

Syntax and Semantics of Programming Languages - Chapter 10: Domain Theory Exercises

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Exercise 10.2.6

Prove that the least upper bound of an ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ of a domain D is unique.

Solution Let y and z be two arbitrary but fixed least upper bounds of some ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$. We show that $y = z$. By the definition of least upper bound, we have that $y \subseteq m$ and $z \subseteq m$ for each upper bound m of $\{x_i \mid i \geq 1\}$. In particular, we have that $y \subseteq z$ and $z \subseteq y$. Thus $y = z$.

Exercise 10.2.10

Prove the following property:

A function f in $\text{Fun}(A, B)$ is continuous if and only if both of the following conditions hold.

- a) f is monotonic
- b) For any ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ in A , $f(\text{lub}_A\{x_i \mid i \geq 1\}) \subseteq \text{lub}_B\{f(x_i) \mid i \geq 1\}$.

Solution Assume f is continuous. This implies, f is monotonic. By definition of continuous, we further have, that: $f(\text{lub}_A\{x_i \mid i \geq 1\}) = \text{lub}_B\{f(x_i) \mid i \geq 1\}$ for any ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ in A . In particular we have $f(\text{lub}_A\{x_i \mid i \geq 1\}) \subseteq \text{lub}_B\{f(x_i) \mid i \geq 1\}$

Now, assume f is monotonic and for any ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ in A , $f(\text{lub}_A\{x_i \mid i \geq 1\}) \subseteq \text{lub}_B\{f(x_i) \mid i \geq 1\}$. It remains to show, that for every ascending chain $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ in A the following property holds: $\text{lub}_B\{f(x_i) \mid i \geq 1\} \subseteq f(\text{lub}_A\{x_i \mid i \geq 1\})$.

Sorry.

Exercise 10.2.12

First inS : We have to consider two cases: $inS : A \rightarrow A + B$ and $inS : B \rightarrow A + B$.

- Case $inS : A \rightarrow A + B$:

Let $a_1 \subseteq a_2 \subseteq a_3 \subseteq \dots$ be an ascending chain in A . We then have:

$$\begin{aligned} & lub\{inS(a_i) \mid i \geq 1\} \\ &= lub\{\langle a_i, 1 \rangle \mid i \geq 1\} \\ &= \langle lub\{a_i \mid i \geq 1\}, 1 \rangle \\ &= inS(lub\{a_i \mid i \geq 1\}) \end{aligned}$$

- Case $inS : B \rightarrow A + B$: analogous to previous case.

First $outA$: Let $x_1 \subseteq x_2 \subseteq x_3 \subseteq \dots$ be an ascending chain in $A + B$. There are three cases to consider: Either $x_i = \perp$ for all $i \geq 1$. Or there is some index k with $1 \leq k$, such that for all $i \geq k$ we have $x_i \in A$ or $x_i \in B$. This is because elements of A are not related to elements of B via \subseteq and vice versa.

- Case $x_i = \perp$ for all $i \geq 1$:

$$\begin{aligned} & lub\{outA(x_i) \mid i \geq 1\} \\ &= lub\{outA(\perp) \mid i \geq 1\} \\ &= lub\{\perp \mid i \geq 1\} \\ &= \perp \\ &= outA(\perp) \\ &= outA(lub\{\perp \mid i \geq 1\}) \\ &= outA(lub\{x_i \mid i \geq 1\}) \end{aligned}$$

- Case $\exists k. 1 \leq k$, s.t. $\forall i. i \geq k. x_i \in B$

$$\begin{aligned} & lub\{outA(x_i) \mid i \geq 1\} \\ &= lub\{\perp \mid i \geq 1\} \\ &= \perp \\ &= outA(lub\{x_i \mid i \geq k\}) \\ &= outA(lub\{x_i \mid i \geq 1\}) \end{aligned}$$

- Case $\exists k. 1 \leq k$, s.t. $\forall i. i \geq k. x_i \in A$

$$\begin{aligned} & lub\{outA(x_i) \mid i \geq 1\} \\ &= lub\{outA(x_i) \mid i \geq k\} \\ &= lub\{x_i \mid i \geq k\} \\ &= outA(lub\{x_i \mid i \geq k\}) \\ &= outA(lub\{x_i \mid i \geq 1\}) \end{aligned}$$

Exercise 10.2.13

Prove that *head* and *tail* are continuous functions.

Solution All the functions used in the definitions of *head* and *tail* are continuous and function composition preserves continuity, hence *head* and *tail* are continuous.

Exercise 10.2.15

Let $N = \{\perp, 0, 1, 2, 3, \dots\}$ be the elementary domain of natural numbers. A function $f : N \rightarrow N$ is called **strict** if $f(\perp) = \perp$. Consider a function $add1 : N \rightarrow N$ defined by $add1(n) = n + 1$ for all $n \in N$ with $n \neq \perp$. Prove that if $add1$ is monotonic, it must also be strict.

