Journal of Real Estate Finance and Economics, 2: 267-283 (1989)

The Probability of Prepayment

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Abstract

Prepayment is a risk of holding a mortgage or derivative security. Incorrect pricing of prepayment risk leads to increased volatility and uncertainty in mortgage security markets. This article prices prepayment risk within an underlying callable bonds model. To price mortgages accurately, a probability of prepayment is required. A mortgage is a callable bond with a package of an option to prepay currently and a sequence of options to prepay up to the date of maturity. This sequence is summarized by a compound option. The probability of prepayment is determined by the prices of the current call and this compound option. These option prices depend on market interest rates and age, and on the contract terms of the originated mortgage.

Key words: mortgage prepayment, mortgage refinancing

1. Introduction

Prepayment after a refinancing or sale of the property is a risk of holding a mortgage or derivative security. Default on an insured mortgage also triggers early receipt of principal.[1] The risk of early receipt of principal increases as market interest rates decline. To reinvest at a given risk level, the direct lender or indirect security holder receives a rate below that on the initial contract. There is risk on maturity and duration, as well as on interest rates.

The risk of prepayment does not disappear with innovation in securities markets. Holders of varying maturity mortgage-backed securities, where payments are allocated between classes or tranches, also face prepayment risk. The risk varies between tranche holders, depending on the rate of prepayment. Incorrect pricing of prepayment risk leads to increased volatility and uncertainty in mortgage security markets. Capital losses for investors and broker-dealers followed sharp declines in long-term interest rates, and subsequent refinancings, in 1986 and 1987. This article prices prepayment risk within an underlying callable bonds model.[2] To price mortgages accurately, a probability of prepayment is required.

A mortgage is a callable bond that comes with a package of an option to prepay currently, and a sequence of options to prepay up to the date of maturity. This sequence is summarized by a compound option. The probability of prepayment is determined by the prices of the current call and this compound option. These option prices depend on market interest rates and age, and on the contract terms of the originated mortgage. The options can be embedded into prices for mortgages and mortgage-backed securities, to reduce unanticipated price volatility.

The model is developed for the fixed interest rate and payment mortgage, but is applicable to adjustable rate financing. The mortgage borrower buys a financial contract containing call options to prepay, with sequential exercise dates up to maturity. Each period, the borrower observes and searches among diverse market contracts, and decides whether to prepay. By not prepaying currently, the borrower receives a sequence of options to prepay in the future.

Prepayment depends on interest and discount rates, transactions costs, and terms of the existing mortgage contract. The probability accommodates autonomous prepayment, resulting from a change in demand for the real estate collateral. Autonomous prepayment can arise from job relocation, changes in household size and composition, and wealth windfalls.

The risk of prepayment has previously been priced by the refinancing experience of mortgage pools. Adjustments based on the observed prepayment of pools are built into mortgage security pricing. The structure is mechanistic, and typically does not have a behavioral model of prepayment.[3]

A mortgage is a callable bond, but its characteristics differ from corporate or public debt.[4] A callable corporate bond is prepaid if the underlying option is priced positively, and uncalled when the option is worthless. A mortgage is not necessarily prepaid, even when interest rates are substantially below those on the contract. It can also be repaid when the call option is out of the money.

The pricing of mortgages and derivative securities is based on a model of the term structure and a stochastic process on interest rates. There remains a prepayment probability to close the model.

In section 2, the specific characteristics that distinguish mortgages from callable bonds are discussed. In section 3 the model of prepayment is developed, with a stochastic specification in section 4. Data and empirical results are presented in section 5. In section 6 are implications for mortgage pricing and public policy. Concluding remarks are given in section 7.

The results indicate that the probability of prepayment affects the pricing and risk of mortgages and derivative securities. The probability of prepayment is unity or zero for a callable bond, but not necessarily for a mortgage. The probability of prepayment during one year, for a sample of direct lenders surveyed over a time series, is 18 percent. The cost of the call option package, pricing both current prepayment and future rights to prepay, is 1.4 percent per year per dollar of loan. This is the charge a risk-neutral lender or security holder must impose to cover the risk of prepayment. Of the cost, 0.9 percent is for the risk of capital from prepayment in the current period. The remaining 0.5 percent is for servicing the sequence of future options to prepay.

The option package purchased with a mortgage is a sequence of calls to prepay over future dates to term. One call option arises for each payment date. There is the potential for error in pricing of mortgages and derivative securities if these prepayment options are excluded.

2. Mortgage contract

Prepayment reduces the duration of a mortgage and derivative securities. The empirical approach to pricing this risk has been based on applying historical experience on prepayment rates. Tables of yields and prices of mortgage securities embed these prepayments, as experienced by pools. Examples are the series produced in the United States by the Federal Housing Administration (FHA) and the Public Securities Association (PSA). Mortgage pools are priced with prepayments as multiples of these series. A pool with 100 percent FHA experience prepays at the same rate as that observed by the FHA. These series do not include a behavioral specification of prepayment.

