

Mathematical Logic and Computability

Lecture 3: Semantics of First-Order Logic

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Outline

Review

Semantics of
First-Order Logic

Structures and
Interpretations

Connectives

Consequence

Coincidence

Satisfaction
Relation

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Semantics of First-Order Logic

Structures and Interpretations

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Substitution

In previous lecture, we learned the syntax of first-order logic.

- ▶ Alphabet of First-order logic
- ▶ Terms
- ▶ Formulas
- ▶ Free and bound variables
- ▶ Substitution

Now, we assign the Mathematical object to each symbol in first-order language.

Before define the semantics of first-order logic, we need to consider our language domain.

Definition

An S -structure is a pair $\mathcal{A} = (A, \alpha)$ with the following properties:

- ▶ A is a non-empty set, called the *domain* or *universe* of \mathcal{A}
- ▶ α is a function that assigns from symbols to following:
 - ▶ for every n -ary relation symbol R in S , $\alpha(R)$ is an n -ary relation on A
 - ▶ for every n -ary function symbol f in S , $\alpha(f)$ is an n -ary function on A
 - ▶ for every constant c in S , $\alpha(c)$ is an element of A

From Ebbinghaus textbook, for convenience, we denote $\alpha(R)$, $\alpha(f)$, $\alpha(c)$ by $R^{\mathcal{A}}$, $f^{\mathcal{A}}$, $c^{\mathcal{A}}$ or R^A , f^A , c^A respectively.

Example of Structures

We write $S = \{R, f\}$, where R is a n -ary relation symbol, f is a n -ary function symbol. The structure of S is denote as $\mathcal{A} = (A, R^{\mathcal{A}}, f^{\mathcal{A}})$.

We consider symbol set of arithmetic as follows:

$$S_{\text{ar}} := \{+, \cdot, 0, 1\} \quad \text{and} \quad S_{\text{ar}}^< := \{+, \cdot, 0, 1, <\} \quad (1)$$

We will use \mathcal{N} as the structure of natural number arithmetic with S_{ar} (equation 1).

$$\mathcal{N} := (\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$$

where, our domain is \mathbb{N} , $+$ and \cdot are addition and multiplication, 0 and 1 are zero and one respectively.

We remain the variable symbols for semantics of first-order logic. We **assign** a value in our domain A to each variable.

Definition

An *assignment* in S -structure \mathcal{A} is a function $\beta : \{v_n | n \in \mathbb{N}\} \rightarrow A$ from the set of variables into the domain A .

Now, we combine structure and interpretations together.

Definition

An S -interpretation \mathcal{I} is a pair (\mathcal{A}, β) , where \mathcal{A} is an S -structure and β is an assignment in \mathcal{A} .

We might consider assignment is a substitution of variables to values in domain.

We can write as follows:

$$\beta_{\frac{a}{x}}(y) := \begin{cases} \beta(y) & \text{otherwise} \\ a & \text{if } y = x \end{cases}$$

Let's define interpretation $\mathcal{I} = (\mathcal{A}, I)$ is given by

$$\mathcal{I} = (\mathbb{N}, +, \cdot, 0, 1, <) \quad \text{and} \quad \beta(v_n) = 2n \text{ for } n \geq 0$$

Example

The formula $v_2 \cdot (v_1 + v_2) \equiv v_4$ reads as $4 \cdot (2 + 4) \equiv 8$.

Question: Interpret the following formulas by \mathcal{I} .

$$\exists v_0 v_0 + v_0 \equiv v_1 \tag{2}$$

$$\forall v_0 \forall v_1 \exists v_2 (v_0 < v_2 \wedge v_2 < v_1) \tag{3}$$

Connectives

As we learned in propositional logic, we need to define the semantics of connectives with truth-table.

