Probability & Statistics for EECS: Homework #09

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Problem 1

1.
$$f_X(x) = \lambda e^{-\lambda x}(x \geq 0) \quad f_Y(y) = \lambda e^{-\lambda y}(y \geq 0)$$

Because X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \lambda^2 e^{-\lambda(x+y)}$$

On the other hand

$$x = \frac{tw}{1+w} \quad y = \frac{t}{1+w}$$

The transformations are $T = X + Y$ and $W = X/Y$. To find the joint PDF of T and W, The Jacobian determinant J is

$$J = \begin{vmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial w} \end{vmatrix} = \frac{1}{(1+w)^2}$$

transform the joint PDF of X and Y to get the joint PDF of T and W:

$$\begin{aligned} f_{T,W}(t,w) &= f_{X,Y}(x,y) \cdot J \\ &= \lambda^2 e^{-\lambda(x+y)} \cdot \frac{1}{(1+w)^2} \\ &= \lambda^2 e^{-\lambda(\frac{tw}{1+w} + \frac{t}{1+w})} \cdot \frac{1}{(1+w)^2} \\ &= \frac{\lambda^2 t e^{-\lambda t}}{(1+w)^2} \end{aligned}$$

For the marginal PDF of T, we integrate the joint PDF over w from 0 to infinity.

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{T,W}(t,w)\, dw \\ &= \int_0^\infty \frac{\lambda^2 t e^{-\lambda t}}{(1+w)^2}\, dw \\ &= \lambda^2 t e^{-\lambda t} \int_0^\infty \frac{1}{(1+w)^2}\, dw \\ &= \lambda^2 t e^{-\lambda t} \left[-\frac{1}{1+w} \right]_0^\infty \\ &= \lambda^2 t e^{-\lambda t} \end{aligned}$$

For the marginal PDF of W, we integrate the joint PDF over t from 0 to infinity.

$$\begin{aligned} f_W(w) &= \int_0^\infty f_{T,W}(t,w)\, dt \\ &= \int_0^\infty \frac{\lambda^2 t e^{-\lambda t}}{(1+w)^2}\, dt \\ &= \frac{1}{(1+w)^2} \int_0^\infty \lambda^2 t e^{-\lambda t}\, dt \\ &= \frac{1}{(1+w)^2} \left[-\frac{e^{-\lambda t}(1+\lambda t)}{\lambda} \right]_0^\infty \\ &= \frac{1}{(1+w)^2} \end{aligned}$$

Problem 2

1. (a)

   The PDFs of X and Y are $f_X(x) = f_Y(y) = 1$ for $x, y \in [0, 1]$ and 0 otherwise.

   The PDF of $U = X + Y$, $f_{X+Y}(u)$, is given by the convolution:

$$f_U(u) = \int_{-\infty}^{\infty} f_X(x) f_Y(u-x)\, dx$$

   Thus we can get that

$$f_U(u) = \int_0^1 f_Y(u-x)\, dx = \begin{cases} \int_0^u dx & \text{for } 0 \le u \le 1, \\ \int_{u-1}^1 dx & \text{for } 1 < u \le 2, \\ 0 & \text{otherwise.} \end{cases} = \begin{cases} u & \text{for } 0 \le u \le 1, \\ 2-u & \text{for } 1 < u \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

   Convolve $f_U(u)$ with $f_Z(z)$ to find $f_W(w)$:

$$f_W(w) = \int_{-\infty}^{\infty} f_U(u) f_Z(w-u)\, du$$

   when $0 \le w \le 1$

$$\begin{aligned} f_W(w) &= \int_0^w u\, du \\ &= \frac{1}{2} w^2 \end{aligned}$$

   when $1 < w \le 2$

$$\begin{aligned} f_W(w) &= \int_{w-1}^w (2-u)\, du \\ &= -w^2 + 3w - \frac{3}{2} \end{aligned}$$

   when $2 < w \le 3$

$$\begin{aligned} f_W(w) &= \int_{w-1}^2 (2-u)\, du \\ &= \frac{1}{2}(3-w)^2 \end{aligned}$$

   For $w < 0$ or $w > 3$, $f_W(w) = 0$.

