FOURIER THEOREM

Given a p.w. continuos $f(x)$, $-L \le x \le L$, the F. S. of $f(x)$ is defined to be $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

The coefficients are $$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

If $f(x)$ is odd then $a_n = 0$ for all $n \ge 0$

If $f(x)$ is even then $b_n = 0$ for all $n \ge 1$

If $f(x)$ is p.w. differentiable on $[-L, L]$ then the F.S. converges to the period extension of $f(x)$ wherever the periodic extension is continuous, and to $\left(f(a^+) + f(a^-)\right)/2$ at each discontinuity a.

If $f(x)$ is given on $[0, L]$ and a sine series is required then
$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$ and
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

If $f(x)$ is given on $[0, L]$ and a cosine series is required then
$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$ and
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx.$$

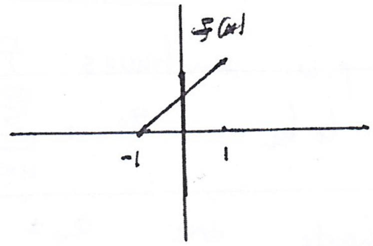
The F.S.S. converges to the periodic extension of the odd extension of $f(x)$.

The F.C.S. converges to the periodic extension of the even extension of $f(x)$.

F.S. 2.1 #5

$f(x) = x+1, \quad -1 \le x \le 1$

Find and graph the F.S. of $f(x)$



Soln $L = 1$

$$a_0 = \frac{1}{L}\int_{-1}^{1} f(x)\,dx = 2$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \int_{-1}^{1}(x+1)\cos n\pi x\,dx$$

$u = x+1 \quad dv = \cos n\pi x$

$du = dx \quad v = \frac{1}{n\pi}\sin n\pi x$

$$= uv\Big|_{-1}^{1} - \frac{1}{n\pi}\int_{-1}^{1}\sin n\pi x\,dx = 0$$

$$b_n = \int_{-1}^{1}(x+1)\sin n\pi x\,dx = \frac{-(x+1)}{n\pi}\cos n\pi x\Big|_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}\cos n\pi x\,dx$$

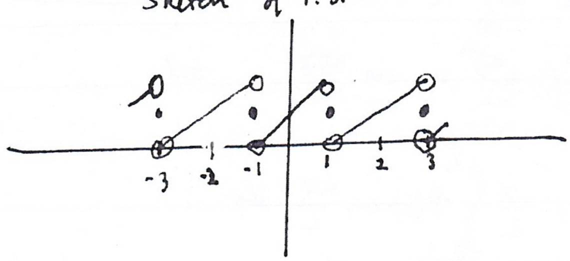
$u = x+1 \quad dv = \sin n\pi x\,dx$

$du = dx \quad v = \frac{-1}{n\pi}\cos n\pi x$

$$= -\frac{2}{n\pi}\cos n\pi + \underbrace{\frac{1}{n^2\pi^2}\sin n\pi x\Big|_{-1}^{1}}_{0} = -\frac{2}{n\pi}\cos n\pi$$

$$f(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{-2}{n\pi}\cos n\pi\right)\sin n\pi x$$

sketch of F.S.

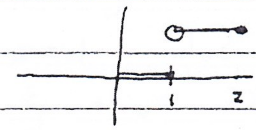


FCS and FSS problems

chapter 2.2

#1 $f(x) = \begin{cases} 0 & 0 \le x \le 1 \\ 1 & 1 < x \le 2 \end{cases}$ Write and sketch its FCS and FSS.

Soln FCS $L = 2$



$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\cdot 1 = 1.$$

$n \ge 1$ $$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \int_1^2 \cos\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{2}\right)$$

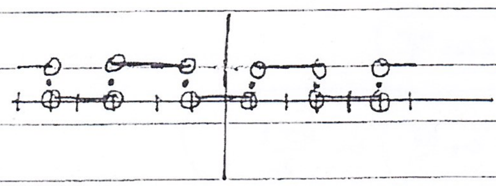
FSS $L = 2$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \int_1^2 \sin\frac{n\pi x}{2}\,dx = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2$$

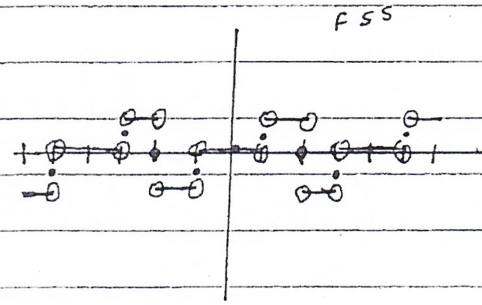
$$= \frac{2}{n\pi}\cos\frac{n\pi}{2} - \frac{2}{n\pi}\cos n\pi$$

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi x}{2}$$

FCS

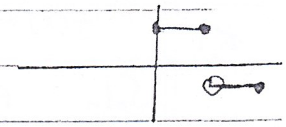


FSS



2.

