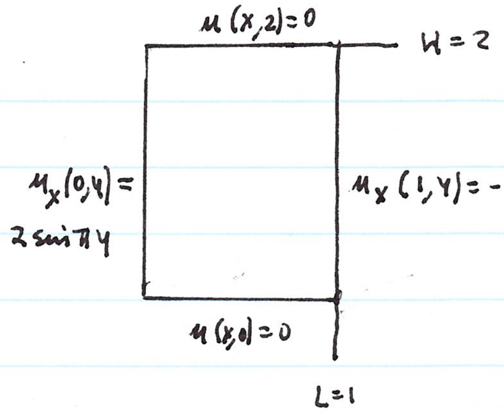
Laplace in a Rectangle   5.3.1 # 2,3

#2



$\Rightarrow$ set up SL BVP by separating

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$$X'' - \lambda X = 0$$

$$\boxed{Y'' + \lambda Y = 0 \qquad Y(0)=0 \quad Y(2)=0}$$

SL BVP in Y: e-vals $\lambda_n = \frac{n^2\pi^2}{4}$

e-funtion's $Y_n = \sin \frac{n\pi y}{2}$

Set up X_n $\qquad X_n = \alpha_n \cosh \frac{n\pi x}{2} + \beta_n \cosh \frac{n\pi(x-1)}{2}$

formal soln $\Rightarrow$

$$u(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi y}{2}\left[a_n \cosh \frac{n\pi x}{2} + b_n \cosh \frac{n\pi(x-1)}{2}\right]$$

$$u_x(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi y}{2}\left[\frac{n\pi a_n}{2} \sinh \frac{n\pi x}{2} + \frac{n\pi b_n}{2} \sinh \frac{n\pi(x-1)}{2}\right]$$

$$u_x(0,y) = 2\sin \pi y \Rightarrow 2 \sin \pi y = \sum_{n=1}^{\infty}\left(\frac{n\pi b_n}{2}\sinh \frac{-n\pi}{2}\right)\sin\frac{n\pi y}{2}$$

match up $\Rightarrow$ $n=2 \qquad b_2 = -\frac{2}{\pi \sinh \pi}$

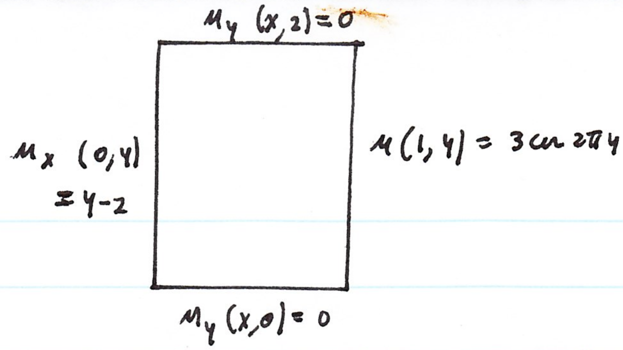
$$u_x(1,y) = -\sin 2\pi y \Rightarrow -\sin 2\pi y = \sum_{n=1}^{\infty}\left(\frac{n\pi a_n}{2}\sinh \frac{n\pi}{2}\right)\sin\frac{n\pi y}{2}$$

match up $\Rightarrow$ $n=4 \qquad a_4 = \frac{-1}{2\pi \sinh 2\pi}$

Assemble

$$u(x,y) = \frac{-2}{\pi \sinh \pi} \sin \pi y \cosh \pi(x-1) - \frac{1}{2\pi \sinh 2\pi} \sin 2\pi y \cosh 2\pi x$$

5.3.1 #3



Soln: Translate boundary conditions as $X''-\lambda X=0$, $\begin{cases} Y''+\lambda Y=0 \\ Y'(0)=0 \quad Y'(2)=0 \end{cases}$

SL BVP in Y: $\lambda_n = \frac{n^2\pi^2}{4}$ $\quad Y_n \sim \cos\frac{n\pi y}{2}$ $\quad n=0,1,2,\ldots$

$$\Rightarrow \underbrace{X_0 = a_0+b_0x}_{n=0} \qquad X_n = \underbrace{\alpha_n\cosh\frac{n\pi x}{2}}_{n>0} + \beta_n\sinh\frac{n\pi(x-1)}{2}$$

formal soln

$$u(x,y) = a_0+b_0x + \sum_{n=1}^{\infty}\cos\frac{n\pi y}{2}\left(a_n\cosh\frac{n\pi x}{2} + b_n\sinh\frac{n\pi(x-1)}{2}\right)$$

$$u_x(x,y) = b_0 + \sum_{n=1}^{\infty}\left(a_n\frac{n\pi}{2}\sinh\frac{n\pi x}{2} + b_n\frac{n\pi}{2}\cosh\frac{n\pi(x-1)}{2}\right)\cos\frac{n\pi y}{2}$$

