Mathematical Logic and Computability

Lecture 3: Semantics of First-Order Logic

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Sep 8, 2023

Review

In previous lecture, we learned the syntax of first-order logic.

- Alphabet of First-order logic
- Terms
- Formulas
- Free and bound variables
- Substitution

Now, we assign the Mathematical object to each symbol in first-order language.

Structures

Before define the semantics of first-order logic, we need to consider our language domain.

Definition

An S-structure is a pair $\mathcal{A} = (A, \mathfrak{a})$ with the following properties:

- A is a non-empty set, called the *domain* or *universe* of $\mathcal{I}$
- $\mathfrak{a}$ is a function that assigns from symbols to following:
  - for every n-ary relation symbol R in S, $\mathfrak{a}(R)$ is an n-ary relation on A
  - for every n-ary function symbol f in S, $\mathfrak{a}(f)$ is an n-ary function on A
  - for every constant c in S, $\mathfrak{a}(c)$ is an element of A

From Ebbinghaus textbook, for convenience, we denote $\mathfrak{a}(R), \mathfrak{a}(f), \mathfrak{a}(c)$ by $R^{\mathcal{A}}, f^{\mathcal{A}}, c^{\mathcal{A}}$ or R^{A}, f^{A}, c^{A}respectively.

Example of Structures

We write $S = \{R, f\}$, where R is a n-ary relation symbol, f is a n-ary function symbol. The structure of S is denote as $\mathcal{A} = (A, R^{\mathcal{A}}, f^{\mathcal{A}})$.
We consider symbol set of arithmetic as follows:

$$S_{\mathrm{ar}} := \{+, \cdot, 0, 1\} \quad \text{and} \quad S_{\mathrm{ar}}^{<} := \{+, \cdot, 0, 1, <\} \tag{1}$$

We will use $\mathcal{N}$ as the structure of natural number arithmetic with S_{ar} (equation 1).

$$\mathcal{N} := (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$$

where, our domain is $\mathbb{N}$, $+$ and $\cdot$ are addition and multiplication, 0 and 1 are zero and one respectively.

Structures and Interpretations

We remain the variable symbols for semantics of first-order logic. We **assign** a value in our domain A to each variable.

Definition

An *assignment* in S-structure $\mathcal{A}$ is a function $\beta : \{v_n | n \in \mathbb{N}\} \to A$ from the set of variables into the domain A.

Structures and Interpretations

Now, we combine structure and interpretations together.

Definition

An S-interpretation $\mathcal{I}$ is a pair $(\mathcal{A}, \beta)$, where $\mathcal{A}$ is an S-structure and β is an assignment in $\mathcal{A}$.

Structures and Interpretations

We might consider assignment is a subtitution of variables to values in domain. We can write as follows:

$$\beta\frac{a}{x}(y) := \begin{cases} \beta(y) & \text{otherwise} \\ a & \text{if } y = x \end{cases}$$

Interpretations

Let's define interpretation $\mathcal{I} = (\mathcal{A}, I)$ is given by

$$\mathcal{I} = (\mathbb{N}, +, \cdot, 0, 1, <) \quad \text{and} \quad \beta(v_n) = 2n \text{ for } n \geq 0$$

Example

The formula $v_2 \cdot (v_1 + v_2) \equiv v_4$ reads as $4 \cdot (2 + 4) \equiv 8$.

Question: Interprete the follwing formulas by $\mathcal{I}$.

$$\exists v_0 v_0 + v_0 \equiv v_1 \tag{2}$$

$$\forall v_0 \forall v_1 \exists v_2 (v_0 < v_2 \wedge v_2 < v_1) \tag{3}$$

Connectives

As we learned in propositional logic, we need to define the semantics of connectives with truth-table.

		$\dot{\vee}$	$\dot{\wedge}$	$\dot{\rightarrow}$	$\dot{\leftrightarrow}$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

	$\dot{\neg}$
T	F
F	T

The Satisfication Relation

Now, we define interprete of our S-formula. Let's given S-formula φ and S-interpretation $\mathcal{I} = (\mathcal{A}, \beta)$. Interpreted result is denoted by $\mathcal{I}(\varphi)$. We define $\mathcal{I}(\varphi)$ by induction on terms

Definition

- For a variable x let $\mathcal{I}(x) = \beta(x)$
- For a constant $c \in S$ let $\mathcal{I}(c) = c^{\mathcal{A}}$
- For n-ary function symbol $f \in S$ and terms $t_1, \cdots, t_n$ let $\mathcal{I}(f(t_1, \cdots, t_n)) = f^{\mathcal{A}}(\mathcal{I}(t_1), \cdots, \mathcal{I}(t_n))$

The Satisfication Relation

For all interpretations $\mathcal{I} = (\mathcal{A}, \beta)$ we define following interpretations

