-1} \times p^{n-1} \times (1-p)^1 (\text{when n is odd, or} \binom{n}{n} \times p^n \times (1-p)^0 \text{when n is even}) \\ &= \sum_{i=2k(k\in\mathbb{N}^+)}^{n} \binom{n}{i} p^i (1-p)^{n-i} \end{aligned}$$

(c)

We can use a to denote

$$\sum_{i=2k(k\in\mathbb{N})}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

and b to denote

$$\sum_{i=2k+1(k\in\mathbb{N})}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

then we have

$$a + b = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = 1$$

and

$$a - b = \sum_{i=0}^{n} \binom{n}{i} (-p)^i (1-p)^{n-i} = (1-2p)^n$$

thus, we can solve that

$$a = \frac{1 + (1-2p)^n}{2}$$

and since

$$a = \sum_{i=2k(k\in\mathbb{N})}^{n} \binom{n}{i} p^i (1-p)^{n-i} = \binom{n}{0} p^0 (1-p)^n + \sum_{i=2k(k\in\mathbb{N}^+)}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (1-p)^n + p$$

So finally we have the probability that the received message has errors which go undetected

$$\begin{aligned} P &= a - (1-p)^n \\ &= \frac{1 + (1-2p)^n}{2} - (1-p)^n \end{aligned}$$

Problem 5

1. (a)

The support of $X \oplus Y$ is 0,1.

$$\begin{aligned}P(X \oplus Y = 1) &= P(X=1)P(Y=0) + P(X=0)P(Y=1) \\ &= p \times \frac{1}{2} + (1-p) \times \frac{1}{2} \\ &= \frac{1}{2}\end{aligned}$$

So the distribution of $X \oplus Y$ is Bern(1/2).

(b)

On one hand, we can learn from the result of (a) that the distribution of $X \oplus Y$ is independent of p (which is a constant $\frac{1}{2}$) , so $X \oplus Y$ is independent of X whether p is $\frac{1}{2}$ or not.
On the other hand, if $p = \frac{1}{2}$, then $X \oplus Y$ is independent of Y. However, if $p \neq \frac{1}{2}$, we can get

$$\begin{aligned}P(X \oplus Y = 1) &= P(X=1)P(Y=0) + P(X=0)P(Y=1) \\ &= p \times (1-y) + (1-p) \times y \\ &= p(1-y) + (1-p)y\end{aligned}$$

which shows that $X \oplus Y$ is not independent of Y.

(c)

Since $X_1, ..., X_n$ ~Bern(1/2), and from the conclusion we get in (a), if a variable P satisfy Bern(1/2), then for any other variables (let's denote it as Q), no matter what kind of distribution it is, it's always true that $P \oplus Q \sim$ Bern(1/2). And let's regard X_{j1} as P, $X_{i2} \oplus ... \oplus X_{jn}$ as Q, then we have $Y_J = X_{j1} \oplus X_{j2} \oplus ... \oplus X_{jn} = P \oplus Q \sim$ Bern(1/2).
For these $2^n - 1$ R.V.s, if J=1,2, then $Y_{1,2} = X_1 \oplus X_2 = Y_1 \oplus Y_2$,so it is obviously to find that they are not independent.
Suppose that $J_m = \{a_1, a_2, ..., a_m\}$, $J_n = \{b_1, b_2, ..., b_n\}$, then our objective is to show that $P(Y_{Jm} = p, P_{Jn} = q) = P(Y_{Jm} = p)P(Y_{Jn} = q)$, and we use A to denote the parts that J_m and J_n intersecting, M to denote the unique part of J_m, N to denote the unique part of J_n, for the common intersecting parts, using the conclusion we get from the previous step $Y_J \sim$ Bern(1/2) to get $Y_A, Y_M, Y_N \sim$ Bern(1/2), thus, on one hand

$$\begin{aligned}P(Y_{Jm} = p, Y_{Jn} = q) &= P(A \oplus M = p, A \oplus N = q | A = 1)P(A=1) + P(A \oplus M = p, A \oplus N = q | A = 0)P(A=0) \\ &= P(1 \oplus M = p, 1 \oplus N = q) \times \frac{1}{2} + P(0 \oplus M = p, 0 \oplus N = q) \times \frac{1}{2} \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\end{aligned}$$

On the other hand,

$$P(Y_{Jm} = p)P(Y_{Jn} = q) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(Y_{Jm} = p, Y_{Jn} = q)$$

So these $2^n - 1$ R.V.s are pairwise independent, but not independent.

Problem 6

1.