Probability & Statistics for EECS: Homework #12

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Problem 1

1. (a)

   Since $X_1, X_2, ..., X_n \sim Bern(p)$, then we have

   $$Pr(X_i = x) = p^x(1-p)^{1-x}$$

   where $x = 0or1$. Then the numerical function

   $$P_X(X_1, .., X_n; p) = \prod_{i=1}^{n} P_{X_i}(X_i; p) = \prod_{i=1}^{n} p^{X_i}(1-p)^{1-X_i} = p^{\sum_{i=1}^{n} X_i}(1-p)^{n-\sum_{i=1}^{n} X_i} = p^{S_n}(1-p)^{n-S_n}$$

   where we use S_n to denote the sum of n tosses, i.e. $S_n = X_1 + X_2 + ... + X_n$

   $$logP_X(X_1, .., X_n; p) = S_n logp + (n - S_n)log(1-p)$$

   denote $f(p) = S_n logp + (n - S_n)log(1-p)$ and let

   $$f'(p) = 0 \quad f''(p) \leq 0$$

   to find

   $$\hat{p} = \text{argmax} f(p) = \frac{X_1 + X_2 + ... + X_n}{n}$$

   (b)

   first find

   $$X|_p \sim Bin(n, p) \quad p|X = k \sim \beta(a + k, b + n - k)$$

   Then

   $$f_{p|X=k}(p) \propto p^{a+k-1}(1-p)^{b+n-k-1}$$

   In order to find

   $$\hat{p} = \text{argmax} f_{p|X=k}(p)$$

   Similarly, let $f'(p) = 0 \quad f''(p) \leq 0$ to find

   $$\hat{p} = \text{argmax} f(p) = \frac{a + k - 1}{a + b + n - 2}$$

   (c)

   $$p|X = k \sim \beta(a + k, b + n - k)$$

   $$\hat{p} = E(p|X = k) = \frac{a + k}{a + b + n}$$

Problem 2

1. (a)

The value ρ reveals how one variable reacts to changes in another, which allows us to infer that ρ corresponds to the line's incline, so it's necessary to consider the line that runs contrary to the initial line when estimating X. And assuming that that the slope of this new line is the inverse of the original one, which is $\frac{1}{\rho}$.

(b)

If V is independent of X. Surely we have $\text{Cov}(X, Y - cX) = 0$, which is

$$\begin{aligned}\text{Cov}(X, Y - cX) &= \text{Cov}(X, Y) - c\text{Var}(X) \\ &= \rho - c = 0\end{aligned}$$

And we can find$c = \rho$.

On the other hand, it's easy to find that $Y - \rho X$ is also normal. Since the Bivariate normal distribution of two normal independent distribution is also independent. Thus we can conclude that X and V are independent.

(c)

Similarly to find that $d = \rho$.

(d)

First to find the conditional PDF

$$\begin{aligned}f_{Y|X}(y|x) &= \frac{f(x,y)}{f(x)} \\ &= \frac{\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right)}{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)} \\ &= \frac{\sqrt{2\pi}\exp\left(\frac{x^2}{2}\right)}{2\pi\sqrt{1-\rho^2}} \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right) \\ &= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\exp\left(\frac{x^2}{2} - \frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right) \\ &= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{1}{2(1-\rho^2)}(y^2-2\rho xy+(\rho x)^2)\right) \\ &= \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\left(-\frac{1}{2(1-\rho^2)}(y-\rho x)^2\right).\end{aligned}$$

So we can find that$Y|X = x \sim N(\rho x, 1-\rho^2)$. So the expectation of the normal distribution is $E(Y|X) = \rho X$. Similarly to find that $E(X|Y) = \rho Y$.

(e)

Since $X, Y \sim N(0,1)$, so we can suppose that $X = kY$.
And Adam's Law gives us $Y = E(Y|X)$, thus we can find that

$$X = kY = kE(Y|X) = k\rho X$$

to conclude that

$$k = \frac{1}{\rho}$$

Problem 3

1. (a)

To find the expected number of games $E(G)$ needed for Vishy to win, we first calculate the expected value of the reciprocal of $p \sim \beta(a, b)$. Using the properties of the Beta distribution, we have:

$$\begin{aligned} E\left(\frac{1}{p}\right) &= \int_0^1 \frac{1}{p} \frac{p^{a-1}(1-p)^{b-1}}{\beta(a,b)}\, dp \\ &= \frac{1}{\beta(a,b)} \int_0^1 p^{(a-1)-1}(1-p)^{b-1}\, dp \\ &= \frac{\beta(a-1,b)}{\beta(a,b)} \\ &= \frac{\Gamma(a-1)\Gamma(b)}{\Gamma(a+b-1)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \\ &= \frac{\Gamma(a-1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b-1)} \\ &= \frac{\Gamma(a-1)(a+b-1)\Gamma(a+b-1)}{(a-1)\Gamma(a-1)\Gamma(a+b-1)} \\ &= \frac{a-1}{a+b-1}. \end{aligned}$$