A mortgage provides the borrower with an option to buy the loan back from the lender on or before the next payment date. If the current call is exercised, the mortgage is prepaid. An unexercised current call is exchanged for a sequence of options to refinance in the future. Not all of the loan need be prepaid, and not all is prepaid when a call is in the money. The prepayment option held by the borrower makes mortgages and their securities callable. Mortgages differ from other callable bonds, with the characteristics summarized in table 1.

The fixed rate mortgage has two characteristics: a fixed payment, and the option to prepay. Options to prepay are sequential calls, giving the borrower the right to buy the loan back from the lender. The pricing of prepayment risk requires an endogenous structure of borrower behavior. Table 2 indicates the notation and symbols used in the model.

The borrower has a contract with provisions (α, γ), where α is the periodic interest rate and γ the number of payments.[5] The balances outstanding are β_t, $t = 0, \ldots, \gamma$ per dollar of loan borrowed. The borrower is able, at payment date t, to purchase one dollar of loan from the lender at β_t, fixed by the mortgage contract. Then β_t is the exercise price of the call option to prepay at date t.

Table 1. Characteristics of callable bonds and mortgages

Characteristic	Callable Bonds	Mortgages
Option	European at call date American thereafter	Current: American Future: compound American
Price	Above par	At par: adjusted Loan balance declines with amortization
Autonomous	No	Yes. Autonomous nonfinancial prepayment.

Table 2. Notation and symbols

Mortgage contract	
α	contract interest rate per payment period
γ	number of payments
$\omega(\alpha,\gamma)$	mortgage payment per dollar of loan
$\beta_t(\alpha,\gamma,t)$	balance outstanding per dollar of loan
Market interest rate	
r	market interest rate
	$f(r)$, density function
p	transactions costs of refinancing (mean λ)
$m(r,\gamma)$	mortgage payment
$A \equiv 1/m$	present value of a dollar annuity
k	relative interest rate, successive periods
	$f(k)$ density function
	$lnk \sim n(\mu,\sigma^2)$
z	continuous interest rate drift
ρ	geometric mean, relative interest rate
Mortgage options	
c	price of current option to prepay
$g \equiv m/\omega$	contract-market payment ratio
h	capital saving from prepayment
q	probability of prepayment
v	price, current-compound prepayment option package maximum θ
Mortgage and mortgage-backed security pricing	
P	price of mortgage or mortgage-backed security
C	cash flow, mortgage
b	proportional risk adjustment term
X	effect of prepayment on P
τ	marginal tax rate, joint federal-state
τ_f	federal marginal tax rate
τ_s	state marginal tax rate

The fixed rate mortgage is described by:

$$\omega(\alpha,\gamma) = \frac{\alpha}{1 - (1 + \alpha)^{-\gamma}}$$

$$\beta_t = \frac{1 - (1 + \alpha)^{-(\gamma - t)}}{1 - (1 + \alpha)^{-\gamma}} \qquad t = 0, \ldots, \gamma.$$

The payment per dollar of loan is ω, dependent on the interest rate α and number of payments γ. The exercise prices of the prepayment options are β_t, $t = 0, \ldots, \gamma$, the outstanding balances per dollar of loan. Time is measured in discrete units of payment dates.

The payment ω is increasing and concave in α. The exercise prices have the properties

$$\beta_t = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t = \gamma \end{cases}$$

$$\frac{\partial \beta}{\partial t} < 0, \qquad \frac{\partial^2 \beta}{\partial t^2} < 0.$$

The price of the call option depends on the relationship between the market interest rate r and the contract rate α. The borrower observes a series of contracts in the market (r,γ). The market distribution of interest rates has density function $f(r)$.

The option to prepay, with price c, is potentially valuable if $r < \alpha$. If $r < \alpha$, the saving from a prepayment is $\omega - m(r)$. The mortgage payment at the market interest rate is $m(r,\alpha) \equiv r/(1 - (1 + r)^{-\gamma})$. The present value of an annuity at rate r and term γ is $A \equiv 1/m$. The capital value of the saving from prepayment is $h = A[\omega - m(r)]$, or

$$\begin{aligned} h(r,\alpha) &= \frac{\omega(\alpha)}{m(r)} - 1 \\ &= g(r,\alpha) - 1. \end{aligned} \tag{1}$$

Here $g(r,\alpha) \equiv \omega/m(r)$ is the ratio of the payment on a mortgage at the market rate to that on the existing contract. For prepayment to reduce costs for the borrower and the call option to have value, $g(r,\alpha) > 1$, with $g_r < 0$ and $g_\alpha > 0$. Subscripts denote partial derivatives. The saving g is decreasing in the market rate r and increasing in the contract rate α.

