Probability & Statistics
Spring 2023
Midterm
2023/03/30
Time Limit: 100 Minutes

Name (Print): ____________________

Advisor Name ____________________

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

Try to answer as many problems as you can. The following rules apply:

- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	20	
4	10	
5	10	
6	10	
7	10	
8	20	
Total:	100	

1. (10 points) You independently toss a fair coin three times.
   (a) (2 points) What is the probability that you observe exactly three heads?
   (b) (3 points) What is the probability that you observe exactly one head?
   (c) (5 points) Given that you have observed at least one head, what is the probability that you observe at least two heads?

2. (10 points) For three events A, B and C, we know that

   - A and C are independent,
   - B and C are independent,
   - A and B are disjoint,
   - $P(A \cup C) = 2/3$, $P(B \cup C) = 3/4$, $P(A \cup B \cup C) = 11/12$.

   Find $P(A)$, $P(B)$, and $P(C)$.

3. (20 points) A family has two children. The genders of the first-born and second-born are independent (with boy and girl equally likely), and which seasons the children were born in are independent, with all 4 seasons equally likely. We also assume gender is independent of season.

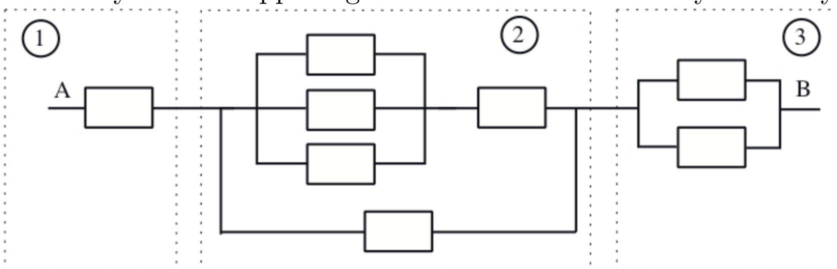
   (a) (5 points) What is the probability that both children are boys, given that the first child is a boy?

   (b) (5 points) What is the probability that both children are boys, given that from the father we know at least one of them is a boy?

   (c) (5 points) What is the probability that both children are girls, given that a randomly chosen one of the two children is a girl who was born in summer?

   (d) (5 points) What is the probability that both children are girls, given that from the father we know at least one of the two children is a girl who was born in summer?

4. (10 points) A computer system consists of identical components, each of which may be broken with probability 1/2, independent of other components. The components are connected in three subsystems, as shown in the following figure. The system is operational if there is a path that starts at point A, ends at point B, and consists of non-broken components. What is the probability of this happening? It is also called reliability of this system.



A: # ① operational.

B: # ② operational

C: # ③ operational.

$P = P(A)\,P(B)\,P(C)$

① $P(A) = \frac{1}{2}$

② B_1: upper circuit operational

B_2: lower circuit operational.

$P(B_1) = (1-(\frac{1}{2})^3) \times \frac{1}{2} = \frac{7}{16}$ $P(\bar{B}_1) = \frac{9}{16}$

$P(B_2) = \frac{1}{2}$ $P(\bar{B}_2) = \frac{1}{2}$

$P(B) = 1 - P(\bar{B}_1)\,P(\bar{B}_2) = 1 - \frac{9}{16} \times \frac{1}{2} = \frac{23}{32}$

③ $P(C) = 1 - (\frac{1}{2})^2 = \frac{3}{4}$

$P = P(A)\,P(B)\,P(C) = \frac{1}{2} \times \frac{23}{32} \times \frac{3}{4} = \frac{69}{256}$

5. (10 points) Let a random variable X be Hypergeometric with parameters w, b, n. The PMF of X is

$$P(X=k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}},$$

for integers k satisfying $0 \le k \le w$ and $0 \le n-k \le b$, and $P(X=k)=0$ otherwise.

$\overset{w+b}{N}$件, $\overset{w}{M}$不合格, 抽n

$E(X) = \frac{nM}{N}$

(a) (2 points) Find $E(X)$ by the method of indicator random variable.

(b) (5 points) Find $E\left[\binom{X}{2}\right]$ by the method of indicator random variable.

(c) (3 points) Use the result of (b) to find the variance of X.

(a) $E(X) = E\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} E(I_j) = \sum_{j=1}^{n} \frac{w}{w+b} = \frac{nw}{w+b}$

(b) $E\left[\binom{X}{2}\right] = \sum_{i<j} P(A_i \cap A_j) = \sum_{i<j} P(A_i)P(A_j) = \binom{n}{2}\frac{w}{w+b}\cdot\frac{w-1}{w+b-1}$

(c) $E(X^2) - E(X) = E[X(X-1)] = 2E\left(\frac{X(X-1)}{2}\right) = 2E\left[\binom{X}{2}\right] = 2\binom{n}{2}\frac{w}{w+b}\cdot\frac{w-1}{w+b-1} = n(n-1)\frac{w}{w+b}\cdot\frac{w-1}{w+b-1}$

$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = n(n-1)\frac{w}{w+b}\cdot\frac{w-1}{w+b-1} + E(X) - (E(X))^2$

$= \frac{n(n-1)w(w-1)}{(w+b)(w+b-1)} + \frac{nw}{w+b} - \left(\frac{nw}{w+b}\right)^2$

6. (10 points) In Monty Hall problem, now suppose the car is not placed randomly with equal probability behind the three doors. Instead, the car is behind door one with probability p_1, behind door two with probability p_2, and behind door three with probability p_3. Here $p_1 + p_2 + p_3 = 1$ and $p_1 \geq p_2 \geq p_3 > 0$. You are to choose one of the three doors, after which Monty will open a door he knows to conceal a goat. Monty always chooses randomly with equal probability among his options in those cases where your initial choice is correct. What strategy should you follow?

7. (10 points) At a movie theater, there is a line (queue) for buying the ticket. A manager announces that she will give a free ticket to the first person in line whose birthday is the same as someone who has already bought ticket. Now You are a VIP of this theater, and you are given the opportunity to choose any position in line. Assuming that you do not know anyone else's birthday and all birthdays are distributed randomly throughout the year (365 days in a year), which position in line gives you the largest chance of getting the free ticket?

I'm on position n.

A: none of the first n-1 share same

B: I share a birthday match with (n-1)

$P(A) = \frac{365 \times 364 \times \cdots 365-(n-2)}{365^{n-1}}$ $P(B) = \frac{n-1}{365}$ $P = P(A)P(B)$

find $P(n) > P(n+1) \Rightarrow \frac{P(n)}{P(n+1)} > 1 \Rightarrow \frac{365}{366-n} \cdot \frac{n-1}{n} > 1$ $365n - 365 > 366n - n^2$

$n^2 - n > 365$ $n = 20$

8. (20 points) Suppose a fair coin is tossed repeatedly, and we obtain a sequence of H and T (H denotes Head and T denotes Tail). Let N denote the number of tosses to observe the first occurrence of the pattern “HH”.

   (a) (5 points) Find the recursive equation for the probability of event “N=n”.

   (b) (5 points) Find $E(N)$.

   (c) (5 points) Let M denote the number of tosses to observe the first occurrence of the pattern “HT”. Find $E(M)$ and provide the intuition for the fact that $E(M) < E(N)$.

   (d) (5 points) Find the probability that pattern “HH” appears before the pattern “HT”.