Problem 3

1. Since X is standardized with a mean of zero, we have $E(X) = 0$ and

$$E(X^2) = Var(X) + (E(X))^2 = Var(X) = \sigma_1^2$$

Besides

$$Cov(X, Y) = Corr(X, Y) \cdot \sigma_X \sigma_Y = \rho \sigma_X \sigma_Y$$

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY)$$

Thus we can get the expected value $\mathrm{E}(XY) = \rho\sigma_1\sigma_2$.

To ensure independence, we need the covariance between $Y - cX$ and X to be zero. The covariance is given by:

$$\begin{aligned}\mathrm{Cov}(Y - cX, X) &= \mathrm{E}[(Y - cX)X] - \mathrm{E}[Y - cX]\mathrm{E}[X] \\ &= \mathrm{E}[YX] - c\mathrm{E}[X^2] \\ &= \rho\sigma_1\sigma_2 - c\sigma_1^2 = 0\end{aligned}$$

Solve that get

$$c = \frac{\rho\sigma_2}{\sigma_1}$$

Thus, when c is $\frac{\rho\sigma_2}{\sigma_1}$, $Y - cX$ is independent of X.

Problem 4

1. (a)

the joint PDF of $U_{(j)}$ and $U_{(k)}$, $f_{U_{(j)},U_{(k)}}(u,v)$, means that $U_{(j)} \leq U_{(k)}$, there are $k-j-1$ random variables between $U_{(j)}$ and $U_{(k)}$ and $n-k$ random variables greater than $U_{(k)}$. So the number of ways to arrange the variables is $\frac{n!}{(j-1)!(k-j-1)!(n-k)!}$,and the probability is $u^{j-1}(v-u)^{k-j-1}(1-v)^{n-k}$ So the joint PDF of $U_{(j)}$ and $U_{(k)}$, $f_{U_{(j)},U_{(k)}}(u,v)$, is given by:

$$f_{U_{(j)},U_{(k)}}(u,v) = \frac{n!}{(j-1)!(k-j-1)!(n-k)!}u^{j-1}(v-u)^{k-j-1}(1-v)^{n-k}$$

(b)

According to the properties of β distribution, Suppose that $U_1,\ldots,U_n$ be i.i.d. Unif$(0,1)$, then the jth order statistic satisfy Beta$(j, n-j+1)$. So $P(B \leq p) = P(U(j) \leq p)$.
Think of these as Bernoulli trials,suppose that if $U_j \leq p$, we can regard this as a successful Bernoulli trial, then for each trial, the probability of success is p. Then $X \geq j$ means that the number of success is higher than j, which is the same as $U_{(j)} \leq p$, so we can get

$$P(X \geq j) = P(B \leq p).$$

(c)

According to the properties of β distribution, Suppose that $U_1,\ldots,U_n$ be i.i.d. Unif$(0,1)$, then the jth order statistic satisfy Beta$(j, n-j+1)$. So we can regard $\int_0^x \frac{n!}{(j-1)!(n-j)!}t^{j-1}(1-t)^{n-j}\,dt$ as $P(U_{(j)} \leq x)$. Think of the right hand side as Bernoulli trials with $U_1,\ldots,U_n$ be i.i.d. Uni$(0,1)$ r.v.s, where x is a constant and we can suppose that if for some U_i satisfy $U_i \leq x$, we can regard this as a successful Bernoulli trial then the right is the probability of having at least j successes n), Then we can realize that having at least j successes is the same thing as having $U_{(j)} \leq x$, thus to prove that

$$\int_0^x \frac{n!}{(j-1)!(n-j)!}t^{j-1}(1-t)^{n-j}\,dt = \sum_{k=j}^{n}\binom{n}{k}x^k(1-x)^{n-k}$$

Problem 5

1. (a)

