E-Function Expansion Problem

7.1.1 #3

$$u_t = u_{xx} + e^{-t}\sin 3\pi x - \sin 5\pi x \qquad 0<x<1 \qquad t>0$$

$$u(0,t)=0 \qquad u(1,t)=0$$

$$u(x,0) = \sin \pi x + 2\sin 3\pi x$$

SOLN e-function assumption $\quad u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin n\pi x$

substitution in the equation gives

$$c_n' + n^2\pi^2 c_n = q_n(t) \quad = n^{th} \text{ coefficient of } \quad q(x,t) = e^{-t}\sin 3\pi x - \sin 5\pi x$$

$$c_n(0) = n^{th} \text{ coefficient of } \quad u(x,0) = f(x) = \sin\pi x + 2\sin 3\pi x$$

computing $q_n(t)$

$$q(x,t) = e^{-t}\sin 3\pi x - \sin 5\pi x = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x$$

matching up $\Rightarrow$ $q_3(t) = e^{-t}$ $\qquad q_5(t) = -1$

computing $c_n(0)$

$$f(x) = \sin\pi x + 2\sin 3\pi x = \sum_{n=1}^{\infty} c_n(0)\sin n\pi x$$

matching up $\Rightarrow$ $c_1(0) = 1 \qquad c_3(0) = 2$

1st order o.d.e.'s to solve

$n=1$	$n=3$	$n=5$
$c_1' + \pi^2 c_1 = 0$	$c_3' + 9\pi^2 c_3 = e^{-t}$	$c_5' + 25\pi^2 c_5 = -1$
$c_1(0)=1$	$c_3(0)=2$	$c_5(0)=0$

$c_n \equiv 0$ all other c_n.

now solve the o.de.'s and assemble the solution

7.1.1. #3 continued - solving the o.d.e.s

$n=1 \quad \Rightarrow \quad c_1 = c_1(0)e^{-\pi^2 t} \qquad c_1 = e^{-\pi^2 t}$

$n=3 \qquad (c_3)_c = k e^{-9\pi^2 t}$

$(c_3)_p = Ae^{-t} \qquad (c_3)_p' = -Ae^{-t}$

$\Rightarrow \quad (-1+9\pi^2)A = 1$

$A = \frac{1}{9\pi^2-1} \qquad \Rightarrow c_3 = ke^{-9\pi^2 t} + \frac{1}{9\pi^2-1}e^{-t}$

$c_3(0)=2 \Rightarrow \quad 2 = k + \frac{1}{9\pi^2-1} \qquad k = 2 - \frac{1}{9\pi^2-1} = \frac{18\pi^2-3}{9\pi^2-1}$

$c_3 = \frac{1}{9\pi^2-1}\left((18\pi^2-3)e^{-9\pi^2 t} + e^{-t}\right)$

$n=5 \qquad (c_5)_c = k e^{-25\pi^2 t}$

$(c_5)_p = \frac{-1}{25\pi^2} \qquad c_5 = ke^{-25\pi^2 t} - \frac{1}{25\pi^2}$

$c_5(0) = 0 \quad \Rightarrow \quad k = \frac{1}{25\pi^2} \quad \Rightarrow \quad c_5 = \frac{1}{25\pi^2}\left(e^{-25\pi^2 t} - 1\right)$

$$u(x,t) = e^{-\pi^2 t}\sin\pi x + \frac{1}{9\pi^2-1}\left((18\pi^2-3)e^{-9\pi^2 t} + e^{-t}\right)\sin 3\pi x$$
$$+ \frac{1}{25\pi^2}\left(e^{-25\pi^2 t} - 1\right)\sin 5\pi x$$

E-funtion Expansion Problems §7

7.1.1 #2

$$\begin{cases} u_t = u_{xx} + \left(3+\pi^2(3t-2)\right)\sin\pi x + \left(9\pi^2 t^2 + 2t\right)\sin 3\pi x \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = f(x) = -2\sin\pi x \end{cases}$$

e-function assumption: $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin n\pi x$

after substitution: $c_n' + n^2\pi^2 c_n = q_n(t)$ $\qquad q_n(t) =$ nth coeff of $q(x,t)$