\# 3 $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -1 & 1 < x \le 2 \end{cases}$ Write its FCS and FSS



Soln $L = 2$

FCS $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 0$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^1 \cos\frac{n\pi x}{2}\,dx - \int_1^2 \cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^1 - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2 = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$$f(x) \sim \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

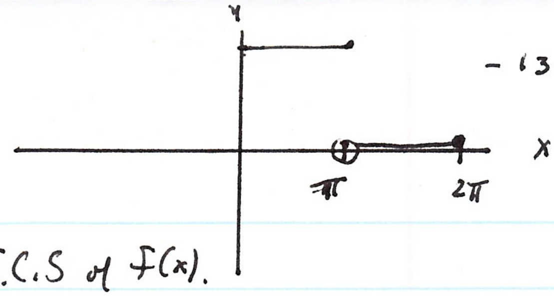
FSS $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \int_0^2 f(x)\sin\frac{n\pi x}{2}\,dx$

$$= \int_0^1 \sin\frac{n\pi x}{2}\,dx - \int_1^2 \sin\frac{n\pi x}{2}\,dx = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_0^1 + \frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2$$

$$= -\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi} + \frac{2}{n\pi}\cos n\pi - \frac{2}{n\pi}\cos\frac{n\pi}{2}$$

$$= \frac{2}{n\pi}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right)\sin\frac{n\pi x}{2}$$



2.2 #2 $\quad f(x) = \begin{cases} 2 & 0 \le x \le \pi \\ 0 & \pi < x \le 2\pi \end{cases}$

Find and graph the F.S.S and F.C.S of $f(x)$.

Soln: $\quad L = 2\pi$

F.C.S. $\quad a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\cdot 2\pi = 2$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{\pi}\int_0^{\pi} 2\cos\frac{nx}{2}\,dx = \frac{4}{n\pi}\sin\frac{nx}{2}\Big|_0^{\pi}$$

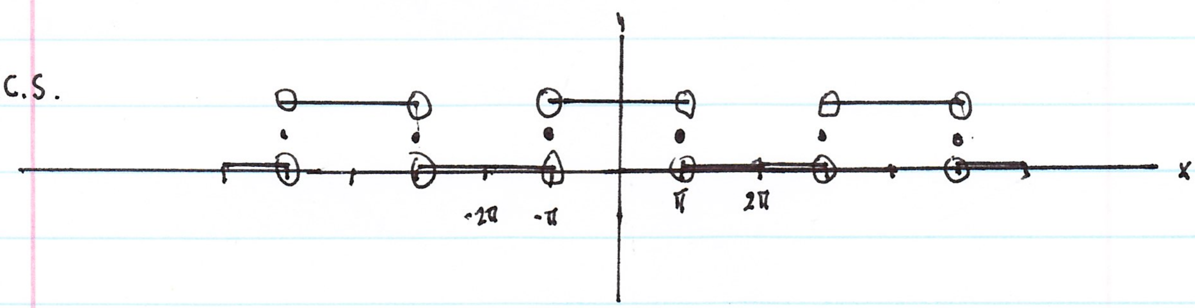
$$= \frac{4}{n\pi}\sin\frac{n\pi}{2} \quad \Rightarrow \text{F.C.S is} \quad f(x) = 1 + \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}\sin\frac{n\pi}{2}\right)\cos\frac{nx}{2}$$

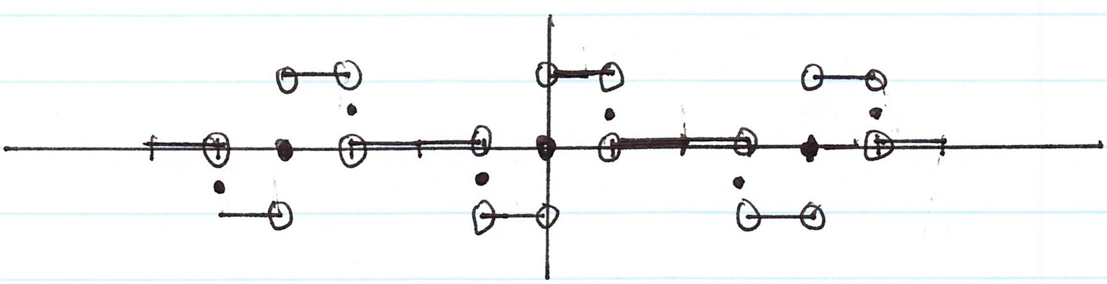
F.S.S. $\quad b_n = \frac{3}{L}\int_0^L f(x)\sin\frac{n\pi x}{2} = \frac{1}{\pi}\int_0^{\pi} 2\sin\frac{nx}{2}\,dx =$

$$= -\frac{4}{n\pi}\cos\frac{nx}{2}\Big|_0^{\pi} = -\frac{4}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n\pi} \quad \Rightarrow$$

F.S.S is $\quad f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n\pi} - \frac{4}{n\pi}\cos\frac{n\pi}{2}\right)\sin\frac{nx}{2}$

F.C.S.





THE FOUR BASIC S-L BVP'S

EQUATION: $f'' + \lambda f = 0$, $f = f(x)$, $0 \le x \le L$

1. BOUNDARY CONDS. $f(0) = 0$ $f(L) = 0$

e-values $\lambda_n = \frac{n^2 \pi^2}{L^2}$ $n = 1, 2, \ldots$

e-fuctions $f_n \sim \sin \frac{n\pi x}{L}$

2) BOUNDARY CONDS. $f'(0) = 0$ $f'(L) = 0$

e-values $\lambda_n = \frac{n^2 \pi^2}{L^2}$ $n = 0, 1, 2, \ldots$

e-fuctions $f_n \sim \cos \frac{n\pi x}{L}$

* note n starts at 0

BOUNDARY CONDS. $f(0) = 0$ $f'(L) = 0$

e-values $\lambda_n = \left[\frac{(2n-1)\pi}{2L}\right]^2$ $n = 1, 2, \ldots$

e-fuctions $f_n \sim \sin \frac{(2n-1)\pi}{2L} \cdot x$

) BOUNDAY CONDS. $f'(0) = 0$ $f(L) = 0$

e-values $\lambda_n = \left[\frac{(2n-1)\pi}{2L}\right]^2$

e-functions $f_n \sim \cos \frac{(2n-1)\pi}{2L} \cdot x$