left boundary condition $\quad u_x(0,y)=y-2 \Rightarrow$

$$y-2 = b_0 + \sum_{n=1}^{\infty}\left(b_n\cdot\frac{n\pi}{2}\cosh\frac{n\pi}{2}\right)\cos\frac{n\pi y}{2}$$

$$\Rightarrow \quad b_0 = \frac{\int_0^2 (y-2)\,dy}{\int_0^2 dy} = -1$$

$$n>0 \quad \frac{n\pi}{2}b_n\cosh\frac{n\pi}{2} = \frac{\int_0^2 (y-2)\cos\frac{n\pi y}{2}\,dy}{\int_0^2 \cos^2\frac{n\pi y}{2}\,dy}.$$

After integrating and solving for b_n

$$b_n = \frac{8}{n^3\pi^3\cosh\frac{n\pi y}{2}}(\cos n\pi - 1) \qquad n>0$$

5.3.1 cont'd

#3

right boundary.

$$u(1,y) = 3\cos 2\pi y \implies$$

$$3\cos 2\pi y = a_0 + b_0 + \sum_{n=1}^{\infty}\left(a_n \cosh\frac{n\pi}{2}\right)\cos\frac{n\pi y}{2}$$

We already know $b_0 = -1$

So matching up

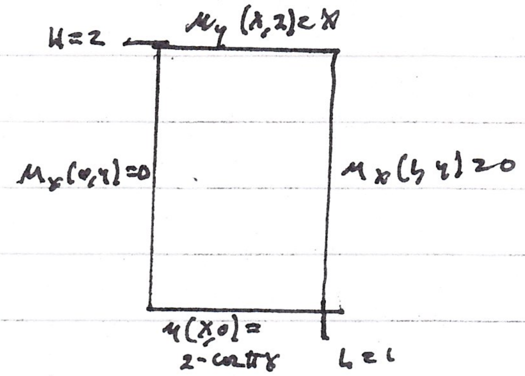
$$3\cos 2\pi y = a_0 - 1 + \sum_{n=1}^{\infty}\left(a_n \cosh\frac{n\pi}{2}\right)\cos\frac{n\pi y}{2}$$

$$\implies a_0 = 1 \qquad a_4 \cosh 2\pi = 3 \qquad a_4 = \frac{3}{\cosh 2\pi}$$

Assembling the solution

$$u(x,y) = 1 - x + \sum_{n=1}^{\infty} \frac{8(\cos n\pi - 1)}{n^3\pi^3 \cosh\frac{n\pi}{2}} \sinh\frac{n\pi(x-1)}{2}\cos\frac{n\pi y}{2} + \frac{3}{\cosh 2\pi}\cosh 2\pi x \cos 2\pi y$$

Laplace in a Rectangle



5.3.1 #. 4 $\quad u_x(0,y)=0 \qquad u_x(1,y)=0$

$u(x,0) = 2-\cos\pi x \qquad u_y(x,2) = x$

Soln: translate boundary conds.

$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$

$$X'(0)=0 \qquad X'(1)=0$$

$$\Rightarrow \lambda_n = n^2\pi^2 \qquad X_n \sim \cos n\pi x \qquad \Rightarrow Y_0 = a_0 y + b_0$$

$n = 0, 1, 2, \ldots \qquad n>0,\ Y_n = \sinh n\pi y + \delta_n \cosh n\pi(y-2)$

formal soln: $u(x,y) = a_0 y + b_0 + \sum_{n=1}^{\infty} \cos n\pi x \left(a_n \sinh n\pi y + b_n \cosh n\pi(y-2)\right)$

$u(x,0) = 2-\cos\pi x \implies$ by matching up $\quad b_0 = 2 \quad b_1 = \frac{-1}{\cosh 2\pi}$

$$u_y(x,y) = a_0 + \sum_{n=1}^{\infty} \cos n\pi x \left(a_n \cdot n\pi \cosh n\pi y + b_n n\pi \sinh n\pi(y-2)\right)$$

$$u_y(x,2) = x \implies x = a_0 + \sum_{n=1}^{\infty} (a_n \cdot n\pi \cosh 2n\pi) \cos n\pi x$$

$$a_0 = \frac{\int_0^1 x\,dx}{\int_0^1 dx} = \frac{1}{2} \qquad n\pi\cosh(2n\pi)\, a_n = \frac{\int_0^1 x\cos n\pi x\,dx}{\int_0^1 \cos^2 n\pi x\,dx}$$

after integration and solving for a_n: $\quad a_n = \frac{2}{n^3\pi^3\cosh 2n\pi}(\cos n\pi - 1)$