If the prepayment option is exercised, transactions costs p are incurred.[6] The price of the prepayment option at the exercise date is

$$c_t = \begin{cases} g(r_t,\alpha) - (1 + p_t) & \text{if } r_t < \alpha \\ 0 & \text{if } r_t \geqslant \alpha \end{cases} \tag{2}$$

for $t = 0, \ldots, \gamma$[7] Prepayment with $c_t < 0$ can arise for nonfinancial reasons, such as relocation, windfalls, or family dislocation.

In the call option, differentiating totally, $dc = g_\alpha d\alpha + g_r dr - dp$. The transactions cost-rate tradeoff, holding the call price constant, is

$$\left.\frac{\partial p}{\partial r}\right|_{dc=0} = g_r = -\frac{gm_r}{m} < 0. \tag{3}$$

The tradeoff is negatively sloped, with slope the derivative of the prepayment ratio to the market rate.

3. Prepayment options

A specification of the dynamics of the call option price, in either discrete or continuous time, does not completely explain the risk of prepayment. Borrowers also have control over the probability of prepayment.

At each payment date, the borrower has a binary choice on whether or not to prepay. The probability of prepaying is q_t for the current option $c_t = g(r_t,\alpha) - (1 + p_t)$. If the borrower does not prepay, with probability $(1 - q_t)$, a compound option is acquired, to refinance at any future date $t + 1, \ldots, \gamma$.

A mortgage is not necessarily prepaid, even when $c_t > 0$, since expectations of future interest rates are embedded in the compound option. The pricing of mortgage prepayment risk requires both an option price c_t and a prepayment probability q_t. The prepayment obeys probability laws, with $q_s \geqslant 0$, $s = t, \ldots, \gamma$, and $\Sigma_{s=t}^{\gamma} q_s = 1$, viewed forward from the current payment t to maturity γ. The expected current return to the borrower from prepayment is $q_t c_t$. For the conventional callable bond, when $q_t = 1$, the expected current return is c_t. Only the option price is required, and not the prepayment probability.

The borrower has a current prepayment option priced at c_t. Options to prepay in the future have expected prices, measured as present values at current date t, of $c_{t+1}, \ldots, c_\gamma$. The probability that the borrower prepays in the immediate next period is $(1 - q_t)q_{t+1}c_{t+1}$, where c_{t+1} is the option price based on the observed cost and rate (p_{t+1}, r_{t+1}). The probability of not prepaying at t is $(1 - q_t)$, and the one period ahead return is $q_{t+1}c_{t+1}$.

Proceeding recursively, the cumulative present value of the return to prepayment at any date prior to maturity is

$$
\begin{aligned}
v_t &= q_t c_t + (1 - q_t)q_{t+1}c_{t+1} + \cdots + \prod_{s=t}^{\gamma-2}(1 - q_s)q_{\gamma-1}c_{\gamma-1} \\
&= q_t c_t + \sum_{j=t+1}^{\gamma-1} q_j c_j \prod_{s=t}^{j-1}(1 - q_s).
\end{aligned} \tag{4}
$$

The borrower rationally need not exercise any positively priced prepayment option, given expectations of future interest rates and refinancing opportunities. By not prepaying currently, a series of options to prepay in the future retains positive prices. The package of prepayment options is worth v_t at date t. This return is composed of a current prepayment option worth $q_t c_t$ and a sequence of options to prepay in the future. Proceeding backward recursively, the value of mortgage prepayment, forward for any $t = 0, \ldots, \gamma$ is

$$v_t = q_t c_t + (1 - q_t)v_{t+1}. \tag{5}$$

The compound option is summarized by v_{t+1}. Its probability of exercise is $(1 - q_t)$. At current date t, the borrower makes a decision on whether or not to exercise the prepayment option. If the decision is to prepay, the return is $q_t c_t$. This return need not be positive, for in a nonfinancial or autonomous prepayment, $c_t < 0$. Since v_{t+1} is positive, and earned in the future, it can be optimal to set the probability of prepayment $q_t < 1$ currently.