Solution Suppose f is monotonic, but not strict.
Then $f(\perp) = n$ for some $n \in N$.
Then $f(\perp) \subseteq f(n)$, because $\perp \subseteq n$ and f monotonic.
But $f(\perp) = n \not\subseteq n + 1 = f(n)$, a contraction!

Exercise 10.3.8

Consider the following functional defined on functions over the natural numbers:

$$G : (N \rightarrow N) \rightarrow (N \rightarrow N)$$
$$G = \lambda g. \lambda n. \text{ if } n = 0 \text{ then } 2 \text{ else } g(n)$$

- Give and justify a recursive definition that corresponds to this functional - that is, an operational definition of a function that will be a fixed point of G .
- Define four different functions, g_0, g_1, g_2, g_3 , that are fixed points of the functional G , including the least fixed point, g_0 . Carefully prove that g_0 and g_1 are fixed points of G .
- Draw a diagram showing the relationship "is less defined than or equal" between these four functions.
- Informally describe the operational behavior of the recursive definition in part a). Which of the four fixed-point functions has the closest behavior to the operational view?

Solution a Using fixed-point construction, we obtain:

$$\begin{aligned}
 G^0(\perp) &= \perp \\
 G^1(\perp) &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } \perp \\
 G^2(\perp) &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } (\lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } \perp)\perp \\
 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } (\text{ if } \perp = 0 \text{ then } 2 \text{ else } \perp) \\
 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } \perp \\
 &= G^1(\perp)
 \end{aligned}$$

Hence, $G^1(\perp) = (\lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } \perp)$ is a fixed point of G . The operational definition is $g = \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } g(n)$

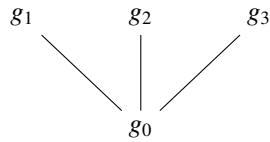
Solution b

$$\begin{aligned}
 g_0 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } \perp \\
 g_1 &= \lambda. 2 \\
 g_2 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } 3 \\
 g_3 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } 4
 \end{aligned}$$

We have already seen, that g_0 is a fixed point of G in part a). Let's turn to g_1 :

$$\begin{aligned}
 G(g_1) &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } (\lambda. 2)\perp \\
 &= \lambda. \text{ if } n = 0 \text{ then } 2 \text{ else } 2 \\
 &= \lambda. 2 = g_1
 \end{aligned}$$

Solution c



Solution d If g is applied to 0 it will evaluate to 2. If g is applied to any other value it will not terminate. This is closest to the minimal fixed-point g_0 of G , which is undefined for values other than 0. In contrast, g_1 , g_2 and g_3 are more defined than g .

Exercise 10.3.11

Use fixed-point induction to prove the equality of the following functions in $N \rightarrow N$:

$$\begin{aligned} f(n) &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(f(n + 13)) \\ g(n) &= \text{if } n > 5 \text{ then } n - 5 \text{ else } g(n + 8) \end{aligned}$$

Solution Consider the respective functionals:

$$\begin{aligned} F f n &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(f(n + 13)) \\ G g n &= \text{if } n > 5 \text{ then } n - 5 \text{ else } g(n + 8) \end{aligned}$$

We show $(F f n = G g n)$ by fixed-point induction on f and g :

- Case \perp, \perp :

$$\begin{aligned} F \perp n &= \text{if } n > 5 \text{ then } n - 5 \text{ else } \perp(\perp(n + 13)) \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } \perp \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } \perp(n + 8) \\ &= G \perp n \end{aligned}$$

- Case f, g with $f \neq \perp \neq g$:

$$\begin{aligned} F f n &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(f(n + 13)) \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(\text{if } n + 13 > 5 \text{ then } n + 13 - 5 \text{ else } f(f(n + 13 + 13))) \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(n + 13 - 5) \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } f(n + 8) \\ &= \text{if } n > 5 \text{ then } n - 5 \text{ else } g(n + 8) \text{ by I.H.} \\ &= G g n \end{aligned}$$

Exercise 10.3.14

Let D be the set of natural numbers. Prove that the fixed-point operator

$$\begin{aligned} \text{fix} : (D \rightarrow D) \rightarrow D \text{ where} \\ \text{fix} = \lambda F. \text{lub}\{F^i(\perp) \mid i \geq 1\} \end{aligned}$$

is monotonic and continuous.