		$\dot{\vee}$	$\dot{\wedge}$	$\dot{\rightarrow}$	$\dot{\leftrightarrow}$		$\dot{\neg}$
T	T	T	T	T	T	T	F
T	F	T	F	F	F	F	T
F	T	T	F	T	F	T	F
F	F	F	F	T	T	F	T

Now, we define interpretation of our S -formula. Let's given S -formula φ and S -interpretation $\mathcal{I} = (\mathcal{A}, \beta)$. Interpreted result is denoted by $\mathcal{I}(\varphi)$. We define $\mathcal{I}(\varphi)$ by induction on terms

Definition

- ▶ For a variable x let $\mathcal{I}(x) = \beta(x)$
- ▶ For a constant $c \in S$ let $\mathcal{I}(c) = c^{\mathcal{A}}$
- ▶ For n -ary function symbol $f \in S$ and terms t_1, \dots, t_n let $\mathcal{I}(f(t_1, \dots, t_n)) = f^{\mathcal{A}}(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n))$

The Satisfaction Relation

For all interpretations $\mathcal{I} = (\mathcal{A}, \beta)$ we define following interpretations

- ▶ $\mathcal{I} \models (t_1 \equiv t_2)$ iff. $\mathcal{I}(t_1) = \mathcal{I}(t_2)$
- ▶ $\mathcal{I} \models (Rt_1 \dots t_n)$ iff. $R^{\mathcal{A}}(\mathcal{I}(t_1), \dots, \mathcal{I}(t_n))$
- ▶ $\mathcal{I} \models (\neg\varphi)$ iff. not $\mathcal{I} \models \varphi$
- ▶ $\mathcal{I} \models (\varphi \wedge \psi)$ iff. $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$
- ▶ $\mathcal{I} \models (\varphi \vee \psi)$ iff. $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- ▶ $\mathcal{I} \models (\varphi \rightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$
- ▶ $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ iff. $\mathcal{I} \models \varphi$ iff. $\mathcal{I} \models \psi$
- ▶ $\mathcal{I} \models (\forall x\varphi)$ iff. for all $a \in A$, $\mathcal{I} \frac{a}{x} \models \varphi$
- ▶ $\mathcal{I} \models (\exists x\varphi)$ iff. there exists $a \in A$, $\mathcal{I} \frac{a}{x} \models \varphi$

The Consequence Relation

Definition

Let Φ be a set of S -formulas and φ be an S -formula. We say that φ is a consequence of Φ (written $\Phi \models \varphi$) iff. for every S -interpretation \mathcal{I} if $\mathcal{I} \models \psi$ for all $\psi \in \Phi$, then $\mathcal{I} \models \varphi$.

Definition

A formula φ is valid (written $\models \varphi$) iff. $\emptyset \models \varphi$.

Definition

A formula φ is *satisfiable* (written $Sat \varphi$) if and only if there is interpretation which is a model of φ . A set of formula Φ is *satisfiable* if and only if there is interpretation which is a model of Φ .

Note. The satisfiability of formula is called SAT problem. It is one of the most important problem in computer science and its complexity is NP-hard.

Example of Consequence Relation

In previous lecture, we consider about left inverse of group. Let's the set of axiom of group formula as Φ_{gr} .

Now we can formulate the left inverse of group as follows:

$$\Phi_{\text{gr}} \models \{\forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e)\} \quad (4)$$

where the axiom of group as follows:

$$\begin{aligned} \Phi_{\text{gr}} = \{ & \forall v_0 \forall v_1 \forall v_2 (v_0 \cdot (v_1 \cdot v_2) \equiv (v_0 \cdot v_1) \cdot v_2), \\ & \forall v_0 (v_0 \cdot e \equiv v_0), \forall v_0 \exists v_1 (v_1 \cdot v_0 \equiv e) \} \end{aligned}$$

Satisfiability and Validity

Lemma

For all Φ and φ ,

$$\Phi \models \varphi \quad \text{iff.} \quad \text{not Sat} \Phi \cup \{\neg\varphi\}$$

Question: Prove it.

