2^{nd} order constant coefficient homogeneous equations

$$ay'' + by' + cy = 0 \qquad y = y(x), \quad a\,b, c \text{ constant}$$

Write the C. E. $\quad ar^2 + br + c = 0$

find the roots r_1, r_2

case 1) $\quad r_1 \neq r_2$ real $\Longrightarrow \boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$ gen'l soln

case 2) $\quad r_1 = r_2 = \rho$ "double root"

$\Longrightarrow \boxed{y = c_1 e^{\rho x} + c_2 \_\_\_\_\_\_}$ gen'l soln

case 3) $\quad r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta \qquad \beta \neq 0$

$\boxed{y = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right)}$ gen'l soln

in general roots are $\quad r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b}{2a} \pm \dfrac{\sqrt{D}}{2a}$

$= \dfrac{-b}{2a} \pm \dfrac{\sqrt{|D|}}{2a} i$

$D > 0 \iff$ case 1)

$D = 0 \iff$ case 2) $\qquad \rho = \dfrac{-b}{2a}$

$D < 0 \iff$ case 3) $\qquad \alpha = \dfrac{-b}{2a} \qquad \beta = \dfrac{\sqrt{-D}}{2a}$

Basic Antidifferentiation

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int u \, dv = uv - \int v \, du$$ int. by parts

Orthogonality Relations

$n, m \geq 1$ integers

1) $$\int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

2) $$\int \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \end{cases}$$

3) $$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\,dx = 0$$

Special cases:

$n = m = 0$ integral 1) is $\int_{-L}^{L} 1\,dx = 2L$

$n = 0$ $m \neq 0$ or $m = 0$ $n \neq 0$

integral 1) is $\int_{-L}^{L} \cos\frac{n\pi x}{L}\,dx = 0$

$\sin bx$ and $\cos bx$ have period $\frac{2\pi}{b}$.

$\Rightarrow \cos\frac{n\pi x}{L}$ and $\sin\frac{2\pi x}{L}$ have period $\frac{2\pi}{n\pi/L} = \frac{2L}{n}$.

<u>The largest</u> period is $2L$ so that's a common period for all.

Fourier Series

Given $f(x)$ on $[-L, L]$

If we write $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

then the coefficients are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

$n = 0, 1, 2, \ldots \qquad n = 1, 2, \ldots$

Convergence Theorem for Fourier Series

The F.S. of $f(x)$ converges to the periodic extension of $f(x)$ wherever $f(x)$ is continuous, and to the average $(f(a^+) + f(a^-))/2$

at every point a

(Look at
www.falstad.com/Fourier)