Focus Tutorial Chapter 4 Exercises

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1 Exercise 4.1

For which stream s do we have that $\mathbf{dom}.s = \{\}$?

Solution Let $s \in M^\omega$ be an untimed steam.
Then $\mathbf{dom}.s = \{\}$ if and only if $s = \langle\rangle$.
By contradiction, assume $\exists m \in M.s(0) = m$
then $m \in \mathbf{dom}.s$, thus $\mathbf{dom}.s \neq \{\}$
Now, let $s \in M^{\underline{\omega}}$ be a timed stream.
Then $\mathbf{dom}.s = \{\}$ if and only if $s = \langle\rangle$.
By contradiction, assume $\exists m \in M \cup \{\checkmark\}.s(0) = m$
then $m \in \mathbf{dom}.s$, thus $\mathbf{dom}.s \neq \{\}$.

2 Exercise 4.2

If $s ^\frown r = r$ for streams s and r, what can we conclude about s and r?

Solution Let $s, r \in M^\omega$ be untimed streams.

1. We observe that $s = \langle\rangle$ satsfies the proposition for every $r \in M^\omega$, becaucse $s ^\frown r = \langle\rangle ^\frown r = r$.

2. If the proposition holds and $s \in M^*$ then $\forall n \in \mathbb{N}.s^n \sqsubseteq r$. We know that $s ^\frown r \sqsubseteq r$ because $s ^\frown r = r$ and $\sqsubseteq$ is reflexive. We can generalize this observation to iterated streams of s like follows: $\forall n \in \mathbb{N}.s^n {}^\frown r \sqsubseteq r$ and therefore $\forall n \in \mathbb{N}.s^n \sqsubseteq r$

3. Furthermore, we observe that if the proposition holds, we know that $s \notin M^\infty$ and $s \neq r$ cannot hold in conjunction.

   By the definition of $^\frown$ we have $s ^\frown r = s$ for all $r \in M^\omega$ and $r \neq s$ by assumption.

4. Corollary: $s \in M^\infty \Rightarrow s = r$.

The same observations can be made for timed streams, with the analagous proves.

3 Exercise 4.3

What is the length of the stream $\langle 1,2\rangle^3$?

Solution $\#\langle 1,2\rangle^3 = \#(\langle 1,2\rangle^\frown\langle 1,2\rangle^\frown\langle 1,2\rangle) = \#\langle 1,2,1,2,1,2\rangle = 6$.

4 Exercise 4.4

Which stream $s \in \mathbb{N}^\omega$ fulfills the equation $s = 1 \ \& \ s$?

Solution The stream $s \in \mathbb{N}^\infty$ with $\textbf{dom}.s = \{1\}$. We show that s is the only fixed point of the equation. First, observe that s is indeed a fixed point, because $s \ \& \ 1 = s$ holds. Moreover, s is the only fixed point. By contraction, assume there was another fixed point s' with $s' \neq s$. Then s' is either finite, in which case s' does not satisfy $s' = 1 \ \& \ s'$. Or s and s' disagree on some message, i.e $\exists n \in \mathbb{N}.s'(n) \neq s(n) = 1$. Again, s' does not satisfy the equation.

5 Exercise 4.5

What can we conclude about a stream s for which $\textbf{rt}.s = s$?

Solution Either $s = \langle\rangle$ or $s \in M^\infty$ and $\exists m \in M.\forall n \in \mathbb{N}.s(n) = m$. For the first case, we have $\textbf{rt}.\langle\rangle = \langle\rangle$ by definition. Now, assume $s \neq \langle\rangle$. If s was finite, then clearly $\#\textbf{rt}.s \neq \#s$, hence $\textbf{rt}.s \neq s$. Thus, if s is not empty it must be infinite. Moreover, the domain of s must be a singelton-set. Assume otherwise, i.e. $\exists m.\wedge s(m) \neq s(m{+}1)$. For every such m we than have $\textbf{rt}.s(m{+}1) \neq s(m{+}1)$, thus $\textbf{rt}.s \neq s$.

6 Exercise 4.6

Prove that the following propositions holds.
$r \sqsubseteq s \Rightarrow t^\frown r \sqsubseteq t^\frown s$

Solution By induction on the length of t. If $t = \langle\rangle$, we have $\langle\rangle^\frown r \sqsubseteq \langle\rangle^\frown s$ and thus $r \sqsubseteq s$ because $\langle\rangle$ is left-neutral with respect to $^\frown$. If $\#t = \infty$, we have $t^\frown r \sqsubseteq t^\frown s$ which simplifies to $t \sqsubseteq t$, which holds because $\sqsubseteq$ is reflexive. Now, assume that $r \sqsubseteq s \Rightarrow t^\frown r \sqsubseteq t^\frown s$ holds for streams t of some fixed length. We need to show $(m \ \& \ t)^\frown r \sqsubseteq (m \ \& \ t)^\frown s$. By definition of $\&$ we obtain $(\langle m\rangle^\frown t)^\frown r \sqsubseteq (\langle m\rangle^\frown t)^\frown s$, and because $^\frown$ is associative we further have $\langle m\rangle^\frown(t^\frown r) \sqsubseteq \langle m\rangle^\frown(t^\frown s)$. We can now apply the indution hypothesis proving the proposition.