- $\mathcal{I} \models (t_1 \equiv t_2)$ iff. $\mathcal{I}(t_1) = \mathcal{I}(t_2)$
- $\mathcal{I} \models (Rt_1 \ldots t_n)$ iff. $R^{\mathcal{A}}(\mathcal{I}(t_1), \ldots, \mathcal{I}(t_n))$
- $\mathcal{I} \models (\neg\varphi)$ iff. not $\mathcal{I} \models \varphi$
- $\mathcal{I} \models (\varphi \wedge \psi)$ iff. $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \vee \psi)$ iff. $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \rightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ iff. $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\forall x\varphi)$ iff. for all $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$
- $\mathcal{I} \models (\exists x\varphi)$ iff. there exists $a \in A$, $\mathcal{I}\frac{a}{x} \models \varphi$

The Consequence Relation

Definition

Let Φ be a set of S-formulas and φ be an S-formula. We say that ϕ *is a consequence of* Φ *(written* $\Phi \models \varphi$*)* iff. for every S-interpretation $\mathcal{I}$ if $\mathcal{I} \models \psi$ for all $\psi \in \Phi$, then $\mathcal{I} \models \varphi$.

Definition

A formula φ is valid (written $\models \varphi$) iff. $\emptyset \models \varphi$.

Definition

A formula φ is *satisfiable* (written $Sat\,\varphi$) if and only if there is interpretation which is a model of φ. A set of formula Φ is *satisfiable* if and only if there is interpretation which is a model of Φ.

Note. The satisfiability of formula is called SAT problem. It is one of the most important problem in computer science and its complexity is NP-hard.

Example of Consequence Relation

In previous lecture, we consider about left inverse of group. Let's the set of axiom of group formula as Φ_{gr}.
Now we can formulate the left inverse of group as follows:

$$\Phi_{\mathrm{gr}} \models \{\forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)\} \tag{4}$$

where the axiom of group as follows:

$$\begin{aligned}\Phi_{\mathrm{gr}} = \{&\forall v_0 \forall v_1 \forall v_2 (v_0 \cdot (v_1 \cdot v_2) \equiv (v_0 \cdot v_1) \cdot v_2), \\ &\forall v_0 (v_0 \cdot e \equiv v_0), \forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)\}\end{aligned}$$

Satisfiability and Validity

Lemma

For all Φ and φ,

$$\Phi \models \varphi \quad \textit{iff.} \quad \textit{not } Sat \Phi \cup \{\neg\varphi\}$$

Question: Prove it.

Coincidence

The satisfication relation between S-formula φ and an S-interpretation $\mathcal{I}$ depends only on interpretation of the symols of S occuring in φ, and on the variable occurring free in φ.

Definition

Two interpretation $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on $k \in S$ on x if $k^{\mathcal{A}_1} = k^{\mathcal{A}_2}$ or $\beta_1(x) = \beta_2(x)$.

Lemma

Let's $\mathcal{I}_1 = (\mathcal{A}_1, \beta)$ be an S_1-interpretation and $\mathcal{I}_2 = (\mathcal{A}_2, \beta)$ be an S_2-interpretation. both with the same domain $A_1 = A_2$. Put $S := S_1 \cap S_2$.

- *Let t be an S-term. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the S-symbols occuring in t and on the variables occuring in t, then $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.*
- *Let φ be an S-formula. If $\mathcal{I}_1$ and $\mathcal{I}_2$ agree on the S-symbols and the variables occuring free in φ, then ($\mathcal{I}_1 \models \varphi$ iff. $\mathcal{I}_2 \models \varphi$).*

Coincidence

Coincidence

Coincidence lemma says that, for an S-formula φ and an S-interpretation $\mathcal{I} = (\mathcal{A}, \beta)$, the validity of φ under $\mathcal{I}$ depends only on the assignments for the *finitely many* variables occuring free in φ.
If these variables among $v_0, v_1, \ldots, v_{n-1}$, the β-values $a_i = \beta(v_i)$ for $i = 0, \ldots, n-1$ which are significant. Thus instead of $(\mathcal{A}, \beta) \models \varphi$, we shall often use the more suggestive notation

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \tag{5}$$

If φ is a sentence, we can choose $n = 0$ and write $\mathcal{A} \models \varphi$ without even mentioning an assignment. We say that $\mathcal{A}$ is a model of φ.

Reduct and Expansion

Definition

Let S and S' be a symbol sets such that $S \subseteq S'$. Let $\mathcal{A} = (A, \mathfrak{a})$ be an S-structure, and $\mathcal{A}' = (A', \mathfrak{a})$ be an S'-structure. we call $\mathcal{A}$ *a reduct* (or the S-reduct) of $\mathcal{A}'$ and write $\mathcal{A} = \mathcal{A}'|_S$ iff $A = A'$ and $\mathfrak{a}$ and $\mathfrak{a}'$ agrees on S. We say that $\mathcal{A}'$ is an *expansion* of $\mathcal{A}$.

Satisfiability on Reduct and Expansion

Φ is satisfiable with respect to S iff Φ is satisfiable with respect to S'.

