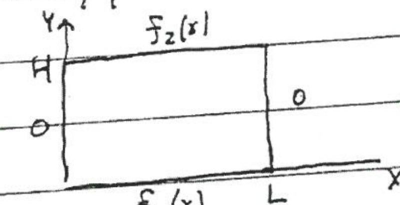
Laplace Eqn in the Plane Basic Version I

1) eqn $u_{xx} + u_{yy} = 0$

2) bdy conds

version (i)



$u(0,y) = u(L,y) = 0$ $u(x,0) = f_1(x)$ $u(x,H) = f_2(x)$

eqn separates as:
side bdy conds traslate as:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases} \qquad Y'' - \lambda Y = 0$$

SL BVP in X $\rightsquigarrow$ e-vals $\lambda_n = \frac{n^2\pi^2}{L^2}$ e-functions: $X_n \sim \sin\frac{n\pi x}{L}$

$$\Rightarrow \quad Y_n \sim \alpha_n \cosh\frac{n\pi y}{L} + \beta_n \sinh\frac{n\pi y}{L}$$

rewrite as $\quad Y_n \sim \gamma_n \sinh\frac{n\pi y}{L} + \delta_n \sinh\frac{n\pi(y-H)}{L}$

produces fund soln

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(a_n \sinh\frac{n\pi y}{L} + b_n \sinh\frac{n\pi(y-H)}{L}\right)$$

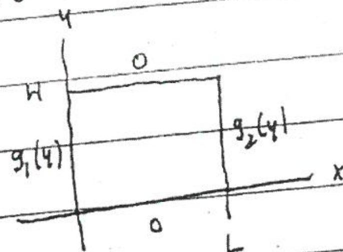
The remaining bdy conds give you the coefficients:

$$u(x,H) = f_2(x) \Rightarrow \quad a_n \cdot \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f_2(x)\sin\frac{n\pi x}{L}\,dx$$

$$u(x,0) = f_1(x) \Rightarrow \quad -b_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}\,dx$$

II

1) eqn $u_{xx} + u_{yy} = 0$



2) bdy conds

version 2)

$u(x,0) = 0$ $\quad u(x,H) = 0$ $\quad u(0,y) = g_1(y)$ $\quad u(L,y) = g_2(y)$

3) translates as $Y(0) = Y(H) = 0$

eqn separates as $X'' - \lambda X = 0$ $\qquad Y'' + \lambda Y = 0$

$Y(0) = 0 \qquad Y(H) = 0$

SLBVP in Y $\rightsquigarrow$ e-vals $\lambda_n = \dfrac{n^2\pi^2}{H^2}$ e-functions $Y_n \sim \sin \dfrac{n\pi y}{H}$

$$\Rightarrow X_n \sim \alpha_n \cosh \frac{n\pi x}{H} + \beta_n \sinh \frac{n\pi x}{H}$$

rewrite as $X_n \sim \gamma_n \sinh \dfrac{n\pi x}{H} + \delta_n \sinh \dfrac{n\pi (x-L)}{H}$

produces final soln

$$u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi y}{H} \left[a_n \sinh \frac{n\pi x}{H} + b_n \sinh \frac{n\pi (x-L)}{H} \right]$$

Remaining bdy conds give you the coeffs

$$u(L,y) = g_2(y) \Rightarrow a_n \sinh \frac{n\pi L}{H} = \frac{2}{H} \int_0^H g_2(y) \sin \frac{n\pi y}{H}\, dy$$

$$u(0,y) = g_1(y) \Rightarrow -b_n \sinh \frac{n\pi L}{H} = \frac{2}{H} \int_0^H g_1(y) \sin \frac{n\pi y}{H}\, dy$$