Coincidence

The satisfaction relation between S -formula φ and an S -interpretation \mathcal{I} depends only on interpretation of the symbols of S occurring in φ , and on the variable occurring free in φ .

Definition

Two interpretation \mathcal{I}_1 and \mathcal{I}_2 agree on $k \in S$ on x if $k^{\mathcal{A}_1} = k^{\mathcal{A}_2}$ or $\beta_1(x) = \beta_2(x)$.

Lemma

Let's $\mathcal{I}_1 = (\mathcal{A}_1, \beta)$ be an S_1 -interpretation and $\mathcal{I}_2 = (\mathcal{A}_2, \beta)$ be an S_2 -interpretation. both with the same domain $A_1 = A_2$. Put $S := S_1 \cap S_2$.

- ▶ Let t be an S -term. If \mathcal{I}_1 and \mathcal{I}_2 agree on the S -symbols occurring in t and on the variables occurring in t , then $\mathcal{I}_1(t) = \mathcal{I}_2(t)$.
- ▶ Let φ be an S -formula. If \mathcal{I}_1 and \mathcal{I}_2 agree on the S -symbols and the variables occurring free in φ , then $(\mathcal{I}_1 \models \varphi \text{ iff. } \mathcal{I}_2 \models \varphi)$.

Coincidence

Me when proving literally anything in
mathematical logic, automata theory,
or computability



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Coincidence lemma says that, for an S -formula φ and an S -interpretation $\mathcal{I} = (\mathcal{A}, \beta)$, the validity of φ under \mathcal{I} depends only on the assignments for the *finitely many* variables occurring free in φ .

If these variables among v_0, v_1, \dots, v_{n-1} , the β -values $a_i = \beta(v_i)$ for $i = 0, \dots, n-1$ which are significant. Thus instead of $(\mathcal{A}, \beta) \models \varphi$, we shall often use the more suggestive notation

$$\mathcal{A} \models \varphi[a_0, \dots, a_{n-1}] \tag{5}$$

If φ is a sentence, we can choose $n = 0$ and write $\mathcal{A} \models \varphi$ without even mentioning an assignment. We say that \mathcal{A} is a model of φ .

Reduct and Expansion

Definition

Let S and S' be a symbol sets such that $S \subseteq S'$. Let $\mathcal{A} = (A, \alpha)$ be an S -structure, and $\mathcal{A}' = (A', \alpha')$ be an S' -structure. we call \mathcal{A} a *reduct* (or the S -reduct) of \mathcal{A}' and write $\mathcal{A} = \mathcal{A}'|_S$ iff $A = A'$ and α and α' agrees on S . We say that \mathcal{A}' is an *expansion* of \mathcal{A} .

Satisfiability on Reduct and Expansion

Φ is satisfiable with respect to S iff Φ is satisfiable with respect to S' .

Horn Formulas

Formulas which are derivable in the following calculus called *Horn formulas*.

$$\frac{}{(\neg\varphi_1 \vee \dots \vee \neg\varphi_n \vee \varphi)} \text{ if } n \in \mathbb{N} \text{ and } \varphi_1, \dots, \varphi_n \text{ are atomic formulas}$$

$$\frac{}{\neg\varphi_0 \vee \neg\varphi_n} \text{ if } n \in \mathbb{N} \text{ and } \varphi_0 \wedge \varphi_1 \wedge \dots \wedge \varphi_n \text{ is an atomic formula}$$

$$\frac{\varphi, \psi}{(\varphi \wedge \psi)} \quad \frac{\varphi}{\forall x \varphi} \quad \frac{\varphi}{\exists x \varphi}$$

Horn formulas without free variables are called *Horn sentences*.

Question: Show that if φ is a Horn sentence and if \mathcal{A}_i is a model of φ for $i \in I$, then $\prod_{i \in I} \mathcal{A}_i \models \varphi$.

