e-function expansion

7.1.2 #3

$u_t = u_{xx} + t - t\cos\pi x \qquad 0 \le x \le 1 \qquad t > 0$

$u_x(0,t) = u_x(1,t) = 0$

$u(x,0) = f(x) = 1 + 3\cos 4\pi x$

e-function exp. assumption $\qquad u(x,t) = \sum_{n=0}^{\infty} c_n(t)\cos n\pi x$

sub in eqn gives $\qquad c_n' + n^2\pi^2 c_n = q_n(t) = n^{th}$ F.C. of $q(x,t)$

$c_n(0) = n^{th}$ F.C. of $f(x)$

for $q(x,t)$ $\qquad t - t\cos\pi x = \sum_{n=0}^{\infty} q_n(t)\cos n\pi x$

$\Rightarrow \quad q_0(t) = t \qquad q_1(t) = -t \qquad$ all others zero

for $c_n(0)$ $\qquad 1 + 3\cos 4\pi x = \sum_{n=0}^{\infty} c_n(0)\cos n\pi x$

$c_0(0) = 1 \qquad c_4(0) = 3 \qquad$ all others zero

eqns:

$n=0$	$c_0' = t$	$c_0(0)=1$
$n=1$	$c_1' + \pi^2 c_1 = -t$	$c_1(0)=0$
$n=4$	$c_4' + 16\pi^2 c_4 = 0$	$c_4(0)=3$

Solutions to O.D.E.'s

$$c_0 = \frac{t^2}{2} + 1$$

$$c_1 = -\frac{1}{\pi^4}e^{-\pi^2 t} - \frac{t}{\pi^2} + \frac{1}{\pi^4}$$

$$c_4 = 3e^{-16\pi^2 t}$$

Assemble

$$u(x,t) = \frac{t^2}{2} + 1 + \left(\frac{1}{\pi^4} - \frac{1}{\pi^4}e^{-\pi^2 t} - \frac{t}{\pi^2}\right)\cos\pi x + 3e^{-16\pi^2 t}\cos 4\pi x$$

Wave Problem 7.2.1 #1

$u_{tt} = u_{xx} + 2\sin 2\pi x \qquad 0 < x < 1 \qquad t > 0$

$u(x,0) = \sin \pi x \qquad u_t(x,0) = -3\sin 2\pi x$

$u(x,0) = \sin \pi x$

Soln: <u>e-function set up</u> $\qquad u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x$

<u>Subbing in eqn and setting up O.D.E.</u> $\Rightarrow$

$c_n'' + n^2\pi^2 c_n = q_n(t) \qquad q_n(t) = n^{th}$ coeff of $q(x,t) = 2\sin 2\pi x$

$c_n(0) = n^{th}$ coeff of $f(x) = \sin \pi x$

$c_n'(0) =$ " " " $g(x) = -3\sin 2\pi x$

for $q_n(t)$: $\quad 2\sin 2\pi x = \sum_{n=1}^{\infty} q_n(t) \sin n\pi x \Rightarrow q_2(t) = 2$ all others are 0

for $c_n(0)$: $\quad \sin \pi x = \sum_{n=1}^{\infty} c_n(0) \sin n\pi x \Rightarrow c_1(0) = 1$ " " " "

for $c_n'(0)$ $\quad -3\sin 2\pi x = \sum_{n=1}^{\infty} c_n'(0) \sin n\pi x \Rightarrow c_2'(0) = -3$ " " " "

O.D.E.s

$n=1$	$n=2$	all other n
$c_1'' + \pi^2 c_1 = 0$	$c_2'' + 4\pi^2 c_2 = 2$	$c_n \equiv 0$
$c_1(0) = 1$	$c_2(0) = 0$	
$c_1'(0) = 0$	$c_2'(0) = -3$	

SOLNS TO ODES: $\quad n=1 \qquad c_1 = \cos \pi t$

$n=2 \qquad c_2 = -\frac{1}{2\pi^2}\cos 2\pi t - \frac{3}{2\pi}\sin 2\pi t + \frac{1}{2\pi^2}$

Assembling the soln:

$$u(x,t) = \cos \pi t \sin \pi x + \left(-\frac{1}{2\pi^2}\cos 2\pi t - \frac{3}{2\pi}\sin 2\pi t + \frac{1}{2\pi^2}\right)\sin 2\pi x$$

