Heat Problems §5

#1,3,5 : $u_t = u_{xx}$ $\quad u(0,t)=0 \quad u(1,t)=0$

$(k=1 \quad L=1)$

The boundary conditions translate to $X(0)=0 \quad X(1)=0$

So we obtain the 1st basic SL-BVP

$$X'' + \lambda X = 0 \qquad X(0)=0 \qquad X(1)=0$$

and we know $\lambda_n = n^2\pi^2 \quad X_n \sim \sin\sqrt{\lambda_n}\, x \quad n=1,2,\ldots$

for each of #1,3,5, and

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \, e^{-n^2\pi^2 t}$$

1. $u(x,0) = \sin 2\pi x - 3\sin 6\pi x \implies$

<u>match up</u> $\sin 2\pi x - 3\sin 6\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x \implies b_2 = 1 \quad b_6 = -3$

all other b_n's $= 0$.

$$\implies u(x,t) = \sin 2\pi x\, e^{-4\pi^2 t} - 3\sin 6\pi x\, e^{-36\pi^2 t}$$

3) $u(x,0) = -2 \implies -2 = \sum_{n=1}^{\infty} b_n \sin n\pi x$

$$b_n = \frac{\int_0^1 -2\sin n\pi x\, dx}{\int_0^1 \sin^2 n\pi x\, dx} \qquad \text{num} = \frac{2}{n\pi}(\cos n\pi - 1) \qquad \text{den.} = 1/2$$

$$\implies u(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(\cos n\pi - 1)\sin n\pi x\, e^{-n^2\pi^2 t}$$

5) $u(x,0) = 2x+1$ $\qquad b_n = \dfrac{\int_0^1 (2x+1)\sin n\pi x\, dx}{\int_0^1 \sin^2 n\pi x\, dx}$

<u>after integrating</u>

$\text{num} = -\dfrac{3}{n\pi}\cos n\pi + \dfrac{1}{n\pi}$ $\qquad$ denom $= \dfrac{1}{2}$

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} - \frac{6}{n\pi}\cos n\pi\right)\sin n\pi x\, e^{-n^2\pi^2 t}$$

E-function expansion problem

This is 3.1.2 #17
Given $f'' + \lambda f = 0$ $\quad f(0) = 0 \quad f'(1) + f(1) = 0 \quad 0 < x < 1$
Write $u(x) = 1$ as a sum of e-functions of this problem - first two terms only.

Soln: We found in an example solved in class that the e-vals were the positive solutions λ_n to $\tan\sqrt{\lambda_n} = -\sqrt{\lambda_n}$ and the e-functions were $f_n \sim \sin\sqrt{\lambda_n}\, x$.
We estimate the first two e-vals: $\sqrt{\lambda_1} \approx 2.03 \quad \sqrt{\lambda_2} \approx 4.91$

$$u(x) \approx a_1 \sin\sqrt{\lambda_1}\, x + a_2 \sin\sqrt{\lambda_2}\, x$$

in general $$a_n = \frac{\int_0^1 f(x) \sin\sqrt{\lambda_n}\, x\, dx}{\int_0^1 \sin^2\sqrt{\lambda_n}\, x\, dx} = \frac{\int_0^1 \sin\sqrt{\lambda_n}\, x\, dx}{\int_0^1 \sin^2\sqrt{\lambda_n}\, x\, dx}$$

num. $$a_n = -\frac{1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\, x \Big|_0^1 = -\frac{1}{\sqrt{\lambda_n}}\left(\cos\sqrt{\lambda_n} - 1\right)$$

den. $$\int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos 2\sqrt{\lambda_n}\, x\right) dx = \left(\frac{1}{2}x - \frac{1}{4\sqrt{\lambda_n}} \sin 2\sqrt{\lambda_n}\, x\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4\sqrt{\lambda_n}} \sin 2\sqrt{\lambda_n} = \frac{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}{4\sqrt{\lambda_n}}$$

$$\Rightarrow \quad a_n = \frac{4\left(1 - \cos\sqrt{\lambda_n}\right)}{2\sqrt{\lambda_n} - \sin 2\sqrt{\lambda_n}}$$

$a_1 \approx 1.2 \qquad a_2 \approx .32$

$$u(x) \approx 1.2 \sin 2.03x + .32 \sin 4.91x$$