Mathematical Logic and Computability

Lecture 1: Syntax of First-Order Logic

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Why we study Mathematical Logic?

In this course, we will study mathematical logic and computability theory. What is the *logic*? Why we need it for us?
Logic is the study of the principles of correct reasoning. Logic consider formal or informal language and prove that with induction or model-theoretic method.
Among logic, mathematical logic is to find reasoning in mathematics (with mathematical reasoning). We will study and explore the *first-order logic* for expressing mathematics formally.

Why we study Computability?

Let's think about another simple question: What is computable? We can say about that "We can compute 1+1=2, 2+2=4, 3+3=6, ...". But, how can we say about that?
How can we construct a machine that can compute 1+1=2, 2+2=4, 3+3=6, ...?
Can we compute all of the functions that we can imagine? We already know about the answer of this question is '**no**' by the Halting Problem. But we need to consider foundation of this answer, computation model.

Outline of Course

In this course, we will study following topics in mathematical logic:

- Syntax of First-Order Logic
- Semantics of First-Order Logic
- Sequent Calculus
- Completeness Theorem
- The Löwenheim-Skolem Theorem and Compactness Theorem

Outline of Course

In this course, we will study following topics in computability theory:

- Introduction to Computable Functions by Unlimited Register Machine
- Build on Computable Functions
- Turing Machines and Church-Turing Thesis
- Numbering of Computable Functions
- s-m-n Theorem, Universal Programs
- Deciability and Undecidability (Halting Problem, Rice's Theorem)

Course in Catalog

Before this course, you have taken following courses:

- Discrete Mathematics
- Programming Languages
- Introduction to Algorithms

After this course, you will discuss following topics:

- Type, Proof, Model Theory
- Set Theory
- Computability Theory
- (further) Computational Complexity Theory

Textbook

In this course, in mathematical logic, we will use following textbook:

- *Mathematical Logic*, by H.-D. Ebbinghaus, J. Flum, and W. Thomas

In this course, in computability theory, we will use following textbook:

- *Computability: An Introduction to Recursive Function Theory*, by N. J. Cutland

Textbook

In this course, I recommend following textbook:

- *Introduction to Mathematical Logic*, by E. Mendelson
- *Mathematical Logic*, by J. Shoenfield
- *Computability and Logic*, by G. S. Boolos, J. P. Burgess, and R. C. Jeffrey
- *Computability and Unsolvability*, by M. Davis

History of Logic: Acient Logic

Euclid's *Elements* (300 BC) is the earliest known uses of logic and proofs. In *Elements*, Euclid uses a deductive axiomatic method to study geometry. Euclid's method of proving mathematical statements is called *axiomatic method*. In axiomatic method, we start with a set of axioms and use rules of inference to derive theorems. In *Elements*, Euclid uses five postulates and five common notions as axioms. Euclid's axioms are not formalized, but they are the first known examples of axioms.

Aristotle (384-322 BC) is the first known logician who studied logic systematically. Aristotle's logic is called *term logic* because it is concerned with terms rather than propositions.

History of Logic: Boolean Logic

In the 19th century, George Boole (1815-1864) and Augustus De Morgan (1806-1871) developed a new kind of logic called *Boolean logic*. Before Boole and De Morgan, we prove statements using predicate logic. But, Boole and De Morgan developed a new kind of logic that is based on algebra. In Boolean logic, we use algebraic operations such as $\wedge$, $\vee$, $\neg$, and $\Rightarrow$ to prove statements.

History of Logic: Hilbert's Program

In the 20th century, David Hilbert (1862-1943) proposed a program to formalize all of mathematics. Hilbert's program is based on the idea that all of mathematics can be formalized in first-order logic. Hilbert's program is called *formalism*.

Key Concepts in Our Journey

Before starting our journey, we will introduce key concepts in our journey. It's hard to look at the forest why you study these contents from the beginning.

Syntax and Semantics of First-order Logic

Example of First-order Logic: Groups

Limitations of First-order Logic: Natural Number

Consequence and Model of First-order Logic

Definition

Model of First-order Logic

Definition

Let Γ be a set of formulas and φ be a formula. We say that φ is a *consequence* of Γ, denoted by $\Gamma \models \varphi$, if every model of Γ is also a model of φ.

Sequent Calculus and Proof

Consistence and Completeness of First-order Logic