So the expected number of games needed in order for Vishy to win a game (including the win) is $1 + \frac{a-1}{a+b-1}$

(b)

It's easy to find that $E(p) = \frac{a}{a+b}$ so

$$\frac{1}{E(p)} = \frac{a+b}{a} = \frac{1}{\frac{a}{a+b}} = E(G)_{\text{Geom}}$$

Due to Jensen's inequality for the convex function $f(x) = \frac{1}{x}$, we have:

$$E\left(\frac{1}{p}\right) \geq \frac{1}{E(p)}$$

Therefore,

$$E(G)_{\text{Beta}} \leq E(G)_{\text{Geom}}$$

(c)

With 7 wins out of 10 games, the beta distribution parameters for p are updated due to the conjugate prior property:

$$p|(7 \text{ wins out of } 10) \sim \text{Beta}(a+7, b+3)$$

This reflects the updated beliefs about Vishy's winning probability after observing the outcomes of the games.

Problem 4

1. (a)

   The chain is irreducible because it is possible to get to any state from any state in a finite number of steps.

   (b)

   The chain is aperiodic because state 1 has a self-loop, which means it can be returned to in one step, making the period 1.

   (c)

   Solving the system of linear equations:

$$\begin{aligned}
\pi_1 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_3 \\
\pi_2 &= \frac{1}{4}\pi_1 + \frac{1}{2}\pi_3 \\
\pi_3 &= \frac{1}{4}\pi_1 + \frac{2}{3}\pi_2 \\
1 &= \pi_1 + \pi_2 + \pi_3
\end{aligned}$$

   Solving that:

$$\pi_1 = \frac{16}{35}, \quad \pi_2 = \frac{9}{35}, \quad \pi_3 = \frac{2}{7}$$

   Thus, the stationary distribution is

$$[\frac{16}{35}, \frac{9}{35}, \frac{2}{7}]$$

   (d)

   A chain is reversible if it satisfies the detailed balance equations $\pi_i P_{ij} = \pi_j P_{ji}$ for all states i and j. Since

$$\begin{aligned}
\frac{16}{35} \cdot \frac{1}{4} &\neq \frac{2}{7} \cdot \frac{1}{2} \quad \text{(between states 1 and 3)} \\
\frac{16}{35} \cdot \frac{1}{4} &\neq \frac{9}{35} \cdot \frac{1}{3} \quad \text{(between states 1 and 2)} \\
\frac{2}{7} \cdot \frac{1}{2} &\neq \frac{9}{35} \cdot \frac{2}{3} \quad \text{(between states 3 and 2)}
\end{aligned}$$

   Thus, the Markov chain is not reversible.

Problem 5

1. the state-transition matrix:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

(a)

Easy to find that

$$P(X_3 = 3|X_2 = 2) = P_{23} = \frac{2}{3}, \quad P(X_4 = 1|X_3 = 2) = P_{21} = \frac{1}{3}$$

(b)

For $P(X_0 = 2, X_1 = 3, X_2 = 1)$, we multiply the given initial probability $P(X_0 = 2) = \frac{2}{5}$ with the transition probabilities:

$$P(X_0 = 2, X_1 = 3, X_2 = 1) = P(X_0 = 2) \cdot P_{23} \cdot P_{31} = \frac{2}{5} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{15}$$

(c)

To find $P(X_2 = i|X_0 = 2)$ for $i = 1, 2, 3$, we first square the matrix P:

$$P^2 = P \cdot P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}^2 = \begin{bmatrix} \frac{11}{24} & \frac{1}{4} & \frac{7}{24} \\ \frac{1}{2} & \frac{5}{12} & \frac{1}{12} \\ \frac{5}{12} & \frac{1}{8} & \frac{11}{24} \end{bmatrix}$$

From P^2, we find the probabilities:

$$P(X_2 = 1|X_0 = 2) = P^2_{21} = \frac{1}{2}, \quad P(X_2 = 2|X_0 = 2) = P^2_{22} = \frac{5}{12}, \quad P(X_2 = 3|X_0 = 2) = P^2_{23} = \frac{1}{12}$$

(d)

$$\begin{aligned} E(X_2|X_0 = 2) &= \sum_{i=1}^{3} i \cdot P(X_2 = i|X_0 = 2) \\ &= 1 \cdot P^2_{21} + 2 \cdot P^2_{22} + 3 \cdot P^2_{23} \\ &= 1 \cdot \frac{1}{2} + 2 \cdot \frac{5}{12} + 3 \cdot \frac{1}{12} = \frac{19}{12} \end{aligned}$$

Problem 6

1.