The current prepayment option c_t depends on market conditions (p_t, r_t). Market interest rates have a variation across suppliers, even at the current date. There is a static, or cross section, and dynamic, or time series interest rate risk of mortgage prepayment. The current market rate is r_t. The rational borrower uses this market rate to link the present and one period ahead packages of prepayment options, or

$$v_{t+1} = (1 + r_t)v_t. \tag{6}$$

Substituting (6) into (5) and deleting time subscripts

$$\begin{aligned} v &= \frac{qc}{q(1 + r) - r} \\ &= \frac{c}{(1 + r) - r/q}. \end{aligned} \tag{7}$$

Since qc is nonnegative, the sequence of future options has a positive price if $q(1 + r) - r > 0$. When this condition fails, the value of v is zero. When $v > 0$, v is decreasing in transactions costs p, the probability of prepayment q and the market rate r. It is increasing in the correct rate α. Defining $y \equiv r/q - 1$,

$$\frac{\partial v}{\partial p} = \frac{-q}{r - y} < 0 \qquad \frac{\partial v}{\partial r} = qg_r + \frac{q(1 - q)c}{r - y} < 0$$

$$\frac{\partial v}{\partial \alpha} = \frac{qg_\alpha}{r - y} > 0 \qquad \frac{\partial v}{\partial q} = \frac{-rc}{r - y} < 0.$$

The cost-rate tradeoff (p, r) had slope g_r where the option price c was constant. The call price does not include future prepayment, as measured in the compound option. If current prepayment occurs with less than certainty, or $q < 1$, the compound option is valuable, and $v > c$. Generally $v > c$, unless $q = 1$ when all future options to prepay are worth zero. The cost-rate tradeoff has a biased slope if the future prepayment, in the compound option, is excluded.

The current call, and the prepayment package covering the added compound option, are capital assets priced at c and v. When prepayment is certain in the current period, $q = 1$, so $1 + r - r/q = 1$. A callable bonds model without a probability of prepayment and without a compound option is applicable.

Since $v = c/(r - y)$, differentiating totally, and holding v constant

$$vdr - vdy = dc. \tag{8}$$

But $c = g(r,\alpha) - (1 + p)$, and α, the contract rate, is parametric, so $dc = g_r - dp$. With $y = (r/q) - 1$, $dy = (1/q)dr$. Substituting

$$vdr - \frac{v}{q}dr = g_r - dp$$

$$dr\left[g_r - v\left(1 - \frac{1}{q}\right)\right] = dp.$$

So

$$\left.\frac{dp}{dr}\right|_{dv=0} = g_r - v\left(1 - \frac{1}{q}\right) < 0. \tag{9}$$

The slope of the transactions costs-rate tradeoff based on the current call option alone is g_r. Augmented by including future prepayment, the slope is $g_r - v(1 - 1/q)$. The term $-v(1 - 1/q)$ measures the effect on the tradeoff of retaining the right to prepay in the future. The greater the current probability of prepayment q, the smaller the weight $(1 - q)$ on the future compound option. If $q = 1$, $1 - (1/q) = 0$, and there is no weight on future prepayment. Only in this special case are the cost-rate slopes identical.

When $q < 1$, the slope of the tradeoff is steeper, or more negative, if future prepayment is included. Failure to include options to prepay at any future date $t + 1, \ldots, \gamma$ causes the market tradeoff dp/dr to be biased downward, toward zero. The cost of refinancing is higher at any given interest rate when the compound option for future prepayment is included. The fraction of the tradeoff slope accounted for by the current call is $g_r/(r_r - v(1 - 1/q))$. The compound option accounts for the remaining $-v(1 - (1/q))/(g_r - v(1 - 1/q))$. Some of the costs charged at any interest rate pay for the current call, and the remainder for the compound option.

The present value of prepayment is v, for a switch in mortgages. In equilibrium, this return is equal to the time and transactions cost invested by the borrower. It is plausible to view this cost as constant over lenders for a specific borrower. The borrower has a constant opportunity cost of time, and resources in contacting a lender are largely fixed. The cost varies over borrowers, dependent also on family structures and borrowing constraints. With this v, the borrower determines the rate of prepayment q.

Rearranging the v equation, the probability of prepayment is

$$q = \frac{rv}{v(1 + r) - (g - (1 + p))}$$

$$= \frac{r}{1 + r - c/v}. \tag{10}$$

This probability depends on the capital value of the option package v, the market rate r, and the price of the current call option c. Prepayment depends on the market mortgage conditions (p,r) and the costs of the borrower in v. The probability of prepayment is increasing in the contract rate α, and decreasing in the market rate r, refinancing costs p, and the value of the compound option v.

The probability of prepayment has an explicit structure that can be used in the pricing of mortgages and derivative securities. Since the probability of prepayment is not unity, there is an additional risk to be included in pricing.

4. Stochastic specification

While the payment on the mortgage occurs at discrete dates, the distribution of market rates $f(r)$ is continuous. The borrower has access to a spectrum of lenders offering terms on mortgages to replace the current contract.