Solution We only show continuity, since it implies monotonicity. That is, we need to show, that for any ascending chain $f_0 \subseteq f_1 \subseteq f_2 \subseteq \dots$ we have $\text{fix}(\text{lub}\{f_i \mid i \geq 0\}) =$

$$\text{lub}\{\text{fix}(f_i) \mid i \geq 0\}.$$

$$\begin{aligned} & \text{fix}(\text{lub}\{f_i \mid i \geq 0\}) \\ &= \text{lub}\{(\text{lub}\{f_i \mid i \geq 0\})^j \mid j \geq 0\} && \text{by def. of } \text{fix} \\ &= \text{lub}\{\text{lub}\{f_i^j \mid i \geq 0\} \mid j \geq 0\} && \forall i. f_i \text{ continuous} \\ &= \text{lub}\{\text{lub}\{f_i^j \mid j \geq 0\} \mid i \geq 0\} && \text{lub associative} \\ &= \text{lub}\{\text{fix}(f_i) \mid i \geq 0\} && \text{by def. of } \text{fix} \end{aligned}$$

Exercise 10.3.16

Contextfree grammars can be viewed as systems of equations where the nonterminals are regarded as variables (or unknowns) over sets of strings; the solution for the start symbol yields the language to be defined. In general, such an equation system has solutions that are tuples of sets, one for each nonterminal. Such solutions can be regarded as fixed points in that when they are substituted in the right-hand side, the result is precisely the solution again. For example (using ϵ for the null string), the grammar

$$\begin{aligned} A &::= aAc \mid B \\ B &::= bB \mid C \\ C &::= \epsilon \mid C \end{aligned}$$

corresponds to the transformation on triples $\langle X, Y, Z \rangle$ of sets defined by

$$f(\langle X, Y, Z \rangle) = \langle \{a\} \bullet X \bullet \{c\} \cup Y, \{b\} \bullet Y \cup Z, \{\epsilon\} \cup Z \rangle,$$

whose fixed point $\langle A, B, C \rangle$ then satisfies the set equations

$$\begin{aligned} A &::= \{a\} \bullet A \bullet \{c\} \mid B \\ B &::= \{b\} \bullet B \mid C \\ C &::= \epsilon \mid C \end{aligned}$$

for appropriate $A, B, C \subseteq \{a, b, c\}^*$. For instance, the equations above are satisfied by the sets $A = \{a^n b^* c^n \mid n \geq 0\}$, $B = b^*$, $C = \{\epsilon\}$. Show that the equation system corresponding to the grammar above has more than one possible solution so that simply seeking an arbitrary solution is insufficient for formal language purposes. However, the *least* fixed point solution provides exactly the language normally defined by the grammar. Illustrate how the first few steps of the ascending chain in the fixed-point construction lead to the desired language elements for the grammar above, and discuss the connection with derivations in the grammar.

Note: We have the natural partial order for tuples of sets where

$$\langle S_1, \dots, S_k \rangle \subseteq \langle T_1, \dots, T_k \rangle \text{ if } S_i \subseteq T_i \text{ for all } i, 1 \leq i \leq k$$

Solution We show the first part by providing just another fixed point of f . Let $\Sigma = \{a, b, c\}$ and $P = \langle \Sigma^*, \Sigma^*, \Sigma^* \rangle$. Then P is a fixed point of f :

$$f(\langle \Sigma^*, \Sigma^*, \Sigma^* \rangle) = \langle \{a\} \bullet \Sigma^* \bullet \{c\} \cup \Sigma^*, \{b\} \bullet \Sigma^* \cup \Sigma^*, \{\epsilon\} \cup \Sigma^* \rangle = \langle \Sigma^*, \Sigma^*, \Sigma^* \rangle$$

Now, let's observe the first few steps of the fixed-point construction.

	X	Y	Z
f^0	\perp	\perp	\perp
f^1	\perp	\perp	$\{\epsilon\} \cup \perp$
f^2	\perp	$\{\epsilon\} \cup \perp$	$\{\epsilon\} \cup \perp$
f^3	$\{\epsilon\} \cup \perp$	$\{\epsilon, b\} \cup \perp$	$\{\epsilon\} \cup \perp$
f^4	$\{\epsilon, b, ac\} \cup \perp$	$\{\epsilon, bb\} \cup \perp$	$\{\epsilon\} \cup \perp$
f^5	$\{\epsilon, bb, abc, aacc\} \cup \perp$	$\{\epsilon, bbb\} \cup \perp$	$\{\epsilon\} \cup \perp$
f^6	$\{\epsilon, bbb, abbc, aabcc, aaacc\} \cup \perp$	$\{\epsilon, bbbb\} \cup \perp$	$\{\epsilon\} \cup \perp$

In general, we observe that $f^i(\langle \perp, \perp, \perp \rangle) = \langle \{w \mid A \Rightarrow^i w\} \cup \perp, \{w \mid B \Rightarrow^i w\} \cup \perp, \{w \mid C \Rightarrow^i w\} \cup \perp \rangle$. Where $T \Rightarrow^i w$ means, that w can be derived with exactly i steps from the nonterminal T .