Wave and Heat by L.T.

9.2.1 #1 $u_{tt} = u_{xx} + 1$

$u(0,t) = t \qquad u_t(\infty, t) = 0$

$u(x,0) = 0 \qquad u_t(x,0) = -1$

Soln: $s^2 U - s u(x,0) - u_t(x,0) = U'' + \frac{1}{s}$

$s^2 U + 1 = U'' + \frac{1}{s}$

$U'' - s^2 U = 1 - \frac{1}{s}$

$U_c = \beta e^{-sx} \qquad U_p = \frac{-1}{s^2} + \frac{1}{s^3}$

$U = \beta e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$

$U(0,s) = \mathcal{L}[u(0,t)] = \mathcal{L}[t] = \frac{1}{s^2}$

$\Rightarrow \frac{1}{s^2} = \beta - \frac{1}{s^2} + \frac{1}{s^3} \qquad \beta = \frac{2}{s^2} - \frac{1}{s^3}$

$\Rightarrow U = \left(\frac{2}{s^2} - \frac{1}{s^3}\right) e^{-sx} - \frac{1}{s^2} + \frac{1}{s^3}$

$\mathcal{L}^{-1}\left(\frac{2}{s^2} - \frac{1}{s^3}\right) = 2t - \frac{1}{2}t^2$

$u(x,t) = \left[2(t-x) - \frac{1}{2}(t-x)^2\right] u(t-x) - t + \frac{1}{2}t^2$

answer in text is written incorrectly

#3 $u_{tt} = 4u_{xx} + e^{-3t}$

$u_x(0,t) = 2t \; ; \quad u(x,0) = 0 \qquad u_t(x,0) = -1$

Soln $s^2 U - s u(x,0) - u_t(x,0) = 4U'' + \frac{1}{s+3}$

$s^2 U + 1 = 4U'' + \frac{1}{s+3}$

$U'' - \frac{s^2}{4} U = \frac{1}{4} - \frac{1}{4(s+3)}$

$U_c = \beta e^{-\frac{s}{2}x}$

$U_p = \frac{-1}{s^2} + \frac{1}{s^2(s+3)} = \frac{-(s+2)}{s^2(s^2+3)}$

pf. $\dfrac{s+2}{s^2(s+3)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3}$

$s+2 = As(s+3) + B(s+3) + Cs^2$

$s=0 \implies B = \frac{2}{3}$

$s=-3 \implies C = -\frac{1}{9}$

comparing s^2 : $A + C = 0 \implies A = \frac{1}{9}$

$$U = \beta e^{-s/2 x} - \frac{\frac{1}{9}}{s} - \frac{\frac{2}{3}}{s^2} + \frac{\frac{1}{9}}{s+3}$$

$\mathcal{L}[u_x(0,t)] = U'(0,s) = \frac{2}{s^2}$

$U' = \beta(-\frac{s}{2}) e^{-\frac{s}{2}x} \implies \beta = \frac{-4}{s^3}$

$$U = -\frac{4}{s^3} e^{-\frac{s}{2}x} - \frac{\frac{1}{9}}{s} - \frac{\frac{2}{3}}{s^2} + \frac{\frac{1}{9}}{s+3} \qquad \mathcal{L}^{-1}\left[-\frac{4}{s^3}\right] = -2t^2$$

$$u(x,t) = -2(t-\tfrac{s}{2})^2 H(t - \tfrac{x}{2}) - \tfrac{1}{9} - \tfrac{2}{3}t + \tfrac{1}{9}e^{-3t}$$

heat problem 2

9.2.2 #1 $\quad u_t = u_{xx} + 1 \qquad u(x,0) = \sin 2x \qquad u(0,t) = t+1$

Soln $\quad sU - \sin 2x = U'' + \frac{1}{s}$

$U'' - sU = -\sin 2x - \frac{1}{s}$

$U_c = \beta e^{-\sqrt{s}x}$

$U_p = \frac{1}{s^2} + \frac{1}{s+4}\sin 2x$ $\quad$ after doing undet. coeffs

$$U = \beta e^{-\sqrt{s}x} + \frac{1}{s^2} + \frac{1}{s+4}\sin 2x$$

$U(0,s) = \mathcal{L}[u(0,t)] = \frac{1}{s^2} + \frac{1}{s} \implies$

$\frac{1}{s^2} + \frac{1}{s} = \beta + \frac{1}{s^2} + \frac{1}{s+4} \qquad \beta = \frac{1}{s}$

$$U = \frac{1}{s}e^{-\sqrt{s}x} + \frac{1}{s^2} + \frac{1}{s+4}\sin 2x$$

$$u(x,t) = \operatorname{erfc}\frac{x}{2\sqrt{t}} + t + e^{-4t}\sin 2x$$