At each date t, the borrower confronts a distribution of lenders and loan conditions. A drawing with interest rate r_t is taken from the distribution. Consider a mortgage newly originated at rate α. In the first period after origination the market rate observed is r_1, and the relative interest rate $k_1 = r_1/\alpha$. In the second period, $k_2 = (r_2/r_1)(r_1/\alpha)$. Proceeding forward to date t

$$k_t = \prod_{s=1}^{t} \frac{r_s}{r_{s-1}} = \frac{r_t}{\alpha}. \tag{11}$$

where $r_0 \equiv \alpha$ at origination. The mortgage payment corresponding to an observed market interest rate r_t is

$$m(r_t) = \frac{r_t}{1 - (1 + r_t)^{-\gamma}}$$

$$= \frac{k_t\alpha}{1 - (1 + k_t\alpha)^{-\gamma}}$$

and $A(r_t) \equiv 1/m(r_t)$ is the present value of an annuity amortizing this loan.

The random variable k_t has range $0 \leqslant k_t < \infty$, and probability density function $f(k_t)$, with associated distribution function F. The conditional expectation of the current prepayment option price is

$$E\left(c{:}A(k_t\alpha) > \frac{1}{\omega}\right) = \int_{1/\omega}^{\infty} [\omega A - (1+p)]f(A)dA$$

$$= \omega\bar{A}(1/\omega) - (1+p)[1 - F(1/\omega)] \tag{12}$$

where $\bar{A}$ is the conditional mean annuity when the market interest rate is below that on the contract. The expected price of the call option is the conditional prepayment ratio $\omega\bar{A}(1/\omega)$ when the observed $A > 1/\omega$, less the cost $(1+p)(1 - F(1/\omega))$.

Relative interest rates are assumed to have independent and identical lognormal distributions. Then $lnk \sim n(\mu,\sigma^2)$, where μ and σ^2 are respectively the geometric mean and variance. Under this form, $lnk_t \sim n(\mu t,\sigma^2 t)$ at any date t.

Since $A \approx 1/k\alpha$ for any relative interest rate k, $lnA = -lnk - ln\alpha$. A linear combination of normally distributed variates is itself normal, so $lnA \sim n(-ln\alpha - \mu,\sigma^2)$, and at any date t, $ln\,A_t \sim n(-ln\alpha - \mu t,\sigma^2 t)$. Randomness in market rates, through lnk, translates in a single valued form to the annuities A.

The expected price of the current call option to prepay is

$$E(c) = \omega N\left(\frac{-ln\alpha - \mu + \frac{1}{2}\sigma^2 + ln(\omega)}{\sigma}\right)$$

$$- (1+p)e^{-ln\alpha-\mu}N\left(\frac{-ln\alpha - \mu - \frac{1}{2}\sigma^2 + ln(\omega)}{\sigma}\right) \tag{13}$$

where N is the cumulative density of a standard normal distribution. The price of the option to prepay depends on the parameters of the distribution of market interest rates μ, σ^2. As market interest rates r shift upward, relative rates k also rise.

The borrower retains the right to future prepayment in the mortgage package. These options have analogous price structure at current date t, for future dates $s = t+1,\ldots,\gamma$

$$E(c_s{:}t) = \omega N\left(\frac{-\rho s + \frac{1}{2}\sigma^2 s + ln(\omega)}{\sigma s^{1/2}}\right) - (1+p)e^{-\rho s}N\left(\frac{-\rho s - \frac{1}{2}\sigma^2 s + ln(\omega)}{\sigma s^{1/2}}\right) \tag{14}$$

where $\rho s \equiv ln\alpha + \mu s$ is the geometric mean of the relative interest rate at future date s. Calculation of each future prepayment option can be performed, based on expected future distributions formed at t. Given the recursion relation in v, the compound option replaces the option sequence.

The value of the mortgage package depends not only on the price of the prepayment options but on the probabilities q that they are exercised. With conventional option pricing models, the sequence c suffices, provided that the future options c_s, $s = t+1,\ldots,\gamma$ are included. The presence alone of the future prepayment options implies that optimal behavior does not require immediate exercise of the current option. The probability of prepayment must be specified. In equilibrium, the value of the options package is equal to the cost of search of the borrower. With these costs specific to the borrower, $v = \theta$, a parameter.

Implementation requires a specification of the probability of prepayment function q when $v = \theta$. The liability of prepayment for the lender depends on both c and q. When $q = 1$ the call is exercised currently with certainty. The probability of prepayment is unity, and the conventional callable bonds model operates. Then $c = \theta$, or $1 = (1 + r)/r - c/\theta r$. This suggests that θ can be priced as the best option available in the market.

Under this specification $q = r/(1 + r - c/\theta)$. This probability is not parametric, but depends on the market interest rate r, the price of the current call option c, the contract interest rate α, and search costs θ. Search costs are fixed for an individual borrower, but an expected value is used for population application.

