Undetermined Coefficients - Review

$$ay'' + by' + cy = g(x) \qquad y = y(x), \; a, b, c \text{ constants}$$

Solve the <u>complementary eqn</u> $ay'' + by' + cy = 0$ for the general solution y_c.

Find <u>any</u> solution y_p to the given equation - this is a <u>particular solution</u>.

Then $y = y_c + y_p$ is the <u>general solution</u> to the given equation.

To find y_p in the "standard" cases.

a) If $g(x)$ is a polynomial of degree n let $y_p = A_n x^n + \cdots + A_1 x + A_0$, a general polynomial of degree n. Substitute and solve for the coefficients

b) If $g(x)$ is an exponential with exponent αx let $y_p = Ae^{\alpha x}$. Substitute and solve for the coefficient.

c) If $g(x)$ is a linear combination of sines and cosines with frequency α let $y_p = A \cos \alpha x + B \sin \alpha x$. Substitute and solve for the coefficients.

<u>This method doesn't always work</u> - (turn over)

First, if $g(x)$ is a product of two or all three of the "types" in a), b) or c) set your y_p equal to a corresponding product.

The method may fail because of <u>duplication</u>

To see if there's duplication first get your y_c. Then set up the <u>standard</u> y_p.

If any of the individual terms in your y_p solve the complementary equation that term will substitute into the left hand side of the given equation to give zero and you won't be able to solve for the coefficient. Ie if "duplicates" the complementary solution

In that case modify the standard y_p by xy_p.

If xy_p also duplicates modify to $x^2 y_p$.

Undetermined Coefficients Exs.

1.3 #5 $\quad y'' - y = x^2 - x + 2$

Soln: for y_c : $y^2 - y = 0$ $\quad$ C.E $\quad r^2 - 1 = 0$ $\quad r = \pm 1$

$$y_c = c_1 \cosh x + c_2 \sinh x$$

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \Rightarrow \begin{aligned} 2A - Ax^2 - Bx - C &= x^2 - x + 2 \\ -Ax^2 - Bx + 2A - C &= x^2 - x + 2 \end{aligned}$$

$$\Rightarrow A = -1 \quad B = 1 \quad -2 - C = 2 \quad C = -4$$

$$y = y_p + y_c \Rightarrow \boxed{y = c_1 \cosh x + c_2 \sinh x - x^2 + x - 4}$$

#6 $\quad y'' - 2y' - 8y = 4 + 4x - 8x^2$

Soln: for y_c : $y^2 - 2y' - 8y = 0$ $\quad$ C.E. $\quad r^2 - 2r - 8 = 0$ $\quad r_1 = 4 \quad r_2 = -2$

$$y_c = c_1 e^{4x} + c_2 e^{-2x}$$

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C \\ y_p' &= 2Ax + B \\ y_p'' &= 2A \end{aligned}\right\} \Rightarrow 2A - 4Ax - 2B - 8Ax^2 - 8Bx - 8C = 4 + 4x - 8x^2$$

$$\left.\begin{aligned} 2A - 2B - 8C &= 4 \\ -4A - 8B &= 4 \\ -8A &= -8 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= 1 \\ B &= -1 \\ C &= 0 \end{aligned}$$

$$y_p = x^2 - x \Rightarrow \boxed{y = c_1 e^{4x} + c_2 e^{-2x} + x^2 - x}$$

#7 $\quad y'' - 25y = 30e^{-5x}$

Soln: for y_c : $y^2 - 25y = 0$ $\quad$ C.E. $\quad r^2 - 25 = 0$ $\quad r = \pm 5$

$y_c = c_1 \cosh 5x + c_2 \sinh 5x$ $\quad$ we have duplication so $y = Ae^{-5x}$ fails

$y_p = Axe^{-5x}$, $\quad y_p' = A(-5xe^{-5x} + e^{-5x})$, $\quad y_p'' = A(25xe^{-5x} - 10e^{-5x}) \Rightarrow$

$A(\cancel{25xe^{-5x}} - 10e^{-5x} - \cancel{25xe^{-5x}}) = 30e^{-5x} \Rightarrow A = -3 \quad y_p = -3xe^{-5x}$

$$\boxed{y = c_1 \cosh 5x + c_2 \sinh 5x - 3xe^{-5x}}$$

#8 $4y'' + y = 8\cos\frac{x}{2}$

Soln: for y_c: $4y'' + y = 0$ C.E. $4r^2 + 1 = 0$ $r = \pm\frac{1}{2}i$

$$y_c = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2}$$

$y_p \sim \cos\frac{x}{2}$ or $y_p \sim \sin\frac{x}{2}$ will fail

let $y_p = x\left(A\cos\frac{x}{2} + B\sin\frac{x}{2}\right)$

$$y_p' = x\left(-\frac{A}{2}\sin\frac{x}{2} + \frac{B}{2}\cos\frac{x}{2}\right) + A\cos\frac{x}{2} + B\sin\frac{x}{2}$$

$$y_p'' = x\left(-\frac{A}{4}\cos\frac{x}{2} \quad \frac{B}{4}\sin\frac{x}{2}\right) + \left(-A\sin\frac{x}{2} + B\cos\frac{x}{2}\right)$$

after subbing: $-4A\sin\frac{x}{2} + 4B\cos\frac{x}{2} = 8\cos\frac{x}{2} \Rightarrow A = 0 \quad B = 2$

$$y_p = 2x\sin\frac{x}{2}$$

$$y = c_1\cos\frac{x}{2} + c_2\sin\frac{x}{2} + 2x\sin\frac{x}{2}$$

1st order eqns can also be solved like this for certain r.h.s.'s

#1 $y' + 2y = 2x + e^{4x}$

for y_c $y' + 2y = 0$ $y = Ce^{-2x}$

let $y_p = Ax + B + Ce^{4x} \Rightarrow y_p' = A + 4Ce^{4x}$

subbing $A + 4Ce^{4x} + 2Ax + 2B + 2Ce^{4x} = 2x + e^{4x}$

$$\Rightarrow \left.\begin{array}{l} 2A = 2 \Rightarrow A = 1 \\ A + 2B = 0 \Rightarrow B = -\frac{1}{2} \\ 6C = 1 \Rightarrow C = \frac{1}{6} \end{array}\right\} \quad y_p = x - \frac{1}{2} + \frac{1}{6}e^{4x}$$

$$y = C_1 e^{-2x} + x - \frac{1}{2} + \frac{1}{6}e^{4x}$$