Two Method of Characteristics Problems

12.2 #1 $u_t - (2u+1)\,u_x = 3$ $u(x,0) = f(x) = 1-x$

Soln $\frac{dx}{dt} = -(2u+1)$ $\frac{du}{dt} = 3$ Let $x_0 = x(0)$

$u = 3t + K$

$\begin{cases} u(x_0,0) = K \\ u(x_0,0) = 1-x_0 \end{cases} \Rightarrow K = 1-x_0$

$\Rightarrow u = 3t + (1-x_0)$

$\frac{dx}{dt} = -(2(3t+1-x_0)+1)$

$= -6t - 2 + 2x_0 + 1$

$\frac{dx}{dt} = -6t + 2x_0 - 1$

$x = -3t^2 + 2x_0 t - t + C$

$x(0) = x_0 \Rightarrow C = x_0$

$x = -3t^2 + 2x_0 t - t + x_0$

$x_0 = \dfrac{x + 3t^2 + t}{2t+1}$

$u = 3t + 1 - \dfrac{x+3t^2+t}{2t+1}$

12.2 #2 $u_t + (u-1)\,u_x = t+1$ $u(x,0) = f(x) = 2x$

Soln: Let $x(0) = x_0$

$\frac{dx}{dt} = u-1$ $\frac{du}{dt} = t+1$

$u = \frac{t^2}{2} + t + K$

$\begin{cases} u(x_0,0) = K \\ u(x_0,0) = 2x_0 \end{cases} \Rightarrow K = 2x_0$

$u = \frac{t^2}{2} + t + 2x_0$

$\frac{dx}{dt} = \frac{t^2}{2} + t + 2x_0 - 1$

over

$$x = \frac{t^3}{6} + \frac{t^2}{2} + 2x_0 t - t + K$$

$$x(0) = x_0 \Rightarrow \quad x = \frac{t^3}{6} + \frac{t^2}{2} + 2x_0 t - t + x_0$$

$$x_0 = \frac{6x - t^3 - 3t^2 + 6t}{6(2t+1)} \Rightarrow$$

$$u = \frac{t^2}{2} + t + \frac{6x - t^3 - 3t^2 + 6t}{3(2t+1)}$$

3 $\quad u_t + (u+t)u_x = 1 \qquad u(x,0) = f(x) = x+1$

done in class

MATH 4545 QUIZ #5 NAME ______________

Solve using Laplace Transforms.

$$u_{tt} = u_{xx} + e^x$$

$$u(x,0) = u_t(x,0) = 0$$

$$u(0,t) = -1 \qquad u(\infty,t) = 0$$

Soln transforming:

$$U'' - s^2 U = -\frac{1}{s} e^x$$

$$U_c = \beta e^{-sx}$$

$$U_p = Ae^x \Rightarrow U_p' = Ae^x \Rightarrow U_p'' = Ae^x \Rightarrow$$

$$Ae^x - s^2 Ae^x = -\frac{1}{s} e^x \Rightarrow A = \frac{1}{s(s^2-1)}$$

$$\text{p.f.} \quad \frac{1}{s(s-1)(s+1)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s-1}$$

$$\Rightarrow \quad a = -1 \qquad b = \tfrac{1}{2} \qquad c = \tfrac{1}{2}$$

$$\Rightarrow U_p = \left(\frac{-1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}\right) e^x$$

$$U = \beta e^{-sx} + \left(-\frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}\right) e^x$$

$$U(0,s) = \mathcal{L}[u(0,t)] = -\frac{1}{s} \Rightarrow \beta = \frac{-\frac{1}{2}}{s+1} + \frac{-1/2}{s-1}$$

$$U = \left(\frac{-\frac{1}{2}}{s+1} - \frac{1/2}{s-1}\right) e^{-sx} + \left(-\frac{1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}\right) e^x$$

$$u = \left[-\frac{1}{2} e^{-(t-x)} - \frac{1}{2} e^{(t-x)}\right] H(t-x) + \left(-1 + \frac{1}{2} e^{-t} + \frac{1}{2} e^{t}\right) e^x$$