5. Empirical implementation

The structure provides estimates of the price of the call option to prepay, the price of the future compound option, and the probability q that the current option is exercised. The probability that the mortgage is not currently prepaid is $1 - q$. The risk of mortgage lending is calculated, and can be incorporated in pricing mortgage-backed securities.

Estimates are presented for originating lenders in San Jose, California, for 1987–1988. The structure is just as applicable to mortgage-backed or derivative securities. Each lender is surveyed weekly on a fixed rate mortgage with standardized conditions. The mortgage has a term of 30 years, compounded monthly, and conforms to loan limits set for secondary market resale. The loan balance is less than \$168,700 in 1987 and \$187,600 in 1988.

For empirical implementation, the search cost, or price of the package θ, the price of the current prepayment option c, and the probability of prepayment q are required. The fixed rate contract is assumed to be based on one originated in the first week of 1987. The contract rate α varies across lenders, and the mortgage payment is $\omega(\alpha,360)$.

In each subsequent week, the borrower faces a distribution of interest rates across lenders. Over the pooled time series and cross-section data, a normal distribution is fitted to the observed relative interest rates. The relative interest rate is fitted to $lnk_t \equiv lnr_t - lnr_{t-1}$. Here r_t is the fixed rate charged at a given lender, during a given week.

Within the market area, conditions of the real estate market and institutions are held constant across borrowers. Any interest rate lower than that on the existing contract potentially puts the option to prepay in the money.

The return to search is $\theta = (g - (1 + p))/(1 + r - r/q)$, and $c = \theta$ for $q = 1$. The price selected for $\theta = \max v$ is the highest available among those lenders in the market that week. The compound option package price varies with each lender.

The mean and variance μ and σ^2 of the fitted relative interest rate distributions are obtained. From the contract rate at the lender α the current call option has mean price

$$\bar{c} = \omega N\left(\frac{-ln\alpha - \mu + \frac{1}{2}\sigma^2 + ln(\omega)}{\sigma}\right)$$

$$- (1 + \lambda)e^{-ln\alpha-\mu}N\left(\frac{-ln\alpha - \mu - \frac{1}{2}\sigma^2 + ln(\omega)}{\sigma}\right) \tag{15}$$

where λ is the mean refinancing cost. The contract rate α is specific to the lender, so there is a variation in the expected current prepayment price $\bar{c}$. The prices $\bar{c}$ and θ, for the current call and current-compound package, as a proportion of a dollar of loan borrowed are reported in table 3. Also in the table is the probability of prepayment.

The probability of prepayment depends on the contract payment ω and interest rate α at the lender. The parameters μ and σ are obtained from the relative interest rate distribution.

The estimated prices of the current call and package options are sample means for each lender. The sample means and standard deviations over all lenders are reported at the end of the table. For lender B, the mean call option to prepay has a price of 2.2 percent, expressed as a percentage of the outstanding loan balance. The mean call option to prepay has a price of 4.5 percent of the outstanding loan, with a standard deviation of 3.4 percent.

Table 3. Prepayment: Option prices and probability (prices $\bar{c}$ and θ proportions per dollar of loan)

Lender	Current Call Option $\bar{c}$	Package Option θ	Prepayment Probability q
A	0.010	0.037	0.107
B	0.022	0.042	0.156
C	0.046	0.069	0.212
D	0.034	0.046	0.255
E	0.075	0.108	0.233
F	0.094	0.114	0.354
G	0.010	0.041	0.103
H	0.082	0.110	0.270
I	0.010	0.034	0.110
J	0.034	0.066	0.155
K	0.027	0.065	0.131
L	0.058	0.085	0.223
M	0.010	0.029	0.118
Mean	0.045	0.070	0.186
Std. deviation	0.034	0.033	0.075

The package option θ is in the next column. Only in the limiting case of immediate current exercise is $c = \theta$. Otherwise, the ability to retain the prepayment option makes θ exceed c. The results confirm this differential, with the package option being worth on average 7.0 percent of the loan balance, with a standard deviation of 3.3 percent.

The probability of prepayment ranges between 10.3 percent and 35.4 percent, with a mean of 18.6 percent, and a standard deviation of 3.5 percent over one year. The probability is obtained endogenously, dependent on market and contract interest rates, and the prices of the call and package options at each lender. To put the estimates in perspective, Green and Shoven (1986) estimate the probability of prepayment in the first year at 6 percent. After the first year, the probability during years 2 and 18 ranges between 9 percent and 33 percent, similar to the estimates in table 3. No systematic pattern of prepayment over the life table emerges, except in the first year.

The results in table 3 indicate variation in probability of prepayment by lender. The variation in risk could arise from selection of clientele, qualification standards, and screening procedures. The risk depends on the weights on the call and package options.