7.1.1. #5 $\quad u_t = u_{xx} + (t-1)\sin \pi x \qquad 0<x<1 \qquad t>0$

$u(0,t) = u(1,t) = 0$

$u(x,0) = \sin \pi x + 2\sin 2\pi x$

Soln:

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin n\pi x$$

$\Rightarrow$

$c_n' + n^2\pi^2 c_n = q_n(t) =$ nth coeff of $(t-1)\sin \pi x$

$c_n(0) =$ nth coeff of $\sin \pi x + 2\sin 2\pi x$

for $q_n(t)$: $\quad (t-1)\sin \pi x = \sum_{n=1}^{\infty} q_n(t) \sin n\pi x \Rightarrow q_1(t) = t-1$

all other $q_n(t) \equiv 0$

for $c_n(0)$: $\quad \sin \pi x + 2\sin 2\pi x = \sum_{n=1}^{\infty} c_n(0) \sin n\pi x \Rightarrow$

$c_1(0) = 1 \quad c_2(0) = 2$

all other $c_n(0) \equiv 0$

eqns

$c_1' + \pi^2 c_1 = t-1 \qquad c_2' + 4\pi^2 c_2 = 0 \qquad$ all other $c_n(t) \equiv 0$

$c_1(0) = 1 \qquad c_2(0) = 2$

$(c_1)_c = k e^{-\pi^2 t} \qquad \Rightarrow c_2 = 2e^{-4\pi^2 t}$

$(c_1)_p = At + B \Rightarrow A + \pi^2(At+B) = t-1$

$\Rightarrow \pi^2 At + A + \pi^2 B = t-1 \Rightarrow A = \frac{1}{\pi^2} \quad B = \frac{-1}{\pi^4}$

$(c_1)_p = \frac{1}{\pi^2}t - \frac{1}{\pi^4}$

$\Rightarrow c_1 = k e^{-\pi^2 t} + \frac{1}{\pi^2}t - \frac{1}{\pi^4}$

$c_1(0) = 1 \Rightarrow 1 = k - \frac{1}{\pi^4} \Rightarrow k = \frac{\pi^4+1}{\pi^4}$

$$c_1 = \frac{\pi^4+1}{\pi^4} e^{-\pi^2 t} + \frac{1}{\pi^2}t - \frac{1}{\pi^4}$$

$$\Rightarrow u(x,t) = \left(\frac{\pi^4+1}{\pi^4} e^{-\pi^2 t} + \frac{t}{\pi^2} - \frac{1}{\pi^4}\right)\sin \pi x + 2e^{-4\pi^2 t}\sin 2\pi x$$

7.1.3 #3
$$\begin{cases} u_t = u_{xx} + t\cos\frac{\pi x}{2} \\ u_x(0,t)=0 \qquad u(1,t)=0 \\ u(x,0) = \cos\frac{\pi x}{2} + 2\cos\frac{5\pi x}{2} \end{cases}$$

Sol: e-function set up: $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\cos\frac{2n-1}{2}\pi x$

subbing and equating coefficients: $c_n' + \frac{(2n-1)^2\pi^2}{4}c_n = q_n(t)$

$\underline{q_n(t)}$ = nth coeff. of $t\cos\frac{\pi x}{2}$; $\underline{c_n(0)}$ = nth coeff of $\cos\frac{\pi x}{2} + 2\cos\frac{5\pi x}{2}$

for $q_n(t)$: $t\cos\frac{\pi x}{2} = \sum_{n=1}^{\infty} c_n(t)\cos\frac{2n-1}{2}\pi x \Rightarrow q_1(t)=t$

all other $q_n(t) \equiv 0$

for $c_n(0)$: $\cos\frac{\pi x}{2} + 2\cos\frac{5\pi x}{2} = \sum_{n=1}^{\infty} c_n(0)\cos\frac{(2n-1)\pi x}{2} \Rightarrow$

$c_1(0)=1 \qquad c_3(0)=2 \qquad$ all other $c_n(0)=0$

eqns:

$n=1$

$c_1' + \frac{\pi^2}{4}c_1 = t$

$c_1(0)=1$

$n=3$

$c_3' + \frac{25\pi^2}{4}c_3 = 0$

$c_3(t) = 2e^{-\frac{25}{4}\pi^2 t}$

all other n

$c_n(t) \equiv 0$

$(c_1)_c = ke^{-\pi^2/4\, t}$

$(c_1)_p = At + B$

sub and solve for A and B $\Rightarrow (c_1)_p = \frac{4}{\pi^2}t - \frac{16}{\pi^4}$

$c_1 = ke^{-\frac{\pi^2}{4}t} + \frac{4}{\pi^2}t - \frac{16}{\pi^4}$

$c_1(0)=1 \Rightarrow k = 1 + \frac{16}{\pi^4} \qquad \Rightarrow c_1 = \frac{\pi^4+16}{\pi^4}e^{-\frac{\pi^2}{4}t} + \frac{4}{\pi^2}t - \frac{16}{\pi^4}$

$$c = \left(\frac{\pi^4+16}{\pi^4}e^{-\frac{\pi^2}{4}t} + \frac{4}{\pi^2}t - \frac{16}{\pi^4}\right)\cos\frac{\pi}{2}x + 2e^{\frac{-25\pi^2 t}{4}}\cos\frac{5\pi x}{2}$$

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A5. Table of Laplace Transforms

	$f(t) = \mathcal{L}^{-1}[F](t)$	$F(s) = \mathcal{L}[f](s)$
1	$f^{(n)}(t)$ (nth derivative)	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
2	$H(t-a)f(t-a)$	$e^{-as}F(s)$
3	$e^{at}f(t)$	$F(s-a)$
4	$(f*g)(t)$	$F(s)G(s)$
5	1	$\frac{1}{s}$ $(s>0)$
6	t^n (n positive integer)	$\frac{n!}{s^{n+1}}$ $(s>0)$
7	e^{at}	$\frac{1}{s-a}$ $(s>a)$
8	$\sin(at)$	$\frac{a}{s^2+a^2}$ $(s>0)$
9	$\cos(at)$	$\frac{s}{s^2+a^2}$ $(s>0)$
10	$\sinh(at)$	$\frac{a}{s^2-a^2}$ $(s>\lvert a\rvert)$
11	$\cosh(at)$	$\frac{s}{s^2-a^2}$ $(s>\lvert a\rvert)$
12	$\delta(t-a)$ $(a \geq 0)$	e^{-as}
13	$e^{a^2 t}\operatorname{erfc}(a\sqrt{t})$ $(a>0)$	$\frac{1}{s+a\sqrt{s}}$
* 14	$\frac{a}{2\sqrt{\pi}} t^{-3/2} e^{-a^2/(4t)}$ $(a>0)$	$e^{-a\sqrt{s}}$
* 15	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$ $(a>0)$	$\frac{1}{s}e^{-a\sqrt{s}}$
16	$-a\sqrt{\frac{t}{\pi}}e^{-a^2/(4t)} + \left(\frac{1}{2}a^2+t\right)\operatorname{erfc}\frac{a}{2\sqrt{t}}$ $(a>0)$	$\frac{1}{s^2}e^{-a\sqrt{s}}$
	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
	$e^{at}\sin bt$	$b/((s-a)^2+b^2)$
	$e^{at}\cos bt$	$s-a/((s-a)^2+b^2)$

Informal Overview of partial Fractions

Any rational Function $\frac{p(x)}{q(x)}$ with degree $p(x) <$ degree $q(x)$ can be written in partial Fractions Form.

To begin $q(x)$ must be factored into real linear factors $(x-\alpha)$ and quadratic factors ax^2+bx+c with no real roots. Repeated factors should be combined.

1) For any factor of the form $(x-\alpha)$ write a term $\frac{A}{x-\alpha}$

2) " " " " " " $(x-\alpha)^n$ write $\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_n}{(x-\alpha)^n}$

3) " " " " " " ax^2+bx+c " $\frac{Ax+B}{ax^2+bx+c}$

4) " " " " " " $(ax^2+bx+c)^n$ " n-terms of the form 3

Sum up all terms of the form 1-4, combine over the common denominator $q(x)$, equate the numerator to $p(x)$ and solve for the coefficients.

You should have the same number of coefficients as the degree of the denominator.