HW #2 Solutions

§7.3.1 #2 highlights:

For your o.d.e.'s after matching up, you should get

$n=1$

$$c_1'' \quad \pi^2 c_1 = -\pi^2(2y+3)$$

$$c_1(0)=3$$

$$c_1'(2)=2$$

$n=2$

$$c_2'' \quad 4\pi^2 c_2 = (1-4\pi^2)e^{-y}$$

$$c_2(0)=1$$

$$c_2'(2) = -e^{-2}$$

all other $c_n \equiv 0$

You should set up

$n=1$ $\quad c_1(y) = C_1 \sinh \pi y + C_2 \cosh \pi(y-2) + Ay + B$

$n=2$ $\quad c_2(y) = C_1 \sinh 2\pi y + C_2 \cosh 2\pi(y-2) + Ae^{-y}$

Apply boundary conditions. The undetermined coefficients can be done by inspection. You get

$$c_1(y) = 2y+3 \qquad c_2(y) = e^{-y}$$

$$u(x,y) = (2y+3)\sin \pi x + e^{-y} \sin 2\pi x$$

HW #2 SOLUTIONS cont'd

#2
$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}\, x \cdot e^{-\lambda_n t}$$

where λ_n are the positive solutions to
$$\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}$$

and
$$a_n = \frac{\int_0^1 100 \sin\sqrt{\lambda_n}\, x\, dx}{\int_0^1 \sin^2\sqrt{\lambda_n}\, x\, dx}$$

estimates $\sqrt{\lambda_1} \approx 2.03$ $\qquad \sqrt{\lambda_2} \approx 4.91$

computing a_n: numerator: $\frac{-100}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\, x \Big|_0^1 = \frac{100}{\sqrt{\lambda_n}}\left(1 - \cos\sqrt{\lambda_n}\right)$

denom $\int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2\sqrt{\lambda_n}\, x\right)dx = \frac{1}{2}x - \frac{1}{4\sqrt{\lambda_n}} \sin 2\sqrt{\lambda_n}\, x \Big|_0^1 = \frac{1}{2}\left(1 - \frac{\sin 2\sqrt{\lambda_n}}{2\sqrt{\lambda_n}}\right)$

$$a_n = \frac{400\left(1 - \cos\sqrt{\lambda_n}\right)}{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}$$

The 1^{st} term of $u(x,t)$ is $\approx \frac{400(1 - \cos 2.03)}{2(2.03) - \sin(2(2.03))} \sin\frac{2.03}{2}\, e^{-4.12} \approx 1.7$

when $x = \frac{1}{2}$ and $t = 1$

The remaining terms in the sum are very, very small.
The next is of order 10^{-10}

So $u\left(\frac{1}{2}, 1\right) \approx 1.7$ and $\lim_{t \to \infty} u(x,t) = 0$

3 $\boxed{\lambda = 0 \text{ is an e-value with e-fction } X_0 = ax}$

i.e. $X_0 \sim x$

The positive e-values satisfy $\tan\sqrt{\lambda_n} = \sqrt{\lambda_n}$

and the formal solution is

$$u(x,t) = a_0 x + \sum_{n=1}^{\infty} a_n \sin\sqrt{\lambda_n}x \cdot e^{-\lambda_n t}$$

There are no negative e-vals.

To compute $a_0 = \dfrac{\int_0^1 100 \cdot x\,dx}{\int_0^1 x^2\,dx} = \dfrac{50}{1/3} = 150$

$$u(x,t) = 150x + \sum_{n=1}^{\infty} \frac{400(1-\cos\sqrt{\lambda_n})}{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}\,x\, e^{-\lambda_n t}$$

$u(\frac{1}{2},1) \approx 75$ (The remaining terms are very small.)

$$\lim_{t\to\infty} u(x,t) = 150x$$

In order to interpret the results take u to be positive. This means thermal energy is positive. In the first case u_x at the right end point is then negative. Heat flow is <u>opposite</u> to the gradient u_x, so heat is flowing to the right and leaving the rod. In the second case the opposite occurs – thermal energy is flowing into the rod. This explains why thermal energy goes to zero in the first case, and increases in the second.