These comparisons are made in table 4. The estimates are reported as carrying charges that would appear on a full information income statement. In the first column is the probability that the mortgage is not currently prepaid. In the second column is the cost of the liability for current prepayment, or qc. This cost has a mean of 0.93 percent. Approximately 100 basis points of the interest rate charged on a fixed rate mortgage pay for the current cost of prepayment.

Table 4. Expenses of holding a mortgage (cols. 3–5 in cents per dollar of loan)

Lender	Probability: Not Prepaid $(1-q)$	Value of Current Call qc	Expense: Current Call rqc	Expense: Compound $r\theta(1+r)(1-q)$
A	0.8932	0.1128	0.0094	0.3096
B	0.8437	0.3416	0.0303	0.3371
C	0.7872	0.9807	0.0881	0.5323
D	0.7450	0.8715	0.0771	0.3311
E	0.7673	1.7438	0.1620	0.8423
F	0.6459	3.3422	0.3170	0.7659
G	0.8967	0.1069	0.0091	0.3444
H	0.7299	2.2216	0.2079	0.8232
I	0.8899	0.1088	0.0097	0.2834
J	0.8450	0.5302	0.0468	0.5388
K	0.8694	0.3517	0.0309	0.5410
L	0.7767	1.2926	0.1181	0.6574
M	0.8820	0.1217	0.0104	0.2396
Mean	0.8132	0.9321	0.0859	0.5036
Std. deviation	0.8132	0.9547	0.0829	0.4634

The probability of survival and the cost of the call option liability qc are expressed as pure numbers. The carrying charges on the current call option rqc and on the package option $r\theta(1 + r)(1 - q)$ are expressed in cents per dollar of original loan. The carrying charge for lender A on the package option is 0.3096 cents per year, or 0.3096 percent.

When a mortgage is prepaid, the capital loss is a cost on the income statement. As a flow cost, the carrying charges on the options package for the lender satisfy $r\theta = rqc + r(1 - q)\theta(1 + r)$. The separate components, for the current call, and the current-compound package, are indicated in the third and fourth columns.

The carrying charge on the compound option ranges from a low of 0.2396 percent, to a high of 0.8423 percent, with a mean of 0.5036 percent. This implies that approximately 50 basis points of the fixed interest rate services future options to prepay. This allocation represents the mean, and there is a dispersion across lenders.

The mean value of the current call option to prepay is 0.93 percent. The carrying charge on the package option is 0.50 percent at the sample mean. The total cost of the current and compound options is 1.43 percent. About 150 basis points of the interest rate charged on a fixed rate mortgage is the cost to a risk-neutral lender of servicing the prepayment options. The call option cost measures the loss if the mortgage is refinanced. The charge on the compound option is the cost of the liability for future prepayment, as an expense in the current period.

6. Implications for mortgage markets

Apart from generating an endogenous probability of prepayment, the results have implications for public policy and mortgage markets. The probability of prepayment depends on interest rates, the prices of the call options, and the distribution of rates. Consequently, the effects of monetary policy, or of a lender attempting to increase market share by lowering rates unmatched by competitors, can be examined.

6.1. Pricing of mortgages and derivative securities

The probability of prepayment enters the fundamental equation of mortgage pricing. In this pricing structure for mortgages and their derivative securities, the market interest rate r is assumed to follow a continuous Markov process given by the stochastic differential equation $dr = \mu(r)dt + \sigma(r)dz$, where $\mu(r)$ is a mean drift, and dz is a Wiener process with $E(dz) = 0$ and $\sigma(r) = \sigma\sqrt{r}$, a constant. The price of the mortgage P is obtained from

$$\tfrac{1}{2}\sigma(r)^2 P_{rr} + [\mu(r) - b(r)\sigma(r)]P_r - P_t - rP + C(t) + q[\beta(t) - P] = 0 \qquad (16)$$

under appropriate hedging assumptions, where $C(t)$ denotes the cash flow at time t, subscripts denote partial derivatives, and $b(r)$ is the proportional risk adjustment term.

The cost of prepayment in this pricing equation is $X \equiv q[\beta(t) - P]$, or

$$X = \frac{r}{1 + r - c/\theta}[\beta(t) - P]. \tag{17}$$

The effect of the call option price on the probability of prepayment, and on the price of the mortgage P can be determined, holding the remaining components of the pricing equation constant.

At lender A, $c = 0.010$, and $\theta = 0.037$, so $c/\theta = 0.27$. For a 12 percent mortgage with once annual compounding, the annual probability of prepayment is $q = 0.12/(1.12 - 0.27) = 0.14$. The effect on the add-on term X is $0.14P$. If the price of the current call option at lender A increases to 0.02, doubling, then $q = 0.2$. The change in the mortgage prepayment component is a reduction of $0.06P$, or by 6 percent of its level, for a doubling in the price of the call option.

