Exercises from Introduction to Functional Programming

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1 Exercise 5.3.4

Prove the laws:

$$\begin{aligned} \mathit{take}\ m\ (\mathit{drop}\ n\ xs) &= \mathit{drop}\ n\ (\mathit{take}\ (m+n)\ xs) \\ \mathit{drop}\ m\ (\mathit{drop}\ n\ xs) &= \mathit{drop}\ (m+n)\ xs \end{aligned}$$

for every natural number m and n and every finite list xs.

Proof $\mathit{take}\ m\ (\mathit{drop}\ n\ xs) = \mathit{drop}\ n\ (\mathit{take}\ (m+n)\ xs)$ By induction on xs:

- Case []

$$\begin{aligned} & \mathit{take}\ m\ (\mathit{drop}\ n\ []) & \\ =\ & \mathit{take}\ m\ [] & \text{by } \mathit{drop}.2 \\ =\ & [] & \text{by } \mathit{take}.2 \\ =\ & \mathit{take}\ (m+n)\ [] & \text{by } \mathit{take}.2 \\ =\ & \mathit{drop}\ n\ (\mathit{take}(m+n)\ []) & \text{by } \mathit{drop}.2 \end{aligned}$$

- Case $(x : xs)$. By induction on n:
  - Case 0

$$\begin{aligned} & \mathit{take}\ m\ (\mathit{drop}\ 0\ (x:xs)) & \\ =\ & \mathit{take}\ m\ (x:xs) & \text{by } \mathit{drop}.1 \\ =\ & \mathit{take}\ (m+0)\ (x:xs) & \text{0 right-neutral w.r.t } + \\ =\ & \mathit{drop}\ 0\ (\mathit{take}\ (m+0)\ (x:xs)) & \text{by } \mathit{drop}.1 \end{aligned}$$

  - Case $(n+1)$

$$\begin{aligned} & \mathit{take}\ m\ (\mathit{drop}\ (n+1)\ (x:xs)) & \\ =\ & \mathit{drop}\ (n+1)\ (\mathit{take}\ (m+(n+1))\ (x:xs)) & \text{by I.H.} \end{aligned}$$

Proof $\mathit{drop}\ m\ (\mathit{drop}\ n\ xs) = \mathit{drop}\ (m+n)\ xs$ By induction on xs:

- Case []:

$$\begin{aligned} & \mathit{drop}\ m\ (\mathit{drop}\ n\ []) & \\ =\ & \mathit{drop}\ m\ [] & \text{by } \mathit{drop}.2 \\ =\ & [] & \text{by } \mathit{drop}.2 \\ =\ & \mathit{drop}(m+n)[] & \text{by } \mathit{drop}.2 \end{aligned}$$

- Case $(x : xs)$, by induction on n:
  - Case 0:

$$\begin{aligned} & \mathit{drop}\ m\ (\mathit{drop}\ 0\ (x:xs)) & \\ =\ & \mathit{drop}\ m\ (x:xs) & \text{by } \mathit{drop}.1 \\ =\ & \mathit{drop}\ (m+0)\ (x:xs) & \text{0 right-neutral w.r.t. } + \end{aligned}$$

  - Case $(n+1)$

$$\begin{aligned} & \mathit{drop}\ m\ (\mathit{drop}\ (n+1)\ (x:xs)) & \\ =\ & \mathit{drop}\ (m+(n+1))\ (x:xs) & \text{by I.H.} \end{aligned}$$

2 Exercise 5.3.5

Prove the laws:

$$\begin{aligned} \mathit{map}\ (f.g)\ xs &= \mathit{map}\ f\ (\mathit{map}\ g\ xs) \\ \mathit{map}\ f\ (\mathit{concat}\ xss) &= \mathit{concat}\ (\mathit{map}\ (\mathit{map}\ f)\ xss) \end{aligned}$$

for every function f and g, finite list xs, and finite list of finite lists xss.