Horn Formulas

Formulas which are derivable in the following calculus called *Horn formulas*.

$$\frac{}{(\neg\varphi_1 \vee \ldots \vee \neg\varphi_n \vee \varphi)} \text{ if } n \in \mathbb{N} \text{ and } \quad \varphi_1, \ldots, \varphi_n \text{ are atomic formulas}$$

$$\frac{}{\neg\varphi_0 \vee \neg\varphi_n} \text{ if } n \in \mathbb{N} \text{ and } \varphi_0 \wedge \varphi_1 \wedge \ldots \wedge \varphi_n \text{ is an atomic formula}$$

$$\frac{\varphi, \psi}{(\varphi \wedge \psi)} \qquad \frac{\varphi}{\forall x \varphi} \qquad \frac{\varphi}{\exists x \varphi}$$

Horn formulas without free variables are called *Horn sentences*.
Question: Show that if φ is a Horn sentence and if $\mathcal{A}_i$ is a model of φ for $i \in I$, then $\prod_{i \in I} \mathcal{A}_i \models \varphi$.
Note. The atomic formula is a formula which is not a compound formula. (from terms)
Further question: How can we prove that the Horn formulas are satisfiable (without quantifier)?

Isomorphism

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be S-Structures

- A map $\pi : A \to B$ is called an *isomorphism* of $\mathcal{A}$ onto $\mathcal{B}$ ($\pi : \mathcal{A} \simeq \mathcal{B}$) iff
  - π is a bijection of A onto B.
  - For n-ary $R \in S$ and $a_1, \ldots, a_n \in A$,

    $$R^{\mathcal{A}}(a_1, \ldots, a_n) \quad \text{iff} \quad R^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For n-ary $f \in S$ and $a_1, \ldots, a_n \in A$,

    $$\pi(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(\pi(a_1), \ldots, \pi(a_n))$$

  - For $c \in S$,

    $$pi(c^{\mathcal{A}}) = c^{\mathcal{B}}$$

- Structure $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic* ($\mathcal{A} \simeq \mathcal{B}$) iff. there is an isomorphism $\pi : \mathcal{A} \simeq \mathcal{B}$.

Isomorphism

Lemma

For isomorphic S-structures $\mathcal{A}$ and $\mathcal{B}$ and every S-sentence φ,

$$\mathcal{A} \models \varphi \quad \textit{iff.} \quad \mathcal{B} \models \varphi$$

Corollary

If $\pi : \mathcal{A} \simeq \mathcal{B}$, then for $\varphi \in L_n^S$ and $a_0, \ldots, a_{n-1} \in A$,

$$\mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}] \quad \textit{iff.} \quad \mathcal{B} \models \varphi[\pi(a_0), \ldots, \pi(a_{n-1})]$$

Note that, isomorphic structures cannot be distinguished in L_0^S. For example, there are structures not isomorphic to the S_{ar}-structure $\mathcal{N}$ of natural numbers in which are the same first-order sentences hold.

Substructure

Definition

Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures. Then $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$ ($\mathcal{A} \subseteq \mathcal{B}$) iff.

- $A \subseteq B$
-
  - for n-ary $R \in S$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^n$
  - for n-ary $f \in S$, $f^{\mathcal{A}}$ is the restriction of $f^{\mathcal{B}}$ to A^n
  - for $c \in S$, $c^{\mathcal{A}} = c^{\mathcal{B}}$

For example the $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{Q}, +, 0)$

Substructure

Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\beta : \{v_n | n \in \mathbb{N}\} \to A$ be an assignment in $\mathcal{A}$. Then the following holds for every S-term t

$$(\mathcal{A}, \beta)(t) = (\mathcal{B}, \beta)(t)$$

and for every quantifier-free S-formula φ:

$$(\mathcal{A}, \beta) \models \varphi \quad \textit{iff.} \quad (\mathcal{B}, \beta) \models \varphi$$

Universal Formula

Definition

The formulas which are derivable by means of the following calculus are called *universal formulas*.

$$\frac{}{\varphi} \text{ if } \varphi \text{ is quantifier-free}$$

$$\frac{\varphi, \psi}{(\varphi \star \psi)} \text{ for } \star \in \{\wedge, \vee\}$$

$$\frac{\varphi}{\forall x \varphi}$$

Substructure and Universal Formulas

Lemma

Let $\mathcal{A}$ and $\mathcal{B}$ be S-structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\varphi \in L_n^S$ be a universal. Then the following holds for all $a_0, \ldots, a_{n-1} \in A$:

$$\textit{If } \mathcal{B} \models \varphi[a_0, \ldots, a_{n-1}], \textit{ then } \mathcal{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

Corollary

If $\mathcal{A} \subseteq \mathcal{B}$, then the following holds for every universal sentence φ:

$$\textit{If } \mathcal{B} \models \varphi, \textit{ then } \mathcal{A} \models \varphi$$