This is an example of how an underlying structure of call options can be used to price the probability of prepayment, mortgages, and their derivative securities. Variation in the return to arbitrage and the moments of the relative interest rate distribution can be similarly analyzed.

6.2. *Taxes, prepayment probabilities, and mortgage options*

Funds borrowed to purchase first and second owner-occupied dwellings have interest deducible against any type of income.[8] The borrower on a fixed rate mortgage is obtaining a loan to finance a property purchase, and a financial package containing a current call and a compound option. The purchase price is partly in the interst premium that the borrower makes over a loan not offering a comparable package, such as an adjustable rate mortgage. The interest premium, although for a financial rather than a strict real estate contract, is fully deductible, and thus a tax expenditure. Mortgage options are subsidized relative to other options contracts.

There are two costs of the options package. These are the capital loss associated with current refinancing, or qc, and the carrying charge on future prepayment of $r(1 - q)\theta(1 + r)$. The portion of the interest rate charged on a fixed rate mortgage to pay for all options to prepay is $qc + r(1 - q)\theta(1 + r)$.

For the residential borrower, the prepayment for the options package, though a financial contract, is deductible in calculating tax liability. The actual cost of the package to the borrower is $(1 - \tau)[qc + r(1 - q)\theta(1 + r)]$ where τ is the marginal tax rate. This rate is $\tau = \tau_f + \tau_s(1 - \tau_f)$, where τ_f and τ_s are the respective marginal rates of federal and state income taxation.[9]

The two principal federal tax rates since 1986 are 15 percent and 28 percnet. With a top rate in California of 9.3 percent, the combined rates are 22.9 percent and 34.6 percent. At the top rate, the tax expenditure associated wtih the current call option *rqc*, is 0.32 of 1 percent at the sample mean, or 0.346 * 0.0093. At the lower rate, the tax expenditure is 0.229 * 0.0093, or 0.21 percent, if there is no response to the policy change. This is a subsidy on the financial option packages, as opposed to the homeownership component of the interest rate. Other options purchasable in the economy do not carry the same tax expenditure. Analogously, the future prepayment options carrying charge is 0.346 * 0.5036 percent at the sample mean, or 0.17 percent, at the higher tax rate, and 0.12 percent at the lower rate. The sum of the tax expenditures, on the call and compound option is 0.49 percent annually at the higher rate, and 0.33 percent at the lower rate.

7. Conclusion

Mortgages do not have the same characteristics as other callable bonds. They are not necessarily called when options are in the money at the exercise date. They are called when options are not in the money, for autonomous reasons. The structure derives a probability of exercise for the call options to prepay. The option to prepay callable corporate or public debt has a nonstochastic decision at the exercise date: if the option is in the money, it is exercised. For a mortgage, the stochastic and probabilistic decision is determined from a model of borrower behavior.

Mortgages and derivative securities are risky partly because of their random call features. The randomness prevents accurate pricing of mortgage contracts, and secondary securities. Pricing tables rely on historical experience, rather than a behavioral model. The probability of prepayment opens the possibility of more accurate pricing, and a reduction of the risk of holding mortgages and their securities.

Acknowledgments

I am grateful to John Byrne, Russell Fuller, Mike Miles, and two referees for their comments, and to Michael Gallagher for research assistance.

Notes

1. There remain timing risks associated with foreclosure, receipt of payment, and incomplete insurance. Default risk is examined in Epperson, Kau, Keenan, and Muller (1985).
2. Callable bonds models are applied to mortgages by Kau, Epperson, Keenan, and Muller (1987) and Brennan and Schwartz (1985) with a continuous time stochastic process. The pricing equation derived by Cox, Ingersoll, and Ross (1980, 1985) underlies this structure.
3. The Salomon Brothers (1986) tables contain these prices. As an alternative, Chen and Ling (1988) derive a behavioral model of refinancing. Dunn and McConnell (1981a, 1981b) use a stochastic difference equation for the price of a mortgage backed security subject to prepayment. A Poisson process is applied to interest rates.

4. Green and Shoven (1986) examine a variable probability of prepayment. Prepayment depends on the age of the loan and a lock-in. The lock-in is positive if the contract rate is below the market rate.

5. Greek letters denote parameters in equations, and English letters variables.

6. Points and taxes are discussed in Kau and Keenan (1987). Points with the loan are traded off against the down payment. Since points are tax-deductible, borrowers rationally accept them in mortgage contracts.

7. The call price will depend on the volatility of interest rates and on the series of exercise prices β_t.

8. There is a total loan limit of $1 million under the 1986 Tax Reform Act.

9. State taxes are deductible in calculating federal tax liability. Federal taxes are not deductible in calculating state tax liability in California.

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