Proof $map\ (f.g)\ xs = map\ f\ (map\ g\ xs)$ By induction on xs:

- Case []:

$$\begin{aligned} map\ (f.g)\ [] &= [] && \text{by definition of } map \\ &= map\ g\ [] && \text{by definition of } map \\ &= map\ f\ (map\ g\ []) && \text{by definition of } map \end{aligned}$$

- Case $(x : xs)$:

$$\begin{aligned} map\ (f.g)\ (x : xs) &= (f.g)(x) : map\ (f.g)\ xs && \text{by definition of } map \\ &= f(g(x)) : map\ (f.g)\ xs && \text{by definition of .} \\ &= f(g(x)) : (map\ f\ (map\ g\ xs)) && \text{by I.H} \\ &= map\ f\ (g(x) : (map\ g\ xs)) && \text{by definition of } map \\ &= map\ f\ (map\ g\ (x : xs)) && \text{by definition of } map \end{aligned}$$

3 Exercise 7.2.1

What is the value of:

$$map\ (3\times)\ [0..] = iterate\ (+3)\ 0$$

when '=' means denotational equality? What is its value when '=' means computable equality?

Solution In the case of denotational equality, we can prove that the claim holds, i.e. evaluates to $True$. Since equality is chain-complete, we use the take lemma and show that the claim holds for all natural numbers. However, a naive structural induction will fail, because the induction hypothesis will not be applicable directly. To see this, let's start with a naive approach:

$$\begin{aligned} map\ (3\times)\ [0..] = iterate\ (+3)\ 0 \text{ iff} \\ take\ n\ (map\ (3\times)\ [0..]) = take\ n\ (iterate\ (+3)\ 0) && \text{by take-lemma} \end{aligned}$$

By induction on n:

- Case 0:

$$\begin{aligned} & take\ 0\ (map\ (3\times)\ [0..]) = take\ 0\ (iterate\ (+3)\ 0) \\ \equiv\ & [] = take\ 0\ (iterate\ (+3)\ 0) && \text{by } (take.1) \\ \equiv\ & [] = [] && \text{by } (take.1) \\ \equiv\ & True && \text{by } (=) \end{aligned}$$

- Case $n + 1$

$$\begin{aligned} & take\ (n+1)\ (map\ (3\times)\ [0..]) = take\ (n+1)\ (iterate\ (+3)\ 0) \\ \equiv\ & take\ (n+1)(((3\times)\ 0) : (map\ (3\times)\ [1..]) = take\ (n+1)\ (iterate\ (+3)\ 0) && \text{by } (map) \\ \equiv\ & ((3\times)\ 0) : (take\ n(map\ (3\times)\ [1..])) = take\ (n+1)\ (iterate\ (+3)\ 0) && \text{by } (take.3) \\ \equiv\ & ((3\times)\ 0) : (take\ n(map\ (3\times)\ [1..])) = take\ (n+1)\ (0 : (iterate\ (+3)\ ((+3)\ 0))) && \text{by } (iterate) \\ \equiv\ & ((3\times)\ 0) : (take\ n(map\ (3\times)[1..])) = 0 : (take\ n\ (iterate\ (+3)\ ((+3)\ 0))) && \text{by } (take.3) \\ \equiv\ & 0 : (take\ n\ (map\ (3\times)\ [1..])) = 0 : (take\ n\ (iterate\ (+3)\ ((+3)\ 0))) && \text{by } (\times) \\ \equiv\ & 0 : (take\ n\ (map\ (3\times)\ [1..])) = 0 : (take\ n\ (iterate\ (+3)\ 3)) && \text{by } (+) \\ \equiv\ & (take\ n\ (map\ (3\times)\ [1..])) = (take\ n\ (iterate\ (+3)\ 3)) && \text{by } (=) \end{aligned}$$

We are now stuck, becuase the I.H. is not applicable

For this reason, we show an even stronger property, namely that the claim holds, if 0 is replaced by m on on the left-hand side of the equation and by $3 \times m$ on the right hand side of the equation. The original claim then follows from the fact that $0 = 3 \times 0$.

$$\begin{aligned} map\ (3\times)\ [0..] = iterate\ (+3)\ 0 \text{ if} \\ map\ (3\times)\ [m..] = iterate\ (+3)\ (3 * m) \text{ iff} \\ take\ n\ (map\ (3\times)\ [m..]) = take\ n\ (iterate\ (+3)\ (3 \times m)) && \text{by take-lemma} \end{aligned}$$

By induction on n:

- Case 0:

$$\begin{aligned} & take\ 0\ (map\ (3\times)\ [m..]) = take\ 0\ (iterate\ (+3)\ (3 \times m)) \\ \equiv\ & [] = take\ 0\ (iterate\ (+3)\ (3 \times m)) && \text{by } (take.1) \\ \equiv\ & [] = [] && \text{by } (take.1) \\ \equiv\ & True && \text{by } (=) \end{aligned}$$