According to the relationship between β distribution and γ distribution, we have

$$\frac{1}{\beta(a,b)} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$\begin{aligned}
E(p^2(1-p)^2) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^2(1-p)^2 p^{a-1}(1-p)^{b-1}\, dp \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{a+1}(1-p)^{b+1}\, dp \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 p^{(a+2)-1}(1-p)^{(b+2)-1}\, dp \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(a+b+4)} \times \frac{\Gamma(a+b+4)}{\Gamma(a+2)\Gamma(b+2)} \int_0^1 p^{(a+2)-1}(1-p)^{(b+2)-1}\, dp \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(a+b+4)}
\end{aligned}$$

According to the property of Γ distribution, we have

$$\Gamma(a+1) = \int_0^{+\infty} x^a e^{-x} dx = -x^a e^{-x}|_0^{+\infty} + a\int_0^{+\infty} x^{a-1}e^{-x}dx = a\Gamma(a)$$

Thus

$$\Gamma(a+2) = (a+1)\Gamma(a+1) = a(a+1)\Gamma(a)$$
$$\Gamma(b+2) = (b+1)\Gamma(b+1) = b(b+1)\Gamma(b)$$

$$\begin{aligned}
\Gamma(a+b+4) &= (a+b+3)\Gamma(a+b+3) = (a+b+2)(a+b+3)\Gamma(a+b+2) \\
&= (a+b+1)(a+b+2)(a+b+3)\Gamma(a+b+1) = (a+b)(a+b+1)(a+b+2)(a+b+3)\Gamma(a+b)
\end{aligned}$$

Then we get

$$\begin{aligned}
E(p^2(1-p)^2) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times \frac{\Gamma(a+2)\Gamma(b+2)}{\Gamma(a+b+4)} \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \times \frac{a(a+1)\Gamma(a)b(b+1)\Gamma(b)}{(a+b)(a+b+1)(a+b+2)(a+b+3)\Gamma(a+b)} \\
&= \frac{a(a+1)b(b+1)}{(a+b)(a+b+1)(a+b+2)(a+b+3)}
\end{aligned}$$

(b)

The posterior distribution derived with the same amount of A and B respectively is identical to those obtained from other sequences. Initiating with a Beta(1, 1) distribution, we plus the first parameter by one following each victory by A, and enhance the second parameter by one subsequent to each victory by B. So the posterior distribution is determined by the number of victories for A and B. The sequence of victories and defeats for A is not necessary; only the counts are of significance.

(c)

$$p\,|\,\text{historical data} \sim \text{Beta}(7,5).$$

Problem 5 continued on next page. . .

(d)

Denote I_1 to be the indicator of A winning the first game of the match, and I_2 be the indicator of A winning the second game of the match.
Since if we have known the probability of A wins (given p), then the coming games can be seen as i.i.d. Bernoulli trails, so I_1 and I_2 are uncorrelated.

If we only condition on the historical data,
then $p \sim \text{Beta}(7, 5)$, we have

$$E(I_1) = P(I_1 = 1) = \int_0^1 P(I_1 = 1 \,|\, p = x) f_p(x)\, dx = \int_0^1 x f_p(x)\, dx = E(p),$$

$$E(I_2) = P(I_2 = 1) = \int_0^1 P(I_2 = 1 \,|\, p = x) f_p(x)\, dx = \int_0^1 x f_p(x)\, dx = E(p),$$

$$\begin{aligned} E(I_1 I_2) &= P(I_1 I_2 = 1) = P(I_1 = 1, I_2 = 1) \\ &= \int_0^1 Pr(I_1 = 1, I_2 = 1 \,|\, p = x) f_p(x)\, dx = \int_0^1 x^2 f_p(x)\, dx = E(p^2) \end{aligned}$$

$$\text{Cov}(I_1, I_2) = E(I_1 I_2) - E(I_1)E(I_2) = E(p^2) - E^2(p) = \text{Var}(p) > 0,$$

So I_1 and I_2 are positively correlated.

(e)

the expected value

$$\binom{4}{2} p^2 (1-p)^2 = \binom{4}{2} \times \frac{8 \times 7 \times 6 \times 5}{15 \times 14 \times 13 \times 12} = \frac{4}{13}$$

Problem 6

1.