Probability & Statistics for EECS: Homework #08

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Problem 1

1. (a)

   To find the value of constant c, we need to make the PDF integrates to 1 over the support of X and Y. The support is given by $0 \leq y \leq x \leq 1$.

$$\int_0^1 dx \int_0^x cx^2 y \, dy \, dx = \int_0^1 \left[\frac{1}{2}cx^2y^2\right]_0^x dx = \int_0^1 \frac{1}{2}cx^4 \, dx = \left[\frac{1}{10}cx^5\right]_0^1 = \frac{1}{10}c = 1$$

   Thus, the value of c is 10.

   (b)

$$\begin{aligned} P(Y \leq \frac{X}{4} | Y \leq \frac{X}{2}) &= \frac{P(Y \leq \frac{X}{4}, Y \leq \frac{X}{2})}{P(Y \leq \frac{X}{2})} \\ &= \frac{P(Y \leq \frac{X}{4})}{P(Y \leq \frac{X}{2})} \\ &= \frac{P(0 \leq Y \leq \frac{X}{4}, X \geq 0)}{P(0 \leq Y \leq \frac{X}{2}, X \geq 0)} \\ &= \frac{\int_0^1 dx \int_0^{\frac{x}{4}} 10x^2 y dy}{\int_0^1 dx \int_0^{\frac{x}{2}} 10x^2 y dy} \\ &= \frac{1}{4} \end{aligned}$$

Problem 2

1. (a)

If $x > 0$, we can find that

$$\begin{aligned}
P_X(x) &= P_{X,Y}(x, x+1) + P_{X,Y}(x, x) + P_{X,Y}(x, x-1) \\
&= \frac{1}{6 \cdot 2^x} + \frac{1}{6 \cdot 2^x} + \frac{1}{6 \cdot 2^{x-1}} \\
&= \frac{1}{3 \cdot 2^x} + \frac{1}{3 \cdot 2^x} \\
&= \frac{1}{3 \cdot 2^{x-1}}
\end{aligned}$$

If $x = 0$, we can find that

$$\begin{aligned}
P_X(x) &= P_{X,Y}(0,0) + P_{X,Y}(0,1) \\
&= \frac{1}{6 \cdot 2^0} + \frac{1}{6 \cdot 2^0} \\
&= \frac{1}{3}
\end{aligned}$$

otherwise,

$$P_X(x) = 0$$

Similarly, we can also find that

$$P_Y(y) = \begin{cases} \frac{1}{3 \cdot 2^{y-1}} & \text{if } y > 0, \\ \frac{1}{3} & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b)

for $x = 0, y = 0$, it's easy to find that

$$P_X(0) = P_Y(0) = \frac{1}{3}$$

However,

$$P_X, Y(0,0) = \frac{1}{6} \neq \frac{1}{3}$$

So X and Y is not independent.

(c)

$$P(X = Y) = \sum_{x=0}^{+\infty} \frac{1}{6 \cdot 2^x} = \frac{1}{6} \cdot \lim_{n \to +\infty} \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \frac{1}{3}$$

Problem 3

1. (a)

   Since X and Y be i.i.d. $N(0,1)$, then $mX + nY$ is also $N(0,1)$, so we can find that

   $$aX + bY + c(X+Y) = (a+c)X + (b+c)Y = mX + nY$$

   also satisfy $N(0,1)$ so $(X, Y, X+Y)$ is Multivariate Normal.

   (b)

   Similarly, we can find that

   $$X + Y + (SX + SY) = (1+S)X + (1+S)Y$$

   When $S = -1$, then $(1+S)X + (1+S)Y = 0$
   Since S be a random sign (1 or -1, with equal probabilities), so there is $\frac{1}{2}$ probability that S$=-1$, so $(X, Y, SX + SY)$ is not Multivariate Normal.

   (c)

   Since the linear combination of 2 normal distribution satisfy

   $$aX + bY \sim N(0, a^2 + b^2)$$

   So we have

   $$a(SX) + b(SY) = S(aX + bY) \sim N(0, a^2 + b^2)$$

   so (SX, SY) is Multivariate Normal.

Problem 4

1. (a) Since Z_1 and Z_2 be i.i.d. $N(0,1)$, then $mZ_1 + nZ_2$ is also $N(0,1)$, since we can find that

$$\begin{aligned} aX + bY &= a(\sigma_X Z_1 + \mu_X) + b(\sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y) \\ &= (a\sigma_X + b\sigma_Y\rho)Z_1 + (b\sqrt{1-\rho^2}Z_2 + a\mu_X + b\mu_Y \end{aligned}$$

have a Normal distribution, so X and Y are bivariate normal.
(b) First to find the *Covriance* of X and Y

$$\begin{aligned} Cov(X,Y) &= E((X - E(X))(Y - E(Y))) \\ &= E((\sigma_X Z_1 + \mu_X - \mu_X)(\sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y - \mu_Y)) \\ &= E(\sigma_X Z_1 \cdot \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2)) \\ &= \sigma_X\sigma_Y E(\rho Z_1^2 + \sqrt{1-\rho^2}Z_1 Z_2) \\ &= \sigma_X\sigma_Y\rho E(Z_1^2) + \sigma_X\sigma_Y\sqrt{1-\rho^2}E(Z_1 Z_2) \\ &= \sigma_X\sigma_Y(Var(Z_1) + 0) + 0 \\ &= \rho\sigma_X\sigma_Y \end{aligned}$$

So the correlation coefficient between X and Y is $Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{\rho\sigma_X\sigma_Y}{\sigma_X\sigma_Y} = \rho$

(c)

$$\begin{aligned} X &= \sigma_X Z_1 + \mu_X; \\ Y &= \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y, \end{aligned}$$

defines a linear transformation of the vector $\mathbf{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ into the vector $\mathbf{V} = \begin{bmatrix} X \\ Y \end{bmatrix}$.
The joint PDF of $\mathbf{Z}$ is the product of the marginal PDFs because Z_1 and Z_2 are independent:

$$f_{Z_1,Z_2}(z_1,z_2) = \frac{1}{2\pi}e^{-\frac{z_1^2+z_2^2}{2}}.$$

To find the joint PDF of X and Y, we need to find the determinant of the Jacobian matrix of the transformation. The Jacobian matrix J for the transformation is given by:

$$J = \begin{bmatrix} \frac{\partial X}{\partial Z_1} & \frac{\partial X}{\partial Z_2} \\ \frac{\partial Y}{\partial Z_1} & \frac{\partial Y}{\partial Z_2} \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y\rho & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix}.$$

The determinant of J is:

$$\det(J) = \sigma_X\sigma_Y\sqrt{1-\rho^2}.$$

The joint PDF of X and Y is then given by:

$$f_{X,Y}(x,y) = f_{Z_1,Z_2}(z_1,z_2) \cdot |\det(J)|^{-1}.$$

where

$$Z_1 = \frac{X-\mu_X}{\sigma_X}, \quad Z_2 = \frac{\frac{Y-\mu_Y}{\sigma_Y} - \rho Z_1}{\sqrt{1-\rho^2}}.$$

so the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right).$$

Problem 5

1.

Problem 6

1.