Note. The atomic formula is a formula which is not a compound formula. (from terms)

Further question: How can we prove that the Horn formulas are satisfiable (without quantifier)?

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Definition

Let \mathcal{A} and \mathcal{B} be S -Structures

- ▶ A map $\pi : A \rightarrow B$ is called an *isomorphism* of \mathcal{A} onto \mathcal{B} ($\pi : \mathcal{A} \simeq \mathcal{B}$) iff

- ▶ π is a bijection of A onto B .
- ▶ For n -ary $R \in S$ and $a_1, \dots, a_n \in A$,

$$R^{\mathcal{A}}(a_1, \dots, a_n) \text{ iff } R^{\mathcal{B}}(\pi(a_1), \dots, \pi(a_n))$$

- ▶ For n -ary $f \in S$ and $a_1, \dots, a_n \in A$,

$$\pi(f^{\mathcal{A}}(a_1, \dots, a_n)) = f^{\mathcal{B}}(\pi(a_1), \dots, \pi(a_n))$$

- ▶ For $c \in S$,

$$\pi(c^{\mathcal{A}}) = c^{\mathcal{B}}$$

- ▶ Structure \mathcal{A} and \mathcal{B} are said to be *isomorphic* ($\mathcal{A} \simeq \mathcal{B}$) iff. there is an isomorphism $\pi : \mathcal{A} \simeq \mathcal{B}$.

Lemma

For isomorphic S -structures \mathcal{A} and \mathcal{B} and every S -sentence φ ,

$$\mathcal{A} \models \varphi \quad \text{iff.} \quad \mathcal{B} \models \varphi$$

Corollary

If $\pi : \mathcal{A} \simeq \mathcal{B}$, then for $\varphi \in L_n^S$ and $a_0, \dots, a_{n-1} \in A$,

$$\mathcal{A} \models \varphi[a_0, \dots, a_{n-1}] \quad \text{iff.} \quad \mathcal{B} \models \varphi[\pi(a_0), \dots, \pi(a_{n-1})]$$

Note that, isomorphic structures cannot be distinguished in L_0^S . For example, there are structures not isomorphic to the S_{ar} -structure \mathcal{N} of natural numbers in which are the same first-order sentences hold.

Definition

Let \mathcal{A} and \mathcal{B} be S -structures. Then \mathcal{A} is called a *substructure* of \mathcal{B} ($\mathcal{A} \subseteq \mathcal{B}$) iff.

- ▶ $A \subseteq B$
- ▶
 - ▶ for n -ary $R \in S$, $R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^n$
 - ▶ for n -ary $f \in S$, $f^{\mathcal{A}}$ is the restriction of $f^{\mathcal{B}}$ to A^n
 - ▶ for $c \in S$, $c^{\mathcal{A}} = c^{\mathcal{B}}$

For example the $(\mathbb{Z}, +, 0)$ is a substructure of $(\mathbb{Q}, +, 0)$

Lemma

Let \mathcal{A} and \mathcal{B} be S -structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\beta : \{v_n | n \in \mathbb{N}\} \rightarrow A$ be an assignment in \mathcal{A} . Then the following holds for every S -term t

$$(\mathcal{A}, \beta)(t) = (\mathcal{B}, \beta)(t)$$

and for every quantifier-free S -formula φ :

$$(\mathcal{A}, \beta) \models \varphi \quad \text{iff.} \quad (\mathcal{B}, \beta) \models \varphi$$

Definition

The formulas which are derivable by means of the following calculus are called *universal formulas*.

