PRACTICE

(A quiz type problem)

SOLVE USING LAPLACE TRANSFORMS

$u_t = u_{xx} \qquad x>0 \qquad t>0$

$u(0,t) = e^t$

$u(x,0) = e^{-2x}$

$\lim_{x\to\infty} u_t(x,t) = 0$

SOLN let $U = \mathcal{L}[u] \Rightarrow U' = \mathcal{L}[u_x]$

$\Rightarrow \quad sU - u(x,0) = U''$

$\boxed{U'' - sU = -e^{-2x}}$ $\qquad U(0,s) = \mathcal{L}[u(0,t)] = \mathcal{L}[e^t] = \frac{1}{s-1}$

$\boxed{U(0,s) = \frac{1}{s-1}}$

$U_c = \beta e^{-\sqrt{s}\,x} \qquad U_p = Ae^{-2x} \Rightarrow 4A\cancel{e^{-2x}} - sA\cancel{e^{-2x}} = -\cancel{e^{-2x}}$

$\Rightarrow (4-s)A = -1$

$A = \frac{1}{4-s}$

$U = \beta e^{-\sqrt{s}\,x} + (1/s-4)\cdot e^{-2x}$

$U(0,s)\ \frac{1}{s-1} \Rightarrow \frac{1}{s-1} = \beta + \frac{1}{s-4} \Rightarrow \beta = \frac{1}{s-1} - \frac{1}{s-4}$

$U = \left(\frac{1}{s-1} \cdot \frac{1}{s-4}\right)e^{-\sqrt{s}\,x} + \frac{1}{s-4}e^{-2x}$

$u = (e^t - e^{4t}) * \frac{x}{2\sqrt{\pi}} t^{-3/2} e^{-x^2/4t} + e^{4t}e^{-2x}$

9.2.2 #3 (not a quiz type problem)

$$u_t = u_{xx} + \begin{cases} -3e^{-3t} + 2(x-1)^2 - 4t & 0 \le x \le 1 \\ -3e^{-3t} & 1 < x \end{cases}$$

$u(x,0) = 1$ $u_x(0,t) = -4t$; u and u_x cont. at $x=1$; $u(\infty,t)=0$

Soln:

Part a) $0 \le x \le 1$ $u_t = u_{xx} - 3e^{-3t} + 2(x-1)^2 - 4t$

transformed equation: $U'' - sU = -1 + \dfrac{3}{s+3} - \dfrac{2(x-1)^2}{s} + \dfrac{4}{s^2}$

$U_c = \beta e^{-\sqrt{s}\,x}$

$U_p = A(x-1)^2 + B(x-1) + C$

solving the undet. coeffs problem: $A = \dfrac{2}{s^2}$ $B = 0$ $C = \dfrac{1}{s+3}$

$$\Rightarrow \quad U = U_c + U_p = \beta e^{-\sqrt{s}\,x} + \frac{2}{s^2}(x-1)^2 + \frac{1}{s+3}$$

Part b) $x > 1$ $u_t = u_{xx} - 3e^{-3t}$

transformed equation $U'' - sU = -1 + \dfrac{3}{s+3}$

soln: $U = \gamma e^{-\sqrt{s}\,x} + \dfrac{1}{s+3}$

Piecing together the solutions $\beta = \gamma$ for continuity at $x=1$

$$U = \beta e^{-s x} + \frac{2}{s^2}(x-1)^2 + \frac{1}{s+3} \quad \text{for } 0 \le x \le 1$$

$U'(0,s) = \mathcal{L}\left[u_x(0,t)\right] = -\dfrac{4}{s^2}$ $\Rightarrow$ $\beta = 0$

So $U = \dfrac{2}{s^2}(x-1)^2 + \dfrac{1}{s+3}$ $0 \le x \le 1$

$$u(x,t) = \begin{cases} 2t(x-1)^2 + e^{-3t} & 0 \le x \le 1 \\ e^{-3t} & x > 1 \end{cases}$$