h.w. 5.1.3 heat problems

$$\begin{cases} u_t = u_{xx} & 0 < x < 1 \quad t > 0 \\ u(0,t) = 0 & u_x(1,t) = 0 \end{cases}$$

Eqn separates to $T' = -\lambda T$ $\quad X'' + \lambda X = 0$

Bdy conditions traslate to $X(0) = 0$ $\quad X'(1) = 0$

The SL BVP in X is the 3rd basic kind.

The e-vals are $\lambda_n = \left(\frac{(2n-1)\pi}{2}\right)^2$; the e-functions are $X_n \sim \sin\frac{(2n-1)\pi x}{2}$

Formal Soln: $u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{4}t}$

#1 $u(x,0) = 3\sin\frac{\pi x}{2} - \sin\frac{5\pi x}{2} \implies$

$$3\sin\frac{\pi x}{2} - \sin\frac{5\pi x}{2} = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)\pi x}{2}$$

match up: $a_1 = 3$ $\quad a_3 = -1$ $\quad$ all other a_n's are 0

$$\implies u(x,t) = 3\sin\frac{\pi x}{2} e^{-\frac{\pi^2}{4}t} - \sin\frac{5\pi x}{2} e^{-\frac{25\pi^2}{4}t}$$

#3 $u(x,0) = f(x) = 2 + x \implies$

$$a_n = \frac{\int_0^1 (2+x)\sin\frac{(2n-1)\pi x}{2}\,dx}{\int_0^1 \sin^2\frac{(2n-1)\pi x}{2}\,dx}$$

$u = 2 + x$ $\quad dv = \sin\frac{(2n-1)\pi x}{2}$

$du = dx$ $\quad v = -\frac{2}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}$

$$\text{num} = \frac{-2(2+x)}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2}\Bigg|_0^1 + \frac{2}{(2n-1)\pi}\int_0^1 \cos\frac{(2n-1)\pi x}{2}\,dx$$

$$= \frac{4}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi x}{2}\Bigg|_0^1 = \frac{4}{(2n-1)\pi} + \frac{4}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}$$

denom: $\frac{1}{2}$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{8}{(2n-1)\pi} + \frac{8}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi}{2}\right)\sin\frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2\pi^2}{4}t}$$

A wave problem 5.2.1 #5

$u_{tt} = u_{xx} \qquad 0 < x < 1 \qquad t > 0$

$u(0,t) = u(1,t) = 0 \qquad u(x,0) = \begin{cases} 1 & 0 \le x \le 1/2 \\ 2 & 1/2 < x \le 1 \end{cases} = f(x)$

$u_t(x,0) = 3 \sin 2\pi x \qquad = g(x)$

Soln $\quad c = 1 \quad L = 1$

Formal Soln: $\quad u(x,t) = \sum_{n=1}^{\infty} (b_{1n} \cos n\pi t + b_{2n} \sin n\pi t) \sin n\pi x$

$u(x,0) = f(x) \implies \sum_{n=1}^{\infty} b_{1n} \sin n\pi x$

$$b_{1n} = 2 \int_0^1 f(x) \sin n\pi x \, dx = 2 \left[\int_0^{1/2} \sin n\pi x \, dx + 2 \int_{1/2}^{1} \sin n\pi x \, dx \right] =$$

$$= 2 \left[-\frac{1}{n\pi} \cos n\pi x \Big|_0^{1/2} - \frac{2}{n\pi} \cos n\pi x \Big|_{1/2}^{1} \right] =$$

$$= 2 \left[-\frac{1}{n\pi} \cos \frac{n\pi}{2} + \frac{1}{n\pi} - \frac{2}{n\pi} \cos n\pi + \frac{2}{n\pi} \cos \frac{n\pi}{2} \right]$$

$$u_t(x,t) = \sum_{n=1}^{\infty} (-n\pi b_{1n} \sin n\pi t + n\pi b_{2n} \cos n\pi t) \sin n\pi x$$

$$u_t(x,0) = g(x) \implies 3 \sin 2\pi x = \sum_{n=1}^{\infty} n\pi b_{2n} \sin n\pi x$$

match up $\quad n = 2 \qquad b_{22} = \frac{3}{2\pi} \qquad$ all other b_{2n}'s $= 0$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \cos \frac{n\pi}{2} + \frac{2}{n\pi} - \frac{2}{n\pi} \cos n\pi \right) \sin n\pi x \cos n\pi t + \frac{3}{2\pi} \sin 2\pi t \sin 2\pi x$$