7 Exercise 4.7

A function $f \in M^\omega \to M^\omega$ on streams is called prefix monotonic, if for all streams r and s: $r \sqsubseteq s \Rightarrow f(r) \sqsubseteq f(s)$
Show that the operator **rt** is prefix monotonic.

Solution Assume the premise $r \sqsubseteq s$ holds. We show the conclusion by induction over the lenght of s.

- Case: $s = \langle\rangle$.
  Then, $r = \langle\rangle$ because its the only prefix of s.
  Thus, $\mathbf{rt}.r = \langle\rangle \sqsubseteq \mathbf{rt}.s$ follows directly, because the empty stream is a prefix of every stream.
- Case: $s = m \mathbin{\&} t$ for some message m and stream t.
  Then, $\mathbf{rt}.s = t$. By induction over the length of r.
  - Case: $r = \langle\rangle$.
    Then $\mathbf{rt}.r = \langle\rangle \sqsubseteq \mathbf{rt}.s$, because the empty stream is a prefix of every stream.
  - Case: $r = n \mathbin{\&} u$ for some message n and stream u.
    Then, $\mathbf{rt}.r = u$. Thus, $r \sqsubseteq s \Rightarrow u \sqsubseteq t \Rightarrow \mathbf{rt}.r \sqsubseteq \mathbf{rt}.s$.

8 Exercise 4.8

Given the stream r such that $r = \langle 1, 2\rangle \frown r$

1. prove that $\#r = \infty$
2. describe the streams denoted by: $\{1\}$ Ⓢ r, $r|_5$, $\propto r$, $\mathbf{map}(r, g)$
   where $g \in \mathbb{N} \to \mathbb{N}$ is defined as $g(n) =$ if $odd(n)$ then 1 else 2 fi

Solution 1 Assume, $\#r \neq \infty$. Then, $\#(\langle 1, 2\rangle \frown r) = 2 + r \neq r$. Thus, $r \neq \langle 1, 2\rangle \frown r$.

Solution 2

- $\{1\}$ Ⓢ r is a possibly infinite stream with $\mathbf{rng}.s = \{1\}$.
- $r|_5$ is a stream with $\#r = 5$ and $r|_5 \sqsubseteq r$.
- $\propto r$ is a stream with $\mathbf{rng}.(\propto r) = \mathbf{rng}.r$, $\mathbf{dom}.(\propto r) = \mathbf{dom}.r$ and $\forall m \in \mathbb{N}.m < \#r \Rightarrow s(m) \neq s(m+1)$.
- $\mathbf{map}(r, g)$ is a stream with $\forall n \in (\mathbf{dom}.r).\mathbf{map}(r, g)(n) = g(n)$ and $\#r = \#\mathbf{map}(r, g)$. In natural language: if m is the message of r at position n, then the stream $\mathbf{map}(r, g)$ has a corresponding message at position n, that is 1 if m is odd and otherwise 2. Moreover, both streams have the same length.

9 Exercise 4.9

Given the timed stream r such that $r = \langle 1, \checkmark, 2\rangle ^\frown r$
describe the elements denoty by: $\overline{r}, r\downarrow_5$, $\mathbf{tm}(r,5)$, $\mathbf{ts}(r)$

Solution

1. $\overline{r}$ is the unique solution to the equation: $r' = \langle 1, 2\rangle ^\frown r'$. In natural language: $\overline{r}$ is the untimed stream, that is like r but with ticks $\checkmark$ removed, or in otherwords: r alternates forever between messages 1 and 2.
2. $r\downarrow_5$ is $\langle 1, 5, \checkmark, 1, 5, \checkmark, 1, 5, \checkmark, 1, 5, \checkmark, 1, 5, \checkmark\rangle$
3. $\mathbf{tm}(r,5)$ is 3
4. $\mathbf{ts}(r)$ is $False$ because r is not time-synchrnonous.

10 Exercise 4.10

Given the stream r such that $r = \checkmark \ \& \ r$
prove that $\overline{r} = \langle\rangle$

Solution By contradiction, assume that $\overline{r} \neq \langle\rangle$. Then $\overline{r}(0) = m$ for some $m \in \mathbf{rng}.r$ with $m \neq \checkmark$. But then, the assumption $r = \checkmark \ \& \ r$ is violated.

11 Exercise 4.11

Which streams $r \in M^{\underline{\infty}}$ fulfill the equation
$\overline{r} = \overline{\checkmark \ \& \ r}$

Solution Every stream r fulfills the equation. $\overline{\checkmark \ \& \ r} = M \circledS (\checkmark \ \& \ r) = M \circledS (\langle\checkmark\rangle ^\frown r) = (M \circledS \langle\checkmark\rangle)^\frown(M \circledS r) = \langle\rangle ^\frown(M \circledS r) = M \circledS r = \overline{r}$.