$c_n(0) =$ nth coeff. of $f(x)$

to get $q_n(t)$: $q(x,t) = \left(3+\pi^2(3t-2)\right)\sin\pi x + \left(9\pi^2t^2+2t\right)\sin 3\pi x = \sum_{n=1}^{\infty} q_n(t)\sin n\pi x$

matching up $\Rightarrow$ $q_1(t) = 3+\pi^2(3t-2)$

$q_3(t) = 9\pi^2 t^2 + 2t$

all other q_n's are 0

to get $c_n(0)$: $f(x) = u(x,0) = -2\sin\pi x = \sum_{n=1}^{\infty} c_n(0)\sin n\pi x$

matching up $\Rightarrow$ $c_1(0) = -2$ all other $c_n(0)$ are 0.

o.d.e.'s: $n=1$: $c_1' + \pi^2 c_1 = 3 + \pi^2(3t-2)$, $c_1(0) = -2$ | $n=3$: $c_3' + 9\pi^2 c_3 = 9\pi^2 t^2 + 2t$, $c_3(0) = 0$

solutions to o.d.e.'s: by inspection or undetermined coeffs: $c_1 = 3t-2$; $c_3 = t^2$

all other $c_n \equiv 0$.

$$u(x,t) = (3t-2)\sin\pi x + t^2\sin 3\pi x$$

7.1.3 # 1

$u_t = u_{xx} + \sin\left(\frac{3}{2}\pi x\right) - 2\sin\left(\frac{5}{2}\pi x\right)$ $\quad 0 < x < 1 \quad t > 0$

$u(0) = u_x(1,t) = 0$

$u(x,0) = \sin \frac{3}{2}\pi x = f(x)$

-fution assumption

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin \frac{(2n-1)\pi x}{2}$$

substitution

$$\sum c_n' \sin \frac{(2n-1)\pi}{2} x = \sum -\frac{(2n-1)^2\pi^2}{4} \cdot c_n \sin \frac{(2n-1)\pi}{2} x + q(x,t)$$

comparing to get o.d.e

$$c_n' + \frac{(2n-1)^2\pi^2}{4} c_n = q_n$$ q_n is the nth coeff of $q(x,t)$

$c_n(0) = n$th coeff of $f(x)$.

for $q_n(t)$

$$\sin \frac{3}{2}\pi x - 2\sin \frac{5}{2}\pi x = \sum_{n=1}^{\infty} q_n(t) \sin \frac{(2n-1)\pi}{2} x$$

matching up $\quad n=2 \Rightarrow q_2(t) = 1$

$\quad n=3 \Rightarrow q_3(t) = -2$ $\quad$ all others are 0

for $c_n(0)$

$$f(x) = u(x,0) = \sin \frac{3}{2}\pi x = \sum_{n=1}^{\infty} c_n(0) \sin \frac{(2n-1)\pi}{2} x$$

matching up $\quad n=2 \Rightarrow c_2(0) = 1$

o.d.e.s

$n=2$

$$c_2' + \frac{9\pi^2}{4} c_2 = 1$$
$$c_2(0) = 1$$

$n=3$

$$c_3' + \frac{25}{4}\pi^2 c_3 = -2$$
$$c_3(0) = 0$$

all others are 0

after solving the o.des. for c_n

$$u(x,t) = \left[\left(\frac{9\pi^2-4}{9\pi^2}\right) e^{-\frac{9\pi^2}{4}t} + \frac{4}{9\pi^2}\right] \sin \frac{3\pi x}{2} + \left[\frac{8}{25\pi^2} e^{-\frac{25\pi^2}{4}t} - \frac{8}{25\pi^2}\right] \sin \frac{5\pi x}{2}$$