– if φ is quantifier-free

$$\frac{\varphi, \psi}{(\varphi \star \psi)} \text{ for } \star \in \{\wedge, \vee\}$$

$$\frac{\varphi}{\forall x \varphi}$$

Lemma

Let \mathcal{A} and \mathcal{B} be S -structures with $\mathcal{A} \subseteq \mathcal{B}$ and let $\varphi \in L_n^S$ be a universal. Then the following holds for all $a_0, \dots, a_{n-1} \in A$:

$$\text{If } \mathcal{B} \models \varphi[a_0, \dots, a_{n-1}], \text{ then } \mathcal{A} \models \varphi[a_0, \dots, a_{n-1}]$$

Corollary

If $\mathcal{A} \subseteq \mathcal{B}$, then the following holds for every universal sentence φ :

$$\text{If } \mathcal{B} \models \varphi, \text{ then } \mathcal{A} \models \varphi$$

Definition

$$x \frac{t_0 \dots t_r}{x_0 \dots x_r} := \begin{cases} x & \text{if } x \neq x_0, \dots, x \neq x_r \\ t_i & \text{if } x = x_i \end{cases}$$

$$c \frac{t_0 \dots t_r}{x_0 \dots x_r} := c$$

$$[ft'_1 \dots t'_n] \frac{t_0 \dots t_r}{x_0 \dots x_r} := f \left(t'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r}, \dots, t'_n \frac{t_0 \dots t_r}{x_0 \dots x_r} \right)$$

Definition

$$\begin{aligned}[t'_1 \equiv t'_2] \frac{t_0 \dots t_r}{x_0 \dots x_r} &:= t'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r} \equiv t'_2 \frac{t_0 \dots t_r}{x_0 \dots x_r} \\ [Rt'_1 \dots t'_n] \frac{t_0 \dots t_r}{x_0 \dots x_r} &:= Rt'_1 \frac{t_0 \dots t_r}{x_0 \dots x_r} \dots t'_n \frac{t_0 \dots t_r}{x_0 \dots x_r} \\ [\neg \varphi] \frac{t_0 \dots t_r}{x_0 \dots x_r} &:= \neg \left[\varphi \frac{t_0 \dots t_r}{x_0 \dots x_r} \right] \\ (\varphi \vee \psi) \frac{t_0 \dots t_r}{x_0 \dots x_r} &:= \left(\varphi \frac{t_0 \dots t_r}{x_0 \dots x_r} \vee \psi \frac{t_0 \dots t_r}{x_0 \dots x_r} \right)\end{aligned}$$

How about quantifier?

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Definition

Suppose x_{i_1}, \dots, x_{i_s} ($i_1 < \dots < i_s$) are exactly the variables x_i among the x_0, \dots, x_r such that

$$x_i \in \text{free}(\exists x \varphi) \quad \text{and} \quad x_i \neq t_i$$

In particular, $x \neq x_{i_1}, \dots, x \neq x_{i_s}$. Then set

$$[\exists x \varphi] \frac{t_0 \dots t_r}{x_0 \dots x_r} := \exists u \left[\varphi \frac{t_{i_1} \dots t_{i_s} u}{x_{i_1} \dots x_{i_s} x} \right]$$

where u is the variable x if x does not occur in $t_{i_1} \dots t_{i_s}$, otherwise u is the first variable in the list v_0, v_1, v_2, \dots which does not occur in $\varphi, t_{i_1}, \dots, t_{i_s}$.

Substitution Lemma

Lemma

For every term t

$$\mathcal{I}(t \frac{t_0 \dots t_r}{x_0 \dots x_r}) = \mathcal{I} \frac{\mathcal{I}(t_0), \dots, \mathcal{I}(t_r)}{x_0, \dots, x_r}(t)$$

And for every formula φ

$$\mathcal{I} \models \varphi \frac{t_0 \dots t_r}{x_0 \dots x_r} \quad \text{iff} \quad \mathcal{I} \frac{\mathcal{I}(t_0), \dots, \mathcal{I}(t_r)}{x_0, \dots, x_r} \models \varphi$$

Rank of Formula

The number of connectives and quantifiers occurring in a formula φ the *rank* of φ , written $\text{rk}(\varphi)$.

Question: How can we define it?

Question: After substitution, the rank of formula is changed?