Assembling the soln:

$$u(x,y) = \frac{1}{2}y + 2 + \sum_{n=1}^{\infty} \frac{2(\cos n\pi - 1)}{n^3\pi^3 \cosh 2n\pi} \cos n\pi x \sinh n\pi y - \frac{1}{\cosh 2\pi}\cos\pi x \cosh\pi(y-2)$$

e-function expansion 7.1.2

#1 $\begin{cases} u_t = u_{xx} + 2 + \cos 2\pi x \\ u_x(0,t) = 0 \quad u_x(1,t) = 0 \\ u(x,0) = 2\cos\pi x - \cos 2\pi x \end{cases}$

soln: e-function assumption $u(x,t) = \sum_{n=0}^{\infty} c_n(t) \cos n\pi x$

after substitution: $c_n' + n^2\pi^2 c_n =$ n^{th} coeff. of $q(x,t) = 2 + \cos 2\pi x$

$c_n(0)$: n^{th} coeff of $f(x) = 2\cos\pi x - \cos 2\pi x$

$q_n(t) =$ n^{th} coeff of $q(x,t) \Longrightarrow$

$$2 + \cos 2\pi x = \sum_{n=0}^{\infty} q_n(t) \cos n\pi x$$

match up $q_0(t) = 2$ $\quad q_2(t) = 1$ all others are 0

$c_n(0) =$ n^{th} coeff of $f(x) \Longrightarrow$

$$2\cos\pi x - \cos 2\pi x = \sum_{n=1}^{\infty} c_n(0) \cos n\pi x$$

match up $n=1$ $\quad c_1(0) = 2$ $\quad n=2$ $\quad c_2(0) = -1$

all others are 0

O.D.E.'s			
	$c_0' = 2$	$c_1' + \pi^2 c_1 = 0$	$c_2' + 4\pi^2 c_2 = 1$
	$c_0(0) = 0$	$c_1(0) = 2$	$c_2(0) = -1$

all other $c_n \equiv 0$

solving: $c_0 = 2t$; $c_1 = 2e^{-\pi^2 t}$; $c_2 = \frac{1}{4\pi^2} - \frac{4\pi^2+1}{4\pi^2} e^{-4\pi^2 t}$

Assembling

$$u(x,t) = 2t + 2e^{-\pi^2 t}\cos\pi x + \left[\frac{1}{4\pi^2} - \frac{4\pi^2+1}{4\pi^2} e^{-4\pi^2 t}\right] \cos 2\pi x$$

heat problems

7.1.3 #1

$u_t = u_{xx} + \sin\left(\frac{3}{2}\pi x\right) - 2\sin\left(\frac{5}{2}\pi x\right) \qquad 0 < x < 1 \qquad t > 0$

$u(0,t) = 0 \qquad u_x(1,t) = 0$

$u(x,0) = \sin \frac{3}{2}\pi x = f(x)$

e-function assumption $\qquad u(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin \frac{(2n-1)\pi}{2} x$

substitution $\qquad \sum c_n' \sin \frac{(2n-1)\pi}{2} x = \sum -\frac{(2n-1)^2\pi^2}{4} \cdot c_n \sin \frac{(2n-1)\pi}{2} x + q(x,t)$

comparing $\qquad c_n' + \frac{(2n-1)^2\pi^2}{4} c_n = q_n \qquad q_n$ is the nth coeff of $q(x,t)$

get o.d.e $\qquad c_n(0) =$ nth coeff of $f(x)$.

on $q_n(t)$ $\qquad \sin \frac{3}{2}\pi x - 2\sin\frac{5}{2}\pi x = \sum_{n=1}^{\infty} q_n(t) \sin \frac{(2n-1)\pi}{2} x$

matching up $\qquad n=2 \Rightarrow q_2(t) = 1$

$n = 3 \Rightarrow q_3(t) = -2 \qquad$ all others are 0

on $c_n(0)$ $\qquad f(x) = u(x,0) = \sin \frac{3}{2}\pi x = \sum_{n=1}^{\infty} c_n(0) \sin \frac{(2n-1)\pi}{2} x$

matching up $\quad n=2 \Rightarrow c_2(0) = 1$

o.d.e.s

$n=2$

$c_2' + \frac{9\pi^2}{4} c_2 = 1$

$c_2(0) = 1$

$n=3$

$c_3' + \frac{25}{4}\pi^2 c_3 = -2$

$c_3(0) = 0$

all others are 0

after solving the o.des. for c_n

$$u(x,t) = \left[\left(\frac{9\pi^2-4}{9\pi^2}\right) e^{-\frac{9\pi^2}{4}t} + \frac{4}{9\pi^2}\right] \sin\frac{3\pi x}{2} + \left[\frac{8}{25\pi^2} e^{-\frac{25\pi^2}{4}t} - \frac{8}{25\pi^2}\right] \sin \frac{5\pi x}{2}$$