- Case $n+1$

$$
\begin{aligned}
& take\ (n+1)\ (map\ (3\times)\ [m..]) = take\ (n+1)\ (iterate\ (+3)\ (3\times m)) & \\
\equiv\ & take\ (n+1)(((3\times)\ m) : (map\ (3\times)[(m+1)..]) = take\ (n+1)\ (iterate\ (+3)\ (3\times m)) & \text{by } (map) \\
\equiv\ & ((3\times)\ m)) : (take\ n\ (map\ (3\times)[(m+1)..])) = take\ (n+1)\ (iterate\ (+3)\ (3\times m)) & \text{by } (take.1) \\
\equiv\ & (3\times m) : (take\ n\ (map\ (3\times)[(m+1)..])) = take\ (n+1)\ (iterate\ (+3)\ (3\times m)) & \text{by } (application) \\
\equiv\ & (3\times m) : (take\ n\ (map\ (3\times)[(m+1)..])) = (take\ (n+1)((3\times m) : iterate\ (+3)\ ((+3)\ (3\times m)))) & \text{by } (iterate) \\
\equiv\ & (3\times m) : (take\ n\ (map\ (3\times)[(m+1)..])) = (3\times m) : (take\ n(iterate\ (+3)\ ((+3)\ (3\times m)))) & \text{by } (take) \\
\equiv\ & take\ n\ (map\ (3\times)[(m+1)..]) = take\ n\ (iterate\ (+3)\ ((+3)\ (3\times m))) & \text{by } (=) \\
\equiv\ & take\ n\ (map\ (3\times)[(m+1)..]) = take\ n\ (iterate\ (+3)\ (3\times (m+1)))) & \\
& \text{by distributivity of } \times \text{ and } + & \\
\equiv\ & True & \text{by I.H.}
\end{aligned}
$$

In the case of computational equality, however, the value will be $\perp$. We show, that the left hand side of the equation suffers from case-exhaustion.

$$map\ (3\times)\ [0..]$$

4 Exercise 7.5.4

In Chapter 5 a proof was given that:

$$take\ n\ xs +\!\!+\ drop\ n\ xs = xs$$

for all finite lists xs. Extend the proof to cover infinite lists xs.

Proof First, notice that equality is chain-complete. It therefore suffices to show that the claim holds for all partial lists xs.

- Case $\perp$, by induction on n:
  - Case 0:

$$
\begin{aligned}
& take\ 0\ \perp +\!\!+\ drop\ 0\ \perp & \\
=\ & [] +\!\!+\ drop\ 0\ \perp & \text{by } (take.1) \\
=\ & drop\ 0\ \perp & \text{by } (+\!\!+) \\
=\ & \perp & \text{by } (drop.0)
\end{aligned}
$$

  - $Case (n+1)$:

$$
\begin{aligned}
& take\ (n+1)\ \perp +\!\!+\ drop\ (n+1)\ \perp & \\
=\ & \perp +\!\!+\ drop\ (n+1)\ \perp & \text{by } (take.0) \\
=\ & \perp +\!\!+ \perp & \text{by } (drop.0) \\
=\ & \perp & \text{by } (+\!\!+)
\end{aligned}
$$

- Case $(x : xs)$ where xs is partial, by induction on n:
  - Case 0:

$$
\begin{aligned}
& take\ 0\ (x : xs) +\!\!+\ drop\ 0\ (x : xs) & \\
=\ & [] +\!\!+\ drop\ 0\ (x : xs) & \text{by } (take.1) \\
=\ & drop\ 0\ (x : xs) & \text{by } (+\!\!+) \\
=\ & (x : xs) & \text{by } (drop.1)
\end{aligned}
$$

  - Case $(n+1)$:

$$
\begin{aligned}
& take\ (n+1)\ (x : xs) +\!\!+\ drop\ (n+1)\ (x : xs) & \\
=\ & x : take\ n\ xs +\!\!+\ drop\ (n+1)\ (x : xs) & \text{by } (take.2) \\
=\ & x : take\ n\ xs +\!\!+\ drop\ n\ xs & \text{by } (drop.2) \\
=\ & x : (take\ n\ xs +\!\!+\ drop\ n\ xs) & : \text{ associative} \\
=\ & x : xs & \text{by I.H}
\end{aligned}
$$