Probability & Statistics for EECS: Homework #06

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Problem 1

1. (a)

   We can find that

$$\int_{-\infty}^{0} f(x)\,dx = 0 \qquad \int_{1}^{+\infty} f(x)\,dx = 1$$

   When $x \in (0,1)$

$$P(X \le x) = \int_{0}^{x} f(x)\,dx = \int_{0}^{x} (12x^2 - 12x^3)\,dx = 4x^3 - 3x^4$$

   Thus, the CDF of X is

$$F(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 4x^3 - 3x^4 & \text{if } x \in (0,1) \\ 1 & \text{if } x \ge 1 \end{cases}$$

   (b)

$$P(0 < X < \frac{1}{2}) = F(\frac{1}{2}) - F(0) = \frac{5}{16}$$

   (c)

$$E(X) = \int_{-\infty}^{+\infty} xf(x)dx = \int_{0}^{1} xf(x)dx = \int_{0}^{1} x(12x^2 - 12x^3)dx = \frac{3}{5}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{0}^{1} x^2 f(x)dx = \int_{0}^{1} x^2(12x^2 - 12x^3)dx = \frac{2}{5}$$

   Thus, we can get

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{5} - (\frac{3}{5})^2 = \frac{1}{25}$$

Problem 2

1. (a)

   If $x \leq 0$, $F(x) = 0$, if $x \geq 1$, $F(x) = 1$,
   If $x \in (0, 1)$, since $X = max\{U_1, ..., U_n\}$, then we can get the CDF of X

   $$F(X) = P(X \leq x) = P(U_1 \leq x, U_2 \leq x, ..., U_n \leq x) = P(U_1 \leq x)P(U_2 \leq x)...P(U_n \leq x) = x^n$$

   Thus, we can get the PDF of X

   $$f(x) = F'(X) = nx^{n-1}$$

   $$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ nx^{n-1} & \text{if } x \in (0, 1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

   (b)

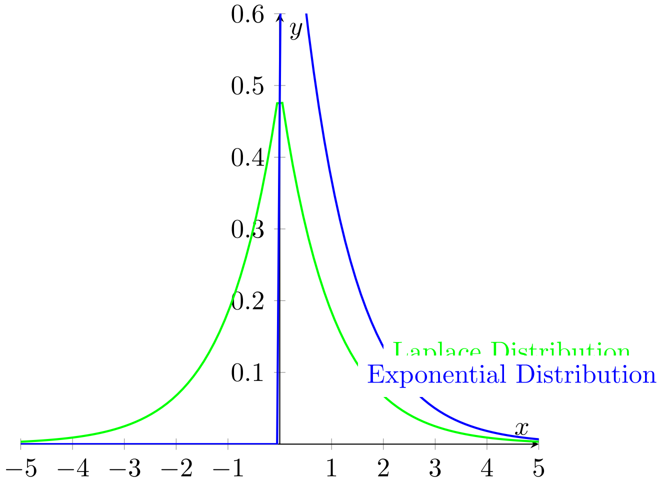
   If $x \in (0, 1)$, then

   $$E(X) = \int_0^1 xnx^{n-1}dx = n\int_0^1 x^n dx = \frac{n}{n+1}$$

   $$E(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{n}{n+1} & \text{if } x \in (0, 1) \\ 0 & \text{if } x \geq 1 \end{cases}$$

Problem 3

1. (a)



Firstly, on the side when $x > 0$, the Laplace distribution is proportional to the Exponential distribution. Secondly, we observe that the Laplace distribution is symmetric about the y-axis. Therefore, the Laplace distribution is also called a symmetrized Exponential distribution.

(b)

First find the CDF of SX

$$\begin{aligned} F(SX) &= P(SX \leq x|S=1)P(S=1) + P(SX \leq x|S=-1)P(S=-1) \\ &= \frac{1}{2}P(X \leq x) + \frac{1}{2}P(-X \leq x) \\ &= \frac{1}{2}(1-e^{-x}) + \frac{1}{2}(1-e^{x}) \end{aligned}$$

So we can find the PDF of SX

$$f(SX) = F'(SX) = \frac{1}{2}(e^{-x} + e^{x})$$

So, the PDF resembles an Exponential distribution that is symmetric about the y-axis.

Problem 4

1. (a)

Denote Y as $Y = -logX$, first we can get $F(x) = 1 - e^{-x}$.
then

$$\begin{aligned}
F(Y \leq y) &= P(-logX \leq y) \\
&= P(logX \geq loge^{-y}) \\
&= P(X \geq e^{-y}) \\
&= 1 - P(X \leq e^{-y}) \\
&= 1 - (1 - e^{-e^{-y}}) \\
&= e^{-e^{-y}}
\end{aligned}$$

(b)

$$\begin{aligned}
F(M_n) &= P(M_n \leq x) \\
&= P(X_1 \leq x, ..., X_n \leq x) \\
&= P(X_1 \leq x)...P(X_n \leq x) \\
&= (F(x))^n \\
&= (1 - e^{-x})^n
\end{aligned}$$

$$\begin{aligned}
F(M_n - logn \leq x) &= P(M_n \leq x + logn) \\
&= (1 - e^{-(x+logn)})^n
\end{aligned}$$

$$\begin{aligned}
\lim_{n\to+\infty} F(M_n - logn \leq x) &= \lim_{n\to+\infty} (1 - e^{-(x+logn)})^n \\
&= \lim_{n\to+\infty} (1 - e^{-x} e^{log\frac{1}{n}})^n \\
&= \lim_{n\to+\infty} (1 + \frac{-e^{-x}}{n})^n \\
&= \lim_{n\to+\infty} ((1 + \frac{-e^{-x}}{n})^{\frac{n}{-e^{-x}}})^{-e^{-x}} \\
&= e^{-e^{-x}} \\
&= F(Y \leq y)
\end{aligned}$$

Thus, we can find that $M_n - logn$ converges in distribution to the Gumbel distribution.

Problem 5

1.

$$
\begin{aligned}
E(max(Z-c,0) &= \int_{-\infty}^{+\infty} max(z-c,0)\phi(z)dz \\
&= \int_{c}^{+\infty} (z-c)\phi(z)dz \\
&= \int_{c}^{+\infty} z\phi(z)dz - \int_{c}^{+\infty} c\phi(z)dz \\
&= -\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}|_c^{+\infty} - c(1-\Phi(c)) \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}} - c(1-\Phi(c))
\end{aligned}
$$

Problem 6

1.